Appendix D

Constrained Hamiltonian Systems, Dirac observables and the Constraint Algebra

D.0.2 Introduction

coordinates of phase space as $(q_1, \ldots, q_n)$ for the position coordinates and $(p_1, \ldots, p_n)$ for the velocity coordinates throughout this thesis we use this notation. Observables in Hamiltonian mechanics are real functions defined on phase space, $\mathbb{R}^{2n}$. Some examples of observables are position, momentum, energy, angular momentum.

The Poisson bracket of two observables f, g is defined as

$$\{f, g\} = \sum_{i=0}^{n} \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p^i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p^i} \right) \tag{D.1}$$

In a system with energy H the time evolution of an arbitrary observable f is defined by Hamilton's equation

$$\frac{df}{dt} = \{f, h\} \tag{D.2}$$

using the theory of vector fields and differential forms on $\mathbb{R}^{2n}$ there exists a two-form, ω, defined as

$$\omega((q, p), (\tilde{q}, \tilde{p})) = q\tilde{p} - p\tilde{q} \tag{D.3}$$

where (q, p) and $(\tilde{q}, \tilde{p})$ are elements of $\mathbb{R}^{2n}$. ω is referred to as the symplectic form on $\mathbb{R}^{2n}$. For a classical observable f the Hamiltonian vector field X_f is the vector field which satisfies

$$df(Y) = \omega(Y, X_f) \tag{D.4}$$

Hamiltonian mechanics is often extended by using a symplectic manifold other than $\mathbb{R}^{2n}$ as phase space.

D.1 Hamiltonian Mechanics

A review of Hamiltonian formalism is in order. to fix ideas and orient the reader before we delve into the more techniqual exposition. Constrained Hamiltonians.

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \tag{D.5}$$

$$\dot{p} = \frac{\partial \mathcal{L}}{\partial q} \tag{D.6}$$

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} + \frac{\partial \mathcal{L}}{\partial t}\delta t \tag{D.7}$$

$$\begin{aligned} \delta\mathcal{L} &= \dot{p}\delta q + p\delta\dot{q} + \frac{\partial \mathcal{L}}{\partial t}\delta t \\ \delta(p\dot{q}) &+ \dot{p}\delta\dot{q} - \dot{q}\delta p + \frac{\partial \mathcal{L}}{\partial t}\delta t \end{aligned} \tag{D.8}$$

$$\delta(p\dot{q} - \mathcal{L}) = -\dot{p}\delta q + \dot{q}\delta p - \frac{\partial \mathcal{L}}{\partial t}\delta t \tag{D.9}$$

$$\delta(p\dot{q} - \mathcal{L}) = \delta p\dot{p} + p\delta\dot{q} - \frac{\partial \mathcal{L}}{\partial q}\delta q - \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q} - \frac{\partial \mathcal{L}}{\partial t}\delta t \tag{D.10}$$

right-hand side implies $\mathcal{H}$ is a function of q, p, and t. That means that $p\dot{q} - \mathcal{L}$ can be expressed in terms of the q's and p's, independent of the velocities.

$$\delta\mathcal{H} = \frac{\partial \mathcal{H}}{\partial q}\delta q + \frac{\partial \mathcal{H}}{\partial p}\delta p + \frac{\partial \mathcal{H}}{\partial t}\delta t \tag{D.11}$$

compare the terms in (D.9) and (D.11)

$$\frac{\partial \mathcal{H}}{\partial q} = -\dot{p}, \quad \frac{\partial \mathcal{H}}{\partial p} = \dot{q}, \quad \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t} \tag{D.12}$$

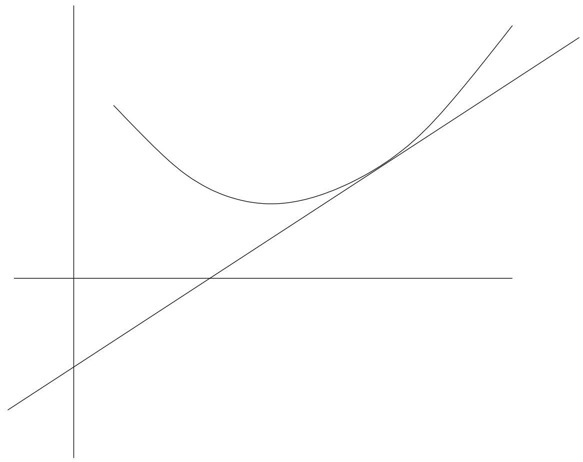


Figure D.1: Legendre transform relating Lagrangian formulation to the Hamiltonian formulation.

The PBs of a functions coordinates wil be made much use of in much of what follows, so we present them here

$$\{\xi^j \tag{D.13}$$

$$F(x,p) = px - f(x)$$

$$\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \dot{q}}\dot{q} - \mathcal{L} \tag{D.14}$$

in the simplified case $p = f'(x)$

we define the curve by the envelope of tangent

each b corresponds to one as long as $f'(x) \neq 0$.

The curve may be specified by b as a function of a

To determine b as a function of a:

What do b and a satisfy

$$\bar{y} = a\bar{x} + b \tag{D.15}$$

Such that:

(a) The line goes through $y = f(x)$

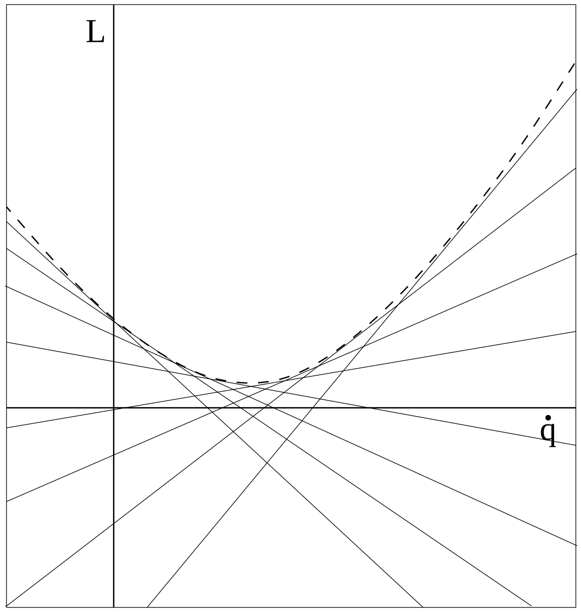


Figure D.2: Legendre transform relating Lagrangian formulation to the Hamiltonian formulation.

(b) has gradient $f'(x)$?

(b) $\rightarrow \frac{d\bar{y}}{d\bar{x}} = a = f'(x)$

$$\begin{aligned}(a) \rightarrow y &= ax + b \\ f(x) &= ax + b \\ b &= f(x) - ax\end{aligned} \tag{D.16}$$

D.1.1 Poisson Brackets

$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial q}\frac{dq}{dt} + \frac{\partial A}{\partial p}\frac{dp}{dt} + \frac{\partial A}{\partial t} \\ &= \frac{\partial A}{\partial q}\frac{\partial \mathcal{H}}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial \mathcal{H}}{\partial q} + \frac{\partial A}{\partial t} \\ &= \{A, \mathcal{H}\} + \frac{\partial A}{\partial t}\end{aligned} \tag{D.17}$$

In general

$$\{A, B\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \tag{D.18}$$

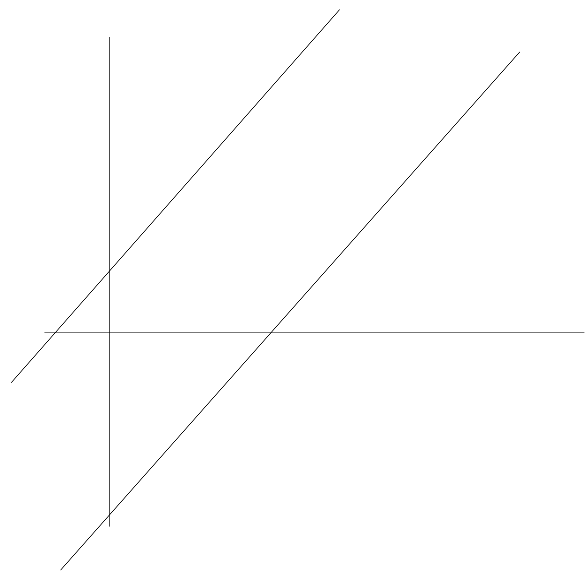


Figure D.3: We cannot solve to get p from... .

The Poisson brackets have some properties that follow from their definition:

(a) $\{f, g\}$ is antisymetric

$$\{f, g\} = -\{g, f\}, \tag{D.19}$$

(b) it is linear in each argument - that is, it is *blinear*

$$\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}, \quad \{f, g_1 + g_2\} = \{f, g_1\} + \{f, g_2\}, \tag{D.20}$$

(c) the Jacobi identity

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} \equiv 0. \tag{D.21}$$

(d) The product law

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + \{f_1, g\} f_2 \tag{D.22}$$

The first three properties antisymmetry, bilinearity and the Jacobi identity appears in the first appendix as the defining properties of a Lie algebra. The space of functions on phase space $f = f(q, p)$ form a Lie algebra under the PB. We will employ this analogy at times when studying the Hamiltonian dynamics. The Lie algebra point of view plays an important role in the transition to quantum mechanics.

see Details page 1008

D.1.2 Sympletic Geometry and Phase Space

$$\dot{q} = \partial_p \mathcal{H}, \quad \dot{p} = -\partial_q \mathcal{H} \tag{D.23}$$

$$\begin{pmatrix} \dot{q}, & \dot{p} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_q \mathcal{H} \\ \partial_p \mathcal{H} \end{pmatrix} \tag{D.24}$$

$$\dot{q}_i = \partial_{p_i} \mathcal{H}, \quad \dot{p}_i = -\partial_{q_i} \mathcal{H}, \quad \text{for } i = 1, 2, \ldots, N. \tag{D.25}$$

Lorentzian tranformations leave scalar product $a^\mu b_\mu$ invariant. The transformations that leave (D.31) invariant will be the subject of the next section and are called *canonical transformations*.

It's sort of like a scalar product. In Minkoskian spacetime the scalar product is

$$\begin{aligned} a^\mu b_\mu &= a^\mu \eta_{\mu\nu} b^\nu \\ &= (a_t \; a_x \; a_y \; a_z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b_t \\ b_x \\ b_y \\ b_z \end{pmatrix} \end{aligned} \tag{D.26}$$

In order to understand the geometry of the situation, we discuss the very simple case: the harmonic oscillator. the Hamilton is $\mathcal{H} = \frac{1}{2}(p^2 + \omega^2 q^2)$ with Poisson bracket $\{p, q\} = 1$.

$$\begin{pmatrix} \dot{q}_1, \dot{p}_2, \dot{q}_2, \dot{p}_2, \cdots, \dot{q}_N, \dot{p}_N \end{pmatrix} = \begin{pmatrix} 0 & 1 & & & \cdots & & 0 \\ -1 & 0 & & & \cdots & & 0 \\ & & 0 & 1 & & & 0 \\ & & -1 & 0 & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & & \\ & & & \cdots & & 0 & 1 \\ 0 & 0 & 0 & \cdots & & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_{q_1} \mathcal{H} \\ \partial_{p_1} \mathcal{H} \\ \partial_{q_2} \mathcal{H} \\ \partial_{p_2} \mathcal{H} \\ \vdots \\ \delta \partial_{q_N} \mathcal{H} \\ \delta \partial_{p_N} \mathcal{H} \end{pmatrix} \tag{D.27}$$

Let $\xi^\mu = (q_1, p_1; q_2, p_2; \ldots; q_N, p_N)$ $(\mu = 1, \ldots, 2N)$

$$\Omega_{\mu\nu} = \begin{pmatrix} 0 & 1 & & & \cdots & & 0 \\ -1 & 0 & & & \cdots & & 0 \\ & & 0 & 1 & & & 0 \\ & & -1 & 0 & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & & \\ & & & \cdots & & 0 & 1 \\ 0 & 0 & 0 & \cdots & & -1 & 0 \end{pmatrix} \tag{D.28}$$

and

$$\partial_\nu = (\partial_{q_1}, \partial_{p_1}; \partial_{q_2}, \partial_{p_2}; \ldots; \partial_{q_N}, \partial_{P_N}) \tag{D.29}$$

Then

$$\frac{d\xi^\mu}{dt} = \sum_{\nu=1}^{2N} \Omega^{\mu\nu} \partial_\nu \mathcal{H} \tag{D.30}$$

$$\{A, B\} = \sum_{\nu=1}^{2N} \partial^\mu A \Omega_{\mu\nu} \partial^\nu B := \partial_\mu A \partial^\mu B \tag{D.31}$$

The matrix Ω plays the role η. Lorentzian tranformations leave scalar product $a^\mu b_\mu$ invariant. The transformations that leave (D.31) invariant will be the subject of the next section and are called *canonical transformations.*

Just as space and time can be treated on an equal footing in special relativity by using Minkowskian spacetime notation, the q and p variables can be treated on an equal footing by using the ξ notation introduced above.

Let $\xi^\mu = (q_1, p_2; q_2, p_2; \ldots; q_N, p_N)$ $(\mu = 1, \ldots, 2N)$

We can use $\Omega_{\alpha\beta}$ to lower indices:

$$\xi_\mu = \Omega_{\mu\nu}\xi^\nu = (p_1, -q_1; p_2, -q_2; \ldots; p_N, -q_N) \tag{D.32}$$

a differential operator $\{\Phi, \ \}$

$$\{\Phi, \ \} = \sum_{\nu=1} \partial^\mu \Phi \Omega_{\mu\nu} \partial^\nu \tag{D.33}$$

$$\{\mathcal{H}, \ \} = \sum_{\nu=1} \partial^\mu \mathcal{H} \Omega^{\mu\nu} \partial^\nu \tag{D.34}$$

$$\Omega(\delta q_i, \delta p_i, \delta \tilde{q}_j, \delta \tilde{p}_j) = \Omega \tag{D.35}$$

$$\Omega_{\mu\nu} d\xi^\mu d\tilde{\xi}^\nu = \left(\delta q_1, \delta p_2, \delta q_2, \delta p_2, \cdots, \delta q_N, \delta p_N \right) \begin{pmatrix} 0 & 1 & & & \cdots & & 0 \\ -1 & 0 & & & \cdots & & 0 \\ & & 0 & 1 & & & 0 \\ & & -1 & 0 & & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & & \\ & & & \cdots & & 0 & 1 \\ 0 & 0 & 0 & \cdots & & -1 & 0 \end{pmatrix} \begin{pmatrix} \delta\tilde{q}_1 \\ \delta\tilde{p}_1 \\ \delta\tilde{q}_2 \\ \delta\tilde{p}_2 \\ \vdots \\ \delta\tilde{q}_N \\ \delta\tilde{p}_N \end{pmatrix}$$

$$= \delta q_1 \delta\tilde{p}_1 - \delta p_2 \delta\tilde{q}_2 + \delta q_2 \delta\tilde{p}_2 - \delta p_2 \delta\tilde{p}_2 + \cdots \delta q_N \delta p_N$$

$$= \sum \left(\delta q_1 \delta\tilde{p}_1 - \delta p_2 \delta\tilde{q}_2 \right) \tag{D.36}$$

$$\begin{aligned} \omega((\delta q^i, \delta p_i), (\delta\tilde{q}^i, \delta\tilde{p}_i)) &= \sum_{i=1}^{N} (\delta p_i \wedge \delta\tilde{q}^i - \delta\tilde{p}_i \wedge \delta q^i) \\ &= \sum_{\mu=1}^{2N} \delta\xi_\mu \wedge \delta\tilde{\xi}^\mu \end{aligned} \tag{D.37}$$

$$\Omega = \Omega_{\mu\nu} d\xi^\mu \wedge d\tilde{\xi}^\nu = \Omega(d\xi^\mu, d\tilde{\xi}^\nu) \tag{D.38}$$

More generally ω_{ij} may depend on the configuration variable q_i, either becase we are not using Cartesian coordinates or because we want to think about a non-Euclidean space.

$$\Omega_{\mu\nu} := \{,\}. \tag{D.39}$$

$$\begin{aligned} \omega^{jk}\omega_{kl} = \delta^j_l, \qquad & \omega^{ki}\omega_{lj}\delta^{il} = -\delta_{kj}, \quad \omega^{ki}\omega^{lj}\delta_{il} - \delta^{kj}, \\ & \omega_{ij} = -\omega_{ji}, \quad \omega^{ij} = -\omega^{ji}. \end{aligned} \tag{D.40}$$

Hamilton flows

if we want the a function of phase space $f(q(t), p(t); t)$ at an infintesmal time later we can do this by using the action of $(1 + t\{\mathcal{H},\ \})$ on f:

$$(1 + \Delta t\{\mathcal{H},\ \})\, f(q(t_0), p(t_0); t_0) = f(q(t_0 + \Delta t), p(t_0 + \Delta t); t_0 + \Delta t) \tag{D.41}$$

$$\begin{aligned} \left(1 + \frac{\Delta t}{2}\{\mathcal{H},\ \}\right)^2 f(q(t_0), p(t_0); t_0) &= 1 + t\{\mathcal{H}, f(q(t_0), p(t_0)\} + \frac{t}{2}\{\mathcal{H}, \{\mathcal{H}, f(q(t_0), p(t_0)\}\} \\ &= f(q(t_0 + 2\Delta t), p(t_0 + 2\Delta t); t_0 + 2\Delta t) \end{aligned} \tag{D.42}$$

$$\lim_{N\to\infty}\left(1+\frac{t}{N}\{\mathcal{H},\ \}\right)^N f(q(t_0),p(t_0);t_0) = f(q(t_0+t),p(t_0+t);t_0+t). \tag{D.43}$$

The Hamilton flow

$$[\alpha(t)](f) := \sum_{n=0}^{\infty}\frac{t^n}{n!}\{\mathcal{H},f\}_{(n)} \tag{D.44}$$

repeated Poisson bracket defined inductively $\{\mathcal{H},f\}_{(n+1)} = \{\mathcal{H},\{\mathcal{H},f\}_{(n)}\}$

Formally for $[\alpha(t)](f)$ we write

$$[\alpha(t)](f) := \exp(t\{\mathcal{H},\ \})f \tag{D.45}$$

The reader is probably more familiar with the corresponding quantum evolution *operator*

$$[\alpha\hat{(t)}](|\psi>) := \exp(it\hat{\mathcal{H}})|\psi> \tag{D.46}$$

We can consider flow generated by other functions on phase space, for example momentum p

$$[\alpha(t)](f) := \exp(t\{p,\ \})f \tag{D.47}$$

These too all called Hamilton flow even though the function isn't the Hamiltonian.

D.1.3 Canonical Transformations

Can we perform a change of variables on phase space

$$\begin{aligned} Q_i &= Q_i(q_i,p^i,t) \\ P^i &= P^i(q_i,p^i,t) \end{aligned} \tag{D.48}$$

such that

$$\dot{Q}_i = \frac{\partial\mathcal{K}}{\partial p^i}, \quad \dot{P}^i = \frac{\partial\mathcal{K}}{\partial q_i} \tag{D.49}$$

and $\mathcal{K}$ is a new Hamiltonian, that is it obeys Hamilton's principle

$$\delta \int_{t_1}^{t_2} \left(\sum_i P^i \dot{Q}_i - \mathcal{K}(Q,P,t) \right) dt = 0. \tag{D.50}$$

It is easily seen that two Lagragians $\mathcal{L}$ and $\mathcal{L}'$ by a total time derivative yield exactly the same equations of motion,

$$\begin{aligned} S'(q,\dot{q}) &= \int_{t_0,q_0}^{t_1,q_1} \mathcal{L}' = \int_{t_0,q_0}^{t_1,q_1} \left(\mathcal{L} + \frac{dF}{dt} \right) dt = S(q,\dot{q}) + [F]_{t_0,q_0}^{t_1,q_1} \\ &= S(q,\dot{q}) + Const'. \end{aligned} \tag{D.51}$$

Which variables does F depend on?

4n variables $\{q,p,Q,P\}$ 2n indices

$$\text{Type 1. } F_1(q,Q,t) \ (q,Q) \text{ are independent.} \tag{D.52}$$
$$\text{Type 2. } F_2(q,P,t) \ (q,P) \text{ are independent.} \tag{D.53}$$
$$\text{Type 3. } F_3(p,Q,t) \ (p,Q) \text{ are independent.} \tag{D.54}$$
$$\text{Type 4. } F_4(p,P,t) \ (p,P) \text{ are independent.} \tag{D.55}$$

Note q, Q may *not* be independent of $Q_i = Q_i(r,t)$ with no dependence on p.

For example, the transformation $Q = q$, $P = q + p$ in

Make

The $\mathbf{F_1(q,Q,t)}$ Generating Function

If we take F to be a function of

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \frac{dF_1}{dt}(q,Q) \tag{D.56}$$

$$\frac{dF_1}{dt}(q,Q) = \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} \tag{D.57}$$

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \sum_i \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} \tag{D.58}$$

So only two terms depend on $\dot{q}_i$ (or $\mathcal{H}$, $\mathcal{K}$) and F_1 do not depend on $\dot{q}_i$. So we may compare coefficients

$$p_i\dot{q}_i = \frac{\partial F}{\partial q_i}\dot{q}_i \text{ so } p_i = \frac{\partial F_1}{\partial q_i} \text{ (1)} \tag{D.59}$$

Now compare coefficients of Q_i:

$$P_i = -\frac{\partial F_1}{\partial Q_i} \text{ (2)} \tag{D.60}$$

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_1}{\partial t} \text{ (3)} \tag{D.61}$$

Solve (1) for $Q_i(q_i, p_i, t)$ use $Q_i(q_i, p_i, t)$ in (2) to give

$$P_i(q_i, p_i, t) \tag{D.62}$$

can either write $\mathcal{K}(Q(q,p), P(q,p), t)$ or invert to get $q(Q,P)$ and $P(q,p)$ to derive $\mathcal{K}(Q,P,t)$

The $\mathbf{F_2(q, P, t)}$ Generating Function

$$\frac{dF_2(q,P)}{dt} = \frac{\partial F_1}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial F_1}{\partial P^i}\dot{P}^i + \frac{\partial F_2}{\partial t} \tag{D.63}$$

So we cannot simply compare. We use a Legendre transform to change variables

$$F_2(q, P, t) = F_1(q, Q, t) + \sum_i P^i Q_i \tag{D.64}$$

Remember $P_i = -\frac{\partial F_1}{\partial Q_i}$

compare $p_i \frac{\partial \mathcal{L}}{\partial q_i}$

$$\mathcal{H} = p\dot{q} - \mathcal{L} \tag{D.65}$$

So replacing F_1 by $F_2 - \sum PQ$ we see:

$$\frac{dF_1}{dt} = \sum_i \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P^i}\dot{P}^i + \frac{\partial F_2}{\partial t} - \sum_i \dot{P}^i Q_i - \sum_i P^i \dot{Q}_i \tag{D.66}$$

So:

$$\sum_i p_i \dot{q}_i - \mathcal{H} = \sum_i P_i \dot{Q}_i - \mathcal{K} + \sum_i \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} - \sum_i \dot{P}^i Q_i - \sum_i P^i \dot{Q}_i \quad \text{(D.67)}$$

$$\sum_i p_i \dot{q}_i - \mathcal{H} = -\sum_i \dot{P}^i Q_i - \mathcal{K} + \sum_i \frac{\partial F_2}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_2}{\partial P_i} \dot{P}_i + \frac{\partial F_2}{\partial t} \quad \text{(D.68)}$$

So comparing terms in $\dot{q}_i$ and $\dot{P}_i$

$$\dot{q}_i : \quad p_i = \frac{\partial F_2}{\partial q_i} \rightarrow \quad \text{(D.69)}$$

$$\dot{P}_i : \quad Q_i = \frac{\partial F_2}{\partial P_i} \rightarrow \quad \text{(D.70)}$$

and $\mathcal{K} = \mathcal{H} + \frac{\partial F_2}{\partial t}$.

Now for some examples

(a)

(b) $F_1 = qQ$ Linear in both

$$p = \frac{\partial F_1}{\partial q} = Q \quad \text{(D.71)}$$

$$P = -\frac{\partial F_1}{\partial Q} = -q \quad \text{(D.72)}$$

interchanges p and q.

Now let us examine $F_2(q, P, t)$

(c) $F_2 = qP$

$$p = \frac{\partial F_2}{\partial q} = P \quad ; \quad Q = \frac{\partial F_2}{\partial P} = q \quad \text{(D.73)}$$

So this is the identity CT.

(d) $F_2 = f(q, t)P$

$f(q, t) = \lambda q, \quad \lambda$ constant:

$$Q = \lambda q \tag{D.74}$$

rescales coordinates.

D.1.4 Infinitesimal Contact Transformations

$$\begin{aligned} Q_i &= q_i + \delta q_i \\ P_i &= p_i + \delta p_i \end{aligned} \tag{D.75}$$

$$F_2 = \sum_i q_i P_i + \epsilon G(q, P) \tag{D.76}$$

$$\frac{\partial F_2}{\partial q_i} = p_i = P_i + \epsilon \frac{\partial G}{\partial q_i} \quad \delta p_i = -\epsilon \frac{\partial G}{\partial q_i} \tag{D.77}$$

Example:

$F_2 = qP + \epsilon P$

$$\delta p = -\epsilon \tag{D.78}$$

$$\delta q = \epsilon \frac{\partial G}{\partial p} = \epsilon \tag{D.79}$$

i.e. infinitesimal translation by ϵ.

change of some function $u(q, p)$ under ϵ

$$\delta u = \sum_i \left(\delta q_i \frac{\partial u}{\partial q_i} + \delta p_i \frac{\partial u}{\partial p_i} \right) \tag{D.80}$$

But

$$\delta q_i = \epsilon \frac{\partial G}{\partial p_i}, \quad \delta p^i = -\epsilon \frac{\partial G}{\partial p_i} \tag{D.81}$$

$$\delta u = \epsilon \sum_i \left(\frac{\partial G}{\partial p^i} \frac{\partial u}{\partial q_i} - \frac{\partial G}{\partial q_i} \frac{\partial u}{\partial p^i} \right) = \epsilon \{u, G\} \tag{D.82}$$

In particular $u = \mathcal{H}$

$$\delta\mathcal{H} = \epsilon\{\mathcal{H}, G\} \tag{D.83}$$

So if we have a vanishing of PB of G with $\mathcal{H}$, then $\delta\mathcal{H}$=0.

Sensible as:

$$\frac{dG}{dt} = \{G, \mathcal{H}\} + \frac{\partial G}{\partial t} \tag{D.84}$$

So if G is not a function of t then G is conserved. Symmetry related to G.

symmetry related to G.

Examples

(a) Translations

$$\delta q_i = \epsilon \frac{\partial G}{\partial p_i} \tag{D.85}$$

So if $G = p_i$ then get translation by G. Is indeed the conserved quantity if the Hamiltonian transformation i.e. translations.

(b) Rotations

$$\underline{\underline{R}}(\theta) = \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & cos\theta \end{pmatrix} \tag{D.86}$$

$$\underline{\underline{R}}(\theta) = \begin{pmatrix} 1 & \delta\theta \\ -\delta\theta & 1 \end{pmatrix} \tag{D.87}$$

So

$$\underline{\underline{R}}(\delta\theta) \cdot \underline{x} = \begin{pmatrix} 1 & \delta\theta \\ -\delta\theta & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\delta\theta \end{pmatrix} = \underline{x} - \delta\theta \underline{y} \tag{D.88}$$

$$\underline{\underline{R}}(\delta\theta) \cdot \underline{x} = \begin{pmatrix} 1 & \delta\theta \\ -\delta\theta & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y\delta\theta \\ y - x\delta\theta \end{pmatrix} = \begin{pmatrix} x + \delta x \\ y + \delta y \end{pmatrix} \tag{D.89}$$

So $x + \delta y = x + \delta\theta y$

So

$$\frac{\partial G}{\partial p_x} = y \quad \frac{\partial G}{\partial p_y} = -x \tag{D.90}$$

Which is consistent with

$$G = yp_x - xp_y = -L_z \tag{D.91}$$

(In fact we can add an arbitary function of (x, y) but this alters the p's.)

D.1.5 Noether's Theorem

We take our Hamiltonian $\mathcal{H}(q, p)$ to be invariant under an infintesimal coordinate tranformation $q_i \to {q'}_i + \epsilon f_i(q)$. Then

$$\mathcal{Q} = \sum_i p^i f_i(q) \tag{D.92}$$

is conserved.

Proof:

One has $\mathcal{H}(q_i, p^j) = \mathcal{H}({q'}_i, {p'}^j)$ by definition. ${q'}_i = q_i + \epsilon f_i(q)$ the Jacobi associated with the coordinate transformaton is

$$J_{ij} = \frac{\partial {q'}_i}{\partial q_i} = \delta_{ij} + \epsilon \frac{\partial f_i(q)}{\partial q_i} + \mathcal{O}(\epsilon^2). \tag{D.93}$$

The momentum transforms under this coordinate change as

$$p \to \sum_j p_j J^{-1}_{ji} = p^i - \epsilon - \epsilon \sum_j p_j \frac{\partial f_j(q)}{\partial q_i} + \mathcal{O}(\epsilon^2). \tag{D.94}$$

Then it follows that

$$\begin{aligned} 0 &= \mathcal{H}({q'}_i, {p'}^i) - \mathcal{H}(q_i, p^i) \\ &= \frac{\partial \mathcal{H}}{\partial q_i} \epsilon f_i(q) - \frac{\partial \mathcal{H}}{\partial p^i} \epsilon p^i \frac{\partial f_i}{\partial q} \\ &= \epsilon \left[\frac{\partial \mathcal{H}}{\partial q_i} f_i(q) - \frac{\partial \mathcal{H}}{\partial p^i} p^i \frac{\partial f_i}{\partial q} \right] \\ &= \epsilon \{\mathcal{H}, \mathcal{Q}\} = \frac{d\mathcal{Q}}{dt}, \end{aligned} \tag{D.95}$$

which shows that $\mathcal{Q}$ is conserved.

The conserved quantity $\mathcal{Q}$ is the *generator* of the transformaton under discussion.

$$\{q_i, \mathcal{Q}\} = \sum_k \left(\frac{\partial q_i}{\partial q_k} \frac{\partial \mathcal{Q}}{\partial p^k} - \frac{\partial q_i}{\partial p^k} \frac{\partial \mathcal{Q}}{\partial q_k} \right) = \sum_k \delta_{ik} f_k(q) \tag{D.96}$$

which shows $\delta q_i = \epsilon f_i(q) = \epsilon\{q_i, \mathcal{Q}\}$.

D.2 Geometry of Configuration Space and Phase Space

D.2.1 Vector Fields on Configuration Space and Phase Space

The tangent bundle TQ consists of the configuration manifold Q and the set of tangent spaces T_qQ, each attached to a point $q \in Q$. The points of TQ are of the form $(q, \dot{q})$, where $q \in Q$ and $\dot{q}$ is a vector in T_qQ. The points of phase space (q, p) are not coordinates of the tangent bundle as p is not a vector in T_qQ.

As we will see $p_\alpha = \partial L/\partial \dot{q}^\alpha$ transform as the components of a one-form, not the components of a vector field, hence p is a one-form not a vector field. As the tangent bundle TQ is forme from Q and its tangent spaces T_qQ, so the cotangent bundle, is formed from Q and and its cotangent spaces T_q^*Q. This is phase space.

D.2.2 The Lie Derivative

Recall the time derivative of a function f along the motion generated by the vector field X is the Lie derivative with respect to X,

$$\dot{f} = X(f) = \mathcal{L}_X f$$

The lie derivative generalises to to vector fields and one-forms and so on.

....

We can now use the above ideas to write the Euler-Lagrange equatoins in a coordinate-free way. Take $\theta_\mathcal{L}$ to be the one-form defined locally by

$$\theta_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} dq^\alpha. \tag{D.97}$$

Then

$$\mathcal{L}_\Delta \theta_\mathcal{L} = \left(\mathcal{L}_\Delta \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \right) dq^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} d(\mathcal{L}_\Delta q^\alpha)$$

The $\mathcal{L}_\Delta$ in both terms of this act on functions in $C^?(T\mathbb{Q})$[1], and so can be replaced by the derviative with respect to time, and then the equation becomes

$$\mathcal{L}_\Delta \theta_\mathcal{L} = \left(\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} \right) dq^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} d\dot{q}^\alpha.$$

Now we insert the Euler-Lagrange equation by rewritting the first term on the RHS. This yields

$$\mathcal{L}_\Delta \theta_\mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^\alpha} dq^\alpha + \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha} d\dot{q}^\alpha.$$

So the Euler-Lagrange equations can be put in the form

$$\mathcal{L}_\Delta \theta_\mathcal{L} = d\mathcal{L}. \tag{D.98}$$

D.2.3 Definition of Hamiltonian System

A Hamiltonian system consisits of

(a) A phase space, which is a differentiable manifold of even dimension.

(b) A closed non-degenerate two-form $\omega_{\mu\nu}(z)$ defined on phase space.

(c) A function $H(z)$ on phase space. By assumption, the time evolution of the Hamiltonian system is defined by the vector field

$$\dot{\xi}^\mu = \omega_{\mu\nu} \partial_\nu H(\xi) \tag{D.99}$$

$$\partial_{[\mu} \omega_{\nu\sigma]} = 0. \tag{D.100}$$

Then the Dirac brackets also have the defining properties of a Lie algebra

See Details page 1009.

See Details page 1010 on the Jacobi identity.

[1]Coordinates are scalar functions as they assign numbers to points of the manifold.

D.2.4 Sympletic Geometry of Phase Space

With the main ideas introduced in the previous sections the purpose of this section is to famili-rarise the reader with differential geometry notions applied sympletic geometry. There are a number of reasons for this it is used extensively throughout the literature.

$$q_V = q_V(q_U)$$
$$\dot{q}_U^j = \sum_i \left(\frac{\partial q_U^j}{\partial q_V^i} \right) \dot{q}_V^i \tag{D.101}$$

Let us see how the p's transform.

$$p_i^V = \frac{\partial \mathcal{L}}{\partial \dot{q}_V^i} = \sum_j \left\{ \left(\frac{\partial \mathcal{L}}{\partial q_U^j} \right) \left(\frac{\partial q_U^j}{\partial \dot{q}_V^i} \right) + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_U^j} \right) \left(\frac{\partial \dot{q}_U^j}{\partial \dot{q}_V^i} \right) \right\} \tag{D.102}$$

However, q_V does not depend on $\dot{q}_U$ and q_U does not depend on $\dot{q}_V$. Therefore the first term in this sum vanishes. Also, from (D.101)

$$\frac{\partial \dot{q}_U^j}{\partial \dot{q}_V^i} = \frac{\partial q_U^j}{\partial q_V^i} \tag{D.103}$$

Thus

$$p_i^V = \sum_j p_j^U \left(\frac{\partial q_U^j}{\partial q_V^i} \right) \tag{D.104}$$

T^*M Phase space

Like the metric tensor, the sympletic 2-form can be used to define a dual to vectors (1-forms) on phase space. Given a vector $\mathbf{V}$, the object $\omega_{\mathbf{V}}$ is a 1-form accepts a vector and produces a scalar. In components, the 6-N quantities form are the components of a unique 1-form associated with the vector $\mathbf{V}$, that is the equation $\omega_\beta = \Omega_{\alpha\beta}(\mathbf{V})^\alpha$ has a unique solution for the components of $\mathbf{V}$. This requires $\Omega_{\alpha\beta}$ have an inverse matrix in other words its determinate is non-zero. Ω is said to be *non-degenerate*.

$$\mathbf{V}_A = \frac{\partial A}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial A}{\partial q^i}\frac{\partial}{\partial p_i} \tag{D.105}$$

$$\mathbf{V}_{\mathcal{H}} = \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial \mathcal{H}}{\partial q^i}\frac{\partial}{\partial p_i} \tag{D.106}$$

$$[\mathbf{V}_A, \mathbf{V}_B] = -\mathbf{V}_C \tag{D.107}$$

where $C = -\{A, B\}_P$.

Since $dA/d\lambda = \{A, A\}_P = 0$, the quantity A is constant along the integral curves of $\mathbf{V}_A$

Thus the integral curves of $\mathbf{V}_A$ and $\mathbf{V}_\mathcal{H}$ mesh together to form surfaces in phase space, and both A and $\mathcal{H}$ constant over any one of these surfaces.

Constraint surface

first class constraints tangental to constraint surface $\Omega^{\mu\nu}\partial_\nu u$

$$X_\mathcal{H} := \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial}{\partial q_i} + \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial}{\partial p^i} \tag{D.108}$$

$$i_{X_\mathcal{H}}\, dq^i = \frac{\partial \mathcal{H}}{\partial p_i}\left(\frac{\partial}{\partial q^i}\, dq^i\right) = \frac{\partial \mathcal{H}}{\partial p_i} \tag{D.109}$$

$$i_{X_\mathcal{H}}\, dp_i = -\frac{\partial \mathcal{H}}{\partial q^i}\left(\frac{\partial}{\partial p_i}\, dp_i\right) = -\frac{\partial \mathcal{H}}{\partial q^i} \tag{D.110}$$

$$\omega = dq^i \wedge dp_i \tag{D.111}$$

$$\begin{aligned} i_{X_\mathcal{H}}\omega &= i_{X_\mathcal{H}}(dq^i \wedge dp_i) \\ &= (i_{X_\mathcal{H}}\, dq^i)\, dp_i - (i_{X_\mathcal{H}}\, dp_i) q^i \\ &= \frac{\partial \mathcal{H}}{\partial p_i}\, dp_i + \frac{\partial \mathcal{H}}{\partial q^i}\, dq^i \\ &= d\mathcal{H} \end{aligned} \tag{D.112}$$

$$i_{X_\mathcal{H}}\omega = d\mathcal{H} \tag{D.113}$$

Poincare one-form

$$\theta = q_i\, dp_i, \tag{D.114}$$

Consider $d\theta$

$$d\theta = d(q_i\, dp_i) = dq^i \wedge dp_i - dq_i \wedge d^2 p_i. \tag{D.115}$$

the symplectic two-form can be expressed as

$$\omega = d\theta \tag{D.116}$$

$$d\omega = d^2\theta = 0 \tag{D.117}$$

$$d\alpha = \frac{\partial \alpha_k}{\partial \xi^j} d\xi^j \wedge d\xi^k. \tag{D.118}$$

$\alpha = \alpha_k d\xi^k$. Globally:

$$i_X i_Y d\alpha = i_X d(i_Y \alpha) - i_Y d(i_X \alpha) - i_{[X,Y]}\alpha. \tag{D.119}$$

Proof:

$$\begin{aligned} i_X i_Y d\alpha &= \frac{\partial \alpha_k}{\partial \xi^j}\, i_X i_Y (d\xi^j \wedge d\xi^k) \\ &= \frac{\partial \alpha_k}{\partial \xi^j}\, (Y^j X^k - X^j Y^k) \end{aligned}$$

$$\begin{aligned} i_X d(i_Y \alpha) &= i_X d(\alpha_k Y^k) \\ &= i_X [\partial_j(\alpha_k Y^k) d\xi^j] \\ &= (Y^k \partial_j \alpha_k + \alpha_k \partial_j Y^k) X^j \end{aligned}$$

$$\begin{aligned} i_{[X,Y]}\alpha &= i_{[X,Y]}(\alpha_k d\xi^k) \\ &= [X,Y]^k \alpha_k \\ &= (X^j \partial_j Y^k - Y^j \partial_j X^k)\alpha_k \end{aligned}$$

D.2.5 Canonical Transformations

Details

we have now covered the mathematics and use this to write the EL

D.2.6 The Hamiltoian Framework: Resumé

The Fibre Bundle Struture of Hamiltoian Equations

sympletic manifold, i.e. a pair $(\Gamma, \Omega_{\alpha\beta})$, where Γ is an even dimensional manifold, and $\Omega_{\alpha\beta}$ a *sympletic form*, i.e. a 2-form which is closed

$$d\Omega_{\alpha\beta} = 0, \qquad \nabla_{[\alpha}\Omega_{\beta\gamma]} = 0 \tag{D.120}$$

and non-degenerate

$$\det \Omega_{\alpha\beta} \neq 0. \tag{D.121}$$

Given any function $f : \Gamma \to \mathbf{R}$, the *Hamiltonian vector field* is defined by $X_f^\alpha = \Omega^{\beta\alpha}\nabla_\beta f$. Given any vector field $v^\alpha \in T_x\Gamma$, we say that v^α is a *symmetry* of the sympletic manifold if it leaves the sympletic form invariant, i.e if $\mathcal{L}_v\Omega_{\alpha\beta} = 0$, in which case the diffeomorphisms generated by v^α are called *canonical transformations.*

Given any two functons $f, g : \Gamma \to \mathbf{R}$, their Poisson bracket is defined by

$$\{f, g\}; = \Omega^{\alpha\beta}\nabla_\alpha f \nabla_\beta g \equiv \mathcal{L}_{X_f} g \equiv -\mathcal{L}_{X_g} f. \tag{D.122}$$

The dynamics of a physical system is given by assigning a phase space $(\Gamma, \Omega_{\alpha\beta}, \mathcal{H})$ on which the evlolution is genrated by the Hamiltonian $\mathcal{H}$ from the intial state. Hence for any observable

D.2.7 Connection to quantum mechanics

Say we are given a wavefunction $\psi(x)$ and we wish to construct the operator $\hat{U}(\epsilon)$ which represent translations by a distance ϵ, then we will get

$$\hat{U}(\epsilon)\psi(x) = \psi(x+\epsilon) \approx \psi(x) + \epsilon\frac{\partial\psi}{\partial x}. \tag{D.123}$$

and assume we can expand

$$\hat{U} = 1 + \epsilon\delta\hat{U} \tag{D.124}$$

$$\hat{U}\psi(x) \approx \psi(x) + \epsilon\delta\hat{U}\psi(x) \tag{D.125}$$

$$\psi(x) + \epsilon\delta\hat{U}\psi(x) = \psi(x) + \epsilon \tag{D.126}$$

$$\begin{aligned}
\epsilon\delta\hat{U}\psi(x) &= \epsilon\frac{\partial\psi}{\partial x} \\
&= \frac{\epsilon}{-i\hbar}\left(-i\hbar\frac{\partial}{\partial x}\right)\psi \\
&= \frac{\epsilon}{-i\hbar}\hat{p}_x
\end{aligned} \tag{D.127}$$

So

$$\delta\hat{U} = \frac{i}{\hbar}\hat{p}_x \tag{D.128}$$

Not G classically $\sim p_x$ for translations can check for operator

D.3 Covarient Phase Space

D.3.1 Space of Solutions

The set of solutions to the equation of motion.

$$I_{01} = \int_{t_0}^{t_1} dt\mathcal{L}(q,\dot{q}) \tag{D.129}$$

Second variation

$$\frac{d}{dt}\left(\frac{\partial^2\mathcal{L}}{\partial q^a\partial q^b}\right) + \left[\frac{\partial^2\mathcal{L}}{\partial q^a\partial \dot{q}^b} - \frac{\partial^2\mathcal{L}}{\partial q^b\partial q^a}\right]u + \lambda u \tag{D.130}$$

$$\left[\frac{d}{dt}(\frac{\partial^2\mathcal{L}_0}{\partial\dot{q}_0^a\partial\dot{q}_0^b} + \frac{\partial^2\mathcal{L}_0}{\partial\dot{q}_0^a\partial q_0^b}u_0^b) - \frac{\partial^2\mathcal{L}_0}{\partial q_0^a\partial\dot{q}_0^b}\dot{u}_0^b - \frac{\partial^2\mathcal{L}_0}{\partial q_0^a\partial q_0^b}u_0^b\right]\Bigg|_{q_0^a=q_0^a(t)}^{u_0^b=u_0^b(t)} = 0. \tag{D.131}$$

The derivative of $I_0(t_2,t_1)$ along U is given by

$$\begin{aligned}
i_u dI_{01} &= \int_{t_0}^{t_1}\left(\frac{\partial\mathcal{L}}{\partial q^a}u^a + \frac{\partial\mathcal{L}}{\partial v^a}\dot{u}^a\right) \\
&= \left[\frac{\partial\mathcal{L}}{\partial v^a}\right]_{t_0}^{t_1} + \int_{t_0}^{t_1}\left(\frac{\partial\mathcal{L}}{\partial q^a}u^a + \frac{\partial\mathcal{L}}{\partial v^a}\right)u^a dt
\end{aligned} \tag{D.132}$$

Here we intrduce the insertion operator $\mathbf{i}_X := \frac{1}{q!}X\wedge u^2\cdots\wedge u^r$

Define θ_t

$$\mathbf{i}_u \theta_t = u^a(t) \frac{\partial \mathcal{L}}{\partial v^a} \tag{D.133}$$

$$\imath_u I_{01} = u^a(t_1) \frac{\partial \mathcal{L}}{\partial v^a} - u^a(t_0) \frac{\partial \mathcal{L}}{\partial v^a} \tag{D.134}$$

implies

$$dI_{01} = \theta_{t_1} - \theta_{t_0} \tag{D.135}$$

$$\omega := d\theta_t \tag{D.136}$$

ω does not depend on time t.

D.3.2 Field Theory

$$\left[\frac{d}{dt} \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^a \partial \dot{\phi}^b} + \frac{\partial^2 \mathcal{L}}{\partial \dot{\phi}^a \partial \phi^b} u^b \right) - \frac{\partial^2 \mathcal{L}}{\partial \phi^a \partial \dot{\phi}^b} \dot{u}^b - \frac{\partial^2 \mathcal{L}}{\partial \phi^a \partial \phi^b} u^b \right] \Bigg|_{\phi^a = \phi^a(t)}^{u^b = u^b(t)} = 0. \tag{D.137}$$

$$\omega(X, Y) = \frac{1}{2} \int \omega^a d\sigma_a \tag{D.138}$$

where X and Y are solutions of the linearized quations and

$$\omega^a = \frac{\partial^2 \mathcal{L}}{\partial \phi^\beta \partial \phi^\alpha_a} (X^\beta Y^\alpha - Y^\beta X^\alpha) + \frac{\partial^2 \mathcal{L}}{\partial \phi^\beta_b \partial \phi^\alpha_a} (Y^\alpha \nabla_b X^\beta - X^\alpha \nabla_b Y^\beta). \tag{D.139}$$

proof.

the derivative of $i_Y \theta$ along X is

$$\begin{aligned} & \int_\Sigma \left[Y^\alpha(\phi) \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_a} \right)_{\phi + tX} + \left(\frac{\partial \mathcal{L}}{\partial \phi^\alpha_a} \right)_\phi \frac{d}{dt} \left(Y^\alpha(\phi + tX) \right) \right]_{t=0} d\sigma_a \\ = & \int_\Sigma \left(Y^\alpha X^\beta \frac{\partial^2 \mathcal{L}}{\partial \phi^a_\alpha \partial \phi^\beta} + Y^\alpha \nabla_b X^\beta \frac{\partial^2 \mathcal{L}}{\partial \phi^\alpha_a \partial \phi^\beta_b} + \frac{\partial \mathcal{L}}{\partial \phi^\alpha_a} \partial_X Y^\alpha \right) d\sigma^a. \end{aligned} \tag{D.140}$$

Therefore,

$$2\omega(X,Y) = X(i_Y\theta) - Y(i_X\theta) - i_{[X,Y]}\theta = \int \omega^a d\sigma_a. \tag{D.141}$$

One can also check that $\nabla_a \omega^a = 0$ as a consequence of the lineralized field equation, and hence that $\omega(X,Y)$ is independent of ? provided that the linerized fields fall off fast enough at spacial infinity.

D.3.3 Hamiltonian-Jacobi Theory

The Hamilton-Jacobi formalism is a window open towards the quantum theory. Schrodinger introduced the Schrodinger equation by interpreting the Hamilton-Jacobi equation as the optical approximation of a wave equation. This means searching for an equation for a wave function ψ, solved to lowest limit i $\hbar$ by a. if S solves the Hamiltion- jacobi equation. On the basis of this idea, Schrodinger found his equaion by simply replacing each partial derivative oof S in the Hamilton-Jacobi function with ($-i\hbar$ times) a partial derivative operator. Tihs same procedure can be used in the covariant formulation of mechanics. The covariant Hamiton-Jacobi equation yields then directly the quatnum dynamical equation of the theroy.

$$\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{D.142}$$

In the last section we saw that canonical transformations were a way of picking phase space coordinates to simplify a problem.

Now, the solution to our problem is to express q and p in terms of the intial conditions q_0 and p_0 $(= q(t=0), p(t=0))$ and time t.

$$q = q(q_0, p_0, t) \tag{D.143}$$
$$p = p(q_0, p_0, t) \tag{D.144}$$

The obvious suggestion is to make $Q = q_0$ and $P = p_0$, i.e. the new variables equal the intial conditions. So the "motion" in the new coordinates is the system remaining staionary at a point $)q_0, p_0)$.

Thus Hamilton's equations in the new variables are:

$$\frac{\partial \mathcal{K}}{\partial p} = \dot{Q} = \dot{q_0} = 0 \tag{D.145}$$
$$-\frac{\partial \mathcal{K}}{\partial Q} = \dot{p} = \dot{p_0} = 0 \tag{D.146}$$

This implies that $\mathcal{K}$ is equal to an arbitary functon of t - we will *choose* the simplest case:

$$\mathcal{K} = 0 \tag{D.147}$$

If $\mathcal{K} = 0$, then the original Hamiltonian must obey :

$$\mathcal{K} = 0 = \mathcal{H} + \frac{\partial F}{\partial t} \tag{D.148}$$

where F is the generating function of the transformation. We choose a CT as type 2:

$$F_2(q, P, t) \tag{D.149}$$

such that

$$\frac{\partial F_2}{\partial q} = p, \quad \frac{\partial F_2}{\partial P} = Q \tag{D.150}$$

and conventually (and for reasons we will see later) we denote F_2 by S which is called *Hamilton's principle function.*

$$\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{D.151}$$

the Hamiltonian-Jacobi equation. It is a partial differential equation for the generating function.

Examples

(a) Free particle:

$$\mathcal{H}(q, p) = \frac{p^2}{2m} \tag{D.152}$$

$$\Rightarrow \mathcal{H}\left(q, \frac{\partial S}{\partial q}\right) = \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 \tag{D.153}$$

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0 \tag{D.154}$$

(b) Harmonic oscilator

$$\mathcal{H}(q, p) = \frac{1}{2}q^2 + p^2\omega$$

$$\Rightarrow \mathcal{H}\left(q, \frac{\partial S}{\partial q}\right) = \frac{1}{2}\left(q^2 + \left(\frac{\partial S}{\partial q}\right)^2\right)$$

$$\Rightarrow \text{ H.J. } \quad \frac{1}{2}\omega\left(q^2 + \left(\frac{\partial S}{\partial q}\right)^2\right) + \frac{\partial S}{\partial t} = 0.$$

□

$$\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{D.155}$$

(D.155) is an equation for S. It is a first order PDE for S and in general we expect $n+1$ constants of integration; n for $(q_1, \ldots, q_n)$ and one from t, $\alpha_1, \ldots, \alpha_{n+1}$. (In fact this is not the most general solution but does not matter here).

So

$$S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_{n+1}, t).$$

But because S itself enters the equation (D.155) only through its derivatives, for one of the α's say we have

$$S(q; \alpha_1, \ldots, \alpha_{n+1}, t) = S^0(q; \alpha_1, \ldots, \alpha_n, t) + \alpha_{n+1} \tag{D.156}$$

We then choose the α_i to be the new momentum p, so that;

$$Q_i = \beta_i = \frac{\partial S^0}{\partial \alpha_i}, \quad i = 1, \ldots, n \tag{D.157}$$

along with

$$p_i = \frac{\partial S^0}{\partial q_i}, \quad i = 1, \ldots, n \tag{D.158}$$

• At $t = 0$ (D.158) determines α_i in terms of $p_i(t=0)$ and $q_i(t=0)$. Note they are not necessary **just** the $q_i(t=0)$.

• Then one solves (D.157) for β_i in terms of $p_i(t=0)$ and $q_i(t=0)$ at $t=0$.

Then substituting α as a function of q_0, p_0.

• Then (D.157) may be written in terms of q, α, β so one can solve for

$$\alpha(q_0, p_0), \quad \beta(q_0, p_0)$$

$q_i(q_i^0, p_i^0, t)$ (Now using equation (D.157)) for $t \neq 0$.

• Finally use (D.158) to get

$$p_i(\alpha_i, \beta_i, t)$$

where

$$\alpha_i = \alpha_i(p_i^0, q_i^0), \quad \beta_i = \beta_i(p_i^0, q_i^0).$$

• Since we know α_i, β_i in terms of the initial conditions the problem has been solved.

D.3.4 Hamilton principal function

An important solution of the Hamilton-Jacobi equations, which gives the reason for S to be denoted by S, the Hamilton principal function. It is defined

$S = \text{action} = \int^t dt'\mathcal{L}$.

We will see that $S(q, q_0, t)$ is the generating function where the time-indepndent canonical variables (α, β) coincide with the initial data (q_0, p_0).

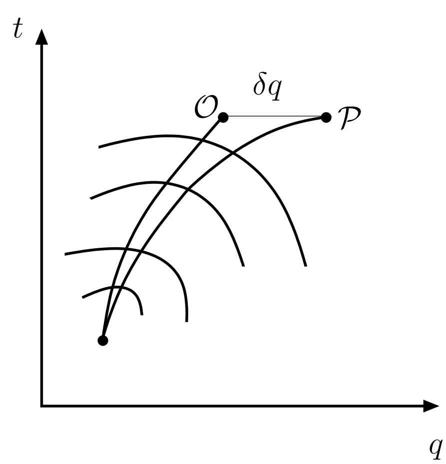


Figure D.4: Hamilton.

$$
\begin{aligned}
\delta S &= \int_{t_0}^{t_1} \mathcal{L}(q(t)+\delta q(t),\dot{q}(t)+\delta\dot{q}(t),t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t),\dot{q}(t),t)dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial q}\delta q + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta \dot{q}\right) dt \\
&= \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \delta q dt + \left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q(t)\right]_{t_0}^{t_1} \\
&= \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q
\end{aligned}
\tag{D.159}
$$

so

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q$$

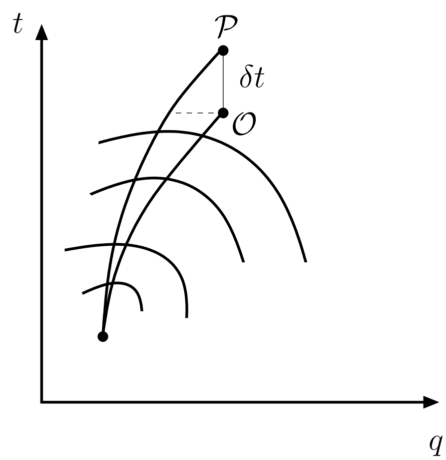


Figure D.5: Hamilton.

$$
\begin{aligned}
\delta S &= \int_{t_0}^{t_1+\delta t} \mathcal{L}(q(t)+\delta q(t),\dot{q}(t)+\delta\dot{q}(t),t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t),\dot{q}(t),t)dt \\
&= \mathcal{L}\delta t + \int_{t_0}^{t_1} \mathcal{L}(q(t)+\delta q(t),\dot{q}(t)+\delta\dot{q}(t),t)dt - \int_{t_0}^{t_1} \mathcal{L}(q(t),\dot{q}(t),t)dt \\
&= \mathcal{L}\delta t + \frac{\partial \mathcal{L}}{\partial \dot{q}}\delta q
\end{aligned}
\tag{D.160}
$$

$$\delta q = -\dot{q}\delta t$$

so

$$\delta S = \left(\mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta t$$

In general we have

$$\delta S = \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q - \left(\dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \mathcal{L} \right) \delta t \tag{D.161}$$

we have

$$\frac{\partial S}{\partial q} = \frac{\partial \mathcal{L}(q, \dot{q}, t)}{\partial \dot{q}} = p \tag{D.162}$$

$$\frac{\partial S}{\partial t} = \mathcal{L} - \dot{q} \frac{\partial \mathcal{L}}{\partial \dot{q}} = -\mathcal{H} \tag{D.163}$$

solve the first equation for $\dot{q}$ so that we have

$$\dot{q} = \dot{q}(q, \frac{\partial S}{\partial q}, t)$$

and substitute the value of $\dot{q}$ into the second giving

$$\frac{\partial S}{\partial t} + \mathcal{H} \left(\frac{\partial S}{\partial q}, q, t \right) = 0, \tag{D.164}$$

that is, Hamilton principal function solves the Hamiltonian-Jacobi equation. Noting that

$$\beta = -\frac{\partial W(q, q_0, t)}{\partial q_0} = -(-p_0).$$

We see that $W(q, q_0, t)$ is the generating function where the time-indepndent canonical variables (α, β) coincide with the initial data (q_0, p_0) is Hamilton principal function.

D.4 Solving for the Dynamics using the HJ Equation

D.4.1 1. Free particle (one-dimension)

1. Free particle (one-dimension)

$$\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$$

implies

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \left(\frac{\partial S}{\partial t}\right) = 0$$

which is a non-linear partial differential equation. We can solve by method of separation of variables using sum:

$$S(q, (\alpha), t) = \tilde{Q}(q) + T(t)$$

implies

$$\begin{aligned} \frac{\partial S}{\partial q} &= \frac{d\tilde{Q}}{dq} \equiv \tilde{Q}' \\ \frac{\partial S}{\partial t} &= \frac{dT}{dt} \equiv \dot{T}. \end{aligned} \tag{D.165}$$

So

$$\frac{1}{2m}(\tilde{Q}')^2 = -\dot{T}$$

The L.H.S. is a function of q only and the R.H.S. is a function of t only both sides must be equal to the same constant:

$$\frac{1}{2m}(\tilde{Q}')^2 = \lambda = -\dot{T}$$

λ called the separartion constant.

$$\dot{T} = -\lambda$$

implies

$$T = -\lambda t + C_1$$

where C_1 is a constant

$$(\tilde{Q}')^2 = 2m\lambda$$

implies

$$\tilde{Q}' = \pm\sqrt{2m\lambda}$$

implies

$$\tilde{Q} = \pm\sqrt{2m\lambda}q + C_2$$

where C_2 is a constant.

So

$$\begin{aligned} T &= -\lambda t + C_1 \\ \tilde{Q} &= \pm\sqrt{2m\lambda}q + C_2 \end{aligned} \tag{D.166}$$

So

$$S = \tilde{Q} + T = \pm\sqrt{2m\lambda}q - \lambda t + C_1 + C_2$$

denote

$$\begin{aligned} \lambda &= \alpha_1 \\ C_1 + C_2 &= \alpha_2 \end{aligned}$$

the two constants of integration ($\alpha_1, \ldots, \alpha_{n+1}$ in the previous jargon)

$$S = \pm\sqrt{2m\alpha_1}q - \alpha_1 t + \alpha_2 \tag{D.167}$$

Now let us follow the second procedure to get $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$

1) $t = 0$

$$\begin{aligned} p(t=0) &= \frac{\partial S}{\partial q}(t=0) \\ &= \pm\sqrt{2m\alpha_1} \end{aligned}$$

therefore

$$\alpha_1 = \frac{p_0^2}{2m}.$$

now to get β_1

$$\begin{aligned}\beta_1 &= \frac{\partial S}{\partial \alpha_1} \\ &= \frac{\partial}{\partial \alpha_1}\left(\pm\sqrt{2m\alpha_1}q - \alpha_1 t + \alpha_2\right) \\ &= \sqrt{2m}\frac{1}{2}\frac{1}{\sqrt{\alpha_1}}q - t. \end{aligned} \tag{D.168}$$

So at $t = 0$;

$$\beta_1 = \sqrt{2m}\frac{1}{2}\frac{1}{\sqrt{\alpha_1}}q_0$$

But $\alpha_1 = p_0^2/2m$ so

$$\beta_1 = \frac{1}{2}\sqrt{2m}\frac{\sqrt{2m}}{p_0}q_0$$

$$\beta_1 = \frac{mq_0}{p_0} \tag{D.169}$$

Now we use (D.168) for $t \neq 0$ such for β_1 from (D.169)

$$\frac{mq_0}{p_0} = \frac{mq}{p_0} - t$$

which implies

$$q = q_0 + \frac{p_0 t}{m}$$

i.e. moves in a straight line.

Finally to get $p(q_0, p_0, t)$ are

$$p = \frac{\partial S}{\partial q} = \sqrt{2m\alpha_1} = \sqrt{2m\frac{p_0^2}{2m}} = p_0$$

i.e. a constant as expected for a free particle.

D.4.2 2. The Harmonic oscillator (one-dimension)

The Harmonic oscillator:

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kq^2$$

implies

$$\frac{\partial S}{\partial t} + \frac{1}{2}kq^2 + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 = 0$$

We suppose that there exists Q and T such that

$$S = \tilde{Q}(q) + T(t)$$

which implies

$$-\dot{T} = \frac{1}{2}\left(kq^2 + \frac{1}{m}(\tilde{Q}')^2\right)$$

both sides must be a constant, say λ. We have

$$\begin{aligned}\dot{T} &= -\lambda \\ (\tilde{Q}')^2 &= 2m\lambda - mkq^2\end{aligned}$$

implying

$$T = -\lambda t + C_1$$

and

$$Q = \sqrt{mk}\int^q dq'\sqrt{2\lambda/k - q'^2}$$

So

$$S = \tilde{Q} + T = \sqrt{mk}\int^q dq'\sqrt{2\lambda/k - q'^2} - \lambda t + C_1$$

denote

$$
\begin{aligned}
\lambda &= \alpha_1 \\
C_1 &= \alpha_2
\end{aligned}
$$

the two constants of integration ($\alpha_1, \ldots, \alpha_{n+1}$ in the previous jargon)

$$
S = \tilde{Q} + T = \sqrt{mk} \int^q dq' \sqrt{2\alpha_1/k - q'^2} - \alpha_1 t + \alpha_2 \tag{D.170}
$$

Now let us follow the second procedure to get $q(q_0, p_0, t)$ and $p(q_0, p_0, t)$

1) $t = 0$

$$
\begin{aligned}
p(t=0) &= \frac{\partial S}{\partial q}(t=0) \\
&= \sqrt{mk}\sqrt{2\alpha_1/k - q_0^2}
\end{aligned}
$$

therefore

$$
\alpha_1 = \frac{p_0^2}{2m} + \frac{k}{2} q_0^2
$$

now to get β_1

$$
\begin{aligned}
\beta_1 &= \frac{\partial S}{\partial \alpha_1} \\
&= \sqrt{\frac{m}{k}} \int \frac{dq}{\sqrt{2\alpha_1/k - q^2}} - t \\
&= \sqrt{\frac{m}{k}} \sin^{-1}\left(\sqrt{\frac{k}{2\alpha_1}} q\right) - t
\end{aligned} \tag{D.171}
$$

So at $t = 0$;

$$
\beta_1 = \sqrt{\frac{m}{k}} \sin^{-1}\left(\sqrt{\frac{k}{2\alpha_1}} q_0\right)
$$

But $\alpha_1 = p_0^2/2m + kq_0^2/2$ so

$$
\beta_1 = \sqrt{\frac{m}{k}} \sin^{-1}\left(\sqrt{\frac{mk}{p_0^2 + mkq_0^2}} q_0\right) \tag{D.172}
$$

Now we use (D.171) for $t \neq 0$ such that for β_1 from (D.172)

$$\beta_1(q_0, p_0) = \sqrt{\frac{m}{k}} \sin^{-1}\left(\sqrt{\frac{mk}{p_0^2 + mkq_0^2}} q_0\right) = \sqrt{\frac{m}{k}} \sin^{-1}\left(\sqrt{\frac{k}{2\alpha_1}} q\right) - t$$

which implies

$$q = \sqrt{\frac{2\alpha_1}{k}} \sin\left(\sqrt{\frac{k}{m}}(t + \beta_1)\right)$$

i.e. motion is oscillatory.

Finally we get $p(q_0, p_0, t)$

$$\begin{aligned} p = \frac{\partial S}{\partial q} &= \sqrt{mk}\sqrt{(2\alpha_1/k) - q^2} \\ &= \sqrt{2\alpha_1 m} \cos\left(\sqrt{\frac{k}{m}}(t + \beta_1)\right) \end{aligned} \tag{D.173}$$

i.e. is oscillatory too.

Then the meaning of the integration constant α_1 can be seen from:

$$\mathcal{H} = \frac{1}{2}kq^2 + \frac{1}{2m}p^2 = \alpha_1.$$

While β_1 is clearly takes the form of some initial time.

D.4.3 Hamiltonian Charateristic Function

Notice that since

$$\mathcal{H}\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0$$

(principal function $S \rightarrow \psi(x, t)$
characteristic function $W \rightarrow \psi(x, E)$)

$$\frac{\partial S}{\partial t} = -E,$$

if $\mathcal{H}$ does not depend on t.

Since $\frac{\partial S}{\partial t} = -E$, we can define a new function which does not depend on t, but on $\partial S/\partial t (= -E)$ instead via a Legendre transformation:

$$W(q, \alpha, E) = S(q, \alpha, t) - \underbrace{(-E)}_{\partial S/\partial t} t \tag{D.174}$$

We see the parallel with Hamiltonian-Lagragian correspondence:

$$\begin{aligned} \mathcal{L} &\rightarrow p = \frac{\partial \mathcal{L}}{\partial \dot{q}} & S \rightarrow -E = \frac{\partial S}{\partial t} \\ \mathcal{H} &= p\dot{q} - \mathcal{L} & -W = -Et - S \\ &= \left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) \dot{q} - \mathcal{L} & = \left(\frac{\partial S}{\partial t}\right) t - S \end{aligned} \tag{D.175}$$

This is the characteristic function. The analogue, the H-J equation for W is

$$\mathcal{H}\left(q, \frac{\partial W}{\partial q}\right) - E = 0$$

or

$$\mathcal{H}\left(q, \frac{\partial W}{\partial q}\right) = E \tag{D.176}$$

and we still have

$$p = \frac{\partial W}{\partial q}, \qquad \beta = \frac{\partial W}{\partial \alpha}$$

This time dependence of W is simpler - as it only depends on t through q.

$$\frac{dW}{dt} = \frac{\partial W}{\partial q} \dot{q}$$

So

$$\begin{aligned} W(q, \alpha, E) &= \int p\dot{q} dt \\ &= \int p dq. \end{aligned} \tag{D.177}$$

Close connection quantunisation

$$\int pdq = n\hbar \tag{D.178}$$

where n is a positive integer.

D.4.4 'Derivation' of Schrodinger's Equation

De Broglie's explanation of the Bohr-Sommerfeld conditions necessitate the waves to posses no volume, and only exist alond the orbit.

Schrodinger's objective was to find a suitable partial differential equation for the hydrogen atom whose corresponding solutions have energies permissible with those given by the Bohr-Sommerfeld condition.

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 - 2m\left(E + \frac{e^2}{r}\right) = 0 \tag{D.179}$$

replace S by $K \ln \Psi$, and the equation reads

$$\left(\frac{\partial \Psi}{\partial x}\right)^2 + \left(\frac{\partial \Psi}{\partial y}\right)^2 + \left(\frac{\partial \Psi}{\partial z}\right)^2 - \frac{2m}{K^2}\left(E + \frac{e^2}{r}\right)\Psi^2 = 0 \tag{D.180}$$

A requirement for the wave theory to be in correspondence with classical dynamics in the limiting case of a wave packet is that the motion of such a packet must coincide with the motion of the corresponding point in configuration space.

Geometric optics had sufficed to acount for the facts of the theory of light because ray optics was inadwquate with the discovery of interference and diffraction. Schrodinger conjected that a similar situation holds in mechanics, where the wave are supposed to be propagated in space described in a way by his relation between dynamics and opits. The classical point of view as only an approximation to a wave packet, comparable to the correspondence of rays and waves in optics.

The ordinary d'Alembert wave equation is

$$\nabla^2\Psi - \frac{1}{u_p^2}\frac{\partial^2\Psi}{\partial t^2} = 0$$

where u_p denotes the phase velocity.

Jacobi Fields

$$\frac{d}{dt}\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}^2}\dot{\phi}\right) + \left[\frac{d}{dt}\left(\frac{\partial^2 \mathcal{L}}{\partial q \partial \dot{q}}\right) - \frac{\partial^2 \mathcal{L}}{\partial q^2}\right]\phi = 0 \tag{D.181}$$

$$q(p, 0) \tag{D.182}$$

$$q(p, 0) = a, \quad \text{for all } p \tag{D.183}$$

$$J(p, t) = \frac{\partial q(p, t)}{\partial p} \tag{D.184}$$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_l}\dot{J}_{lk}\right) + \left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial q_l} - \frac{\partial^2 \mathcal{L}}{\partial q_i \partial \dot{q}_l} -\right)\dot{J}_{lk} \\ + \left[\frac{d}{dt}\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial q_l}\right) - \frac{\partial^2 \mathcal{L}}{\partial q_i \partial \dot{q}_l}\right] J_{lk} = 0 \end{aligned} \tag{D.185}$$

$$J_{ik}(p, T) = 0 \quad \frac{\partial J_{ik}(p, 0)}{\partial t} = \left(\frac{1}{m}\right)\delta_{ik} \tag{D.186}$$

$$\det J_{ik}(p, T) = 0 \text{ when } q_i(T) \text{ is a focal point} \tag{D.187}$$

Equation of geodesic deviation as is used too in general relativity.

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{D.188}$$

Worked example: Integrability condition

$$\mathcal{H} = \frac{1}{2m}p^2 + \frac{\omega}{2}q^2 \tag{D.189}$$

$$\{\mathcal{H}, q\} = \frac{1}{2m}\{p^2, q\} = -\frac{p}{m}, \quad \{\mathcal{H}, p\} = \frac{\omega}{2}\{q^2, p\} = \omega q \tag{D.190}$$

$$\begin{aligned} \{\mathcal{H}, q\}_2 &= \{\mathcal{H}, \{\mathcal{H}, q\}\} = -\frac{1}{m}\{\mathcal{H}, p\} = -\frac{\omega}{m}q \\ \{\mathcal{H}, q\}_3 &= \{\mathcal{H}, \{\mathcal{H}, q\}_2\} = -\frac{\omega}{m}\{\mathcal{H}, q\} = \frac{\omega}{m^2}p \\ \{\mathcal{H}, q\}_4 &= \{\mathcal{H}, \{\mathcal{H}, q\}_3\} = \frac{\omega}{m^2}\{\mathcal{H}, p\} = -\frac{\omega^2}{m^2}q \end{aligned} \tag{D.191}$$

$$\{\mathcal{H}, q\}_{2r-1} = (-1)^r \frac{\omega^{r-1}}{m^r} p, \quad \{\mathcal{H}, q\}_{2r} = (-1)^r \frac{\omega^r}{m^r} q \quad r = 1, 2, 3, \dots \tag{D.192}$$

$$\begin{aligned}
\alpha^t_{\mathcal{H}}(q) &= q + t\{\mathcal{H}, q\}_1 + \frac{t^2}{2!}\{\mathcal{H}, q\}_2 + \frac{t^3}{3!}\{\mathcal{H}, q\}_3 + \frac{t^4}{4!}\{\mathcal{H}, q\}_4 + \dots \\
&= q - t\frac{1}{m}p - \frac{t^2}{2!}\frac{\omega}{m}q + \frac{t^3}{3!}\frac{\omega}{m^2}p + \frac{t^4}{4!}\frac{\omega^2}{m^2}q \\
&= q(1 - \frac{t^2}{2!}\frac{\omega}{m} + \frac{t^4}{4!}\frac{\omega^2}{m^2} + \dots) + p(t\frac{1}{m} + \frac{t^3}{3!}\frac{\omega}{m^2} + \dots) \\
&= q\cos(\sqrt{\frac{\omega}{m}}t) + \frac{p}{\omega}\sin(\sqrt{\frac{\omega}{m}}t)
\end{aligned} \tag{D.193}$$

$\{\mathcal{H}, q\}_0 := q$

$$\begin{aligned}
\{\mathcal{H}, p\}_2 &= \{\mathcal{H}, \{\mathcal{H}, p\}\} = -\frac{1}{m}\{\mathcal{H}, p\} = -\frac{\omega}{m}q \\
\{\mathcal{H}, p\}_3 &= \{\mathcal{H}, \{\mathcal{H}, p\}_2\} = -\frac{\omega}{m}\{\mathcal{H}, q\} = \frac{\omega}{m^2}p \\
\{\mathcal{H}, p\}_4 &= \{\mathcal{H}, \{\mathcal{H}, p\}_3\} = \frac{\omega}{m^2}\{\mathcal{H}, p\} = -\frac{\omega^2}{m^2}q
\end{aligned} \tag{D.194}$$

D.5 Constrained Hamiltonian Systems

In this report we are mainly interested in gauge field theories, but to outline the basic ideas we will discuss the main ideas with the context of a finite system. We consider a system given by a Lagrangian $\mathcal{L}$ which is a function of the coordinates q_i ($i = 1, \dots, N$) ans their first time derivatives.

The Lagrangian is *singular* if its *Hessian matrix* $\partial\mathcal{L}/\partial\dot{q}_i\partial\dot{q}_i$ has zero determinate

$$\det\left(\frac{\partial^2\mathcal{L}}{\partial\dot{q}_i\partial\dot{q}_j}\right) = 0. \tag{D.195}$$

In this case not all of the equations that define the momenta p_i,

$$p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}, \tag{D.196}$$

can be solved for the velocities $\dot{q}_i$. Instead, some of the relations yield the *primary constraints*

$$\phi_{a_1}(q,p) = 0, \quad a_1 = 1, \ldots, \mathcal{K}?? \tag{D.197}$$

We demonstrate this with an example.

Simple example

Consider the fairly general action

$$\mathcal{L}(q,\dot{q}) = \frac{1}{2}\sum_{ij} \dot{q}_i A_{ij}(q)\dot{q}_j - \sum \eta_i(q)\dot{q}_i - V(q) \tag{D.198}$$

where the matrix $A_{ij}(q)$ is symmetric $A_{ij} = A_{ji}$. This matrix is evidently the Hessian of (D.198),

$$A_{ij}(q) = \frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}. \tag{D.199}$$

From (D.196) we find the conjugate momentum

$$p_i = \sum_i A_{ij}(q)\dot{q}_i - \eta_i(q). \tag{D.200}$$

One can solve for the velocity $\dot{q}_i$ if and only if the matrix $A_{ij}(q)$ is non-singular.

If this is the case then we have that

$$\dot{q}_i = \sum_j A^{-1}_{ij}(q)(p_i + \eta_i(q)) \tag{D.201}$$

and one easily constructs the Hamiltonian

$$\mathcal{H}(q,p) = \frac{1}{2}\sum_{ij}(p_i + \eta_i)A^{-1}_{ij}(q)(p_j + \eta_j) + V(q). \tag{D.202}$$

Now let us consider the case where one or more of the eigenvalues of $A_{ij}(q)$ are zero. Being a symmetric matrix it has orthogonal eigenvectors $v_i^{(n)}$ and real eigenvalues $\lambda^{(n)}$. We construct a matrix from the eigenvectors

$$O_{ij} = v_i^{(j)} \tag{D.203}$$

whatsmore we order the eigenvectors so that the ones that have non-zero eigenvalues appear as the first columns on the left-hand side of the matrix O.

$$O_{ik}^{T}(q)A_{kl}^{-1}(q)O_{lj} = D_{ij} \tag{D.204}$$

where D

$$\underline{\underline{D}} = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{D.205}$$

$$D_{ij}(O_{jk}^{T}\dot{q}_k) = O_{ij}^{T}(p_j + \eta_j(q)) \tag{D.206}$$

For the zero eigenvalue subspace we have the set of conditions between the q's and q's

$$O_{ij}^{T}(p_j + \eta_j(q)) = 0, \quad \text{for} \;\; i = R, \ldots, N. \tag{D.207}$$

The equations of motion, of Newton and Lagrange, are of second order. They tell us what the accelaration given the position and velocity. We are able to calculate the motion of the particle at later times. The second order nature Lagranges equations can be more clearly seen if we rewrite them explicitly expanding the time derivative

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_i + \frac{\partial^2 \mathcal{L}}{\partial q_i \partial \dot{q}_j}\dot{q}_i + \frac{\partial^2 \mathcal{L}}{\partial t \partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \tag{D.208}$$

it is of the form

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_i = \mathcal{F}_i(q, \dot{q}, t) \tag{D.209}$$

where the function $\mathcal{F}_i$ and the second partial derivatives $\partial^2\mathcal{L}/\partial\dot{q}_i\partial\dot{q}_j$, (*Hessian matrix*), on the left hand side depend only on q, $\dot{q}$ and t. The initial accelaration $\ddot{q}_i(t_0)$ is obtained by inserting the initial $q(t_0)$ and $\dot{q}(t_0)$ into the expressions for f_i and $\partial^2\mathcal{L}/\partial\dot{q}_i\partial\dot{q}_j$ and applying the inverse of the Hessian matrix to both sides of the equation. We are then know the position this way to find the motion at later times. For this prcoedure to work, requires that

$$\det\left(\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}\right) \neq 0 \tag{D.210}$$

Equations of motion

As we saw in the last section unconstrained case, the equations of motion followed from the minimizaton of

$$\delta I = \delta(p_n \dot{q}_n - \mathcal{H}(q,p)). \tag{D.211}$$

However in this case, the variations of q_i and p_i, are nolonger independent of each other; they are restricted by the constraints ().

with variations of q_n and p_n such that $\phi_m(q_n, p_n) = 0$ that is:

$$\delta I = \delta(p_n \dot{q}_n - \mathcal{H}(q,p)) = \left(-\dot{p}_n - \frac{\partial \mathcal{H}}{\partial q_n}\right)\delta q_n + \left(\dot{q}_n - \frac{\partial \mathcal{H}}{\partial p_n}\right)\delta p_n = 0, \quad \text{subject to } \phi_m(q_n, p_n) = 0. \tag{D.212}$$

But this is just a standard minimizaton problem with constraints, and are dealt with using Lagrangian multipliers,

$$I = \int \left[p_n \dot{q}_n - \mathcal{H}(p_n, q_n) - \sum_m u_m(t)\phi_m(q,p)\right] dt \tag{D.213}$$

$$\delta I = p\delta q|_{q',t'}^{q'',t''} + \int_{q',t'}^{q'',t''} \left[\left(\dot{q_n} - \frac{\partial H}{\partial p_n} - \sum_m u_m \frac{\partial \phi_m}{\partial p_n}\right)\delta p_n + \left(-\dot{p_n} - \frac{\partial H}{\partial q_n} - \sum_m u_m \frac{\partial \phi_m}{\partial q_n}\right)\delta q_n\right] dt. \tag{D.214}$$

$$\dot{q}_n = \frac{\partial \mathcal{H}}{\partial p_n} + u_m \frac{\partial \phi_m}{\partial p_n} \tag{D.215}$$

$$\dot{p}_n = -\frac{\partial \mathcal{H}}{\partial q_n} - u_m \frac{\partial \phi_m}{\partial q_n} \tag{D.216}$$

defining the *total Hamiltonian* $\mathcal{H}_T$ by

$$\mathcal{H}_T = \mathcal{H} + \sum_m u_m \phi_m \tag{D.217}$$

equations (D.215) and (D.216) can be expressed simply as

$$\dot{q}_n = \frac{\partial \mathcal{H}_T}{\partial p_n}, \quad \dot{p}_n = -\frac{\partial \mathcal{H}_T}{\partial q_n}. \tag{D.218}$$

D.5.1 Examples

Example 1: Free Relativistic Particle

We introduce a labelling of the points of the curve by an arbitrary parameter τ, so that we are run through every point on the curve $\{x^0(\tau), x^1(\tau), x^2(\tau), x^3(\tau)\}$ as we run through every τ from τ_i to τ_f. The parameter τ represents an unobservable and hence unphysical labelling of the curve.

The action is proportional to the arc length from initial point x_i^μ to x_f^μ. If $x_i^\mu = x^\mu(\tau_i)$ and $x_f^\mu = x^\mu(\tau_f)$ then the action can be written

$$S = -m \int_{x_i}^{x_f} \sqrt{dx^\mu dx_\mu} = -m \int_{\tau_i}^{\tau_f} \sqrt{\left(\frac{dx^\mu}{d\tau}\right)\left(\frac{dx_\mu}{d\tau}\right)} d\tau. \tag{D.219}$$

$$S = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^2} \tag{D.220}$$

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = -\frac{m\dot{x}_\mu}{\sqrt{\dot{x}^2}} \tag{D.221}$$

has constraints

$$p_\mu p^\mu = \frac{m^2 \dot{x}_\mu \dot{x}^\mu}{\dot{x}^2} \tag{D.222}$$

$$\begin{aligned} H_0 &= p_\mu \dot{x}^\mu - \mathcal{L} \\ &= -\frac{m\dot{x}_\mu \dot{x}^\mu}{\sqrt{\dot{x}^2}} - (-m\sqrt{\dot{x}^2}) \\ &= 0. \end{aligned} \tag{D.223}$$

$$\begin{aligned} H &= H_0 + u_i \phi_i \\ &= \gamma(p^2 - m^2) \end{aligned} \tag{D.224}$$

$$\dot{x}^\mu = \frac{\partial H}{\partial p_\mu} \quad \text{so that} \quad \frac{dx^\mu}{d\tau} = \kappa(2p^\mu) \tag{D.225}$$

This arbitrariness in $\dot{x}^\mu$ is a reflection of it being unphysical as the freely chosen τ is unphysical.

$$S = -m \int d\tau \sqrt{\frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau}} \tag{D.226}$$

$$\tau \to \tau(\tau'), \qquad d\tau = \frac{d\tau}{d\tau'}d\tau'$$

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tau'}\frac{d\tau'}{d\tau} \tag{D.227}$$

$$S = -m \int d\tau' \sqrt{\frac{dx_\mu}{d\tau'}\frac{dx^\mu}{d\tau'}} \tag{D.228}$$

Gauge fixing in this case corresponds to choice of the parameter τ.

Equation of motion

$$\mathcal{L} = -m\sqrt{\dot{x}^2} \tag{D.229}$$

$$0 = \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} \tag{D.230}$$

Thus,

$$\frac{-m\dot{x}_\mu}{\sqrt{\dot{x}^2}} = 0$$
$$\frac{m\ddot{x}^\mu}{\sqrt{\dot{x}^2}} = 0$$

Now how do the primary constraints change with time? Do we need to impose additional secondary constraints?

Example 1: Double Harmonic Oscillator

Electrodynamics

$$\partial_t^2 A_\alpha - \nabla^2 A_\alpha - \partial \cdot \dot{A} = 0 \tag{D.231}$$

$$\partial^\alpha \partial_\alpha A_t - \partial^\alpha \dot{A}_\alpha = 0 \tag{D.232}$$

$$\ddot{A}_\alpha - \partial_a \partial_a A_t - \ddot{A}_t + \partial_a \dot{A}_a = 0 \tag{D.233}$$

imposes a constraint on initial conditions.

$$\partial_a(\partial_a A_t - \dot{A}_a) = 0 \tag{D.234}$$

$$\ddot{A}_t = \ddot{A}_t(A_\alpha, \dot{A}_\alpha) \tag{D.235}$$

does not exist.

$$A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \lambda(x) \tag{D.236}$$

smearing with test functions $f(x), g(x), \ldots$. If we define

$$\varphi[f] := \int d^3x f(x) \varphi(x) \tag{D.237}$$

We can rewrite the Poisson brackets as

$$\{\varphi[f], \pi[g]\} = f[g] = g[f]. \tag{D.238}$$

This is often convenient, especially when one wants to keep track of partial integrations in a calculation.

The Hamiltonian systems whose dynamics is described by a Lagragian. Primary constraints have no analog on the Lagragian level, requiring their persistence in time leads to equations relating the coordinates to the velocities. These are part of the Euler-Lagrange equations of motion.

D.5.2 Dirac's Procedure for Constrained Hamiltonian Systems

The time derivative of any quantitiy $g(q,p)$ is given by

$$\dot{g} = \frac{\partial g}{\partial q_n} \dot{q}_n + \frac{\partial g}{\partial p_n} \dot{p}_n \tag{D.239}$$

Using (D.218) this becomes

$$\dot{g} = \frac{\partial g}{\partial q_n}\frac{\partial \mathcal{H}_T}{\partial p_n} - \frac{\partial g}{\partial p_n}\frac{\partial \mathcal{H}_T}{\partial q_n} = \{g, \mathcal{H}_T\} \tag{D.240}$$

(**??**) (D.218)

where $\mathcal{H}_T = \mathcal{H} + u_m\phi_m$ is the total Hamiltonian.

We require the primary constraints to persistent in time, so they must be consistent with the equations of motion, i.e. the time derivative has to vanish

$$\dot{\phi}_m \approx 0 \approx \{\phi_m, \mathcal{H}\} + u_{m'}\{\phi_m, \phi_{m'}\}. \tag{D.241}$$

This condition leads to one of three possibilities (that is, if we have a physically sensible theory on our hands):

• They are fulfilled already with the help of the existing constraints. They could be trivially satisfied, that is, we just get the identity $0 = 0$.

• They serve to determine some of the unknown Lagrange multipliers u_m.

• We could generate additional conditions relating the coordinates and momenta. These constraints will be independent of the primary constraints (otherwise, they would be identically zero and so be of the first type). We therefore have a new set of constraints:

$$\phi_k(q,p) \approx 0 \quad k = \mathcal{M}+1, \mathcal{M}+2, \ldots, \mathcal{K}. \tag{D.242}$$

These are called *secondary constraints*.

This procedure has to be iterated, i.e. we require the time derivatives of the secondary constraints to vanish which in turn can generate further *tertiary constraints*,

$$\{\phi_k, \mathcal{H}\} + u_m\{\phi_k, \phi_m\} \approx 0, \tag{D.243}$$

where $k > \mathcal{M}$, and so on, until one finally obtains a set of constraints consistent with the equations of motion. In the end, we are left with our original primary constraints ϕ_m, a new set of secondary constraints, and a set of equations of the thrid type relating the u_m's.

$$\{f, g\}_D = \{f, g\} - \{f, \phi_m\}C^{-1}_{mn}\{\phi_n, g\}, \tag{D.244}$$

where we have employed the inverse of the constraint matrix

$$C_{mn} := \{\phi_n, \phi_m\}. \tag{D.245}$$

We notice that the Dirac bracket of any function on phase space with any second class constraint vanishes *strongly*,

$$
\begin{aligned}
\{f, \phi_p\}_D &= \{f, \phi_p\} - \{f, \phi_m\} C^{-1}_{mn} \{\phi_n, \phi_p\} \\
&= \{f, \phi_p\} - \{f, \phi_m\} C^{-1}_{mn} C_{np} = 0.
\end{aligned} \tag{D.246}
$$

Let us further examine these equations of the third type. These can be considered as a system of linear non-homogeneous equations in the unknown u's, with coefficients in terms of the Poison brackets of primary and secondary constraints. Since these coeficients are in terms are functions of the q's and p's the u's must also be functions of the q's and p's:

$$u_m = U_m(q, p) \tag{D.247}$$

A complete solution for these variables includes arbitrary linear combinations of solutions $V_m(q, p)$ to the homogeneous equations associated with (D.243):

$$V_m\{\phi_i, \phi_m\} = 0 \tag{D.248}$$

Hence, the general solution to this system of equations is

$$u_m = U_m + v_a V_{am}, \tag{D.249}$$

where a sums over the solutions found in (D.248).

$$\Psi_m \approx 0 \tag{D.250}$$

an operator is *first class* if its Poisson brackets with every constraint are weakly zero:

$$\{R, \phi_m\} \approx 0 \tag{D.251}$$

$$\{R, \phi_m\} = C_{mm'} \phi_{m'} \tag{D.252}$$

If R does not satisfy for each ϕ_m, we say that R is *second class*. A special case is when all the system of constraints are first class - we then have a closed constraint algebra.

$$\dot{\Psi}_m \equiv \{\Psi_m, \mathcal{H}\} \tag{D.253}$$

$$\{\Psi_m, \Psi_n\} = C_{mn}^{\ \ r} \Psi_r \approx 0, \tag{D.254}$$

where the "structure functions" $C_{mn}^{\ \ r}$ may depend on the q's and p's; life is much easier when they are constants, (this is important for when we come onto problems with the Hamiltonian constraint of quantum GR). We define an extended Hamiltonian,

$$\mathcal{H}_{ext} = \mathcal{H} + \lambda_m \Psi_m \tag{D.255}$$

where the λ_m's are Lagrange mulitipliers.

The phase space action

The second class constraints result from irrelevant phase space variables which play no role in the dynamics of the system. This can be illustrated by a simple example. Consider the set of constraints

$$q^1 \approx 0, \quad p_1 \approx 0. \tag{D.256}$$

They are obviously second class because $\{q^1, p_1\}$.

We simply ignore them and redifine Poisson brackets that exclude the $n = 1$ degree of freedom,

$$\{f, g\}' = \sum_{n=2}^{N} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right). \tag{D.257}$$

$\phi_1 = q^1$ and $\phi_2 = p_1$

$$C_{12} = \{\phi_1, \phi_2\} = \{q^1, p_1\} = 1, \;\; C_{21} = -1 \tag{D.258}$$

$C_{12}^{-1} = -1$, $C_{21}^{-1} = 1$

$$\begin{aligned} \{f, g\}_D &= \{f, g\} - \{f, \phi_1\} C_{11}^{-1} \{\phi_1, g\} - \{f, \phi_2\} C_{21}^{-1} \{\phi_1, g\} \\ &\quad - \{f, \phi_1\} C_{12}^{-1} \{\phi_2, g\} - \{f, \phi_2\} C_{22}^{-1} \{\phi_2, g\} \\ &= \{f, g\} - \{f, \phi_2\}\{\phi_1, g\} + \{f, \phi_1\}\{\phi_2, g\} \end{aligned} \tag{D.259}$$

$$\begin{aligned} \{f, g\}_D &= \{f, g\} - \{f, p_1\}\{q^1, g\} + \{f, q^1\}\{p_1, g\} \\ &= \sum_{n=1}^{N} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right) - \frac{\partial f}{\partial q^1} \frac{\partial g}{\partial p_1} + \frac{\partial f}{\partial p_1} \frac{\partial g}{\partial q^1} \\ &= \{f, g\}'. \end{aligned} \tag{D.260}$$

D.5.3 First Class Constraints and Gauge Symmetries

Let us consider the arbittrary coeficients v_a in (). We may use an extended Hamiltonian

$$\mathcal{H}_E = \mathcal{H}_T + v_a \phi_a \tag{D.261}$$

This should be an equally good Hamiltonian to use in making physical predictions. However, the time evolution of any function of phase space is partly arbitrary

$$\dot{g} \approx \{g, \mathcal{H}_T + v_a \phi_a\} \tag{D.262}$$

This may at first sight seems unsatisfactory as our theory had better be deterministic. However, this is precisely the situation we encountered for Maxwell's equations.

$$\begin{aligned} \delta_t g &= g_0 + \dot{g}\delta t \\ &= g_0 + \{g, \mathcal{H}_T\}\delta t \\ &= g_0 + \delta t(\{g, \mathcal{H}'\} + v_a\{g, \phi_a\}) \end{aligned} \tag{D.263}$$

$$\begin{aligned} \Delta g &= \delta t(v_a - {v'}_a)\{g, \phi_a\} \\ &:= \epsilon_a\{g, \phi_a\} \end{aligned} \tag{D.264}$$

$$\begin{aligned} g &\to g + \epsilon_a\{g, \phi_a\} \\ &\to g + \epsilon_a\{g, \phi_a\} + \gamma_{a'}\{g + \epsilon_a\{g, \phi_a\}, \phi_{a'}\} \equiv g' \end{aligned} \tag{D.265}$$

If we apply them in the opposite order instead, we get

$$g \to g + \gamma_{a'}\{g, \phi_{a'}\} + \epsilon_a\{g + \gamma_{a'}\{g, \phi_{a'}\}, \phi_a\} \equiv g'' \tag{D.266}$$

$$\begin{aligned} g' - g'' &= \epsilon_a\{g, \phi_a\} + \gamma_{a'}\{g, \phi_{a'}\} + \gamma_{a'}\epsilon_a\{\{g, \phi_a\}, \phi_{a'}\} \\ &\quad - \gamma_{a'}\{g, \phi_{a'}\} - \epsilon_a\{g, \phi_a\} + \epsilon_a\gamma_{a'}\{\{g, \phi_{a'}\}, \phi_a\} \\ &= \epsilon_a\gamma_{a'}(\{\{g, \phi_a\}, \phi_{a'}\} + \{\{\phi_{a'}, g\}, \phi_a\}) \\ &= \epsilon_a\gamma_{a'}(\{g, \{\phi_a, \phi_{a'}\}\}). \end{aligned} \tag{D.267}$$

where we have used the Jacobi indentity to arrive at the final answer.

undetermined Lagragian multipliers.

by introducing additional *gauge fixing conditions*

$$\chi_a(q,p) = 0 \tag{D.268}$$

with the number of conditions being equal to the number of first class constrints.

D.5.4 Dirac Method and Electrodynamics

$$S = \frac{1}{2}\int d^4x A^\mu P_{\mu\nu}\nabla^\lambda\nabla_\lambda A^\nu \tag{D.269}$$

The total Hamiltonian of the electromagnetic field is then given by

$$\mathcal{H}_T \int \frac{1}{4}F^{ab}F_{ab} + \frac{1}{2}E^a E_a\, d^3x - \int v(x)E^0\, d^3x, \tag{D.270}$$

$$\mathcal{H}_E = \mathcal{H}_T + \int u(x)\partial_a E^a\, d^3x \tag{D.271}$$

where $u(x)$ is another arbitrary function.

$$\begin{aligned}
&\dot{E}^0(x) \approx 0\\
&[E^0, H] \approx 0\\
&\frac{\partial E^0}{\partial A_0(x)}\frac{\partial H}{\partial E^0(x)} - \frac{\partial E^0}{\partial E^0(x)}\frac{\partial H}{\partial A_0(x)} = 0\\
&\frac{\partial}{\partial A_0(x)}\left(\int A_0(x)\partial_a E^a(x) d^3x\right) = 0\\
&\int\left(\frac{\partial A_0(x)}{\partial A_0(x)}\partial_a E^a + A_0(x)\frac{\partial}{\partial A_0(x)}(\partial_a E^a(x))\right)
\end{aligned} \tag{D.272}$$

secondary constraint associated with is

$$\partial_a E^a(x) \approx 0 \tag{D.273}$$

which is the Gauss's law in the absence of charge.

Details

D.5.5 Quantization of Constrained Hamiltonian Systems

D.5.6 Dirac Observables

Redundancy of the mathematical formulation.

A classical observable A_0 is a function on the constraint surface which is invariant under gauge transformations. Its Dirac bracket with the first-class constraints γ_a vanish weakly,

$$\{A_0, \gamma_a\}_{DB} \approx 0, \tag{D.274}$$

or, equivalently,

$$\{A_0, \gamma_a\}_{DB} = A_a^{\ b}\gamma_b. \tag{D.275}$$

Since the system is constraint surface should be identified,

$$A'_0 \sim A_0 \ \text{ iff } \ A'_0 + k^a\gamma_a. \tag{D.276}$$

restriction to the constraint surface and gauge invariant, i.e. the observable be first class with respect to the Dirac bracket.

we can make this homomorphism exact by dividing both $C^\infty(\mathcal{M})$ and $D^\infty(\mathcal{M})$ by the ideal under pointwise addition and multiplication of smooth functions vanishing on the constraint surface

D.5.7 Darboux's Theorem

The formal definition of a sympletic manifold.

Definition As sympletic structure for a differential manifold $\mathcal{M}$ is a non-degenerate, closed two-form ω. The pair $(\mathcal{M}, \omega)$ is called a sympletic manifold.

Given a function $f \in \mathcal{F}(\mathcal{M})$ (with condition that $df \neq 0$ in U), there exists another function $g \in \mathcal{F}(\mathcal{M})$ whose PB with f is $\{f, g\} = 1$. The vector field X_f associated with function f's generates motion in U. We show this by constructing in U the function g such that along the motion of f so that $\{f, g\} = 1$. The curves in fig () represent the integral curves of X_f. At some point ξ_0 in U set $f = 0$ (by adding some constant to f) and take some surface N through ξ_0 that intersects all the integral curves of X_f in U. Pick a point $\xi \in U$ not in N; it necessarily lies on a integral curve of X_f and belongs to some value of the parameter λ of the associated one-parameter group. If we set $\lambda = 0$ on N, we define λ to be the distance from N to ξ. Now define the function $g(\xi)$ for each $\xi \in U$ as the distance λ of ξ from N, so that the rate of change of g along the integral curves is $dg/d\lambda = 1$. As we know, the rate of change of g along the integral curves of X_f is $\{f, g\}$. Hence $\{f, g\} = 1$. This would complete the proof if $n = 1$, for f could be taken as q^1, and g to be p^1. For higher n we proceed by induction, after the first induction we are left to consider a lower dimensional submanifold $\mathbb{L}$. Part of the proof requires establishing that the symplectic form ω on $\mathcal{M}$ can be broken into two parts:

$$\omega = df \wedge dg + \omega_{|_{\mathbb{L}}}, \tag{D.277}$$

where $\omega_{|_{\mathbb{L}}}$ is a symplectic form on $\mathbb{L}$. We do not complete this version of the proof, instead we give the following proof.

Construction of symplectic coordinates by induction on n

We will assum that Darboux's theorem is already proved for $\mathbb{R}^{2n-2}$. Consider the set M given by the equations $p_1 = q_1 = 0$. The differentials dp_1 and dq_1 are linearly independent at $\mathbf{x}$ since $\omega^2(I\ dp_1, I\ dq_1) = (q_1, p_1) \equiv 1$. Thus, by the implicit function theorem, the set M is a manifold of dimension $2n - 2$ in a neighbourhood of $\mathbf{x}$; we will denote it by M^{2n-2}.

Lemma. The symplectic structure ω^2 on $\mathbb{R}^{2n}$ induces a symplectic structure on some neighbourhood of the point $\mathbf{x}$ on M^{2n-2}.

Proof: For the proof we need only the nondegeneracy of ω^2 on TM_x. Consider the symplectic vector space $T\mathbb{R}^{2n}_x$. The vectors $\mathbf{P_1}(\mathbf{x})$ and $\mathbf{Q_1}(\mathbf{x})$ of the Hamiltonian vector fields with Hamiltonian functions p_1 and q_1 belong to $T\mathbb{R}^{2n}_x$. Let $\xi \in TM_x$. The derivatives of p_1 and q_1 in the direction

□

Theorem D.5.1 (Darboux). *Let $(\mathcal{M}, \omega)$ be a sympletic manifold. Then for a neighbourhood Z of each point p one can choose canonical coordinates $(x^\mu)_{\mu=1}^{2m} = (q^a, p_a)_{a=1}^{m}$ such that $\omega = dp_\alpha \wedge dq^\alpha$. The coordinates q, p are called configuration and momentum variables respectively.*

Proof:

The push-forward $\phi^*\omega$ of a two-form ω is defined by

$$[\phi^*\omega](X, Y) = \omega(\phi_* X, \phi_* Y)$$

where

$$(\phi_* X)^\mu = \frac{\partial x^\mu}{\partial \xi^\nu} X^\nu$$

with $x = \phi(\xi)$.

Symplectic Gram-Schibt orthonormaisation

First say $k = 1$. Let e_1 be any vector in V. We can find f_1 such that $\omega(f_1, e_1) \neq 0$ (the existence of f_1 follows from the non-degeneracy of ω). Let M_1 be a subspace of V spanned by $\{e_1, f_1\}$ and set $V_1 = M_1^{\perp}$. Note

(a) since $\omega(f_1, e_1) \neq 0$ we have $V_1 \cap M_1 = 0$.

(b) The restriction ω_1 of ω to V_1 is non-degenerate: if $X_1 \in V_1$ is such that $\omega_1(X_1, Y) = 0$ for all $Y \in V_1$, then $X_1 \in V_1^{\perp} = M_1$ and hence the only element in V_1 satisfying the condition is 0. Thus (V_1, ω_1) is by itself a symplectic vector space.

Now choose some vector $e_2 \in V_1$, we can find $f_2 \in V_1$ such that $\omega_1(f_2, e_2) \neq 0$ by the non-degeneracy of ω_1.

We can normailse so that we have

$$\omega(e_1, e_2) = \omega(f_1, f_2) = 0, \quad \omega(f_i, e_j) = \delta_{ij} \text{ for } i, j = 1, 2.$$

The vectors $\{e_1, e_2; f_1, f_2\}$ are linearly independent:

$$\omega(e_1, a_1 e_1 + a_2 e_2 + b_1 f_1 + b_2 f_2) = 0$$

implies $b_1 = 0$, we similarly get $b_2 = 0, a_1 = 0, a_2 = 0$ by contracting with e_2, f_1, f_2 respectively. Hence E_2 has dimension 4.

Continuing this construction another $n-2$ times we obtain a set of linearly independent vectors $\{e_1, \ldots, e_n; f_1, \ldots, f_n\}$ in V.

□

Darboux's theorem is important in reducing the dimension of a dynamical system.

D.5.8 Symplectic Reduction

A distribution on $\mathcal{M}$ assigns in a smooth way to each point $p \in \mathcal{M}$ a k-dimensional subspace of the tangent space to $p \in \mathcal{M}$.

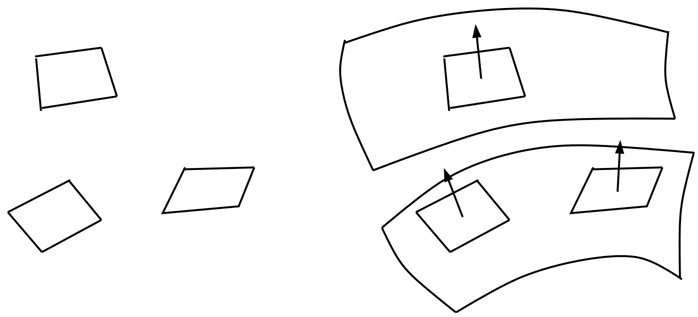
Definition Let $\mathcal{M}$ be a smooth manifold. A distribution $D : \mathcal{M} \mapsto E_0^1(\mathcal{M})$ is an assignment of a subspace $D_p(\mathcal{M}) \subset T_p(\mathcal{M})$ of the tangent space for each point $p \in \mathcal{M}$ such that

(1) $\dim(D_p(\mathcal{M})) = n = const.$ and

(2) for each point $\mathcal{M}$ there is a neighbourhood U of p and n vector fields $v_k \in T^1(\mathcal{M})$, $k = 1, \ldots, n$ such that $D_p(\mathcal{M})$ is spanned by them for each $q \in U$.

□

Given a smooth family of two-planes, can they be identified with the tangent vector space of a family of non-intersecting surfaces that fill up a region everwhere?



(a) (b)

Figure D.6: Here a smooth family of 2-planes coincides with the tangent spaces of a nonintersecting space filling surfaces. Suprisingly this is not always the case.

Definition A submanifold $\mathbb{L} \subset \mathcal{M}$ is called an **integral manifold** of the distribution D provided that $T_p(\mathbb{L}) = D_p(\mathcal{M})$ for all $p \in \mathbb{L}$.

□

A related notion is the following: a distribution is said to be integrable if the set of vector fields tangent to the distribution are closed under the Lie bracket. One can phrase this as:

Definition The distribution D is said to be **integrable** provided that the subspace $T^1(\mathcal{M}, D)$ of vector fields which are everywhere tangential to D is a subalgebra of $T^1(\mathcal{M})$. An integrable distribution is called a **foliation**.

□

By Frobenius' theorem, integral manifolds exist if and only if D is integrable.

An integral manifold that is not properly contained in another integral manifold is called a maximal integral manifold.

Definition Maximal integral manifolds of a foliation are called **leaves**.

□

If the integral curves of the various vector fields are to define a submaifold, they must remain tangent to it. This remaining tangent is guaranteed if all the Lie brackets are themselves tangent, since the Lie brackets are simply the derivatives of the various vector fields along one another.

Frobenius theorem says that a necessary and sufficient condition for a distribution (i.e. an $n-p$ dimensional sub-bundle of the tangent bundle of the manifold $\mathcal{M}$) to be tangent to leaves of a foliation, is the set of vector fields tangent to the distribution are closed under the Lie bracket.

Our interest in this section is reduced phase spaces comming about from constrants.

In appendix O when we come to geometric quantisation, our interest will be in submanifolds determined by differential equations. Solutions of differential equations are usually relations, for example $\{y_i = f_i(x^1, \ldots, x^m), i = 1, \ldots, p\}$, which can be thought of as submanifolds with coordinates $\{y_1, \ldots, y_p, x^1, \ldots, x^m\}$. Frobenius' theorem in terms of differential forms provides a fundamental theorem which gives conditions for the existence of solutions to partial differential equations 'integrability conditions'.

We note that several steps in the proof only work locally: that is, they involve assumptions or known results which may only hold in a neighbourhood of a point, So it guarantees the existence of an integral submanifold through every point, but only in a neighbourhood of the point.

Before proving Frobenius' theorem we establish the equivalence of the two forms of the theorem with a couple of lemmas.

Lemma D.5.2 *A distribution of an n-dimensional tangent spaces can be equivalently described by the specification of n vector fields v_k which are everywhere tangent to D or by $m-n$ one-forms θ_α which satify $(\theta^\alpha[v])(p) = 0$ for all v tangent to D.*

Proof:

Any basis for the constraint one-forms $\{\theta^\alpha\}$ may be extended to a basis of one-forms in $\mathcal{M}$; the first m members of the dual basis of vector fields will do. Any vector field belonging to the distribution may be uniquely expressed as a linear combination (with variable coefficients) of basis vector fields.

□

Lemma D.5.3 *The following conditions are equivalent:*

1. *The v_k form a subalgebra of $T^1(\mathcal{M})$, that is, $[v_j, v_k] = f_{jk}{}^l v_l$ for some functions $f_{jk}{}^l$.*

2. *The θ^α form a closed Pfaff system, that is, $d\theta^\alpha = \omega^\alpha_\beta \wedge \theta_\beta$ for some one-forms ω^α_β.*

Proof: We have

$$i_{v_j} i_{v_k}(d\theta^\alpha) = v_j[\theta^\alpha[v_k]] - v_k[\theta^\alpha[v_j]] - \theta^\alpha[[v_j, v_k]] = -\theta^\alpha[[v_j, v_k]] \tag{D.278}$$

since $\theta^\alpha[v_k] = 0$ by definition. Now criterion (1) implies $i_{v_j} i_{v_k}(d\theta^\alpha) = 0$, i.e., $d\theta^\alpha = 0$ when restricted to the distribution. Complete $\theta_1, \ldots, \theta_{m-n}$ locally to a basis of one-forms by adjoining $\theta_{m-n+1}, \ldots, \theta_m$ such that $\theta_{(m-n)+k}(v_{k'}) = \delta_{kk'}$. Now expand $d\theta^\alpha$ in terms of the basis $\{\theta_1, \ldots, \theta_m\}$.

$$
\begin{aligned}
d\theta^\alpha &= \sum_{\beta=1}^{m}\sum_{\gamma=1}^{m} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma \\
&= \left(\sum_{\beta=1}^{m-n}\sum_{\gamma=1}^{m-n} + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=1}^{m-n} + \sum_{\beta=1}^{m-n}\sum_{\gamma=m-n+1}^{m} + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m}\right) f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma \\
&= \left(\sum_{\beta=1}^{m-n}\sum_{\gamma=1}^{m-n} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=1}^{m-n} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma + \sum_{\beta=1}^{m-n}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma\right) \\
&\quad + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma \\
&= \sum_{\beta=1}^{m-n}\left(-\sum_{\gamma=1}^{m-n} f^\alpha_{\beta\gamma}\theta^\gamma - 2\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta^\gamma\right)\wedge\theta^\beta + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma \\
&= \sum_{\beta=1}^{m-n}\omega^\alpha_\beta\wedge\theta^\beta + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma
\end{aligned}
$$

where

$$
\omega^\alpha_\beta = -\sum_{\gamma=1}^{m-n} f^\alpha_{\beta\gamma}\theta^\gamma - 2\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}
$$

$$
\begin{aligned}
0 &= i_{v_j} i_{v_k}(d\theta^\alpha) \\
&= i_{v_j} i_{v_k}\left(\sum_{\beta=1}^{m-n}\omega^\alpha_\beta\wedge\theta^\beta\right) + i_{v_j} i_{v_k}\left(\sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta^\beta\wedge\theta^\gamma\right) \\
&= \sum_{\beta=1}^{m-n}(\omega^\alpha_\beta[v_j]\theta_\beta[v_k] - \omega^\alpha_\beta[v_k]\theta_\beta[v_j]) + \sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}(\theta_\beta[v_j]\theta_\gamma[v_k] - \theta_\beta[v_k]\theta_\gamma[v_j]) \\
&= 2\sum_{\beta=m-n+1}^{m}\sum_{\gamma=m-n+1}^{m} f^\alpha_{\beta\gamma}\theta_\beta[v_j]\theta_\gamma[v_k] \\
&= 2f^\alpha_{(m-n)+j,(m-n)+k}
\end{aligned}
$$

Therefore $f^\alpha_{\beta\gamma} = 0$ for $\beta, \gamma > m-n$, hence

$$
d\theta^\alpha = \sum_{\beta=1}^{m-n}\omega^\alpha_\beta\wedge\theta^\beta.
$$

Conversely, if (2) is satisfied then

$$i_{v_j} i_{v_k}(d\theta^\alpha) = i_{v_j} i_{v_k}(\omega^\alpha_\beta \wedge \theta_\beta) = \omega^\alpha_\beta[v_j]\theta_\beta[v_k] - \omega^\alpha_\beta[v_k]\theta_\beta[v_j] = 0$$

hence $\theta^\alpha[[v_j, v_k]] = 0$ which means that the commutator has the form $[v_j, v_k] = f_{jk}{}^l v_l$.

□

Theorem D.5.4 Frobenius. *A necessary and sufficient condition for a distribution D to be integrable is one of the equivalent criteria from the previous lemma.*

Proof:

FIRST PROOF

We prove the converse by construction of n fields built from linear combinations of the v_ks which all commute and which therefore form a coordinate basis for a submanifold of dimension n.

These curves are obtained by solving the ordinary differential equations

$$\frac{dx^\alpha}{du} = v^\alpha(x(u)).$$

The existence and uniqueness theorem for ordinary differential equations guarantees a solution, at least for some small enough open neighbourhood. Therefore the theorem is trivial when there is only one vector field.

The theorem for $n \geq 2$ will be proved by induction. Select one vector, say $v_n = d/d\lambda_n$. The value of the parameter λ_n along the v_n congruence is defined at at every point. Define $(n-1)$ vector fields X_k which are linear combinations of all the original v_ks and which satisfy

$$d\lambda_n(X_k) = 0, \quad k = 1, \ldots, n-1. \tag{D.279}$$

We write

$$[X_i, X_j] = \sum_{k=1}^{n-1} \tilde{f}_{ijk} X_k + g_{ij} v_n \tag{D.280}$$

$$[v_n, X_i] = \sum_{j=1}^{n-1} m_{ij} X_j + n_i v_n \tag{D.281}$$

where $\tilde{f}_{ijk}, g_{ij}, m_{ij}$, and n_i are functions on U. We will now nee the following formula

$$d\lambda_n([V, W]) = \mathcal{L}_V d\lambda_n(W) - d(\mathcal{L}_V \lambda_n)(W) \tag{D.282}$$

which is easily derived. Liebnitz rule for the Lie derivative implies

$$\mathcal{L}_V d\lambda_n(W) = (\mathcal{L}_V d\lambda_n)(W) + d\lambda_n(\mathcal{L}_V W)$$

as $\mathcal{L}_V W = [V, W]$ and that $\mathcal{L}_V$ and d commute we arrive at (D.282). We will also need the simply identity

$$d\lambda_n(v_m) = \mathcal{L}_{v_m}\lambda_n = \frac{d\lambda_n}{d\lambda_n} = 1.$$

Contracting the left hand side of both equations (D.280) and (D.281) using (D.282)

$$\begin{aligned} d\lambda_n([X_i, X_j]) &= \mathcal{L}_{X_i}(d\lambda_n(X_j)) - d(\mathcal{L}_{X_i}\lambda_n)(X_j) \\ &= \mathcal{L}_{X_i}(0) - d(d\lambda_n(X_i))(X_j) \\ &= 0 \end{aligned} \tag{D.283}$$

$$\begin{aligned} d\lambda_n([v_i, X_j]) &= \mathcal{L}_{v_i}(d\lambda_n(X_j)) - d(\mathcal{L}_{v_i}\lambda_n)(X_j) \\ &= \mathcal{L}_{v_i}(0) - d(1)(X_j) \\ &= 0 \end{aligned} \tag{D.284}$$

This implies that $g_{ij} = n_i = 0$ and the commutators dont involve v_n.

We now use the inductive hypothesis - that any set of $(n-1)$ in involotion form an $(n-1)$-dimensional submanifold. This applies to the set $\{X_k, k = 1, \ldots, n-1\}$, which is therefore assumed to form a family of $(n-1)$-dimensional submanifolds filling U. Define a set of vectors $\{Y_k, k = 1, \ldots, n-1\}$ which form a coordinate basis for one of the submanifolds, say $\mathbb{L}$, so that these vector fields commute on $\mathbb{L}$. Let us see if there exist linear combinations

$$Z_i = \sum_j \alpha_{ij} X_j. \tag{D.285}$$

such that the vector fields $\{Z_k, k = 1, \ldots, n-1\}$ result from Lie dragging along v_n away from $\mathbb{L}$:

$$\begin{aligned} Z_k &= Y_k \quad \text{on } \mathbb{L} \\ \mathcal{L}_{v_n} Z_k = [v_n, Z_k] &= 0 \text{ in } U \end{aligned} \tag{D.286}$$

It follows from the Jacobi identity that the Z_k would commute among themselves as they do on $\mathbb{L}$:

$$\begin{aligned}\mathcal{L}_{v_n}([Z_i, Z_j]) &= [v_n, [Z_i, Z_j]] \\ &= [Z_i, [Z_j, v_n]] + [Z_j, [v_n, Z_i]] = 0 \end{aligned} \tag{D.287}$$

We would then have constructed the set $\{v_n, Z_k, k = 1, \ldots, n-1\}$ of fully commuting vectors and proved the theorem.

We must establish that vectors defined by (D.286) Z_i are a linear combination of the v_ks. Each Z_k is unique, given the postulate (D.285) we must have

$$\begin{aligned} 0 &= [v_n, Z_k] \\ &= \mathcal{L}_{v_n} Z_k \\ &= (\sum_j \alpha_{ij}) X_j + \sum_j \alpha_{ij} [v_n, X_j] \\ &= \sum_j \frac{d\alpha_{ij}}{d\lambda_n} X_j + \sum_{jk} n_{ij} X_k, \end{aligned} \tag{D.288}$$

or

$$0 = \sum_j \left(\frac{d\alpha_{ij}}{d\lambda_n} + \sum_k \alpha_{ik} n_{kj} \right) X_j$$

Since the X_ks are linearly independent, this requires

$$\frac{d\alpha_{ij}}{d\lambda_n} + \sum_k \alpha_{ik} n_{kj} = 0. \tag{D.289}$$

The initial conditions (at $\mathbb{L}$) that α_{ij} give the appropriate combination of X_ks to form Y_k, determine a unique solution, which always exists. Therefore, at every point the Z_ks are linear combinations of the X_ks.

SECOND PROOF

We demostrate the proof using condition (2) of the previous lemma.

To prove the converse we must show that if we have a Pfaff system of rank $r = m - n$ then there are local coordinates $(x^\mu) = (y^j, z^\alpha)$, $\mu = 1, \ldots, m$, $j = 1, \ldots, m-n$ such that $\theta^\alpha = \theta^\alpha_\beta dz^\beta$ where θ^α_β is an invertible matrix.

We proceed by induction, that is, we assume that if there is a Pffaf system in the manifold with local coordinates $x^1, \ldots, x^{m-1}$ then there are local coordinates $(x^\mu) = (y^j, \tilde{z}^\alpha, x^m)$ such that $\tilde{\theta}^\alpha = \tilde{\theta}^\alpha_\beta d\tilde{z}^\beta$ where $\tilde{\theta}^\alpha_\beta$ is an invertible matrix independent of x^m.

For $m > r$ let us write

$$\theta^\alpha = \tilde{\theta}^\alpha + f^\alpha dx^m$$

where

$$\tilde{\theta}^\alpha(x) = \sum_{\mu=1}^{m-1} \theta^\alpha_\mu(x) dx^\mu$$

and $f^\alpha(x)$ is some function on $\mathcal{M}$. Then

$$\begin{aligned}
d\theta^\alpha &= d\,[\sum_{\nu=1}^{m-1} \theta^\alpha_\nu(x) dx^\nu + f^\alpha(x) dx^m] \\
&= \sum_{\mu=1}^{m} \partial_\mu\,[\sum_{\nu=1}^{m-1} \theta^\alpha_\nu(x) dx^\mu \wedge dx^\nu + f^\alpha(x) dx^\mu \wedge dx^m] \\
&= \left[\sum_{\mu=1}^{m-1}\sum_{\nu=1}^{m-1} (\partial_\mu \theta^\alpha_\nu) dx^\mu \wedge dx^\nu\right] + \left[\sum_{\mu=1}^{m-1} (-\partial_m \theta^\alpha_\mu + \partial_\mu f^\alpha) dx^\mu\right] \wedge dx^m \\
&=: \tilde{d}\tilde{\theta}^\alpha + \xi \wedge dx^m \\
&= \omega^\alpha_\beta \wedge \theta^\beta = [\omega^\alpha_\beta \wedge \tilde{\theta}^\beta] + \omega^\alpha_\beta f^\beta \wedge dx^m
\end{aligned} \tag{D.290}$$

We conclude

$$\tilde{d}\tilde{\theta}^\alpha = \omega^\alpha_\beta \wedge \tilde{\theta}^\beta$$

by the inductive assumption

$$\tilde{\theta}^\alpha(x) = \tilde{\theta}^\alpha_\beta d\tilde{z}^\beta$$

$$\begin{aligned}
\theta^{\alpha'} &:= (\tilde{\theta}^{-1})^\alpha_\beta \theta^\beta \\
&= (\tilde{\theta}^{-1})^\alpha_\gamma (\tilde{\theta}^\gamma_\beta d\tilde{z}^\beta + f^\gamma dx^m) \\
&= d\tilde{z}^\alpha + (\tilde{\theta}^{-1})^\alpha_\beta f^\beta dx^m \\
&=: d\tilde{z}^\alpha + f^{\alpha'} dx^m
\end{aligned} \tag{D.291}$$

$$
\begin{aligned}
d\theta^{\alpha'} &= d[(\tilde{\theta}^{-1})^{\alpha}_{\beta}\theta^{\beta}] \\
&= d(\tilde{\theta}^{-1})^{\alpha}_{\beta} \wedge \theta^{\beta} + (\tilde{\theta}^{-1})^{\alpha}_{\beta} d\theta^{\beta} \\
&= (d(\tilde{\theta}^{-1})^{\alpha}_{\gamma}\tilde{\theta}^{\gamma}_{\beta}) \wedge ((\tilde{\theta}^{-1})^{\beta}_{\delta}\theta^{\delta}) + (\tilde{\theta}^{-1})^{\alpha}_{\delta}(\omega^{\delta}_{\gamma} \wedge \theta^{\gamma}) \\
&= (d(\tilde{\theta}^{-1})^{\alpha}_{\gamma}\tilde{\theta}^{\gamma}_{\beta}) \wedge \theta^{\beta'} + (\tilde{\theta}^{-1})^{\alpha}_{\delta}\omega^{\delta}_{\gamma}\tilde{\theta}^{\gamma}_{\beta} \wedge (\tilde{\theta}^{-1})^{\beta}_{\sigma}\theta^{\sigma} \\
&= ([d(\tilde{\theta}^{-1})^{\alpha}_{\gamma} + (\tilde{\theta}^{-1})^{\alpha}_{\delta}\omega^{\delta}_{\gamma}]\tilde{\theta}^{\gamma}_{\beta}) \wedge \theta^{\beta'} \\
&=: \omega^{\alpha'}_{\beta} \wedge \theta^{\beta'}
\end{aligned}
\tag{D.292}
$$

Thus

$$
d\theta^{\alpha'} = df^{\alpha'} \wedge dx^m = \omega^{\alpha'}_{\beta} \wedge [d\tilde{z}^{\beta} + f^{\beta'} dx^m]
$$

$(y^1, \ldots, y^{m-r-1}, \ldots, \tilde{z}^1, \ldots, \tilde{z}^r, x^m)$

thus

$$
\theta^{\alpha'}(x) = d\tilde{z}^{\alpha} + f^{\alpha'}(\tilde{z}, x^m)dx^m.
$$

Define the 'time' x^m-dependent 'Hamiltonian'

$$
H(\{\tilde{z}^{\beta}, p_{\beta}\}^{r}_{\beta=1}; x^m) := -f^{\alpha'}(\{\tilde{z}\}, x^m)p_{\alpha} \tag{D.293}
$$

with 'Hamiltonian equations'

$$
\begin{aligned}
\frac{d\tilde{z}^{\alpha}}{dx^m} &= \frac{\partial H}{\partial p_{\alpha}} \\
\frac{d\tilde{p}_{\alpha}}{dx^m} &= -\frac{\partial H}{\partial \tilde{z}^{\alpha}}
\end{aligned}
\tag{D.294}
$$

These are equaivalent to

$$
\begin{aligned}
\frac{d\tilde{z}^{\alpha}}{dx^m} &= -f^{\alpha'}(\{\tilde{z}\}, x^m) \\
\frac{dp_{\alpha}}{dx^m} &= \frac{\partial f^{\alpha'}}{\partial \tilde{z}^{\alpha}}(\{\tilde{z}\}, x^m)p_{\alpha}
\end{aligned}
\tag{D.295}
$$

Note that p_{α} has no effect on the solution of the first set of these equations. Now the 'Hamiltonian equations' can be rexpressed by an associated Hamilton-Jacobi equation

Recall that Hamilton's equations which represent a system of first order ODEs two for each coordinate, can be rewritten of the HJE which is a single non-linear PDE of one variable for each coordinate.

$$\frac{\partial S}{\partial x^m}(\tilde{z}, x^m) + H\left(\left\{\tilde{z}^\beta, p_\beta = \frac{\partial S}{\partial \tilde{z}^\beta}\right\}; x^m\right) = 0 \tag{D.296}$$

Solves our problem elegantly.

From the theory of partial differential equations, we have the result that all first order equations have a solution that is unique up to an arbitrary function. Although the solutions of ordinary differential equations depend on constants of integration, the solution of a partial differential equation depends on functions. It is not clear how to obtain a solution S of the HJ equation that will depend on the n constants. There are many such solutions, called complete solutions.

A complete integral of the Hamilton-Jacobi equation is an n-parameter family of solutions satisfying

$$\det\left(\frac{\partial^2 S}{\partial q^a \partial q'^b}\right) \neq 0.$$

Diversion into general integrals:

Recall that every first order partial differential equation has a solution depending on an arbtitrary function: such a solution is called the general integral of the equations.

The independent variables in the Hamilton-Jacobi equation are the time and the coordinates qs. For a system with s degrees of freedom, therefore, a complete integral of this equation must contain $s+1$ arbitrary constants. Since the function S enters the equation only through its derivatives, one of these constants is additive, so that a complete integral of the Hamilton-Jacobi equation is

$$S = F(t, q_1, \ldots, q_s; \alpha_1, \ldots, \alpha_s) + A,$$

where $\alpha_1, \ldots, \alpha_s$ and A are arbitrary constants **reference**.

Let us now asertain the relation between a complete integral of the Hamilton-Jacobi equation and the solution of the equations of motion which is of interest. To do this, we effect a canonical transformation from the variables q and p to new variables, taking the function $F(t, q; \alpha)$ as the generating function, and the quantities $\alpha_1, \ldots, \alpha_s$ as the new moenta. Let the new coordinates be $\beta_1, \beta_2, \ldots, \beta_s$. Since the generating function depends on the old coordinates and the new momenta, we use the formula for F_2 type:

$$p_i = \frac{\partial f}{\partial q_i}, \quad \beta_i = \frac{\partial f}{\partial \alpha_i}, \quad \mathcal{H}' = \mathcal{H} + \frac{\partial f}{\partial t}.$$

But since the function f satisfies the Hamilton-Jacobi equation, we see that the new Hamiltonian is zero: $\mathcal{H}' = \mathcal{H} + \partial f/\partial t = \mathcal{H} + \partial S/\partial t = 0$. Henec the canonical equations in the new variables are $\dot{\alpha}_i = 0$, $\dot{\beta}_i = 0$, whence

$$\alpha_i = const. \qquad \beta_i = const.$$

By means of the s equations $\partial f/\partial \alpha_i = \beta_i$, the s coordinates q can be expressed in terms of the time and the $2r$ constants α and β. This gives the general integral of the equations of motion.

Thus the solution of the problem of the motion of a mechaniacl systmem by the Hamilton-Jacobi method proceeds as follows. From the Hamiltonian, we form the Hamilton-Jacobi equation, and find its complete integral. Differentiating this with respect to the arbitrary constants α and equating the derivatives to new constants β, we obtain s algebraic equations

$$\partial S/\partial \alpha_i = \beta_i,$$

whose solution gives the coordinates q as a function of time and of the $2s$ arbitrary constants. The momenta s functions of time may them be found from the equations

$$p_i = \frac{\partial S}{\partial q_i}.$$

refering:

The general solution of the Hamilton-Jacobi equation can be found from the complete integral. To do this we regard A as an arbtrary function of the remaining constants:

$$S = F(x^m, z^1, \ldots, z^s; \alpha_1, \ldots, \alpha_s) + A(\alpha_1, \ldots, \alpha_s).$$

Replacing the α_i by functions of coordinates and time given by the s conditions

$$\partial S/\partial \alpha_i = 0,$$

we obtain the general integral in terms of the arbitrary function $A(\alpha_1, \ldots, \alpha_s)$. For, when the function S is obtained in this way, we have

$$\frac{\partial S}{\partial q_i} = \left(\frac{\partial S}{\partial q_i}\right)_\alpha + \sum_k \left(\frac{\partial S}{\partial \alpha_k}\right)_q \frac{\partial \alpha_k}{\partial q_i} = \left(\frac{\partial S}{\partial q_i}\right)_\alpha$$

The quantities $(\partial S/\partial q_i)_\alpha$ satisfy the Hamilton-Jacobi equation, since the function $S(t, q; \alpha)$ is assumed to be a complete integral of that equation. The quantities $\partial S/\partial q_i$ therefore satify the same equation.

Contiuing with the proof of Frobenius' theorem.

The general integral is obtained from the complete integral as follows: prescribe an arbitrary function $C = F(c^\alpha)$ and solve the system of algebraic equations

$$\frac{\partial S}{\partial c^\alpha} + \left(\frac{\partial S}{\partial C}\right)_{C=F(c)} \frac{\partial C}{\partial c^\alpha} = 0 \tag{D.297}$$

for $c^\alpha = f^\alpha_C(\tilde{z}, x^m)$, which is always possible by the implicit function theorem. Specialise to the case that $F = c^\alpha =: F^\alpha(c)$ and define

$$z^{\alpha'}(\tilde{z}, x^m) = S(\tilde{z}, x^m; C = F^\alpha(c(\tilde{z}, x^m)), \quad c = f_{F^\alpha}(\tilde{z}, x^m) \tag{D.298}$$

(D.298) still solves the Hamilton-Jacobi equation:

These r solutions are algebraically independent:

Hence

$$dz^{\alpha'} = \frac{\partial z^{\alpha'}}{\partial \tilde{z}^\beta}(\tilde{z}, x^m)\theta^{\beta'} \tag{D.299}$$

accomplishes our task since $\partial z'/\partial \tilde{z}$ is invertible.

□

Subspaces of symplectic vector spaces

Let us also denote a foliation of D as D.

We mention without proof that a quotient space of a Hausdorff space may not be Hausdorff.

Definition A foliation is called **reducible** provided that the space of leaves $\mathcal{M}/D = \{[p] : p \in \mathcal{M}\}$, $[p] = \{p' \in \mathcal{M} : p, p'$ lie in the same leaf$\}$ is a Hausdorff manifold with smooth projection $\pi : \mathcal{M} \to \mathcal{M}/D$.

□

Definition Let $\mathcal{N}$ be a submanifold of $(\mathcal{M}, \omega)$. Given a closed two-form σ on $\mathcal{N}$ we call $(\mathcal{N}, \sigma)$ a presymplectic (or Poisson) submanifold.

□

Definition If σ is degenerate we call

$$K : \mathcal{N} \to T(\mathcal{N}); \qquad K_p(\mathcal{N}) = \{v \in T_p(\mathcal{N}) : i_v\sigma = 0\}$$

the characteristic distribution of $(\mathcal{N}, \sigma)$. $\mathcal{N}$ is then said to be reducible if K is reducible.

□

Lemma D.5.5 *Every presymplectic manifold is integrable. If $\mathcal{N}$ is reducible then $\mathcal{N}/K$ carries a natural symplectic structure.*

Proof:

□

Definition Let (V, ω) be a sympletic vector space (i.e., $\mathcal{M} = V$ is a vector space) and let F be a subspace of V. Then complement of F is the subspace

$$F^{\perp} = \{X \in V : \omega(X, Y) = 0 \text{ for all } Y \in F\}$$

and is called the annihilator of F.

□

Definition A subspace F of a sympletic vector space is said to be

(i) isotropic whenever $F \subset F^{\perp}$

(ii) coisotropic whenever $F^{\perp} \subset F$

(iii) Lagrangian whenever $F = F^{\perp}$

(iv) sympletic whenever $F \cap F^{\perp} = \{0\}$.

□

Lemma D.5.6 *Let F and G be subspaces of a sympletic vector space (V, ω). Then*

(1) $F^{\perp} \supset G^{\perp}$ whenever $F \subset G$,

(2) $(F^{\perp})^{\perp} = F$,

(3) $(F + G)^{\perp} = F^{\perp} \cap G^{\perp}$,

(4) $(F \cap G)^{\perp} = F^{\perp} + G^{\perp}$

Proof:

(1): As G is a larger set including F, it puts a greater restriction on the elements of V that can be in the complement than does F.

(2): $(F^{\perp})^{\perp}$ is defined by $\{X \in V : \omega(X,Y) = 0 \text{ for all } Y \in F^{\perp}\}$. Obviously, if $X \in F$ then $X \in (F^{\perp})^{\perp}$. Say we had $X \in (F^{\perp})^{\perp}$ but $X \notin F$ then there is $G \supset F$ such that $Y \in G^{\perp}$ but this contradicts (1).

(3): $(F+G)^{\perp}$ is defined by $\{X \in V : \omega(X,Y_1) = 0 \text{ for all } Y_1 \in F \text{ and } \omega(X,Y_2) = 0 \text{ for all } Y_2 \in G\} \equiv F^{\perp} \cap G^{\perp}$.

(4): We prove this using (2) and (3):

$$\begin{aligned} F^{\perp} + G^{\perp} &= (F^{\perp} + G^{\perp})^{\perp\perp} \\ &= (F^{\perp\perp} \cap G^{\perp\perp})^{\perp} \\ &= (F \cap G)^{\perp}. \end{aligned}$$

□

Lemma D.5.7 *If dim$(V) = 2m$, dim$(F) = k$, that is, dim$(F^{\perp}) = 2m - k$ then*

(1) isotropic implies $k \leq m$,

(2) co-isotropic implies $k \geq m$,

(3) Lagrangian implies $k = m$,

(4) symplectic implies $k = 2n$ is even.

Proof: $\dim(F^{\perp}) = 2m - k$ follows from the non-degenercay of ω. Let $\{e_1, \ldots, e_k\}$ be a basis of F; we have

$$F^{\perp} = \{X \in V : \omega(X, e_j) = 0 \ \text{ for } \ j = 1, \ldots, k\}$$

Let $\{e_1, \ldots, e_{2m}\}$ be a basis of V and write

$$X = \sum_{k=1}^{2m} \alpha_k e_k$$

then $F^{\perp}$ is defined by the equations

$$\sum_{k=1}^{2m} \omega(\alpha_k e_k, e_j) = 0 \ \text{ for } \ j = 1, \ldots, k.$$

which are k independent linear equations, for $\dim(V)$ unknowns, hence

$$\dim(F^{\perp}) = \dim(V) - k = 2m - k.$$

(1) $\dim F \leq \dim F^{\perp}$ implies $k \leq 2m - k$.

(2) $\dim F \geq \dim F^{\perp}$ implies $k \geq 2m - k$.

(3) $\dim F = \dim F^{\perp}$ implies $k = 2m - k$.

(4) First say $k = 1$. Let e_1 be any vector in F. We can find f_1 such that $\omega(f_1, e_1) \neq 0$ (the existence of f_1 follows from the non-degeneracy of ω) but this contracdicts that $F \cap F^{\perp} = 0$. Now say $k = 2n + 1$. Let M_1 be a subspace of V spanned by $\{e_1, f_1\}$ and set $V_1 = M_1^{\perp}$. Note

(a) since $\omega(f_1, e_1) \neq 0$ we have $V_1 \cap M_1 = 0$.

(b) The restriction ω_1 of ω to V_1 is non-degenerate: if $X_1 \in V_1$ is such that $\omega_1(X_1, Y) = 0$ for all $Y \in V_1$, then $X_1 \in V_1^{\perp} = M_1$ and hence the only element in V_1 satisfying the condition is 0. Thus (V_1, ω_1) is by itself a symplectic vector space.

(c) As $e_1, f_1 \in F$, $F^{\perp} \subset V_1$.

If $k = 3$

Now choose some vector $e_2 \in V_1$, we can find $f_2 \in V_1$ such that $\omega_1(f_2, e_2) \neq 0$ by the non-degeneracy of ω_1.

We can normailse so that we have

$$\omega(e_1, e_2) = \omega(f_1, f_2) = 0, \quad \omega(f_i, e_j) = \delta_{ij} \text{ for } i, j = 1, 2.$$

The vectors $\{e_1, e_2; f_1, f_2\}$ are linearly independent:

$$\omega(e_1, a_1 e_1 + a_2 e_2 + b_1 f_1 + b_2 f_2) = 0$$

implies $b_1 = 0$, we similarly get $b_2 = 0, a_1 = 0, a_2 = 0$ by contracting with e_2, f_1, f_2 respectively. Hence E_2 has dimension 4.

Continuing this construction another $n - 2$ times we obtain a set of linearly independent vectors $\{e_1, \ldots, e_n; f_1, \ldots, f_n\}$ in V and arrives at a symplectic vector space (V_n, ω_n) containing $F^{\perp}$ which is of dimension $\dim(F^{\perp}) + 1$. But we have shown we cannot have this, therefore we must have $k = 2n$ for some integer n.

□

Note tha Lagrangian subspaces always have half the dimension of the original symplectic vector space.

Notice that every symplectic vector space has subspaces of either category but that the categories (a)-(d) are not exhaustive, for example, $F \cap F^{\perp} \neq \{0\}$ is possible while neither $F \subset F^{\perp}$ nor $F^{\perp} \subset F$.

Dirac's formalism of constraints and classification of constraints

We can now generalise these defintions from symplectic vector spaces to syplectic manifolds.

Definition A submanifold C of a symplectic manifold $(\mathcal{M}, \omega)$ is isotropic, coisotropic, Lagrangian or syplectic if its tangent space is of the corresponding type as a subspace of $T_p\mathcal{M}$ at every $p\ inC$.

□

The connection of this terminology with Dirac's formulism of constraints is as follows: if $(\mathcal{M}, \omega)$ is a given unconstrained phase space with a system of constraints

$$C_I, \quad I = 1, \dots, r$$

then we define the constraint submanifold as

$$\mathcal{N} = \{p \in \mathcal{M} : C_I(p) = 0\}$$

We will see that, in Dirac's terminology that a coisotropic submanifold is a first class constraint (or, rather, a family of first class constraints), and a symplectic submanifold is a second class constraint.

Projection $\rho : \mathcal{M} \to \mathcal{N}$ which is obtained by using the C_I as local coordinates (owing to the implicit function theorem) and defining ρ to be the map that sets $C_I = 0$.

Definition When K is coisotropic, the subspaces $\mathcal{K}_p = (T_p(K))^{\perp}$ comprise a sub-bundle $(T(K))^{\perp}$ of $T(K)$ known as the **characteristic distribution** of K.

□

If K is the characteristic distribution of $(\mathcal{N}, \sigma)$ then $\mathcal{N}/K$ with symplectic structure τ is called the reduced phase space.

We can now move on to the classification of constraints.

(a) isotropic $T_p(\mathcal{N}) \subset (T_p(\mathcal{N}))^{\perp}$

(b) co-isotropic $(T_p(\mathcal{N}))^{\perp} \subset T_p(\mathcal{N})$

Then

Since we can always find a basis of $T_p(\mathcal{M})$ consisting of Hamiltonian vector fields because ω is non-generate we can always find a basis of K_p consisting of Hamiltonian vector fields χ_f.

Constrints are first class if the Hamiltonian constraint vector fields are tangential to the constraint surface.

(c) symplectic $(T_p(\mathcal{N}))^\perp \cap T_p(\mathcal{N}) = \{0\}$

D.5.9 Poisson Reduction

$(\mathcal{M})$ - Poisson manifold that's our phase space
$f : (\mathcal{M}) \to \mathbb{R}$ smooth

$C^\infty(\mathcal{M})$ the space of smooth phase space functions

$D^\infty(\mathcal{M})$ the set of smooth, weak Dirac observables.

Poisson Reduction: 2 steps process:

$(\mathcal{M}$ "kinematic phase space" $f = 0 \subset \mathcal{M}$ constraint surface

"kinematic observables" $C^\infty(\mathcal{M})$ Poisson algebra $\to$ Quotient algebra $C^\infty(\mathcal{M})/[f]$

Equivalence class of functions modulo functions which vanish on the constraint surface $\mathcal{C}$. $g \sim g'$ if

$$g = g' + g''f \quad \text{for every } g`` \in C^\infty(\mathcal{M}) \tag{D.300}$$

We denote the equivalence class of g as $[g]$

The space of smooth functions on phase space which Poisson commute with the constraint function f modulo all functions which vanish on the constraint surface:

$$\{g \in C^\infty(\mathcal{M}) : \{f, g\} = 0\}/[f] \tag{D.301}$$

D.5.10 Symplectic Group Actions

Lie$(G)^*$ is the space of linear forms on Lie(G) is called a momentum map.

A central extension of a Lie algebra Lie(G) is a Lie algebra E together with a homomorphism $\pi : E \to \text{Lie}(G)$ such that $\text{Ker}(\pi) \subset Z(E)$ where $Z(E) = \{A \in E : [A, B] = 0 \text{ for all } B \in E\}$ is the centre of E.

D.6 Worked Examples

Worked example: Alternative sets of constraints

One can replace the set of constraints $\{C_1,\ldots,C_n\}$ by another set of constraints $\{\tilde{C}_1,\ldots,\tilde{C}_n\}$ as long as there constraint hypersurfaces $\mathcal{C}$ and $\tilde{\mathcal{C}}$ coincide.

(a) Show that this is guaranteed if one can write

$$\tilde{C}_j = \sum_k A_{jk} C_k \tag{D.302}$$

where A_{jk} is a phase space matrix function with non-vanishing determinant on $\mathcal{C}$.

(b) Show that the two sets of constraints lead to the same gauge orbits $\mathcal{G}_x$ if $x \in \mathcal{C}$.

(c) Prove that Weak Dirac observables with respect to one set of constraints are also weak Dirac observables with respect to the second and vice versa.

(d) Why does this not hold for strong Dirac observables?

Proof:

(a) As A_{jk} is invertible we can write

$$C_k = \sum_k (A^{-1})_{jk} \tilde{C}_k. \tag{D.303}$$

The condition $C_k = 0$ implies

$$\sum_k (A^{-1})_{jk} \tilde{C}_k = 0. \tag{D.304}$$

Multiplying this on both sides by A_{lj} and summing over j gives

$$\delta_{lk} \tilde{C}_k = \tilde{C}_l = 0 \tag{D.305}$$

for all $l = 1,\ldots,n$.

(b)

(c) Recall that a Dirac observable F satisfies the condition $\{C_j, F\} = 0$ on the constraint hypersurface $\mathcal{C}$. By the product rule of differentiation,

$$\{\tilde{C}_j, F\} = \{A_{jk} C_k, F\} = A_{jk}\{C_k, F\} + C_k\{A_{jk}, F\}. \tag{D.306}$$

Hence on the constraint surface $\mathcal{C}$ we have

$$\{\tilde{C}_j, F\} \approx A_{jk}\{C_k, F\} \tag{D.307}$$

($\approx$ means equality holds on the constraint surface). As A_{jk} is invertible the condition $\{\tilde{C}_j, F\} = 0$ implies $\{C_j, F\} = 0$.

(e)

$$F_{[f,T]}(\tau, x) \approx \tilde{F}_{[f,T]}(\tau, x) \tag{D.308}$$

Worked example: Dirac bracket as the induced Poisson bracket on the submanifold $\mathcal{T}$.

define Hamiltonian vector

$$X_\alpha^\gamma := \Omega^{\gamma\beta}\partial_\beta \chi_\alpha \tag{D.309}$$

The induced Poisson bracket on the submanifold $\mathcal{T}$ is given by the Dirac bracket.

No linear combination $a^\alpha X_\alpha^\gamma$ can be tangent to the constraint hypersurface.

Proof:

(a)

The proof is by contradiction: we assume that $a^\alpha X^\alpha$ is a vector tangental to the constraint hypersurface and then show that this can only be the case if every coefficient a^α is weakly zero. A vector is tangent to the surface $\chi_\alpha = 0$ if and only if $Y^\beta \partial_\beta \chi_\alpha = 0$. $Y^\beta = a^\alpha$

for soof

$$\begin{aligned} 0 &\approx a^\beta X_\beta^\gamma \partial_\gamma \chi_\alpha \\ &\approx a^\beta \partial_\beta \chi_\alpha \Omega^{\gamma\beta} \partial_\gamma \chi_\alpha \\ &\approx a^\beta f_{\beta\gamma}{}^\alpha \end{aligned} \tag{D.310}$$

$$a^\beta f_{\beta\gamma}{}^\alpha \approx 0 \quad \Rightarrow \quad a^\beta \approx 0 \tag{D.311}$$

(b)

For arbitrary phase space function f define

$$f^* = f - \{f, \chi_j\}(B^{-1})_{jk}\chi_k \tag{D.312}$$

$$\{f, g\}^* = \{f, g\} - \{f, \cdot_\mu\}A^{\mu\nu}\{\cdot_\nu, \cdot\} \tag{D.313}$$

D.7 Open Constraint Algebras

when one has structure functions (functions on phase space) instead of structure constants.

D.7.1 The Master Constraint

$$\{C_I, C_J\} = f_{IJ}{}^K(q,p)C_K \tag{D.314}$$

The Master constraint $\mathbf{M} = 0$ imposes all the individual constraints $C_j = 0$ simultaneously.

$$M : \frac{1}{2}C_a Q^{ab}(q,p)C_b \tag{D.315}$$

where $Q^{ab} = Q^{ba}$.

$$\begin{aligned} \{C_I, \mathbf{M}\} &= \frac{1}{2}\{C_I, C_a Q^{ab}(q,p)C_b\} \\ &= \{C_I, C_a\}Q^{ab}(q,p)C_b + \{C_I, Q^{ab}(q,p)\}C_a C_b \\ &\approx 0. \end{aligned} \tag{D.316}$$

$$\{\mathbf{M}, \mathbf{M}\} = 0. \tag{D.317}$$

D.8 Partial, Complete and Dirac Observables for Hamiltonian Constrained Systems

If we

$\alpha_C^t(x)$ is the flow generated by the constraint C starting from the point x.

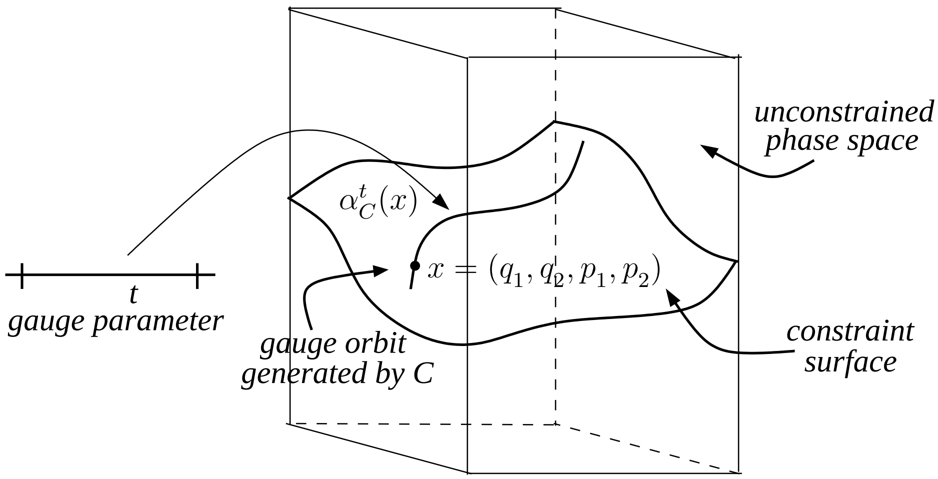


Figure D.7: partComptDitt1.

The value of the function $\alpha_C^t(f)$ at the point x is given by

$$\alpha_C^t(f)(x) := f(\alpha_C^t(x)) \tag{D.318}$$

It can be calculated with the series

$$\alpha_C^t(f(x)) = \sum_{r=0}^{\infty} \frac{1}{r!}\{C, f(x)\}_r \tag{D.319}$$

Partial observables

"a physical quantity to which we can associate a (measuring) procedure leading to a number", [32]. If we assume that one can associate to an arbitrary phase space function such a measuring procedure, then any phase space function is a partial observable. Partial observables need not be a Dirac observable.

Complete observables

A complete observable is "a quantity whose value can be predicted by the theory (in classical theory)".

$$F_{[f,T]}(\tau, x) := \alpha_C^t(f(x))_{|\alpha_C^t(T(x))=\tau} \tag{D.320}$$

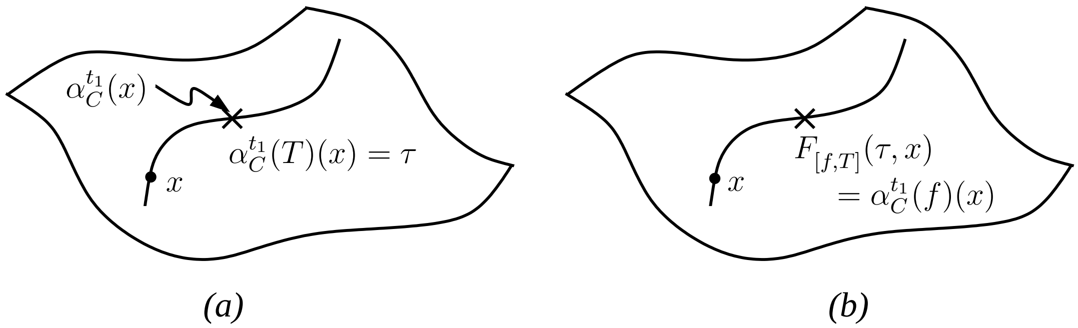


Figure D.8: partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^t(x))$ assumes the value τ. (b) The function $F_{[f,T]}(\tau, x)$ gives the value that the function $f(\alpha_C^t(x))$ assumes if the function $T(\alpha_C^t(x))$ assumes the value τ. $F_{[f,T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$.

Complete observables as Gauge invariant extensions of gauge restricted functions

through each point x of the constraint hypersurface there is given a gauge orbit $\mathcal{G}_x$ and on each gasuge orbit there is exactly one point y with $T(y) = \tau$, i.e. which satisfies the gauge condition.

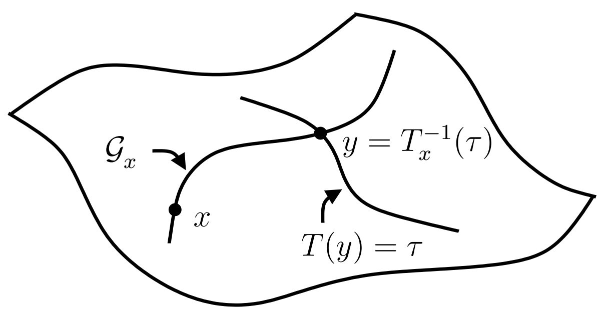


Figure D.9: comptasGIE.

Hence complete observables are simply gauge invarint extensions of gauge restricted functions.

D.8.1 Weak Dirac Observables

Theorem D.8.1 *Let f, T be two phase space functions and $x \in \mathcal{M}$ a phase space point, fulfilling the condition: $\alpha_C^t(f)(x) = \alpha_C^s(f)(x)$ for all values $s, t \in \mathbb{R}$ for which $\alpha_C^t(T)(x) = \alpha_C^s(T)(x)$. Then $F_{[f,T]}(\tau, x)$ is invariant under the flow generated by $\mathcal{C}$.*

Proof

Obviously we have

$$\alpha_C^t\big(\alpha_C^\epsilon(f)(x)\big) = \alpha_C^{t+\epsilon}(f)(x) \quad \text{and} \quad \alpha_C^t(T)(\alpha_C^\epsilon(x)) = \tau \quad \Longleftrightarrow \quad \alpha_C^{t+\epsilon}(T)(x) = \tau.$$

So that

$$\begin{aligned}
\alpha_C^\epsilon\big(F_{[f,T]}(\tau,x)\big) = F_{[f,T]}(\tau,\alpha_C^\epsilon(x)) &= \alpha_C^t(f)(\alpha_C^\epsilon(x))_{|\alpha_C^t(T)(\alpha_C^\epsilon(x))=\tau} \\
&= \alpha_C^{t+\epsilon}(f)(x)_{|\alpha_C^{t+\epsilon}(T)(x)=\tau} \\
&= \alpha_C^s(f)(x)_{|\alpha_C^s(T)(x)=\tau} \\
&= F_{[f,T]}(\tau,x)
\end{aligned} \tag{D.321}$$

The last line follows from the fact that $s = t + \epsilon$ is just a dummy variable.

Examples

Example 0. Parameterized clock harmonic + oscillator.

$$\begin{aligned}
\alpha_C^t(T) &= q_1 + t \\
\alpha_C^t(f) &= q_2 \cos t + \frac{p_2}{\omega} \sin t = \sqrt{q_2^2 + (p_2^2/\omega^2)} \sin(t + \arctan \frac{q_2 \omega}{p_2})
\end{aligned} \tag{D.322}$$

Example 1.

$$C = q_1 p_2 - q_2 p_1 \tag{D.323}$$

$$\begin{aligned}
\alpha_C^t(T) &= q_1 \cos t + q_2 \sin t = \sqrt{q_1^2 + q_2^2} \sin(t + \arctan \frac{q_1}{q_2}) \\
\alpha_C^t(f) &= q_2 \cos t - q_1 \sin t = \sqrt{q_1^2 + q_2^2} \cos(t + \arctan \frac{q_1}{q_2})
\end{aligned} \tag{D.324}$$

Example 2. Two harmonic oscillators with constrained energy difference.

$$C = \frac{1}{2}(p_1^2 + \omega_1^2 q_1^2) - \frac{1}{2}(p_2^2 + \omega_2^2 q_2^2)$$

$$\begin{aligned}
\alpha_C^t(T) &= q_1 \cos(\omega_1 t) + p_1 \sin(\omega_1 t) = \sqrt{q_1^2 + (p_1/\omega_1)^2} \sin(\omega_1 t + \arctan \frac{\omega_1 q_1}{p_1}) \\
\alpha_C^t(f) &= q_2 \cos(\omega_2 t - p_2 \sin(\omega_2 t) = \sqrt{q_2^2 + (p_2/\omega_2)^2} \sin(\omega_2 t + \arctan \frac{\omega_2 q_2}{p_2})
\end{aligned} \tag{D.325}$$

D.8.2 Backreaction

A key feature of this work is, that we keep the zeroth order variables as fully dynamical phase space variables and not just as parameters describing the background universe as one would do in a perturbation around a fixed phase space point. Indeed we have to keep the zeroth order variables as canonical variables to allow for a consistent gauge invariant framework to higher than linear order. Moreover this provides a very natural description for backreaction effects: these arise as higher order corrections to observables arising through averaging of (time evolved) phase space variables. Since this approach is gauge invariant it could shed some light on the discussion whether these backreactions are measurable effects or caused by a specific choice of gauge, see for instance [??, ??, ??]. As already mentioned we have to choose clocks, which define also the hypersurfaces (by physical criteria, e.g. by demanding that a scalar field is constant on these hypersurfaces) over which the averaging is performed. Therefore the observables describing the backreacion effects depend on the choice of clocks. However, as we will see, one can find relations between the gauge invariant observables corresponding to one choice of clocks and the gauge invariant observables corresponding to another choice of clocks.

D.8.3 Different Observers

We consider the transformation between complete observables defined with respect to different choices of clocks (which can be understood as representing different families of observers).

We consider the transformation between complete observables defined with respect to different choices of clocks (which can be understood as representing different families of observers). We define lapse and shift functions, which allows us to compare this canonical approach to covariant approaches. Lapse and shift function determine foliations of spacetimes. These foliations are defined by the choice of clock variables, i.e. by physical conditions.

D.8.4 Systems with Several Constraints

f and T_j functions on phase space.

Consider the flow generated by $\alpha^t_{\beta_k C_k}(x)$ the linear combination of constraints $\sum_k \beta_k C_k$ starting from the phase space point x. The function $F_{[f,T_i]}(\tau_i, x)$ gives the value that the function $f_x(t) := f(\alpha^t_{\beta_k C_k}(x))$ assumes if the functions $T_i(\alpha^t_{\beta_k C_k}(x))$, $i = 1, \ldots, n$, assume the values τ_i respectively.

$$F_{[f,T]}(\tau, x) := \alpha^t_{\beta_k C_k}(f(x))_{|\alpha^t_C(T_i(x))=\tau_i} \tag{D.326}$$

$$\alpha_{\beta_i C_i}(T_i(x)) = \tau_i, \quad i = 1, \ldots, n \tag{D.327}$$

for $\beta_1, \ldots, \beta_n$.

Alternative linear set of constraints

Finally we would like to mention that one can replace the set of constraints $C_1, \ldots, C_n$ by another set of constraints $\{\tilde{C}_1, \ldots, \tilde{C}_n\}$ as long as the constraint hypersurfaces defined by these two sets coincide. This is guaranteed if one can write

$$\tilde{C}_j = \sum_k A_{jk} C_k$$

where $(A_{jk}), k = 1$ is a matrix of phase space functions with non-vanishing determinant on $\mathcal{C}$. The two sets of constraints lead also to the same gauge orbits $\mathcal{G}_x$ if $\mathcal{C}$. Moreover weak Dirac observables with respect to the first set are also weak Dirac observables with respect to the second and vice versa. This does not hold for strong Dirac observables.

D.8.5 Partial Differential Equations for Complete Observables

1st Derivation

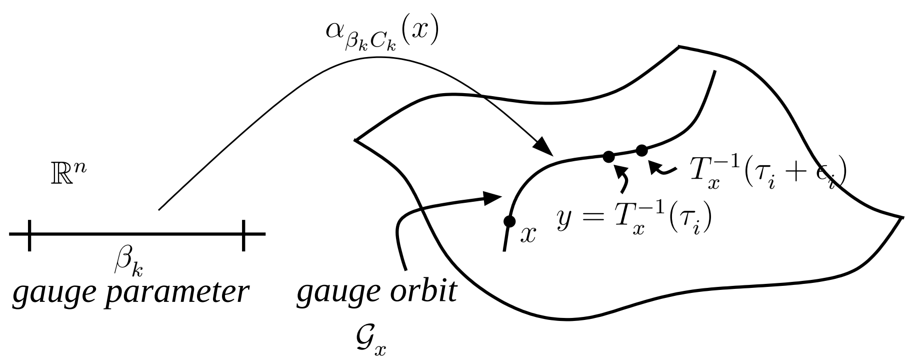


Figure D.10: partComptDitt4.

We wish to find $F_{[f,T_i](\tau_i+\epsilon_i)}$, that is, the value of f for $\alpha_{\beta_k C_k}(T_i) = \tau_i + \epsilon_i$. We need γ_k such that for $y = \mathbf{T_x^{-1}}(\tau_i)$

$$\tau_i + \epsilon_i = \alpha_{\gamma_k C_k}(y) = \tau_i + \sum_k \gamma_k \{C_k, T_j\}(y) + \mathcal{O}(\epsilon^2) \tag{D.328}$$

we define a matrix

$$A_{kj} := \{C_k, T_j\} \tag{D.329}$$

with inverse A^{-1}_{jm} by $A_{kj} A^{-1}_{jm} = \delta_{km} = A^{-1}_{kj} A_{jm}$. (This inverse exists because of the assumptions we made about the map T^{-1}.) The solution of the equations (4.3.4) can then be written as

$$\gamma_k = \sum_j \epsilon_j \; A^{-1}_{jk}(y) + \mathcal{O}(\epsilon^2). \tag{D.330}$$

Inserting these values into

$$\alpha_{\gamma_k C_k}(f)(y) = f(y) + \sum_k \gamma_k \{C_k, f\}(y) + \mathcal{O}(\gamma^2) \tag{D.331}$$

we arrive at

$$\begin{aligned} F_{[f,T_i]}(\tau_i + \epsilon_i) &= f(y) + \sum_{j,k} \epsilon_j \; \left(A^{-1}_{jk}\{C_k, f\}\right)(y) + \mathcal{O}(\epsilon^2) \\ &= F_{[f,T_i]}(\tau_i, x) + \sum_{j,k} \epsilon_j \; \left(A^{-1}_{jk}\{C_k, f\}\right)(y) + \mathcal{O}(\epsilon^2) \end{aligned} \tag{D.332}$$

This gives us the differential equation for $F_{[f,T_i]}(\tau_i)$ in τ_m

$$\frac{\partial}{\partial \tau_m} F_{[f,T_i]}(\tau_i, x) = \left(\sum_k A^{-1}_{mk}\{C_k, f\}\right)(\mathbf{T_x^{-1}}(\tau_i)) =: g_m(y)(\mathbf{T_x^{-1}}(\tau_i)). \tag{D.333}$$

2nd Derivation

By definition of the complete observable $F_{[f,T_i]}$ we have the equation

$$F_{[f,T_i]}(\tau_i = T_i(y), x) = f(y) \tag{D.334}$$

where y is a point in the gauge orbit $\mathcal{G}_x$ of through x. Hence we can also write

$$F_{[f,T_i]}(T_i(\alpha^t_{\gamma_k C_k}(y)), x) = f(T_i(\alpha^t_{\gamma_k C_k}(y)) \tag{D.335}$$

Differentiating both sides with respect to t gives

$$\sum_m \frac{\partial}{\partial \tau_m} F_{[f,T_i]}(T_i(\alpha^t_{\gamma_k C_k}(y)), x) \frac{dT_m}{dt}(\alpha^t_{\gamma_k C_k}(y)) = \frac{dT_m}{dt} f(T_i(\alpha^t_{\gamma_k C_k}(y)) \tag{D.336}$$

and set $t = 0$:

$$\sum_{m,k} \frac{\partial}{\partial \tau_m} F_{[f,T_i]}(T_i(y), x) \gamma_k \{T_k, f\}(y) = \sum_k \gamma_k \{C_k, f\}(y) \tag{D.337}$$

This equation has to hold for an arbitrary set of $\gamma_k \in \mathbf{R}; k = 1, \ldots, n$. Hence we conclude that

$$\sum_m \frac{\partial}{\partial \tau_m} F_{[f,T_i]}(T_i(y), x)\{T_k, f\}(y) = \{C_k, f\}(y) \tag{D.338}$$

holds for $k = 1, \ldots, n$. This gives

$$\frac{\partial}{\partial \tau_m} F_{[f,T_i]}(\tau_i, x) = \left(\sum_k A^{-1}_{mk} \{C_k, f\} \right) (T_x^{-1}(\tau_i)) \tag{D.339}$$

$$g_{(k_1,\ldots,k_n)} = X_1^{k_1} \cdot \cdots X_n^{k_n} \cdot f \tag{D.340}$$

Closed system of higher order PDE's

However, one can limit the number of necessary iterations if one realizes that if a function g is composed of m phase space functions f_h the associated complete observable is ....

Consistency condition on PDE's

consistency condition on PDE's:

$$\frac{\partial^2}{\partial \tau_i \partial \tau_j} F_{[f,T_i]} = \frac{\partial^2}{\partial \tau_j \partial \tau_i} F_{[f,T_i]} \tag{D.341}$$

Weak Dirac observable there are n T_j's

$$F^{\tau}_{[f,T]} := \sum_{k_1,\ldots,k_n=0} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} (X_1)^{k_1} \ldots (X_n)^{k_n} \cdot f. \tag{D.342}$$

where $X_r \cdot f$ is defined as

$$X_j \cdot f := \{(A^{-1})_{jk} C_k, f\}, \quad A_{jk} := \{C_j, T_k\}. \tag{D.343}$$

The map $F_{[T_i]}(\tau_i) : f \mapsto F_{[f,T_i]}(\tau_i, x)$ is an algebra homomorphism:

The map from the space of partial observables to the space of complete observables $F_{[f,T_i]}(\tau_i, x)$ is an algebra homomorphism with respect to multiplication and addition; that is to say, it preserves the multiplication and addition properties of the partial observables f in the sense that

$$F_{[f,T_i]}(\tau_i,x)F_{[f',T_i]}(\tau_i,x) = F_{[ff',T_i]}(\tau_i,x), \tag{D.344}$$

and

$$F_{[f,T_i]}(\tau_i,x) + F_{[f',T_i]}(\tau_i,x) = F_{[f+f',T_i]}(\tau_i,x), \tag{D.345}$$

(exercise). As noted in [108], we can make this homomorphism exact by dividing both $C^\infty(\mathcal{M})$ and $D^\infty(\mathcal{M})$ by the ideal under pointwise addition and multiplication of smooth functions vanishing on the constraint surface, (see section ??).

$$F_{[g(f_1,\dots,f_m),T_i]}(\tau_i,x) = g(F_{[f_1,T_i]}(\tau_i,x),\dots,F_{[f_1,T_i]}(\tau_i,x)) \tag{D.346}$$

Example:

$$F_{[q^2+p^2,T_i]}(\tau_i,x) = (F_{[q,T_i]}(\tau_i,x))^2 + (F_{[p,T_i]}(\tau_i,x))^2 \tag{D.347}$$

An over-complete bases for the space of Dirac observables!

The above result suggests complete observables that form a basis for the space of weak Dirac observables:

associated with a Dirac observable $d(x)$ is weakly equal to itself:

$$F_{[d,T_i]}(\tau_i,x) \simeq d(x) \tag{D.348}$$

This is easily seen from the expansion (D.342) and from the definition of a Dirac observable, i.e. $X_j \cdot d(x) \approx 0$ for all $j = 1,\dots,n$. the associated complete observables vanish weakly or are constants respectively:

$$\begin{aligned} F_{[C_k,T_i]}(\tau_i,x) &\simeq 0 \\ F_{[T_k,T_i]}(\tau_i,x) &\simeq \tau_k, \end{aligned} \tag{D.349}$$

(exercise). The point x is labelled by the canonical variables, i.e. $x = (q_a,p_a)$. The complete observables associated with the partial observables (q_a,p_a) provides an over-complete basis of the space of Dirac observables: If d any Dirac observable then it can be expressed in terms of the Dirac observables associated with the canonical variables q_a,p_a; that is $F_{[q_a,T_i]}$ and $F_{[p_a,T_i]}$ respectively

$$F_{[d,T_i]} \simeq d(x = (q_a,p_a)) \simeq d(F_{[q_a,T_i]},F_{[p_a,T_i]}) \tag{D.350}$$

Hence d is not a Dirac observable that could not already be constructed from $F_{[q_a,T_i]}$ and $F_{[a,T_i]}$. These Dirac observables are over-complete as they have to satisfy the conditions

$$F_{[C_k,T_i]}(\tau_i,x) \simeq 0 \simeq C_k(F_{[q_a,T_i]},F_{[p_a,T_i]})$$
$$F_{[T_k,T_i]}(\tau_i,x) \simeq 0 \simeq T_k(F_{[q_a,T_i]},F_{[p_a,T_i]}), \tag{D.351}$$

that it we have $2n$ relations between the $2p$ complete observables.

Formal solution of the PDE's

consider the (formal) power series of $F_{[f,T_i]}(\tau_i,x)$ in the τ_i's around the point $\tau_i = T_i(y)$ where y is a point in the gauge orbit $\mathcal{G}_x$ through x:

$$F_{[f,T_i]}(\tau_i,x) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{k_1!\cdots k_n!} \frac{\partial^{k_1\ldots k_n}}{\partial^{k_1}\tau_1\cdots\partial^{k_n}\tau_n} F_{[f,T_i]}(T_i(y),x)\ (\tau_1 - T_1(y))^{k_1}\cdots(\tau_n - T_n(y))^{k_n} \tag{D.352}$$

We know that the partial derivatives appearing in (4.3.4) can be written as complete observables associated to some phase space function $g_{k_1,\ldots,k_n}$

$$\frac{\partial^{k_1\ldots k_n}}{\partial^{k_1}\tau_1\cdots\partial^{k_n}\tau_n} F_{[f,T_i]} =: F_{[g_{k_1,\ldots k_n},T_i]} \tag{D.353}$$

$$F_{[f,T_i]}(\tau_i,x) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{k_1!\cdots k_n!} g_{(k_1,\ldots,k_n)}(y)\ (\tau_1 - T_1(y))^{k_1}\cdots(\tau_n - T_n(y))^{k_n} \tag{D.354}$$

D.8.6 Partially Invariant Partial Observables

$$\mathcal{C} = (\mathcal{C}_{\mathbf{2}};\mathcal{C}_{\mathbf{1}}) = (C_1,\ldots,C_m;C_{m+1},\ldots,C_n) \tag{D.355}$$

$$\begin{aligned}
&f \text{ is } \mathcal{C}_{\mathbf{1}} - \text{invariant}: \quad \{C_i,f\} \approx 0 \text{ for } i = m+1,\ldots,n;\\
&\text{the first } m \text{ clock variables } T_j \text{ are } \mathcal{C}_{\mathbf{1}} - \text{invariant}:\\
&\{C_i,T_j\} \approx 0 \text{ for } i = m+1,\ldots,n, \text{ and } j = 1,\ldots,m.
\end{aligned} \tag{D.356}$$

the determinant of A is no where vanishing (on the constraint hypersurface). So that $A_{ij} = 0$ for $i = m+1,\ldots,n$, and $j = 1,\ldots,m$. The determinant of the submatrices A' and A'' are nowhere vanishing. We summarise all this below:

$$\begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix} \begin{pmatrix} \partial_\tau F_{[f,T]_{(1)}} \\ \partial_\tau F_{[f,T]_{(2)}} \end{pmatrix} = \begin{pmatrix} \{C_k, f\} \\ 0 \end{pmatrix} \quad \det(A') \neq 0;\ \det(A'') \neq 0$$

where

$$\partial_\tau F_{[f,T]_{(1)}} := \begin{pmatrix} \partial_{\tau_1} F_{[f,T]} \\ \vdots \\ \partial_{\tau_m} F_{[f,T]} \end{pmatrix}, \ \partial_\tau F_{[f,T]_{(2)}} := \begin{pmatrix} \partial_{\tau_{m+1}} F_{[f,T]} \\ \vdots \\ \partial_{\tau_n} F_{[f,T]} \end{pmatrix}, \text{ and } \{C_k, f\} := \begin{pmatrix} \{C_1, f\} \\ \vdots \\ \{C_m, f\} \end{pmatrix}. \tag{D.357}$$

The second set of $(n-m)$ PDE's in (D.357) are:

$$\sum_{j=m+1}^{n} A''_{kj}(T_x^{-1}(\tau_i)) \frac{\partial}{\partial \tau_j} F_{[f,T_i]}(\tau_i, x) = \{C_k, f\}(T_x^{-1}(\tau_i)) = 0 \text{ for } k = m+1, \ldots, n. \tag{D.358}$$

Since the determinant of A''_{kj} is nowhere vanishing the unique solution to equation (D.358)

$$\frac{\partial}{\partial \tau_j} F_{[f,T_i]}(\tau_i, x) = 0 \text{ for } k = m+1, \ldots, n. \tag{D.359}$$

Hence the complete observable $F_{[f,T_i]}$ does not depend on the last $(n-m)$ of the parameters τ_j. Since the determinant of the submatrix A' is non-vanishing the first set of m PDE's in (D.357) can be expressed as

$$\frac{\partial}{\partial \tau_j} F_{[f,T_i]}(\tau_i, x) = \left(\sum_{k=1}^{m} A'^{-1}_{jk} \{C_k, f\} \right) \text{ for } j = 1, \ldots, m \tag{D.360}$$

where A'^{-1}_{kl} is the inverse of A'_{kl}. Now it could a priori happen, that the righthand side of (D.360) depends on the parameters $\{\tau_l | l = m+1, \ldots, n\}$ through the argument $(T_x^{-1}(\tau_i))$. However this is excluded by the consistency condition (D.341),

$$\frac{\partial}{\partial \tau_l} \left(\frac{\partial}{\partial \tau_j} F_{[f,T_i]} \right) = \frac{\partial}{\partial \tau_j} \left(\frac{\partial}{\partial \tau_l} F_{[f,T_i]} \right) = 0, \text{ where } j = 1, \ldots, m \text{ and } l = m+1, \ldots, n. \tag{D.361}$$

Another way to see this is to remember that the right hand side of (D.359) is again a complete observable

$$F_{[g_j,T_i]}(\tau_i, x) \tag{D.362}$$

associated to the partial observable $g_j = A'^{-1}_{jk}\{C_k, f\}$.

In (exercise F?.2) we show that g_j is again a (weakly) $\mathcal{C}_{\mathbf{1}}-$invariant phase space function. From this it follows that $F_{[g_j,T_i]}(\tau_i, x)$ does not depend on $\{\tau_l; l = m{+}1, \ldots, n\}$. Iterating this argument we find, that all the functions $g_{(k_1,\ldots,k_n)}$ defined in (D.340) are weakly $\mathcal{C}_1-$invariant. Hence when f is $\mathcal{C}_1-$invariant the formal series (D.342) is (weakly) truncated:

$$F_{[f,T_i]} \approx \sum_{k_1,\ldots,k_m=0}^{\infty} \frac{1}{k_1!\ldots k_m!} g_{(k_1,\ldots,k_m)}(\tau_1 - T_1(y))^{k_1} \ldots (\tau_m - T_m(y))^{k_m} \tag{D.363}$$

Now as both $g_{(k_1,\ldots,k_m)}$ and T_m in the above series are both weakly $\mathcal{C}_{\mathbf{1}}-$invariant the formal power series (D.363) is weakly $\mathcal{C}_{\mathbf{1}}-$invariant.

D.9 A Perturbative Approach to Dirac Observables and Their Spacetime Algebra

D.9.1 Introduction

1. Approximate Dirac observables
2. Application to General relativity
3. Interpretation of propagators and the interection processes
4. Scalar field coupled to gravity
5. Commutator algebra of fields: How does the choice of clock variable matter?
6. Outlook and summary

D.9.2 The Approximate Dirac Observable

Fluctuation variables

1. X_0 "background" phase space point on constraint hypersurface.
2. Introduce fluctuations $x = X - X_0$ around background; consider x^a as first order quantities.

Notation: for a phase space function g

$$\begin{array}{ll} {}^{(m)}g & \text{all terms of order } m \text{ in } g \\ {}^{[m]}g & \text{all terms of order } \leq m \text{ in } g \\ {}^{(m+)}g & \text{all terms of order } > m \text{ in } g \end{array} \tag{D.364}$$

Expand (first class) constraints

$$C_j = \sum_{m=1} {}^{(m)}C_j \tag{D.365}$$

One can refine the constraints such that the new constraints have particularly nice properties, which are key for the construction of complete observables. In the second part we will introduce a method to obtain an Abelian set of constraints.

$$\tilde{C}_K = (A^{-1})^j_K C_j \tag{D.366}$$

These weakly commute with themselves

$$\{\tilde{C}_K, \tilde{C}_J\} \simeq 0 \tag{D.367}$$

$$\{T^K, \tilde{C}_M\} = \delta^K_M + \lambda^{KN}_M \tilde{C}_N \simeq \delta^K_M \tag{D.368}$$

$$\{f, \tilde{C}_L\} \simeq \{f, \tilde{C}_L + \mu^{KM}_L \tilde{C}_K \tilde{C}_M\} \tag{D.369}$$

We are able to define a set of constraints that have vanishing Poisson brackets up to terms of arbitrary high order in the constraints.

$$\tilde{C}_M \rightarrow \tilde{C}_M + \mu^{L_1 \ldots L_{r+1}}_K \tilde{C}_{L_1} \cdots \tilde{C}_{L_{r+1}} \tag{D.370}$$

If the iteration procedure converges it will result in a set of Abelian constraints. This gives us another constraint operator $\breve{C}_M$ which satisfies

$$\{T^K, \breve{C}_M\} = \delta^K_M \tag{D.371}$$

exactly, i.e. also commutes away from the constraint hypersurface.

$$F_{[f;T]}(\tau) = \sum_{r=0}^{\infty} \{\cdots\{f, \tilde{C}_{K_1}\}, \cdots, \}, \tilde{C}_{K_r}\}(\tau^{K_1} - T^{K_1}) \cdots (\tau^{Kr} - T^{K_r}) \tag{D.372}$$

The approximate Dirac observable

$${}^{[q]}F_{[f;T]}(\tau \equiv T(X_0), x). \tag{D.373}$$

If the power series (D.372) for the complete observable converges it defines an exact Dirac observable which coincides ${}^{[q]}F_{[f;T]}(\tau \equiv T(X_0), x)$ modulo terms of order $(q+1)$.

We have started at a phase space point X_0, which represents a background spacetime, and then constructed a phase space function which is approximately invariant under active diffeomorphisms.

$$(A^{-1})^j_M = \delta^j_M - \delta^j_L B^L_i (A^{-1})^i_M \tag{D.374}$$

substituting this into itself we find a series solution to any required order of approximation in B^K_j. Hence we can determine $\tilde{C}_K$ up to finite order in the fluctuations in a finite number of steps. To calculate the constraints C_K to some order r we have to perform $(r-1)$ times the iteraction steps (D.370). For instance the second order of the constraints C_K is given by

$${}^{(2)}\check{C}_K = {}^{(2)}\tilde{C}_K + \sum_{L_1 L_2} {}^{(0)}\mu_K^{L_1 L_2}\ {}^{(1)}\tilde{C}_{L_1}\ {}^{(1)}\tilde{C}_{L_2} \tag{D.375}$$

where

$${}^{(0)}\mu_K^{L_1 L_2} = -\frac{1}{2}\{T^{L_1}, \{T^{L_2}, {}^{(2)}\tilde{C}_K\}\}. \tag{D.376}$$

This can be used to calculate the complete observable $F_{[f;T]}$ with parameter choice $\tau^K = 0$ to arbitrary finite order in m.

$${}^{[m]}F_{[f;T]}(\tau = 0) = \sum_{r=0}^{m} \frac{1}{r!}\{\cdots\{f, {}^{[m]}C_{K_1}\}, \cdots, \}, {}^{[m-r+1]}C_{K_r}\}(-1)^r T^{K_1} \cdots T^{K_r} \tag{D.377}$$

However we are also interested in dymanical questions, that is complete observabes for varing clock parameters τ. Introducing non-vanishing clock parameters into the series for the complete observable (D.372) we see that it is now a power series in $(\tau^K - T^K)$ which includes the zeroth order term τ^K. Hence the complete observable to the $m-$th order is no longer a finite summation.

D.9.3 Abelianization

$$\begin{aligned}
\{T^K, \tilde{C}^J\} &= \{T^K, (A^{-1})^j_M C_j\} \\
&= (A^{-1})^j_M \{T^K, C_j\} + \{T^K, (A^{-1})^j_M\} C_j \\
&= \delta^K_M + (\{T^K, (A^{-1})^j_M\} A^N_j)\ (A^{-1})^l_N C_l \\
&= \delta^K_M + \lambda^{KN}_M \tilde{C}_N \simeq \delta^K_M
\end{aligned} \tag{D.378}$$

where $\lambda^{KN}_M = \{T^K, (A^{-1})^j_M\} A^N_j$.

□

We compute $\{\{T^K, \tilde{C}_M\}, \tilde{C}_N\}$ first directly and then using the Jacobi identity. Comparing the two results one can conclude that the structure functions $\tilde{f}^M_{KJ}$ defined by $\tilde{f}^M_{KJ} = \{\tilde{C}_K, \tilde{C}_J\}$ have to vanish on the constraint surface.

$$\begin{aligned} \{\{T^K, \tilde{C}_M\}, \tilde{C}_N\} &= \{\delta^K_M + \lambda^{KN}_M \tilde{C}_N, \tilde{C}_N\} \\ &= \lambda^{KJ}_M \{\tilde{C}_J, \tilde{C}_N\} + \tilde{C}_N \{\lambda^{KN}_M, \tilde{C}_N\} \\ &\simeq \lambda^{KJ}_M \tilde{f}^P_{JN} \tilde{C}_P \end{aligned} \tag{D.379}$$

Now using the Jacobi identity

$$\begin{aligned} \{\{T^K, \tilde{C}_M\}, \tilde{C}_N\} &= \{\{T^K, \tilde{C}_N\}, \tilde{C}_M\} + \{\{\tilde{C}_M, \tilde{C}_N\}, T^K\} \\ &= \{\lambda^{KL}_N \tilde{C}_L, \tilde{C}_M\} + \{\tilde{f}^J_{NM} \tilde{C}_J, T^K\} \\ &= \tilde{C}_L \{\lambda^{KL}_N, \tilde{C}_M\} + \lambda^{KL}_N \{\tilde{C}_L, \tilde{C}_M\} + \tilde{C}_J \{\tilde{f}^J_{NM}, T^K\} + \tilde{f}^J_{NM} \{\tilde{C}_J, T^K\} \\ &\simeq (\lambda^{KL}_N \tilde{f}^P_{LM} - \tilde{f}^J_{NM} \lambda^{KP}_J) \tilde{C}_P \end{aligned} \tag{D.380}$$

we have

$$\lambda^{KL}_M \tilde{f}^P_{LN} \tilde{C}_P \simeq (\lambda^{KL}_N \tilde{f}^P_{LM} - \tilde{f}^J_{NM} \lambda^{KP}_J) \tilde{C}_P$$

Hence

$$\tilde{f}^K_{NM} \simeq 0$$

or

$$\{\tilde{C}_N, \tilde{C}_M\} \simeq 0.$$

□

D.9.4 Approximations

$$A^K_j = \{T^K, C_j\} = \delta^K_j + B^K_j$$

$$(A^{-1})^j_M = \delta^j_M - \delta^j_L B^L_i (A^{-1})^i_M \tag{D.381}$$

substituting this into itself we find a series solution to any required order of approximation in B^K_j

$$(A^{-1})^j_M = \delta^j_M - B^j_M + B^i_M(\delta^j_L B^L_i \delta^i_M) - \dots \tag{D.382}$$

where $B^i_M := \delta^j_K B^K_i \delta^i_M$.

Formula for $\lambda^{KL_1\dots L_r}_M$

$$\mathcal{O}(C) = \nu^K \tilde{C}_K, \qquad \mathcal{O}(C^r) = \nu^{K_1\dots K_r}\tilde{C}_{K_1}\dots\tilde{C}_{K_r}$$

with some smooth phase space functions ν^K and $\nu^{K_1\dots L_r}$.

$$\lambda^{KL_1\dots L_r}_M = \frac{1}{r!}\{,T^{L_1},\{T^{L_2},\{\cdots,\{T^{L_r},\{T^K,\tilde{C}_M\}\cdots\} + \mathcal{O}(C). \tag{D.383}$$

Proof:

Let us try $\lambda^{KL_1L_2}_M$, so that $\{T^K,\tilde{C}_M\} = \delta^K_M + \lambda^{KK_1K_2}_M\tilde{C}_{K_1}\tilde{C}_{K_2}$. Let us first calculate $\{T^{L_2},\{T^K,\tilde{C}_M\}\}$,

$$\begin{aligned}
\{T^{L_2},\{T^K,\tilde{C}_M\}\} &= \{T^{L_2},\delta^K_M + \lambda^{KK_1}_M\tilde{C}_{K_1}\} \\
&= \lambda^{KK_1}_M\{T^{L_2},\tilde{C}_{K_1}\} + \{T^{L_2},\lambda^{KK_1}_M\}\tilde{C}_{K_1} \\
&= \lambda^{KK_1}_M(\delta^{L_2}_{K_1} + \lambda^{L_2P}_{K_1}\tilde{C}_P) + \{T^{L_2},\lambda^{KK_1}_M\}\tilde{C}_{K_1} \\
&= \lambda^{KL_2}_M + (\lambda^{L_2P}_{K_1} + \{T^{L_2},\lambda^{KP}_M\})\tilde{C}_P \\
&= \lambda^{KL_2}_M + \mathcal{O}(C)
\end{aligned} \tag{D.384}$$

$$\begin{aligned}
\{T^{L_2},\{T^K,\tilde{C}_M\}\} &= \{T^{L_2},\delta^K_M + \lambda^{KK_1K_2}_M\tilde{C}_{K_1}\tilde{C}_{K_2}\} \\
&= \{T^{L_2},\delta^K_M + \lambda^{KK_1K_2}_M\tilde{C}_{K_2}\}\tilde{C}_{K_1} + \{T^{L_2},\delta^K_M + \lambda^{KK_1K_2}_M\tilde{C}_{K_1}\}\tilde{C}_{K_2} \\
&= \lambda^{KK_1L_2}_M\tilde{C}_{K_1} + \lambda^{KL_2K_2}_M\tilde{C}_{K_2} + (\nu^{K_1K_1}_1 + \nu^{K_1K_2}_2)\tilde{C}_{K_1}\tilde{C}_{K_2}
\end{aligned} \tag{D.385}$$

where $\nu^{K_1K_2}_1$ and $\nu^{K_1K_2}_2$ can be read off from (D.384)

$$\begin{aligned}
\{T^{L_1},\{T^{L_2},\{T^K,\tilde{C}_M\}\} &= \{T^{L_1},\lambda^{KK_1L_2}_M\tilde{C}_{K_1} + \lambda^{KL_1K_2}_M\tilde{C}_{K_2} + (\nu^{K_1K_1}_1 + \nu^{K_1K_2}_2)\tilde{C}_{K_1}\tilde{C}_{K_2}\} \\
&= \lambda^{KK_1L_2}_M\{T^{L_1},\tilde{C}_{K_1}\} + \lambda^{KL_2K_2}_M\{T^{L_1},\tilde{C}_{K_2}\} + \mathcal{O}(C) \\
&= \lambda^{KL_1L_2}_M + \lambda^{KL_2L_1}_M + \mathcal{O}(C) \\
&= 2\lambda^{KL_1L_2}_M + \mathcal{O}(C)
\end{aligned} \tag{D.386}$$

where in going to the last line we have assumed that the clock variables are Abelian.

Proceeding in the same way in general we find

$$\{T^{L_1},\{T^{L_2},\{\cdots,\{T^{L_r},\{T^K,\tilde{C}_M\}\cdots\} = \underbrace{\lambda_M^{KL_1L_2L_3\dots L_r} + \lambda_M^{KL_2L_1L_3\dots L_r} + \dots}_{\text{over all permutations of } L_1L_2L_3\dots L_r} + \mathcal{O}(C). \quad \text{(D.387)}$$

Again, as the clock variables are Abelian we have

$$\{T^{L_1},\{T^{L_2},\{\cdots,\{T^{L_r},\{T^K,\tilde{C}_M\}\cdots\} = r!\lambda_M^{KL_1L_2L_3\dots L_r} + \mathcal{O}(C)$$

□

Formula for $\{T^K,\breve{C}_M\}$

R.H.S:

$$\begin{aligned}
\{T^K,\breve{C}_M\} &= \{T^K,(\tilde{C}_M + \mu_M^{L_1\dots L_rL_{r+1}}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r}\tilde{C}_{L_{r+1}})\} \\
&= \{T^K,\tilde{C}_M\} + \mu_M^{L_1\dots L_rL_{r+1}}\{T^K,\tilde{C}_{L_1}\cdots\tilde{C}_{L_r}\tilde{C}_{L_{r+1}})\} + \mathcal{O}(C^{r+1}) \\
&= \delta_M^K + \lambda_M^{KL_1\dots L_r}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r} + \mu_M^{L_1\dots L_rL_{r+1}}\big(\{T^K,\tilde{C}_{L_1}\}\tilde{C}_{L_2}\cdots\tilde{C}_{L_{r+1}} \\
&\quad + \tilde{C}_{L_1}\{T^K,\tilde{C}_{L_2}\}\cdots\tilde{C}_{L_{r+1}} + \dots\tilde{C}_{L_1}\tilde{C}_{L_2}\cdots\{T^K,\tilde{C}_{L_{r+1}}\}\big) \\
&= \delta_M^K + \lambda_M^{KL_1\dots L_r}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r} + \mu_M^{K\dots L_rL_{r+1}}\tilde{C}_{L_2}\cdots\tilde{C}_{L_{r+1}} + \dots \\
&\quad \mu_M^{L_1\dots L_rK}\tilde{C}_{L_2}\cdots\tilde{C}_{L_r} + \mathcal{O}(C^{r+1}) \\
&= \delta_M^K + \lambda_M^{KL_1\dots L_r}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r} + (r+1)\mu_M^{KL_1\dots L_r}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r} + \mathcal{O}(C^{r+1})
\end{aligned} \quad \text{(D.388)}$$

Hence, we arrive at

$$\{T^K,\breve{C}_M\} = \delta_M^K + \lambda_M^{KL_1L_2\dots L_r}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r} + (r+1)\mu_M^{KL_1\dots L_r}\tilde{C}_{L_1}\cdots\tilde{C}_{L_r} + \mathcal{O}(C^{r+1}) \quad \text{(D.389)}$$

L.H.S:

$$\begin{aligned}
\{T^K,\breve{C}_M\} &= \delta_M^K + \lambda_M^{KL_1\dots L_rL_{r+1}}(\tilde{C}_{L_1} + \mu_{L_1}^{P_1\dots P_{r+1}}\tilde{C}_{P_1}\cdots\tilde{C}_{P_{r+1}}) \\
&\qquad \times\cdots\times(\tilde{C}_{L_{r+1}} + \mu_{L_{r+1}}^{Q_1\dots Q_{r+1}}\tilde{C}_{Q_1}\cdots\tilde{C}_{Q_{r+1}}) \\
&= \delta_M^K + \mathcal{O}(C^{r+1})
\end{aligned} \quad \text{(D.390)}$$

Therefore we have to choose

$$\mu_K^{L_1 L_2 \ldots L_r} \simeq -\frac{1}{r+1}\lambda_M^{L_1 L_2 \ldots L_r} \simeq -\frac{1}{r+1}\{T^{L_1}, \{T^{L_2}, \{\cdots, \{T^{L_r}, \{T^K, \tilde{C}_M\} \cdots\} \qquad \text{(D.391)}$$

in order to satisfy (??).

□

We now show that the new constriants commute up to $\mathcal{O}(C^{r+2})$ terms. To this end consider the jacobi identitiy

$$\{\{T^K, \breve{C}_M\}, \breve{C}_N\} - \{\{T^K, \breve{C}_N\}\breve{C}_M\} = \{\{\breve{C}_N, \breve{C}_M\}, T^K\}. \qquad \text{(D.392)}$$

because $\{T^K, \breve{C}_M\} = \delta_M^K + \mathcal{O}(C^{r+1})$ the left hand side is of order $\mathcal{O}(C^{r+1})$

Using the definition (??) of $\breve{C}_K$ in terms of the $\hat{C}_K$'s and the assumption (??) for the Poisson bracket between the constraints $\hat{C}_K$'s the right hand side can be written as

$$\begin{aligned} \{\{\breve{C}_N, \breve{C}_M\}, T^K\} &= \{g_{NM}^{L_1 \ldots L_{r+1}} \breve{C}_{L_1} \cdots \breve{C}_{L_{r+1}}, T^K\} + \mathcal{O}(C^{r+1}) \\ &= (r+1) g_{NM}^{L_1 \ldots L_r K} \breve{C}_{L_1} \cdots \breve{C}_{L_r} + \mathcal{O}(C^{r+1}) \end{aligned} \qquad \text{(D.393)}$$

where $g_{NM}^{L_1 \ldots L_{r+1}}$ is some set of phase space functions symmetric in the $L_1 \ldots L_{r+1}$-indices. Rienserting this into (D.392) we see that all terms are of the order $\mathcal{O}(C^{r+1})$ except for $g_{NM}^{L_1 \ldots L_r K} \breve{C}_{L_1} \cdots \breve{C}_{L_r}$, that is,

$$g_{NM}^{L_1 \ldots L_r K} \breve{C}_{L_1} \cdots \breve{C}_{L_r} = \mathcal{O}(C^{r+1}). \qquad \text{(D.394)}$$

Applying the Poisson bracket with clock variables T^{P_r} on both sides of this equation we find

$$\{T^{P_r}, g_{NM}^{L_1 \ldots L_r K}\} \breve{C}_{L_1} \cdots \breve{C}_{L_r} + r g_{NM}^{L_1 \ldots L_r K} \{T^{P_r}, \breve{C}_{L_r}\} \breve{C}_{L_1} \cdots \breve{C}_{L_{r-1}} = \{T^{P_r}, \mathcal{O}(C^{r+1})\}. \qquad \text{(D.395)}$$

As $\{T^K, \breve{C}_M\} = \delta_M^K + \mathcal{O}(C^{r+1})$, $\{T^K, \hat{C}_M\} = \delta_M^K + \mathcal{O}(C)$ and $\breve{C}_{L_1} \cdots \breve{C}_{L_r} = \hat{C}_{L_1} \cdots \hat{C}_{L_r} + \mathcal{O}(C^{r+1})$, we find

$$g_{NM}^{L_1 \ldots L_{r-1} P_r K} \breve{C}_{L_1} \cdots \breve{C}_{L_{r-1}} = \mathcal{O}(C^r).$$

Applying the Poisson bracket with clock variables T^{P_j} on both sides of the equation (D.394) r times we conclude

$$g_{NM}^{L_1 \ldots L_{r+1}} \simeq 0.$$

□

First form of approximations

$$ {}^{[2]}F_{[f;T]}(\tau=0) = g + \{g,\ {}^{[2]}\breve{C}_{K_1}\}(-T^{K_1}) + \frac{1}{2}\{\{g,\ {}^{[2]}\breve{C}_{K_1}\},\ {}^{[1]}\breve{C}_{K_2}\}(-T^{K_1})(-T^{K_2}). \tag{D.396} $$

$$ \begin{aligned} {}^{[2]}F_{[f;T]}(\tau=0) \quad &= \quad {}^{[2]}F_{[f;T]}(\tau=0) + \frac{1}{3!}\{\{\{g,\ {}^{[3]}\breve{C}_{K_1}\},\ {}^{[2]}\breve{C}_{K_2}\},\ {}^{[1]}\breve{C}_{K_3}\} \times \\ &\times (-T^{K_1})(-T^{K_2})(-T^{K_3}). \end{aligned} \tag{D.397} $$

This can be used to calculate the complete observable $F_{[f;T]}$ with parameter choice $\tau^K = 0$ to arbitrary finite order in m.

$$ {}^{[m]}F_{[f;T]}(\tau=0) = \sum_{r=0}^{m} \frac{1}{r!}\{\cdots\{f,\ {}^{[m]}C_{K_1}\},\cdots,\},\ {}^{[m-r+1]}C_{K_r}\}(-1)^r T^{K_1}\cdots T^{K_r} \tag{D.398} $$

D.9.5 Dynamics

First rewriting of $F_{[f;T]}(\tau^K)$

$$ \frac{1}{r!}\sum_{q=1}^{r} \tau^{K_1}\tau^{K_2}\cdots(-T^{K_q})\cdots\tau^{K_r} \rightarrow \frac{1}{(r-1)!}\tau^{K_1}\tau^{K_2}\cdots\tau^{K_{r-1}}(-T^{K_q}) \tag{D.399} $$

When $p=2$, $q=1$, when $p=3$, p can be 1 and 2, and so on. Adding these $1+2+\cdots+r-1$ we find there $r(r-1)/2$ permutations.

$$ \frac{1}{r!}\sum_{q=1,p=2,q<p}^{r} \tau^{K_1}\tau^{K_2}\cdots(-T^{K_q})\cdots(-T^{K_p})\cdots\tau^{K_r} \rightarrow \frac{1}{2!(r-2)!}\tau^{K_1}\tau^{K_2}\cdots(-T^{K_q})\cdots(-T^{K_p}) \tag{D.400} $$

$$ \begin{aligned} F_{[f;T]}(\tau^K) \quad \simeq \quad & \sum_{r=0}^{\infty}\frac{1}{r!}\{\cdots\{f,\breve{C}_{K_1}\},\cdots,\breve{C}_{K_r}\}\ \tau^{K_1}\tau^{K_2}\cdots\tau^{K_r} + \\ & \sum_{r=1}^{\infty}\frac{1}{(r-1)!}\{\cdots\{f,\breve{C}_{K_1}\},\cdots,\breve{C}_{K_r}\}\ \tau^{K_1}\tau^{K_2}\cdots\tau^{K_{r-1}}(-T^{K_r}) + \\ & \sum_{r=1}^{\infty}\frac{1}{2!(r-2)!}\{\cdots\{f,\breve{C}_{K_1}\},\cdots,\breve{C}_{K_r}\}\ \tau^{K_1}\tau^{K_2}\cdots\tau^{K_{r-2}}(-T^{K_{r-1}})(-T^{K_r}) \\ & +\cdots \end{aligned} \tag{D.401} $$

which after relabelling of the summation index r can be recognized as

$$F_{[f;T]}(\tau^K) \simeq F_{[\alpha^{\tau^K}_{\breve{C}_K}(f);T]}(\tau^K = 0). \tag{D.402}$$

Here $\alpha^{\tau^K}_{\breve{C}_K}(f)$ denotes the evolution of the f with respect to the constraints $\breve{C}_K$ and the parameters τ^K given by the first summand in (??).

Second rewriting of $F_{[f;T]}(\tau^K)$

In the same way one can arrive at

$$F_{[f;T]}(\tau, x) \simeq \frac{1}{q!}\sum_{q=0}^{\infty} \alpha^{\tau^M}_{\breve{C}_M}\big(\{\cdots\{f, \breve{C}_{K_1}\}, \cdots, \breve{C}_{K_q}\}\big)\ (-T^{K_1})(-T^{K_2})\cdots(-T^{K_q}). \tag{D.403}$$

One τ parameter dynamics

D.9.6 The second Order Approximation

$$^{[2]}\alpha^t_H = \alpha^t_{(2)H}(f) + \sum_{r=1}^{\infty}\frac{t^r}{r!}\sum_{s=0}^{s}\{\{\{f,\ ^{(2)}H\}_p,\ ^{(3)}H\}\ ^{(2)}H\}_{r-s-1} \tag{D.404}$$

$$\sum_{r=1}^{\infty}\sum_{s=0}^{r} a_{s,r} = \sum_{s=0}^{\infty}\sum_{r=s}^{\infty} a_{s,r}$$

which is the same as

$$\sum_{s=0}^{\infty}\sum_{p=0}^{\infty} a_{s,p+s+1}.$$

So the above expression for $^{[2]}\alpha^t_H$ can be rewritten as

$$^{[2]}\alpha^t_H = \alpha^t_{(2)H}(f) + \sum_{p,q=0}^{\infty}\frac{t^{(p+q+1)}}{(p+q+1)!}\{\{\{f,\ ^{(2)}H\}_q,\ ^{(3)}H\}\ ^{(2)}H\}_p \tag{D.405}$$

Using the identity

$$\frac{t^{(p+q+1)}}{(p+q+1)!} = \int_0^t \frac{(t-t')^q}{q!}\frac{t'^p}{p!}dt'$$

$$
\begin{aligned}
{}^{[2]}\alpha^t_H &= \alpha^t_{(2)H}(f) + \int_0^t dt' \sum_{p,q=0}^{\infty} \frac{(t-t')^q}{q!}\frac{t'^p}{p!}\{\{\{f,\ {}^{(2)}H\}_q,\ {}^{(3)}H\}\ {}^{(2)}H\}_p \\
&= \alpha^t_{(2)H}(f) + \int_0^t dt' \alpha^{(t')}_{(2)H}(\{\alpha^{(t-t')}_{(2)H}(f),\ {}^{(3)}H\})
\end{aligned} \tag{D.406}
$$

D.9.7 Application to General Relativity

(complex) connection variables:

$$A_a^{\ j} = \Gamma_a^{\ j} + \beta K_a^{\ j}, \quad E^b{}_k$$

where $\beta = 1/2$

$$\{A_a^{\ j}(\sigma), dE^b{}_k(\sigma')\} = \kappa \delta^a_b \delta^j_k \delta(\sigma,\sigma')$$

Minkowski background fluctuation variables:

first order constraints

$$
\begin{aligned}
{}^{(1)}G_b &= \kappa^{-1}(\partial_a\ {}^{(LT+LL)}e^a{}_b + \beta\epsilon_{bdc}\ {}^{(LT+TL+AT)}a^{dc}) && \text{Gauss} \\
{}^{(1)}V_a &= \kappa^{-1}(\partial_a\ {}^{T}a_b{}^b - \partial_c\ {}^{TL}a_a{}^c) && \text{vector} \\
{}^{(1)}C &= \kappa^{-1}(2\beta\epsilon^{abd}\partial_a\ {}^{AT}a_{bd}) && \text{scalar}
\end{aligned} \tag{D.407}
$$

LL left and right long. mode
T transverse trace part mode
AT antisymmetric transverse mode
LT left long. right transverse modes
TL left transverse right long. modes
STT symmetric transverse trace-free modes

D.9.8 ADM Clock Variables

For every constraint $C_j(\sigma)$ we choose a clock variable $T^K(\sigma')$.

Convenient:

$${}^{(0)}A^K_j(\sigma,\sigma') := {}^{(0)}\{T^K(\sigma), C_j(\sigma')\} = \delta^K_j \delta(\sigma,\sigma')$$

In ADM variables:

vector: $(LT+TL)$, LL mode of the 3-metric

scalar: T mode of the momentum

In connection variables:

$$\begin{aligned}
{}^{(G)}T^a &= \beta^{-1}\epsilon^a{}_{bc}\,{}^{LT}e^{bc} - \Delta^{-1}(\partial^a\,{}^{LL}a^b_b + \frac{1}{2}\,{}^{T}a^b_b) \\
{}^{(V)}T^a &= \Delta^{-1}(-\partial_b\,{}^{LT}e^{ba} - \partial_b\,{}^{TL}e^{ab} + \frac{1}{2}\,{}^{T}e^b_b) - \frac{1}{2}\partial^a\,{}^{LL}e^b_b \\
{}^{(C)}T^a &= (4\beta)^{-1}\Delta^{-1}(\epsilon_{cab}\partial^c\,{}^{AT}e^{ab} + \beta(\,{}^{T}a^b{}_b + 2\,{}^{LL}a^b{}_b))
\end{aligned} \tag{D.408}$$

satisfy

$$\{T^K(\sigma),\ {}^{(1)}C_j(\sigma') = \delta^K_j\,\delta(\sigma,\sigma').$$

D.9.9 Gravity Coupled to a Scalar Field

In this section we will consider gravity coupled to a scalar field and compute the complete observable associated to the scalar field to second order. Here the scalar field is assumed to have only small deviations from the zero value, that is the scalar field and its conjugated momentum will be counted as phase space functions of first order. As we will see, the first order complete observable coincides with the expression for the scalar field on a fixed Minkowski background. Hence we will compute the lowest order gravitational correction to this expression.

• Commplete observations associated to matter field can be understood as an expansion in $\kappa^{1/2}$

• First order complete observable ${}^{(1)}F_{[\phi(\sigma);T^K]}(t) = \phi(t,\sigma)$ coincides with observable of field theory in Minkowski space.

• Second order complete observable includes (one) scattering between matter field on gravitons; to lowest order on graviton background.

• Justifies (up to second order) to work with an effective matter Hamiltonian, where gravitonal variables are time dependent but non-dynamical.

• Poisson brackets between second order matter fields reflect to lowest order the causality structure of the gravitn background.

• Higher order terms include backreaction but also non-local expressions.

• Higher order Poisson brackets will be difficult to interprete because of non-local terms.

D.9.10 Control of Gauge Dependence

arXiv:gr-qc/0702093v1 15 Feb 2007 Gauge invariant perturbations around symmetry reduced sectors of general relativity: applications to cosmology

A gauge invariant canonical perturbation theory could be also fruitful in classical applications, such as second (and higher) order perturbation theory around cosmological solutions [??] or black holes. The main difficulty here is to control the gauge dependence of the results. This gauge dependence can be understood from the fact, that one has to identify spacetime points in the "physical" (non symmetric) universe with spacetime points in the "background" universe, around which the perturbation is taken. This identification can be related to a choice of coordinates for the "physical" universe. One might wonder why we attempt to develop a perturbation theory in the canonical formalism, where one would expect the problem to be even worse due to the foliation for the "physical" and "background" universe one has to choose in the canonical framework.

The resolution is that we use observables as central objects, i.e. we attempt to approximate directly a gauge invariant observable of the full theory and do not consider (the difference of) fields on two different manifolds representing the perturbed and unperturbed spacetime. Observables in the canonical formalism correspond to phase space functions, gauge invariant observables are invariant under the action of the constraints (the gauge generators).

The phase space of general relativity is just a representation of the space of all spacetimes (i.e. solutions of the Einstein equations). Gauge invariant phase space functions give the same value on spacetimes which are related by a diffeomorphism. Hence by considering gauge invariant phase space functions we do not need to worry about the identification process between points in the perturbed and unperturbed spacetime.

D.9.11 Outlook and Summary

• Approximate Dirac observables with a dynamical interpretation can be calculated explicitly to an arbitrary order

• Precise inderstanding of linearized theory and (quantum) field theory on a fixed background as approximations to full general relativity

• Formulism can be used to address construction and interpretation of Dirac observables

• Can be generalized to expansion around symmetry reduced/cosmological sectors $\Rightarrow$ use knowledge on symmetry reduced sectors to construct approximate Dirac observables for full theory

• Need a better understanding of the (quantum) interpretation of complete observables, in particular role of clock variables.

D.10 Reduced Phase Space Quantization of Constrained Theories

However, gauge invariance can also be secured by adopting a method in which only gauge-invariant objects are quantized. The key idea of this method is to find an algebra of phase-space functions whose Poisson brackets with all the first-class constraints vanish; such functions are therefore constant on the phase-space orbits of the (function) group generated by the constraints. Furthermore, this algebra is required to be large enough to generate all gauge-invariant functions in an appropriate sense. Quantization of the system then consists in finding irreducible self-adjoint representations of this algebra of physical observables or, essentially equivalently, finding irreducible, unitary representations of the associated canonical group.

arXiv: gr- qc/ 9510034 v1 17 Oct 1995 IMPERIAL/TP/95 96/2 Perennials and the Group-Theoretical Quantization of a Parametrized Scalar Field on a Curved Background P. Haj1cek C.J. Isham

$C^\infty(\mathcal{M})$ the space of smooth phase space functions

$D^\infty(\mathcal{M})$ the set of smooth, weak Dirac observables.

D.10.1 Reduced Phase Space Quantization with Dirac Observables

$$\begin{aligned}
\{f(q^a,p_b),f'(q^a,p_b)\}^* \; = \; & \{f(q^a,p_b),f'(q^a,p_b)\} - \{f(q,pq^a,p_b),C_j\}\underbrace{\{C_j,C_k\}}_{\simeq 0}\{C_k,f'(q^a,p_b)\} - \\
& \{f(q^a,p_b),C_j\}\{C_j,T_k\}\underbrace{\{T_k,f'(q^a,p_b)\}}_{\simeq 0} - \\
& \underbrace{\{f(q^a,p_b),T_j\}}_{\simeq 0}\{T_j,C_k\}\{C_k,f'(q^a,p_b)\} \\
& \underbrace{\{f(q^a,p_b),T_j\}}_{\simeq 0}\{T_j,T_k\}\underbrace{\{T_k,f'(q^a,p_b)\}}_{\simeq 0}
\end{aligned} \tag{D.409}$$

where we used the first class property in the first line.

under the assumption the Dirac bracket is equal to the Poisson bracket for functions depending only on q^a and p_a,

$$\{f(q^a,p_b),f'(q^a,p_b)\}^* = \{f(q^a,p_b),f'(q^a,p_b)\} \tag{D.410}$$

$$F_T(\ .\ ) := F^{\tau=0}_{\ .\ ,T} \tag{D.411}$$

Alissa Crans

$$Q^a := F_T(q^a), \quad P_a := F_T(p_a) \tag{D.412}$$

$$F_{T_j,T} \simeq \tau_j, \quad F_{C_j,T} \simeq 0. \tag{D.413}$$

$$F_T(P_j) \simeq E_j(F_T(q^a), F_T(p_a), F_T(T_k)) \simeq E_j(Q^a, P_a, \tau_k) \tag{D.414}$$

$$\alpha^\tau(Q^a) = \alpha^\tau(F_T(q^a)) = F_{\alpha'_\tau(q^a),T} \tag{D.415}$$

$$\alpha^\tau(Q^a) \quad = \quad \sum_{k_1,k_2,\cdots=0}^{\infty} \frac{\tau_{j_1}}{k_1!} \cdots \frac{\tau_{j_n}}{k_n!} F_{\prod_j X_j^{k_j} \cdot q^a, T} \tag{D.416}$$

$$Q^a := F_T(q^a), \quad P^a := F_T(p_a) \tag{D.417}$$

$$\{P_a, Q^b\} \simeq F^0_{\{p_a,q^b\}} = \delta_{a^b}, \quad \{Q^a, Q^b\} \simeq \{P_a, P_b\} \simeq 0 \tag{D.418}$$

thus does not give rise to a Dirac observable which we could not already construct from P_a, Q^b.

$$\pi : \mathcal{D} \to \mathcal{L}(\mathcal{H}) \tag{D.419}$$

of that subalgebra of $\mathcal{D}$

$$\pi((\alpha^\tau(Q^a)) = U(\tau)\pi(Q^a)U(t)^{-1} \tag{D.420}$$

$$\tilde{C}_j = P^j + E_j(q^a, p_a, T_k) \tag{D.421}$$

The functions that vanish on $\mathcal{C}$ form an ideal in $C^\infty(\mathcal{M})$ for the ordinary multiplication. That is, the product of an arbitrary phase space function with a function that vanishes on $\mathcal{C}$ also vanishes on $\mathcal{C}$.

D.11 Biblioliographical notes

In this chapter I have relied on the following reffferences:

Robert Bartnik, Jim Isenberg, *The Constraint Equations*??

D.12 Worked Examples

D.12.1 Poisson Brackets

Details

(a) The first is very obvious

$$\begin{aligned}\{f,g\} &= \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q}\frac{\partial f}{\partial p} \\ &= -\left(\frac{\partial g}{\partial q}\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right) \\ &= -\{g,f\}\end{aligned} \tag{D.422}$$

(b)

$$\begin{aligned}\{f_1+f_2,g\} &= \frac{\partial(f_1+f_2)}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial(f_1+f_2)}{\partial p}\frac{\partial g}{\partial q} \\ &= \frac{\partial f_1}{\partial q}\frac{\partial g}{\partial p} + \frac{\partial f_2}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f_1}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f_2}{\partial p}\frac{\partial g}{\partial q} \\ &= \{f_1,g\} + \{f_2,g\}\end{aligned} \tag{D.423}$$

(c)

$$\begin{aligned}&\{f,\{g,h\}\} \\ &-\{g,\{f,h\}\} = \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\right)\left(\frac{\partial g}{\partial p_j}\frac{\partial}{\partial q^j} - \frac{\partial g}{\partial q^j}\frac{\partial}{\partial p_j}\right)h - f\leftrightarrow g \\ &= \underbrace{\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial q^i\partial q^j} - f\leftrightarrow g\right)}_{=0} + \underbrace{\left(\frac{\partial f}{\partial q^i}\frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial p_i\partial p_j} - f\leftrightarrow g\right)}_{=0} \\ &- \underbrace{\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^j}\frac{\partial^2 h}{\partial q^i\partial p_j} + \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_i\partial q^j} - f\leftrightarrow g\right)}_{=0} \\ &+ \underbrace{\left(\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q^i\partial p_j}\frac{\partial h}{\partial q^j} - f\leftrightarrow g\right)}_{(a)} + \underbrace{\left(\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q^i\partial q^j}\frac{\partial h}{\partial p_j} - f\leftrightarrow g\right)}_{(b)} \\ &- \underbrace{\left(\frac{\partial f}{\partial q^i}\frac{\partial^2 g}{\partial p_i\partial p_j}\frac{\partial h}{\partial q^j} - f\leftrightarrow g\right)}_{(c)} + \underbrace{\left(\frac{\partial f}{\partial q^i}\frac{\partial^2 g}{\partial p_i\partial q^j}\frac{\partial h}{\partial p_j} - f\leftrightarrow g\right)}_{(d)}\end{aligned} \tag{D.424}$$

Consider the first term of (a) and the second term of (c) together,

$$\left(\frac{\partial f}{\partial p_i}\frac{\partial^2 g}{\partial q^i \partial p_j} - \frac{\partial g}{\partial q^i}\frac{\partial^2 f}{\partial p_i \partial p_j}\right)\frac{\partial}{\partial q^j}, \tag{D.425}$$

this is just

$$\frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}\right)\frac{\partial}{\partial q^j}. \tag{D.426}$$

The sum of the second term of (a) and the first term of (c) is

$$-\frac{\partial}{\partial p_i}\left(\frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j}\right)\frac{\partial h}{\partial q^j} \tag{D.427}$$

Equations (D.426) and (D.426) combined is simply

$$-\frac{\partial \{f,g\}_P}{\partial p_i}\frac{\partial h}{\partial q^j} \tag{D.428}$$

The terms (b) and (d) are treated similarly. Putting it all together we finally arrive at

$$\{f,\{g,h\}\} - \{g,\{f,h\}\} = \frac{\partial \{f,g\}_P}{\partial p_i}\frac{\partial h}{\partial q^i} - \frac{\partial \{f,g\}_P}{\partial q^i}\frac{\partial h}{\partial p_i} = \{\{f,g\},h\} \tag{D.429}$$

From antisymmetry $\{f,h\} = -\{h,f\}$ and $\{\{f,g\},h\} = -\{h,\{f,g\}\}$, we have

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0. \tag{D.430}$$

(d)

$$\begin{aligned}
\{f_1 f_2, g\} &= \frac{\partial (f_1 f_2)}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial (f_1 f_2)}{\partial p}\frac{\partial g}{\partial q} \\
&= f_1\frac{\partial f_2}{\partial q}\frac{\partial g}{\partial q} - f_1\frac{\partial f_2}{\partial p}\frac{\partial g}{\partial q} + \frac{\partial f_1}{\partial q}\frac{\partial g}{\partial q}f_2 - \frac{\partial f_1}{\partial p}\frac{\partial g}{\partial q}f_2 \\
&= f_1\{f_2,g\} + \{f_1,g\}f_2
\end{aligned} \tag{D.431}$$

Detailes

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = 0 \tag{D.432}$$

as for a general matrix $\underline{\underline{M}}$ that $\det(\underline{\underline{M}}^T) = det(\underline{\underline{M}})$

$$\det\left(\underline{\underline{A}}^T - \lambda \underline{\underline{I}}^T\right) = 0 \tag{D.433}$$

from the antisymmetry of the matrix $\underline{\underline{A}}$, this condition is the same as

$$\det\left(\underline{\underline{A}} + \lambda \underline{\underline{I}}\right) = 0 \tag{D.434}$$

Therefore if λ is an eigenvalue of then $-\lambda$ is also an eigenvalue. If there are an odd number of eigenvalues then at least one of them has to be zero. And we conclude that in the case of the odd-dimension the matrix $\underline{\underline{A}}$ doesn't have an inverse.

Detailes

x , y , z	u_m
a, b, c	$\{\phi_k, \phi_m\}$
d	$\{\phi_k, \mathcal{H}\}$
X, Y, Z	V_m
v_x, v_y, v_z	v_m
$x \rightarrow x + v_x X$	$u_m \rightarrow u_m + v_m V_m$

Detailes

The Jacobi identity.

$$\{f, \{g, h\}_D\}_D + \{g, \{h, f\}_D\}_D + \{h, \{f, g\}_D\}_D = 0 \tag{D.435}$$

$$\{f(z), g(z)\}_D = \partial_\mu f(z) \omega^{\mu\nu} \partial_\nu g(z) \quad \rightarrow \{z^\mu, z^\nu\} = \omega^{\mu\nu} \tag{D.436}$$

$$\begin{aligned} \{f(\xi), \{g(\xi), h(\xi)\}_D\}_D &= \partial_\gamma f(z) \omega^{\gamma\delta} \partial_\delta \{g(\xi), h(\xi)\} \\ &= \partial_\gamma f(\xi) \omega^{\gamma\delta} \partial_\delta \left[\partial_\mu g(\xi) \omega^{\mu\nu} \partial_\nu h(\xi)\right] \\ &= \underbrace{\partial_\gamma f(\xi) \partial_\delta \left[\partial_\mu g(\xi) \partial_\nu h(\xi)\right] \omega^{\mu\nu} \omega^{\gamma\delta}}_{(a)} + \underbrace{\left[\partial_\gamma f(\xi) \partial_\mu g(\xi) \partial_\nu h(\xi)\right] \omega^{\gamma\delta} \partial_\delta \omega^{\mu\nu}}_{(b)} \end{aligned} \tag{D.437}$$

If we write $\omega^{\mu\nu}(\xi) =: \omega(\xi)\epsilon^{\mu\nu}$ the term (a) can be written as an ordinary Poisson bracket $\{f(\xi), \{g(\xi), h(\xi)\}\}(\omega(\xi))^2$. The sum of terms of type (a) is identically zero by (D.21). Before summing the terms of type (b) we first prove an identity

$$\partial_\delta \omega^{\mu\tau} = \omega^{\mu\sigma}\omega^{\nu\tau}\partial_\delta\omega_{\sigma\nu}. \tag{D.438}$$

We start by noting

$$\partial_\delta(\omega_{\mu\sigma}\omega^{\sigma\nu}) = \partial_\delta(\delta^\nu_\mu) = 0. \tag{D.439}$$

This implies

$$(\partial_\delta\omega^{\mu\sigma})\omega_{\sigma\nu} = -\omega^{\mu\sigma}\partial_\delta\omega_{\sigma\nu} \tag{D.440}$$

we multiply both sides by $\omega^{\nu\tau}$

$$(\partial_\delta\omega^{\mu\sigma})\omega_{\sigma\nu}\omega^{\nu\tau} = -\omega^{\mu\sigma}(\partial_\delta\omega_{\sigma\nu})\omega^{\nu\tau} \tag{D.441}$$

and arrive at

$$\partial_\delta \omega^{\mu\tau} = \omega^{\mu\sigma}\omega^{\nu\tau}\partial_\delta\omega_{\sigma\nu}. \tag{D.442}$$

We now sum over terms of type (b)

$$\begin{aligned}
&\{f, \{g, h\}_D\}_D + \{g, \{h, f\}_D\}_D + \{h, \{f, g\}_D\}_D \\
&= [\partial_\gamma f(z)\partial_\mu g(z)\partial_\nu h(z)]\left(\omega^{\gamma\delta}\partial_\delta\omega^{\mu\nu} + \omega^{\mu\delta}\partial_\delta\omega^{\nu\gamma} + \omega^{\nu\delta}\partial_\delta\omega^{\gamma\mu}\right) \\
&= [\partial_\gamma f(z)\partial_\mu g(z)\partial_\nu h(z)]\,\omega^{\mu\sigma}\omega^{\nu\tau}\omega^{\delta\gamma}(\partial_\gamma\omega_{\sigma\nu} + \partial_\delta\omega_{\nu\gamma} + \partial_\nu\omega_{\gamma\sigma})
\end{aligned} \tag{D.443}$$

So the Jacobi identity is guaranteed if

$$\omega^{\mu\sigma}\omega^{\nu\tau}\omega^{\delta\gamma}(\partial_\gamma\omega_{\sigma\nu} + \partial_\delta\omega_{\nu\gamma} + \partial_\nu\omega_{\gamma\sigma}) = 0 \tag{D.444}$$

As $\omega^{\mu\nu}$ has an inverse the above condition is equivalent

$$\partial_\gamma\omega_{\sigma\nu} + \partial_\delta\omega_{\nu\gamma} + \partial_\nu\omega_{\gamma\sigma} = 0 \tag{D.445}$$

Using the antisymmetry of $\omega^{\mu\nu}$ this condition is equivalent to

$$
\begin{aligned}
0 &= \partial_\gamma \omega_{\sigma\nu} + \partial_\delta \omega_{\nu\gamma} + \partial_\nu \omega_{\gamma\sigma} \\
&\quad -\partial_\gamma \omega_{\nu\sigma} - \partial_\delta \omega_{\gamma\nu} - \partial_\nu \omega_{\sigma\gamma} \\
&= 3!\ \partial_{[\gamma} \omega_{\sigma\nu]}
\end{aligned} \tag{D.446}
$$

$$
\partial_{[\gamma} \omega_{\sigma\nu]} = 0 \tag{D.447}
$$

D.12.2 Worked Examples: Dittrich

Dittrich example 0. Parameterized clock harmonic + oscillator.

(i)

$$
\begin{aligned}
\alpha_C^t(T) &= q_1 + t \\
\alpha_C^t(f) &= q_2 \cos t + \frac{p_2}{\omega} \sin t = \sqrt{q_2^2 + (p_2^2/\omega^2)} \sin(t + \arctan \frac{q_2 \omega}{p_2})
\end{aligned} \tag{D.448}
$$

The constraint

$$
C = p_1 + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) = 0
$$

We take as the partial obseverables T the reading on the clock and f the elongation of the pendulum - that is, $T = q_1$ and $f = q_2$.

(ii) Prove for $T_x(t) = \tau$ on the interval

For $T = \tau$

Proof

(i)

For $\alpha_C^t(T)$

$$
\{C, q_1\} = 1, \quad \{C, q_1\}_2 = 0, \quad \ldots
$$

$$
\alpha_C^t(T) = q_1 + t \tag{D.449}
$$

For $\alpha_C^t(f)$

$$\begin{aligned}\{C,q_2\} &= -p_2,\\ \{C,p_2\} &= \omega^2 q_2\end{aligned} \tag{D.450}$$

$$\{C,q_2\}_1 = -p_2, \quad \{C,q_2\}_2 = \omega^2 - q_2, \quad \{C,q_2\}_3 = \omega^2 p_2, \quad \{C,q_2\}_4 = \omega^4 q_2 \tag{D.451}$$

$$\begin{aligned}\alpha_C^t(f) &= q_2 - tp_2 + \frac{\omega^2 t^2}{2!}q_2 - \frac{\omega^4 t^3}{3!}p_2 + \frac{\omega^6 t^4}{4!}q_2 + \dots\\ &= q_2\Big(1 - \frac{\omega^2 t^2}{2!} + \frac{\omega^4 t^4}{4!} - \dots\Big) + p_2\Big(t - \frac{\omega^2 t^3}{3!} + \frac{\omega^4 t^5}{5!} - \dots\Big)\\ &= q_2 \cos t + \frac{p_2}{\omega}\sin t\end{aligned} \tag{D.452}$$

$$\begin{aligned}\alpha_C^t(f) &= \sqrt{q_2^2 + (p_2^2/\omega^2)}\Big(\frac{q_2}{\sqrt{q_2^2 + (p_2^2/\omega^2)}}\cos t + \frac{p_2/\omega}{\sqrt{q_2^2 + (p_2^2/\omega^2)}}\sin t\Big)\\ &= \sqrt{q_2^2 + (p_2^2/\omega^2)}(\sin a \cos t + \cos a sin t)\\ &= \sqrt{q_2^2 + (p_2^2/\omega^2)}\sin(t + \arctan\frac{q_2\omega}{p_2})\end{aligned} \tag{D.453}$$

Identity $\cos A \cos B + \sin A \sin B = \sin(A+B)$ with a determined by $\tan a = \sin a/\cos a = q_2/(p_2/\omega)$.

(ii) For $\alpha_C^t(T)(x) = T_x(t) = \tau$ we invert $T_x(t)$. The equation $T = \tau$ is uniquely solvable.

$$t = \tau - q_1$$

(iv)

Evaluating $f_x(t) := \alpha_C^t(f)(x)$ gives

$$f_x(t) = \alpha_C^t(f)(x) = \sqrt{q_2^2 + (p_2^2/\omega^2)}\sin(\tau - q_1 + \arctan\frac{q_2\omega}{p_2})$$

the corresponding $F_{[f,T]}(\tau, x)$

$$F_{[f,T]}(\tau, x) = \sqrt{q_2^2 + (p_2^2/\omega^2)}\sin(\tau - q_1 + \arctan\frac{q_2\omega}{p_2}) \tag{D.454}$$

In the usual variables $q_1 = t, p_1 = E, q_2 = x, p_2 = p$ - this reads

$$F_{[f,T]}(\tau; t, E, x, p) = \sqrt{x^2 + (p^2/\omega^2)}\sin(\tau - t + \arctan\frac{x\omega}{p}) \tag{D.455}$$

$\tan\theta = x\omega/p$ with constraint

$$C = E + \frac{1}{2}(p^2 + \omega^2 x^2) = 0$$

Dittrich example 1.

(i)

$$\begin{aligned} \alpha_C^t(T) &= q_1\cos t + q_2\sin t = \sqrt{q_1^2+q_2^2}\sin(t+\arctan\frac{q_1}{q_2}) \\ \alpha_C^t(f) &= q_2\cos t - q_1\sin t = \sqrt{q_1^2+q_2^2}\cos(t+\arctan\frac{q_1}{q_2}) \end{aligned} \tag{D.456}$$

(ii) Prove for $T_x(t) = \tau$ on the interval

$$t_{10} = \arcsin\frac{\tau}{\sqrt{q_1^2+q_2^2}} - \arctan\frac{q_1}{q_2}.$$

$$\begin{aligned} t_{1k} &= t_{i0} + 2\pi k \\ t_{2k} &= \pi - t_{10} - 2\arctan\frac{q_1}{q_2} + 2\pi k, \quad k \in \mathbb{Z}. \end{aligned} \tag{D.457}$$

(iii)

$$f_x(t_{1k}) = \sqrt{q_1^2+q_2^2-\tau^2} \quad \text{and} \quad f_x(t_{2k}) = -\sqrt{q_1^2+q_2^2-\tau^2} \tag{D.458}$$

Proof

(i)

$$C = q_1 p_2 - q_2 p_1 \tag{D.459}$$

$T = q_1$ $f = q_2$

$$\{C, q_1\} = -q_2, \quad \{C, q_2\} = q_1 \tag{D.460}$$

$$\begin{aligned}\{C,q_1\} &= \{q_1p_2 - q_2p_1, q_1\} \\ &= -\{q_2p_1, q_1\} \\ &= q_2\end{aligned} \tag{D.461}$$

$$\{C,q_1\}_1 = q_2, \quad \{C,q_1\}_2 = q_1, \quad \{C,q_1\}_3 = -q_2, \quad \{C,q_1\}_4 = -q_1 \tag{D.462}$$

$$\begin{aligned}\alpha^t_C(T) &= q_1 + tq_2 - \frac{t^2}{2!}q_1 - \frac{t^3}{3!}q_2 + \frac{t^4}{4!}q_1 - \ldots \\ &= q_1\Big(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots\Big) + q_2\Big(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots\Big) \\ &= q_1\cos t + q_2\sin t\end{aligned} \tag{D.463}$$

$$\begin{aligned}\alpha^t_C(T) = q_1\cos t + q_2\sin t &= \sqrt{q_1^2+q_2^2}\Big(\frac{q_1}{\sqrt{q_1^2+q_2^2}}\cos t + \frac{q_2}{\sqrt{q_1^2+q_2^2}}\sin t\Big) \\ &= \sqrt{q_1^2+q_2^2}\Big(\sin a\cos t + \cos a\sin t\Big) \\ &= \sqrt{q_1^2+q_2^2}\Big(\sin(a+t)\Big) \\ &= \sqrt{q_1^2+q_2^2}\sin(t + \arctan\frac{q_1}{q_2})\end{aligned} \tag{D.464}$$

where we have used the identity $\cos A\sin B + \sin A\cos B = \sin(A+B)$ with a determined by $\tan a = \sin a/\cos a = q_1/q_2$.

A similar calculation for $\alpha^t_C(f)$ leads to

$$\begin{aligned}\alpha^t_C(f) = q_1\cos t - q_2\sin t &= \sqrt{q_1^2+q_2^2}\Big(\frac{q_1}{\sqrt{q_1^2+q_2^2}}\cos t - \frac{q_2}{\sqrt{q_1^2+q_2^2}}\sin t\Big) \\ &= \sqrt{q_1^2+q_2^2}\cos(t + \arctan\frac{q_1}{q_2})\end{aligned} \tag{D.465}$$

where we have used the identity $\cos^2 A - \sin^2 B = \cos(A+B)$

(ii) solutions to $T_x(t) = \tau$ are

(iii)

$$
\begin{aligned}
f_x(t_{1k}) = \alpha_C^{t_{1k}}(f)(x) &= \sqrt{q_1^2+q_2^2}\ \cos\Big(\arcsin(\frac{\tau}{\sqrt{q_1^2+q_2^2}})\Big) \\
&= \sqrt{q_1^2+q_2^2}\ \left[1-\sin^2\Big(\arcsin(\frac{\tau}{\sqrt{q_1^2+q_2^2}})\Big)\right]^{1/2} \\
&= \sqrt{q_1^2+q_2^2}\ \left[1-\frac{\tau^2}{q_1^2+q_2^2}\right]^{1/2} \\
&= \sqrt{q_1^2+q_2^2-\tau^2}. \qquad \text{(D.466)}
\end{aligned}
$$

Dittrich example 2. Two harmonic oscillators with constrained energy difference.

$$
C = \frac{1}{2}(p_1^2+\omega_1^2 q_1^2) - \frac{1}{2}(p_2^2+\omega_2^2 q_2^2)
$$

$T(x) = q_1$ $f(x) = q_2$

$$
\begin{aligned}
\{C, q_1\} &= \frac{1}{2}\{p_1^2, q_1\} = -p_1 \\
\{C, p_1\} &= -\frac{1}{2}\{\omega_1^2 q_1^2, q_1\} = \omega_1^2 q_1 \qquad \text{(D.467)}
\end{aligned}
$$

$$
\{C,q_1\} = -p_1, \quad \{C,q_1\}_2 = -\omega_1^2 q_1, \quad \{C,q_1\}_3 = \omega_1^2 p_1, \quad \{C,q_1\}_4 = \omega_1^4 q_1.
$$

$$
\begin{aligned}
\alpha_C^t(T)(x) &= q_1 - p_1 t - \frac{\omega_2 q_1 t^2}{2!} + \frac{\omega_1^2 p_1 t^3}{3!} + \frac{\omega_1^4 p_1 t^4}{4!} - \dots \\
&= q_1\Big(1 - \frac{\omega_2 q_1 t^2}{2!} + \frac{\omega_1^4 p_1 t^4}{4!} - \dots\Big) - \frac{p_1}{\omega_1}\Big(t + \frac{\omega_1^2 p_1 t^3}{3!} - \frac{\omega_1^5 p_1 t^5}{5!} + \dots\Big) \\
&= q_1 \cos(\omega_1 t) - \frac{p_1}{\omega_1}\sin(\omega_1 t) \qquad \text{(D.468)}
\end{aligned}
$$

(ii) $T_x(t) := \alpha_C^t(T)(x)$ is uniquely invertible on the interval $\big[-\frac{\pi}{2\omega_1} + \arctan(\omega_1 q_1/p_1)\ ,\ \frac{\pi}{2\omega_1} + \arctan(\omega_1 q_1/p_1)\big]$. The solutions of $T_x(t) = \tau$ are

$$
\begin{aligned}
t_{1k} &= \frac{1}{\omega_1}\left(\arcsin\,(\frac{\omega_1\tau}{\sqrt{\omega_1^2 q_1^2+p_1^2}}) + \arctan\left(\frac{\omega_1 q_1}{p_1}\right) + 2\pi k\right) \\
t_{2k} &= \frac{1}{\omega_1}\left(\pi - \arcsin\,(\frac{\omega_1\tau}{\sqrt{\omega_1^2 q_1^2+p_1^2}}) + \arctan\left(\frac{\omega_1 q_1}{p_1}\right) + 2\pi k\right) \qquad \text{(D.469)}
\end{aligned}
$$

(iii)

$$
\begin{aligned}
F_{[f,T]}(\tau,x)_{1k} &= \sqrt{q_2^2+(\frac{p_2}{\omega_2})^2}\sin\left(\frac{\omega_2}{\omega_1}\left(\arcsin(\frac{\omega_1\tau}{\sqrt{\omega_1^2q_1^2+p_1^2}})+\arctan\left(\frac{\omega_1 q_1}{p_1}\right)+2\pi k\right)\right. \\
&\qquad \left.+\arctan\left(\frac{\omega_2 q_2}{p_2}\right)\right) \\
F_{[f,T]}(\tau,x)_{1k} &= \sqrt{q_2^2+(\frac{p_2}{\omega_2})^2}\sin\left(\frac{\omega_2}{\omega_1}\left(\pi-\arcsin(\frac{\omega_1\tau}{\sqrt{\omega_1^2q_1^2+p_1^2}})+\arctan\left(\frac{\omega_1 q_1}{p_1}\right)+2\pi k\right)\right. \\
&\qquad \left.+\arctan\left(\frac{\omega_2 q_2}{p_2}\right)\right)
\end{aligned}
\tag{D.470}
$$

Dittrich example 3.

Dittrich example 4.

$$
C_i=\sum_{j,k=1}^{3}\epsilon_{ijk}\,q_jp_k,\quad i=1,2,3 \tag{D.471}
$$

$T_1=q_1$ and $T_2=q_2$

$$
\begin{aligned}
\alpha_{(\beta_1C_1+\beta_2C_2)}(T_1)(x) &= q_1+\frac{\beta_2(\beta_2q_1-\beta_1q_2)}{\beta_1^2+\beta_2^2}\Big(\cos(\sqrt{\beta_1^2+\beta_2^2})-1\Big)-\frac{\beta_2q_3}{\sqrt{\beta_1^2+\beta_2^2}}\sin(\sqrt{\beta_1^2+\beta_2^2}) \\
\alpha_{(\beta_1C_1+\beta_2C_2)}(T_2)(x) &= q_2+\frac{\beta_1(\beta_1q_2-\beta_2q_1)}{\beta_1^2+\beta_2^2}\Big(\cos(\sqrt{\beta_1^2+\beta_2^2})-1\Big)-\frac{\beta_1q_3}{\sqrt{\beta_1^2+\beta_2^2}}\sin(\sqrt{\beta_1^2+\beta_2^2})
\end{aligned}
\tag{D.472}
$$

$$
C_1=q_2p_3-q_3p_2,\qquad C_2=q_3p_1-q_1p_3 \tag{D.473}
$$

$$
\begin{aligned}
\{\beta_1C_1+\beta_2C_2,q_1\} &= \beta_1\{C_1,q_3\}+\beta_2\{C_2,q_3\}=-\beta_2q_3 \\
\{\beta_1C_1+\beta_2C_2,q_2\} &= \beta_1\{C_1,q_2\}+\beta_2\{C_2,q_2\}=\beta_1q_3 \\
\{\beta_1C_1+\beta_2C_2,q_3\} &= \beta_1\{C_1,q_2\}+\beta_2\{C_2,q_2\}=\beta_2q_1-\beta_1q_2
\end{aligned}
\tag{D.474}
$$

$$\begin{aligned}
\{\beta_1 C_1 + \beta_2 C_2, q_1\} &= [-\beta_2 q_3], \\
\{\beta_1 C_1 + \beta_2 C_2, q_1\}_2 &= -[\beta_2(\beta_2 q_1 - \beta_1 q_2)], \\
\{\beta_1 C_1 + \beta_2 C_2, q_1\}_3 &= -(\beta_1^2 + \beta_2^2)[-\beta_2 q_3], \\
\{\beta_1 C_1 + \beta_2 C_2, q_1\}_4 &= (\beta_1^2 + \beta_2^2)[\beta_2(\beta_1 q_2 - \beta_2 q_1)]
\end{aligned} \tag{D.475}$$

So that $\{\beta_1 C_1 + \beta_2 C_2, q_1\}_5 = -(\beta_1^2 + \beta_2^2)[-\beta_2 q_3]^2$ and $\{\beta_1 C_1 + \beta_2 C_2, q_1\}_7 = (\beta_1^2 + \beta_2^2)^2[-\beta_2 q_3]^3$ and so on.

So that $\{\beta_1 C_1 + \beta_2 C_2, q_1\}_6 = -(\beta_1^2 + \beta_2^2)^2[\beta_2(\beta_1 q_2 - \beta_2 q_1)]$ and $\{\beta_1 C_1 + \beta_2 C_2, q_1\}_8 = (\beta_1^2 + \beta_2^2)^3[\beta_2(\beta_1 q_2 - \beta_2 q_1)]$ and so on.

$$\begin{aligned}
\alpha_{\beta_1 C_1 + \beta_2 C_2}(T_1)(x) &= q_1 + \beta_2(\beta_1 q_2 - \beta_2 q_1)\Big[-\frac{1}{2!} + \frac{1}{4!}(\beta_1^2 + \beta_2^2) - \frac{1}{6!}(\beta_1^2 + \beta_2^2)^2 + \dots \Big] \\
&- \beta_2 q_3 \Big[1 - \frac{1}{3!}(\beta_1^2 + \beta_2^2)^2 + \frac{1}{5!}(\beta_1^2 + \beta_2^2)^3 - \dots \Big]
\end{aligned} \tag{D.476}$$

Noting that

$$\begin{aligned}
\cos\sqrt{\beta_1^2 + \beta_2^2} - 1 &= -\frac{1}{2!}(\beta_1^2 + \beta_2^2) + \frac{1}{4!}(\beta_1^2 + \beta_2^2)^2 - \frac{1}{6!}(\beta_1^2 + \beta_2^2)^3 + \dots \\
\sin\sqrt{\beta_1^2 + \beta_2^2} &= (\beta_1^2 + \beta_2^2)^{1/2} - \frac{1}{3!}(\beta_1^2 + \beta_2^2)^{3/2} + \frac{1}{5!}(\beta_1^2 + \beta_2^2)^{5/2} - \dots
\end{aligned} \tag{D.477}$$

we obtain the first equation of (D.472). From (D.473) and (D.474) it should be clear that the second equation can be obtained from the first by making the interchanges $q_1 \leftrightarrow q_2$ and $\beta_1 \leftrightarrow \beta_2$.

Dittrich example 5.

$$\begin{aligned}
C_1 &= q_2^{n_2} p_3^{m_3} - q_3^{n_3} p_2^{m_2} \\
C_2 &= q_3^{n_3} p_1^{m_1} - q_1^{n_1} p_3^{m_3} \\
C_3 &= q_1^{n_1} p_2^{m_2} - q_2^{n_2} p_1^{m_1}
\end{aligned} \tag{D.478}$$

$$\{C_i, C_j\} = \sum_k \epsilon_{ijk} n_k m_k q_k^{n_k - 1} p_k^{n_k - 1} C_k \tag{D.479}$$

The set of constraints aren't independent because of the anti-symmetry of ϵ_{ijk} we have the relations

$$q_1^{n_1}C_1 + q_2^{n_2}C_2 + q_3^{n_3}C_3 = p_1^{m_1}C_1 + p_2^{m_2}C_2 + p_3^{m_3}C_3 = 0. \tag{D.480}$$

(ii) Derive

$$\frac{\partial}{\partial \tau_1} q_3(\tau_1,\tau_2) = -\frac{m_3 p_3^{m_3-1} q_1^{n_1}}{m_1 p_1^{m_1-1} q_3^{n_3}} \qquad \frac{\partial}{\partial \tau_2} q_3(\tau_1,\tau_2) = -\frac{m_3 p_3^{m_3-1} q_2^{n_2}}{m_2 p_2^{m_2-1} q_3^{n_3}} \tag{D.481}$$

(iii) Derive

$$\begin{aligned}
\frac{\partial}{\partial \tau_1} p_1(\tau_1,\tau_2) &= \frac{n_1 q_1^{n_1-1} p_3^{m_3}}{m_1 p_1^{m_1-1} q_3^{n_3}}(\tau_1,\tau_2) & \frac{\partial}{\partial \tau_2} p_1(\tau_1,\tau_2) &= 0 \\
\frac{\partial}{\partial \tau_1} p_2(\tau_1,\tau_2) &= 0 & \frac{\partial}{\partial \tau_2} p_2(\tau_1,\tau_2) &= \frac{n_2 q_2^{n_2-1} p_3^{m_3}}{m_2 p_2^{m_2-1} q_3^{n_3}}(\tau_1,\tau_2) \\
\frac{\partial}{\partial \tau_1} p_3(\tau_1,\tau_2) &= -\frac{n_3 p_1}{m_1 q_3}(\tau_1,\tau_2) & \frac{\partial}{\partial \tau_2} p_3(\tau_1,\tau_2) &= -\frac{n_3 p_2}{m_2 q_3}(\tau_1,\tau_2)
\end{aligned} \tag{D.482}$$

(iv) Integrate the above equations on the constraint surface.

(v) Find the two Dirac observables.

Proof

(i)

(ii)

We need to calculate $A_{kj} = \{C_k, T_j\}$, its inverse and $\{C_k, f\}$.

The PDE's for

$$F_{[f,T_1,T_2]}(\tau_1,\tau_2,\cdot) = q_3(\tau_1,\tau_2) \tag{D.483}$$

$$\{C_1,q_1\} = \{C_2,q_2\} = \{C_3,q_3\} = 0$$

$$\begin{aligned}
\{C_1,q_2\} &= \{q_2^{n_2}p_3^{m_3} - q_3^{n_3}p_2^{m_2}, q_2\} \\
&= -q_3^{n_3}\{p_2^{m_2}, q_2\} \\
&= m_2 q_3^{n_3} p_2^{m_2-1}
\end{aligned} \tag{D.484}$$

$$\begin{aligned}\{C_2, q_1\} &= \{q_3^{n_3} p_1^{m_1} - q_1^{n_1} p_3^{m_3}, q_1\} \\ &= q_3^{n_3}\{p_1^{m_1}, q_2\} \\ &= -m_1 q_3^{n_3} p_1^{m_1-1} \end{aligned} \tag{D.485}$$

$$\begin{aligned}\{C_1, q_3\} &= \{q_2^{n_2} p_3^{m_3} - q_3^{n_3} p_2^{m_2}, q_3\} \\ &= q_2^{n_2}\{p_3^{m_3}, q_3\} \\ &= -m_3 q_2^{n_2} p_3^{m_3-1} \end{aligned} \tag{D.486}$$

$$\begin{aligned}\{C_2, q_3\} &= \{q_3^{n_3} p_1^{m_1} - q_1^{n_1} p_3^{m_3}, q_3\} \\ &= -q_1^{n_1}\{p_3^{m_3}, q_2\} \\ &= m_3 q_1^{n_1} p_3^{m_3-1} \end{aligned} \tag{D.487}$$

$$A = \begin{pmatrix} \{C_1, q_1\} & \{C_1, q_2\} \\ \{C_2, q_1\} & \{C_2, q_2\} \end{pmatrix} = \begin{pmatrix} 0 & m_2 q_3^{n_3} p_1^{m_2-1} \\ -m_1 q_3^{n_3} p_1^{m_1-1} & 0 \end{pmatrix} \tag{D.488}$$

The inverse matrix is

$$A^{-1} = \begin{pmatrix} 0 & -1/m_1 q_3^{n_3} p_1^{m_1-1} \\ 1/m_2 q_3^{n_3} p_1^{m_2-1} & 0 \end{pmatrix} \tag{D.489}$$

$$\frac{\partial}{\partial \tau_m} F_{[f,T_1,T_2]}(\tau_1, \tau_2, x) = A^{-1}_{m1}\{C_1, f\} + A^{-1}_{m2}\{C_2, f\}$$

$$\begin{aligned}\frac{\partial}{\partial \tau_1} q_3(\tau_1, \tau_2) &= A^{-1}_{11}\{C_1, q_3\} + A^{-1}_{12}\{C_2, q_3\} \\ &= -\frac{m_3 q_1^{n_1} p_3^{m_3-1}}{m_1 q_3^{n_3} p_1^{m_1-1}} \end{aligned} \tag{D.490}$$

$$\begin{aligned}\frac{\partial}{\partial \tau_2} q_3(\tau_1, \tau_2) &= A^{-1}_{21}\{C_1, q_3\} + A^{-1}_{22}\{C_2, q_3\} \\ &= -\frac{m_3 q_2^{n_2} p_3^{m_3-1}}{m_2 q_3^{n_3} p_1^{m_2-1}} \end{aligned} \tag{D.491}$$

(iii) p_i's are complete observables (section). We find the partial derivatives of the function $p_1(\tau_1, \tau_2), p_2(\tau_1, \tau_2)$ and $p_3(\tau_1, \tau_2)$.

$$\{C_1, p_1\} = \{C_2, p_2\} = \{C_3, p_3\} = 0$$

$$\{C_2, p_1\} = -n_1 q_1^{n_1-1} p_3^{m_3}, \quad \{C_1, p_2\} = n_2 q_2^{n_2-1} p_3^{m_3}, \quad \{C_1, p_3\} = -n_3 q_3^{n_3-1} p_2^{m_2}, \quad \{C_2, p_3\} = n_3 q_3^{n_3-1} p_1^{m_1}$$

$$\begin{aligned} \frac{\partial}{\partial \tau_1} p_1(\tau_1, \tau_2) &:= \frac{\partial}{\partial \tau_1} F_{[p_1,T_1,T_2]}(\tau_1, \tau_2, x) \\ &= A_{11}^{-1}\{C_1, p_1\} + A_{12}^{-1}\{C_2, p_1\} \\ &= \frac{n_1 q_1^{n_1-1} p_3^{m_3}}{m_1 q_3^{n_3} p_1^{m_1-1}} \end{aligned} \tag{D.492}$$

$$\begin{aligned} \frac{\partial}{\partial \tau_1} p_2(\tau_1, \tau_2) &:= \frac{\partial}{\partial \tau_1} F_{[p_2,T_1,T_2]}(\tau_1, \tau_2, x) \\ &= A_{11}^{-1}\{C_1, p_2\} + A_{12}^{-1}\{C_2, p_2\} \\ &= 0 \end{aligned} \tag{D.493}$$

$$\begin{aligned} \frac{\partial}{\partial \tau_1} p_3(\tau_1, \tau_2) &:= \frac{\partial}{\partial \tau_1} F_{[p_3,T_1,T_2]}(\tau_1, \tau_2, x) \\ &= A_{11}^{-1}\{C_1, p_3\} + A_{12}^{-1}\{C_2, p_3\} \\ &= -\frac{n_3 q_3^{n_3-1} p_1^{m_1}}{m_1 q_3^{n_3} p_1^{m_1-1}} \\ &= -\frac{n_3 p_1}{m_1 q_3} \end{aligned} \tag{D.494}$$

(iv)

On the constraint surface we have from (D.478)

$$\frac{p_3^{m_3}}{q_3^{n_3}} \simeq \frac{p_2^{m_2}}{q_2^{n_2}} \simeq \frac{p_1^{m_1}}{q_1^{n_1}}, \tag{D.495}$$

hence we can replace in (4.3.4) the term $p_3^{m_3}/q_3^{n_3}$ according to (4.3.4).

$$\begin{aligned} \frac{\partial}{\partial \tau_1} p_1(\tau_1, \tau_2) \simeq \frac{n_1 p_1}{m_1 \tau_1}(\tau_1, \tau_2) \qquad & \frac{\partial}{\partial \tau_2} p_1(\tau_1, \tau_2) = 0 \\ \frac{\partial}{\partial \tau_1} p_2(\tau_1, \tau_2) = 0 \qquad & \frac{\partial}{\partial \tau_2} p_2(\tau_1, \tau_2) \simeq \frac{n_2 p_2}{m_2 \tau_2}(\tau_1, \tau_2) \end{aligned} \tag{D.496}$$

$$\frac{\tau_1^{n_1+1}}{p_1^{m_1-1}}\frac{\partial}{\partial\tau_1}\left[\frac{p_1^{m_1}}{\tau_1^{n_1}}\right] = m_1\frac{\partial p_1}{\partial\tau_1} - n_1 p_1 \simeq 0 \tag{D.497}$$

implies

$$\frac{p_1^{m_1}}{\tau_1^{n_1}} \simeq Const + G(\tau_2)$$

$G(\tau_2)$ is constant as $\partial p_1/\partial\tau_2 = 0$, so that

$$\frac{p_1^{m_1}}{q_1^{n_1}}(\tau_1,\tau_2) = \frac{p_1^{m_1}(\tau_1,\tau_2)}{\tau_1^{n_1}} \simeq \frac{p_1^{m_1}(\tau_{10},\tau_{20})}{\tau_{10}^{n_1}} \tag{D.498}$$

where τ_{10} and τ_{20} are any fixed values of τ_1 and τ_2. Similarly

$$\frac{p_2^{m_2}}{q_2^{n_2}}(\tau_1,\tau_2) = \frac{p_2^{m_2}(\tau_1,\tau_2)}{\tau_2^{n_2}} \simeq \frac{p_2^{m_2}(\tau_{10},\tau_{20})}{\tau_{20}^{n_1}} \tag{D.499}$$

Hence $F_1 := p_2^{m_2}/q_2^{n_2} \simeq p_1^{m_1}/q_1^{n_1} \simeq p_3^{m_3}/q_3^{n_3}$ is conserved and indeed it commutes weakly with the constraints.

$$\begin{aligned}\frac{\partial}{\partial\tau_1}q_3(\tau_1,\tau_2) &= -\frac{m_3}{m_1}\frac{q_1^{n_1}}{p_1^{m_1-1}}\frac{p_3^{m_3-1}}{q_3^{n_3}}(\tau_1,\tau_2)\\ &= -\frac{m_3}{m_1}\frac{p_1}{p_3}(\tau_1,\tau_2)\end{aligned} \tag{D.500}$$

$$p_1(\tau_1,\tau_2) \simeq \frac{p_1(\tau_{10},\tau_{20})}{\tau_{10}^{n_1/m_1}}\,\tau_1^{n_1/m_1}$$

$$\frac{p_3^{m_3}(\tau_1,\tau_2)}{q_3^{n_3}(\tau_1,\tau_2)} = Const. = \frac{p_3^{m_3}(\tau_{10},\tau_{20})}{q_3^{n_3}(\tau_{10},\tau_{20})}$$

$$\begin{aligned}\frac{\partial}{\partial\tau_1}q_3(\tau_1,\tau_2) &= -\frac{m_3}{m_1}\frac{p_1(\tau_{10},\tau_{20})}{p_3(\tau_{10},\tau_{20})}\frac{\tau_1^{n_1/m_1}}{\tau_{10}^{n_1/m_1}}\;\frac{p_3(\tau_{10},\tau_{20})}{p_3(\tau_1,\tau_2)}\\ &= -\frac{m_3}{m_1}\;\frac{p_1(\tau_{10},\tau_{20})}{p_3(\tau_{10},\tau_{20})}\;\frac{\tau^{n_1/m_1}}{\tau_{10}^{n_1/m_1}}\;\frac{q_3^{n_3/m_3}(\tau_{10},\tau_{20})}{q_3^{n_3/m_3}(\tau_1,\tau_2)}\end{aligned} \tag{D.501}$$

And similarly for $\frac{\partial}{\partial\tau_2}q_3(\tau_1,\tau_2)$.

$$\frac{\partial}{\partial \tau_1} q_3(\tau_1, \tau_2) = -C_1 \frac{\tau_1^{n_1/m_1}}{q_3^{n_3/m_3}(\tau_1, \tau_2)} \tag{D.502}$$

$$\frac{\partial}{\partial \tau_2} q_3(\tau_1, \tau_2) = -C_2 \frac{\tau_1^{n_2/m_2}}{q_3^{n_3/m_3}(\tau_1, \tau_2)} \tag{D.503}$$

where

$$C_1 = \frac{A p_1(\tau_{10}, \tau_{20})}{m_1 \tau_{10}^{n_1/m_1}}, \qquad C_2 = \frac{A p_2(\tau_{10}, \tau_{20})}{m_2 \tau_{10}^{n_2/m_2}}, \qquad A = \frac{m_3 q_3^{n_3/m_3}(\tau_{10}, \tau_{20})}{p_3(\tau_{10}, \tau_{20})}$$

First we treat τ_2 as constant.

$$\frac{m_3}{m_3 + n_3} \frac{\partial}{\partial \tau_1} q_3^{n_3/m_3+1}(\tau_1, \tau_2) = -C_1 \tau_1^{n_1/m_1}$$

integrating this we get

$$\frac{q_3^{n_3/m_3+1}(\tau_1, \tau_2)}{m_3 + n_3} = -\frac{C_1 m_1}{m_3} \frac{\tau_1^{n_1/m_1+1}}{m_1 + n_1} + \frac{G(\tau_2)}{m_3},$$

where $G(\tau_2)$ is an arbitrary function of τ_2. Similarly

$$\frac{q_3^{n_3/m_3+1}(\tau_1, \tau_2)}{m_3 + n_3} = -\frac{C_2 m_2}{m_3} \frac{\tau_2^{n_2/m_2+1}}{m_2 + n_2} + \frac{H(\tau_1)}{m_3}.$$

Comparing the two we have

$$G(\tau_2) = -C_2 m_2 \frac{\tau_2^{n_2/m_2+1}}{m_2 + n_2} + B, \qquad H(\tau_1) = -C_1 m_1 \frac{\tau_1^{n_1/m_1+1}}{m_1 + n_1} + B$$

we have

$$\begin{aligned} \frac{q_3^{n_3/m_3+1}(\tau_1, \tau_2)}{m_3 + n_3} &= \frac{2B}{m_3} - \frac{C_1 m_1}{m_3} \frac{\tau_1^{n_1/m_1+1}}{m_1 + n_1} - \frac{C_2 m_2}{m_3} \frac{\tau_2^{n_2/m_2+1}}{m_2 + n_2} \\ &= \frac{2B}{m_3} - \frac{A p_1(\tau_{10}, \tau_{20})}{m_3 \tau_{10}^{n_1/m_1}} \frac{\tau_1^{n_1/m_1+1}}{m_1 + n_1} - \dots \end{aligned} \tag{D.504}$$

$$\frac{m_3}{A}\frac{q_3^{n_3/m_3+1}(\tau_1,\tau_2)}{m_3+n_3} = \frac{2B}{A} - \frac{\tau_1^{(n_1/m_1)+1}}{(n_1+m_1)\tau_{10}^{n_1/m_1}}p_1(\tau_{10},\tau_{20}) - \frac{\tau_2^{(n_2/m_2)+1}}{(n_2+m_2)\tau_{20}^{n_2/m_2}}p_2(\tau_{10},\tau_{20}) \tag{D.505}$$

Setting $\tau_1 = \tau_{10}$ and $\tau_2 = \tau_{20}$ the above equation yeilds

$$\begin{aligned}\frac{q_3(\tau_{10},\tau_{20})p_3(\tau_{10},\tau_{20})}{n_3+m_3} &= \frac{2B}{A} - \frac{\tau_{10}}{(n_1+m_1)}p_1(\tau_{10},\tau_{20}) - \frac{\tau_{20}}{(n_2+m_2)}p_2(\tau_{10},\tau_{20}) \\ &\simeq \end{aligned} \tag{D.506}$$

$$\begin{aligned}\frac{q_3^{(n_3/m_3)+1}(\tau_1,\tau_2)}{(n_3+m_3)q_3^{n_3/m_3}(\tau_{10},\tau_{20})}p_3(\tau_{10},\tau_{20}) &\simeq \frac{q_3(\tau_{10},\tau_{20})p_3(\tau_{10},\tau_{20})}{n_3+m_3} - \frac{\tau_1^{(n_1/m_1)+1}}{(n_1+m_1)\tau_{10}^{n_1/m_1}}p_1(\tau_{10},\tau_{20}) \\ &\quad - \frac{\tau_2^{(n_2/m_2)+1}}{(n_2+m_2)\tau_{20}^{n_2/m_2}}p_2(\tau_{10},\tau_{20})\end{aligned} \tag{D.507}$$

(v)

$$F_2 := \frac{q_1p_1}{n_1+m_1} + \frac{q_2p_2}{n_2+m_2} + \frac{q_3p_3}{n_3+m_3} \tag{D.508}$$

is a Dirac observable.

Worked example: Integrability condition

PDE's are consistent if integrability condition

$$\frac{\partial^2}{\partial\tau_l\partial\tau_m}F_{[f,T_i]}(\tau_i,x) = \frac{\partial^2}{\partial\tau_l\partial\tau_m}F_{[f,T_i]}(\tau_i,x) \tag{D.509}$$

Worked example: $\mathcal{C}_{\mathbf{1}}-$invariance of g_j

Prove the $\mathcal{C}_{\mathbf{1}}-$invariance of g_j.

$$
\begin{aligned}
A'_{lj}\{C_h, g_j\} &= A'_{lj}A'^{-1}_{jk}\{C_h,\{C_k,f\}\} + A'_{lj}\{C_h, A'^{-1}_{jk}\}\{C_k,f\} \\
&= \{C_h,\{C_l,f\}\} - A'^{-1}_{jk}\{C_h, A'_{lj}\}\{C_k,f\} \\
&= -\{f\{C_h,C_l\}\} - \{C_l,\{f,C_h\}\} - A'^{-1}_{jk}\{C_h,\{C_l,T_j\}\}\{C_k,f\} \\
&= -\{f\{C_h,C_l\}\} - \{C_l,\{f,C_h\}\} - A'^{-1}_{jk}()\{C_k,f\} \qquad \text{(D.510)}
\end{aligned}
$$

Worked example: Partially invariant partial observers.

$$
f(x) = \sum_{n=1}^{\infty} a_n x^n \qquad \text{(D.511)}
$$

such that the series converges absolutely

$$
\left| \frac{f(x+h)-f(x)}{h} - \sum_{n=1}^{\infty} n a_n x^{n-1} \right| < K|h| \qquad \text{(D.512)}
$$

Proof:

$$
\left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n|. \qquad \text{(D.513)}
$$

D.12.3 Worked Examples: Brute force Thiemann

Worked example:

Worked example:

For clarity

$$
\begin{aligned}
F^{\tau}_{f,T} &= \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j\in\mathcal{I}} \frac{(\tau_j - T_j)^{k_j}}{k_j!} f_{\{k_j\}_{j\in\mathcal{I}}} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\frac{(\tau_2 - T_2)^{k_2}}{k_2!}\ldots\frac{(\tau_n - T_n)^{k_n}}{k_n!} f_{k_1 k_2 \ldots k_n} \qquad \text{(D.514)}
\end{aligned}
$$

$$
F^{\tau}_{f,T} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\frac{(\tau_2 - T_2)^{k_2}}{k_2!} f_{k_1 k_2} \qquad \text{(D.515)}
$$

$$
\begin{aligned}
\{C_l, F^{\tau}_{f,T}\} &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \{C_l, \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\frac{(\tau_2 - T_2)^{k_2}}{k_2!} f_{k_1 k_2}\} \\
&= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \{C_l, \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\} \times \frac{(\tau_2 - T_2)^{k_2}}{k_2!} f_{k_1 k_2} + \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \times \{C_l, \frac{(\tau_2 - T_2)^{k_2}}{k_2!}\} f_{k_1 k_2} \\
&\quad + \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\frac{(\tau_2 - T_2)^{k_2}}{k_2!}\{C_l, f_{k_1 k_2}\} \qquad \text{(D.516)}
\end{aligned}
$$

Take the first term

$$
\begin{aligned}
\sum_{k_1=0}^{\infty}\{C_l, \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\} f_{k_1 k_2} \times \ldots &= \sum_{k_1=1}^{\infty} -\frac{(\tau_1 - T_1)^{k_1-1}}{(k_1-1)!}\{C_l, T_1\} f_{k_1 k_2} \times \ldots \\
&= \sum_{k_1=0}^{\infty} -\frac{(\tau_1 - T_1)^{k_1}}{(k_1)!}\{C_l, T_1\} f_{k_1+1 k_2} \times \ldots \qquad \text{(D.517)}
\end{aligned}
$$

$$
A_{l,m} := \{C_l, T_m\} \qquad \text{(D.518)}
$$

$$
\begin{aligned}
\{C_l, F^{\tau}_{f,T}\} &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{(\tau_1 - T_1)^{k_1}}{k_1!}\frac{(\tau_2 - T_2)^{k_2}}{k_2!} \times \\
&\times \ [-A_{l,1} f_{k_1+1 k_2} - A_{l,2} f_{k_1 k_2+1} + \{C_l, f_{k_2 k_1}\}] \qquad \text{(D.519)}
\end{aligned}
$$

We set (D.519) weakly to zero. Differentiating both sides of (D.519) with respect to τ_1 k_1 times and with respect to τ_2 k_2 times, then setting $\tau_1 = T_1$ and $\tau_2 = T_2$ (we are keeping x fixed) we arrive at the recursion relation

$$
-A_{l1} f_{k_1+1 k_2} - A_{l2} f_{k_1 k_2+1} + \{C_l, f_{k_1 k_2}\} \approx 0 \quad \text{for } k_1, k_2 = 0, 1, \ldots, \infty, \ l = 1, 2. \qquad \text{(D.520)}
$$

$$f_{k_1k_2} = (X_1')^{k_1} \cdot (X_2')^{k_2} \cdot f \tag{D.521}$$

As A is invertible condition (D.520) is equivalent to

$$\begin{aligned} &(A^{-1})_{il}(-A_{l1}f_{k_1+1k_2} - A_{l2}f_{k_1k_2+1} + \{C_l, f_{k_1k_2}\}) \\ &= -\delta_{i1}f_{k_1+1k_2} - \delta_{i2}f_{k_1k_2+1} + (A^{-1})_{il}\{C_l, f_{k_1k_2}\} \approx 0 \end{aligned} \tag{D.522}$$

that is

$$\begin{aligned} f_{k_1+1k_2} &\approx (A^{-1})_{1l}\{C_l, f_{k_1k_2}\} \\ f_{k_1k_2+1} &\approx (A^{-1})_{2l}\{C_l, f_{k_1k_2}\} \end{aligned} \tag{D.523}$$

Has the formal solution

$$f_{\{k\}} = f_{k_1k_2} = (X_1')^{k_1}(X_1')^{k_1} \cdot f, \qquad X_1' \cdot f = (A^{-1})_{jk}\{C_k, f\} \tag{D.524}$$

for $k_1, k_2 = 0, 1, \ldots, \infty$, $l = 1, \ldots, n$. Formal because we have not demonstrated that

$$(X_1') \cdot (X_2') \cdot f \approx (X_2') \cdot (X_1') \cdot f. \tag{D.525}$$

Before we do that we generalize (D.523) to the case of any finite integer n.

$$\begin{aligned} \{C_l, F^{\tau}_{f,T}\} &= \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j=1}^{n} \frac{(\tau_j - T_j)^{k_j}}{k_j!} \times [-A_{l,m}\{C_l, A_{l,1}f_{k_1+1\ldots 1+k_r\ldots k_n} + \ldots \\ &\quad A_{l,r}f_{k_1\ldots k_r+1\ldots k_n} + \cdots + A_{l,n}f_{k_1\ldots k_{n-1}k_n+1}\} + \{C_l, f_{k_1k_2\ldots k_n}\}] \\ &= \sum_{k_1,\ldots,k_n=0}^{\infty} \prod_{j=1}^{n} \frac{(\tau_j - T_j)^{k_j}}{k_j!} \times [\sum_{m=1}^{n} -A_{l,m}\{C_l, f_{\{k_j'(m)\}}\} + \{C_l, f_{\{k\}}\}] \end{aligned} \tag{D.526}$$

where $k_j'(m) = k_j$ for $j \neq m$ and $k_j'(m) = k_j + 1$.

recursion relation

$$A_{l,1}f_{k_1+1\ldots 1+k_r\ldots k_n} + \ldots A_{l,r}f_{k_1\ldots k_r+1\ldots k_n} + \cdots + A_{l,n}f_{k_1\ldots k_{n-1}k_n+1} \approx \{C_l, f_{k_1k_2\ldots k_n}\}, \tag{D.527}$$

$$\delta_{i,1} f_{k_1+1\ldots 1+k_r\ldots k_n} + \ldots \delta_{i,r} f_{k_1\ldots k_r+1\ldots k_n} + \cdots + \delta_{i,n} f_{k_1\ldots k_{n-1}k_n+1} \approx \sum_l (A^{-1})_{il} \{C_l, f_{k_1k_2\ldots k_n}\} \tag{D.528}$$

that is

$$f_{k_1\ldots k_r+1\ldots k_n} \approx \sum_l (A^{-1})_{rl} \{C_l, f_{k_1k_2\ldots k_n}\} \quad \text{for } 1 \geq r \geq n \tag{D.529}$$

Has the formal solution

$$f_{\{k\}} = f_{k_1\ldots k_n} = (X_1')^{k_1} \cdot \ldots (X_n')^{k_n} \cdot f, \qquad X_j' \cdot f = (A^{-1})_{jk} \{C_k, f\} \tag{D.530}$$

(d)

$$C_j' := \sum_{k_1,\ldots,k_n} (A^{-1})_{jk} C_k \tag{D.531}$$

The Hamiltonian vector fields

$$X_j \cdot f = \{C_j', f\} = \{(A^{-1})_{jk} C_k, f\} \tag{D.532}$$

prove we have

$$X_{j_1}' \ldots X_{j_n}' \cdot f \approx X_{j_1} \ldots X_{j_n} \cdot f \tag{D.533}$$

Consistency requires

$$X_q \cdot X_r \cdot f \approx X_r \cdot X_q \cdot f \quad \text{that is} \quad \{C_j', \{C_k', f\}\} - \{C_k', \{C_j', f\}\} \approx 0. \tag{D.534}$$

We will repeatedly use that

$$\{B_{ij} C_k, \, . \,\} = \{B_{ij}, \, . \,\} C_k + B_{ij} \{C_k, \, . \,\} \approx B_{ij} \{C_k, \, . \,\} \tag{D.535}$$

$$
\begin{aligned}
& \{C'_j, \{C'_k, f\}\} - \{C'_k, \{C'_j, f\}\} \approx \sum_{m,n} B_{jm}[\{C_m, B_{kn}\{C_n, f\} + B_{kn}\{C_m, \{C_n, f\}\}] - j \leftrightarrow k \\
\approx \quad & \sum_{m,n} B_{jm}[\{C_m, B_{kn}\}\{C_n, f\} + B_{kn}\{C_m, \{C_n, f\}\}] - j \leftrightarrow k \\
= \quad & \sum_{m,n} B_{jm}[\sum_{l,i} B_{kl}\{C_n, A_{li}\}B_{in}\{C_n, f\} + B_{kn}\{C_m, \{C_n, f\}\}] - j \leftrightarrow k \\
= \quad & \sum_{m,n} B_{jm}[\sum_{l,i} B_{kl}B_{in}\{C_n, f\}(\{C_m, \{C_n, T_i\}\} - \{C_l, \{C_m, T_i\}\}) + (\{C_m, \{C_n, f\}\} - \{C_l, \{C_m, f\}\}) \\
= \quad & \sum_{m,n} B_{jm}[-\sum_{l,i} B_{kl}B_{in}\{C_n, f\}\{T_i, \{C_m, C_l\}\} - B_{kn}\{f, \{C_m, C_l\}\}] \\
\approx \quad & \sum_{m,n} B_{jm}[-\sum_{l,i,p} B_{kl}B_{in}\{C_n, f\}f_{ml}{}^{p}A_{pi} + B_{kn}\sum_{l} f_{mn}{}^{l}\{C_l, f\}] \\
= \quad & \sum_{m,n,l} B_{jm}[-B_{kl}\{C_n, f\}f_{mn}{}^{l} + B_{kn}f_{ml}{}^{n}\{C_l, f\}] \\
= \quad & 0.
\end{aligned}
\tag{D.536}
$$

the Dirac observables generated by f, T_j indeed are

$$
F^{\tau}_{f,T} = \sum_{k_1 \ldots k_n = 0}^{\infty} \prod_j \frac{(\tau_j - T_j)^{k_j}}{k_j!} \prod_j (X_j)^{k_j} \cdot f \tag{D.537}
$$

(f)

Let $\alpha'_\beta(f) := \exp\left(\sum_j \beta_j X_j\right)$ be the gauge flow generated by the new constraints C'_j for real valued gauge parameters β_j. Prove

$$
\alpha'_\beta(T_j) \approx T_j + \beta_j. \tag{D.538}
$$

Proof:

$$
X_j \cdot T_k = \{(A^{-1})_{jl}C_l, T_k\} \approx (A^{-1})_{jl}\{C_l, T_k\} = (A^{-1})_{jl}A_{lk} = \delta_{jk} \tag{D.539}
$$

Two applications of an X_r on T_k is obviously weakly zero:

$$
X_q \cdot X_r \cdot T_k \approx X_q \cdot \delta_{jk} \approx 0. \tag{D.540}
$$

This is all we need to establish (D.538). Now, by definition,

$$
\begin{aligned}
\alpha'_\beta(T_j) &:= \sum_{m=0}^{\infty} \frac{1}{m!}(\beta_1 X_1 + \dots \beta_n X_n)^m \cdot T_j \\
&= T_j + (\beta_1 X_1 + \dots \beta_n X_n) \cdot T_j + \sum_{m=2}^{\infty} \frac{1}{m!}(\beta_1 X_1 + \dots \beta_n X_n)^m \cdot T_j \\
&\approx T_j + \sum_r \beta_r \delta_{rj} + 0 \\
&\approx T_j + \beta_j
\end{aligned} \tag{D.541}
$$

Worked example: Algebra of Dirac observables.

It is easy to see that

$$
\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\tau - T)^k}{k!}\frac{(\tau - T)^l}{l!} F(k,l) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{(\tau - T)^n}{n!} \begin{pmatrix} n \\ j \end{pmatrix} F(j, n-j) \tag{D.542}
$$

by noting that for each term in the summation on the left there is exactly one term in the summation on the right that is identical to it:

$$
\begin{aligned}
\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(\tau - T)^k}{k!}\frac{(\tau - T)^l}{l!} F(k,l) &= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (\tau - T)^{k+l} \frac{1}{k!l!} F(k,l) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{(\tau - T)^n}{n!} \frac{n!}{(n-j)!j!} F(j, n-j).
\end{aligned} \tag{D.543}
$$

Make use of (D.542) to prove

$$
\{F^\tau_{f,T}, F^\tau_{f',T}\} \approx F^\tau_{\{f,f'\}^*,T} \tag{D.544}
$$

where $\{f, f'\}^*$ is the Dirac bracket

$$
\{f, f'\}^* := \{f, f'\} - \{f, C_\mu\} K^{\mu\nu} \{C_\nu, f'\}, \qquad K^{\mu\nu} = \{C_\mu C_\nu\}, \;\; K^{\mu\rho} K_{\rho\nu} = \delta^\mu_\nu. \tag{D.545}
$$

(a)

By definition of the Hamiltonian vector field we have (basiscally the Jacobi identity)

$$
X_j \{f, f'\} = \{X_j f, f'\} + \{f, X_j f'\} \tag{D.546}
$$

$$
\begin{aligned}
\prod_{j=1}^{m}(X_{l_j})^{n_j}\{f,f'\} &= X_{l_1}^{n_1}\dots X_{l_m}^{n_m}\cdot\{f,f'\}\\
&= X_{l_1}^{n_1}\dots X_{l_{m-1}}^{n_{m-1}}(X_{l_m}^{n_m}\cdot\{f,f'\})\\
&= X_{l_1}^{n_1}\dots X_{l_{m-2}}^{n_{m-2}}(\sum_{k_m=0}^{n_m}\begin{pmatrix} n_m\\ k_m\end{pmatrix}\cdot\{X_{l_m}^{k_m}f,X_{l_m}^{n_m-k_m}f'\})\\
&= X_{l_1}^{n_1}\dots X_{l_{m-3}}^{n_{m-3}}(\sum_{k_{m-1}=0}^{n_{m-1}}\cdot\begin{pmatrix} n_{m-1}\\ k_{m-1}\end{pmatrix}\sum_{k_m=0}^{n_m}\begin{pmatrix} n_m\\ k_m\end{pmatrix}\times\\
&\quad\times\{X_{l_{m-1}}^{k_{m-1}}X_{l_m}^{k_m}\cdot f,X_{l_{m-1}}^{n_{m-1}-k_{m-1}}X_{l_m}^{n_m-k_m}f'\})\\
&= \sum_{k_1}^{n_1}\cdots\sum_{k_m}^{n_m}\prod_{l}\begin{pmatrix} n_l\\ k_l\end{pmatrix}\{X_l^{k_l}\cdot f,X_l^{n_l-k_l}\cdot f'\}
\end{aligned}
\tag{D.547}
$$

that is

$$
\prod_{j=1}^{m}(X_{l_j})^{n_j}\{f,f'\}=\sum_{k_1}^{n_1}\cdots\sum_{k_m}^{n_m}\prod_{l}\begin{pmatrix} n_l\\ k_l\end{pmatrix}\{f_{\{k\}},\cdot f'_{\{n-k\}}\}
\tag{D.548}
$$

(b)

We introduce some abbreviations

$$
\begin{aligned}
Y_{\{k\}}=\prod_j\frac{(\tau_j-T_j)^{k_j}}{k_j!},\quad f_{\{k\}}=\prod_j(X_j)^{k_j}\cdot f,\\
\sum_{\{k\}}=\sum_{k_1,\dots,k_n=0}^{\infty},\quad \sum_{\{k\}}{}':=\sum_{k_1\dots k_{j-1}k_{j+1}\dots k_n=0}
\end{aligned}
\tag{D.549}
$$

$$
\begin{aligned}
\{F^{\tau}_{f,T},F^{\tau}_{f',T}\} &= \sum_{k_1,\dots,k_n=0}^{\infty}\sum_{l_1,\dots,l_n=0}^{\infty}\{\prod_j\frac{(\tau_j-T_j)^{k_j}}{k_j!}\prod_j(X_j)^{k_j}\cdot f,\prod_l\frac{(\tau_l-T_l)^{k_l}}{k_l!}\prod_l(X_l)^{k_l}\cdot f'\}\\
&= \sum_{\{k\},\{l\}}\{Y_{\{k\}}f_{\{k\}},Y_{\{l\}}f'_{\{l\}}\}\\
&= \sum_{\{k\},\{l\}}[Y_{\{k\}}Y_{\{l\}}\{f_{\{k\}},f'_{\{l\}}\}+Y_{\{l\}}f_{\{k\}}\{Y_{\{k\}},f'_{\{l\}}\}+\\
&\quad +Y_{\{k\}}f'_{\{l\}}\{f_{\{k\}},Y_{\{l\}}\}+f_{\{k\}}f'_{\{l\}}\{Y_{\{k\}}Y_{\{l\}}\}].
\end{aligned}
\tag{D.550}
$$

The first term is already in the form we want, so we move to the second term

$$
\begin{aligned}
& \sum_{\{k\},\{l\}} \{Y_{\{k\}}, f'_{\{l\}}\} f_{\{k\}} Y_{\{l\}} \\
= \quad & \sum_{\{k\}} \sum_{\{l\}} \{ \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \ldots \frac{(\tau_m - T_m)^{k_m}}{k_m!} \ldots \frac{(\tau_n - T_n)^{k_n}}{k_n!}, f'_{\{l\}} \} f_{\{k\}} Y_{\{l\}} \\
= \quad & \sum_{\{k\}}{}' \sum_{k_m=0}^{\infty} \sum_{\{l\}} \sum_{m=0}^{n} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \ldots \{ \frac{(\tau_m - T_m)^{k_m}}{k_m!}, f'_{\{l\}} \} \ldots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \\
\times \quad & \qquad (X_1^{k_1} \cdot \ldots X_m^{k_m} \cdot \ldots X_n^{k_n} \cdot f) Y_{\{l\}} \\
= \quad & \sum_{\{k\}}{}' \sum_{\{l\}} \underbrace{\sum_{k_j=1}^{\infty}} \sum_{j=0}^{n} - \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \ldots \underbrace{\frac{(\tau_j - T_j)^{k_j - 1}}{(k_j - 1)!}} \{T_j, f'_{\{l\}}\} \ldots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \times \\
\times \quad & \qquad (X_1^{k_1} \cdot \ldots \underbrace{X_j^{k_j} \cdot} \ldots X_n^{k_n} \cdot f) Y_{\{l\}} \\
= \quad & -\sum_{\{k\}}{}' \sum_{\{l\}} \underbrace{\sum_{k_m=0}^{\infty}} \sum_{j=0}^{n} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \ldots \underbrace{\frac{(\tau_j - T_j)^{k_j}}{(k_j)!}} \ldots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \times \\
\times \quad & \qquad (X_1^{k_1} \cdot \ldots \underbrace{X_j^{k_j+1} \cdot} \ldots X_n^{k_n} \cdot f) Y_{\{l\}} \{T_j, f'_{\{l\}}\} \\
= \quad & -\sum_{\{k\}\{l\}} Y_{\{k\}} Y_{\{l\}} \sum_{j=0}^{n} (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{l\}}\}
\end{aligned}
\tag{D.551}
$$

where we have defined $(X_m \cdot f)_{\{k\}}$ to be

$$
(X_j \cdot f)_{\{k\}} := X_1^{k_1} \cdot \ldots X_j^{k_j+1} \cdot \ldots X_n^{k_n} \cdot f \tag{D.552}
$$

Similarly we get

$$
\sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} \{Y_{\{k\}}, f'_{\{l\}}\} = \sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} \sum_{j=0}^{n} (X_j \cdot f')_{\{l\}} \{T_j, f_{\{k\}}\} \tag{D.553}
$$

for the last term we need $\{Y_{\{k\}}, Y_{\{l\}}\}$:

$$
\{ \frac{(\tau_j - T_j)^{k_j}}{k_j!}, \frac{(\tau_m - T_m)^{k_m}}{k_n!} \} = \frac{(\tau_j - T_j)^{k_j - 1}}{(k_j - 1)!} \frac{(\tau_m - T_m)^{k_m - 1}}{(k_m - 1)!} \{T_j, T_m\} \tag{D.554}
$$

$$
\begin{aligned}
& \sum_{\{k\},\{l\}} f_{\{k\}} f'_{\{l\}} \{Y_{\{k\}}, Y_{\{l\}}\} \\
= & \sum_{\{k\},\{l\}}{}' \underbrace{\sum_{k_j,l_m=1}^{\infty}} \sum_{j=1}^{n} \sum_{m=1}^{n} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \underbrace{\frac{(\tau_j - T_j)^{k_j - 1}}{(k_m - 1)!}} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \times \\
& \quad \times \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \underbrace{\frac{(\tau_m - T_m)^{k_m - 1}}{(k_m - 1)!}} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \{T_j, T_m\} \\
\times & \; X_1^{k_1} \cdot \ldots \underbrace{X_j^{k_j}} \cdot \ldots X_n^{k_n} \cdot f X_1^{l_1} \cdot \ldots \underbrace{X_m^{l_m}} \cdot \ldots X_n^{l_n} \cdot f' \\
= & \sum_{\{k\},\{l\}}{}' \underbrace{\sum_{k_j,l_m=0}^{\infty}} \sum_{j=1}^{n} \sum_{m=1}^{n} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \underbrace{\frac{(\tau_j - T_j)^{k_j}}{(k_m)!}} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \times \\
& \quad \times \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \underbrace{\frac{(\tau_m - T_m)^{k_m}}{(k_m)!}} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} \{T_j, T_m\} \\
\times & \; X_1^{k_1} \cdot \ldots \underbrace{X_j^{k_j+1}} \cdot \ldots X_n^{k_n} \cdot f \times X_1^{l_1} \cdot \ldots \underbrace{X_m^{l_m+1}} \cdot \ldots X_n^{l_n} \cdot f' \\
= & \sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} \sum_{j,m=1}^{n} (X_j \cdot f)_{\{k\}} (X_m \cdot f')_{\{l\}} \{T_j, T_m\}
\end{aligned}
\tag{D.555}
$$

Putting it all together we have

$$
\begin{aligned}
& \{F^{\tau}_{f,T}, F^{\tau}_{f',T}\} = \sum_{\{k\},\{l\}} \{Y_{\{k\}} f_{\{k\}}, Y_{\{l\}} f'_{\{l\}}\} \\
\approx & \sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} \left[\{f_{\{k\}}, f'_{\{l\}}\} - \sum_j (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{l\}}\} \right. \\
+ & \left. \sum_j (X_j \cdot f')_{\{l\}} \{T_j, f_{\{k\}}\} + \sum_{j,m} (X_j \cdot f)_{\{k\}} (X_m \cdot f)_{\{l\}} \{T_j, T_m\} \right]
\end{aligned}
\tag{D.556}
$$

Using the multi version of (D.542), i.e.

$$
\begin{aligned}
& \sum_{\{k\}} \sum_{\{l\}} \frac{(\tau_{k_1} - T_{k_1})^{k_1}}{k_1!} \cdots \frac{(\tau_{k_n} - T_{k_n})^{k_n}}{k_1!} \frac{(\tau_{l_1} - T_{l_1})^{l_1}}{l_1!} \cdots \frac{(\tau_{l_n} - T_{l_n})^{l_n}}{l_n!} F(k_1, l_1, \ldots k_n, l_n) \\
= & \sum_{\{n\}} \sum_{j_1=0}^{n_1} \cdots \sum_{j_n=0}^{n_n} \frac{(\tau_1 - T_1)^{n_1}}{n_1!} \cdots \frac{(\tau_1 - T_1)^{n_n}}{n_n!} \begin{pmatrix} n_1 \\ j_1 \end{pmatrix} \cdots \begin{pmatrix} n_n \\ j_n \end{pmatrix} F(j_1, n_1 - j_1 \ldots j_n, n_n - j_n),
\end{aligned}
\tag{D.557}
$$

we rewrite this involving a sum over $Y_{\{n\}}$ rather than a sum involving the product $Y_{\{k\}} Y_{\{l\}}$,

$$\begin{aligned}
&\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\} \\
\approx\ & \sum_{\{n\}} Y_{\{n\}} \sum_{k_1}^{n} \cdots \sum_{k_n}^{n} \prod_{l} \begin{pmatrix} n_l \\ k_l \end{pmatrix} [\{f_{\{k\}}, f'_{\{n-k\}}\} - \sum_j (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{n-k\}}\} + \\
+\ & \sum_j (X_j \cdot f')_{\{n-k\}} \{T_j, f_{\{k\}}\} + \sum_{j,m} (X_j \cdot f)_{\{k\}} (X_m \cdot f')_{\{n-k\}} \{T_j, T_m\}]
\end{aligned} \tag{D.558}$$

$$\prod_l (X_l)^n_l [\{f, f'\}^* - \{f, f'\}] \approx \sum_{k_1}^{n} \cdots \sum_{k_n}^{n} \prod_l \begin{pmatrix} n_l \\ k_l \end{pmatrix} \tag{D.559}$$

$$\sum_{L,l} K^{Jj,Ll} K_{Ll,Kk} = \delta^J_K \delta^j_k \tag{D.560}$$

therefore

$$K_{1j,1k} = \{C_j, C_k\} \approx 0, \quad K_{1j,2j} = \{C_j, T_k\} = A_{jk} = -K_{2k,1j}, \quad K_{2k,2j} = \{T_j, T_k\}. \tag{D.561}$$

$$K^{1j,1k} \approx \sum_{m,n} (A^{-1})_{mj} \{T_m, T_n\} (A^{-1})_{nk}, \quad K^{1j,2k} \approx -(A^{-1})_{kj} \approx -K^{2k,1j}, \quad K^{2j,2k} \approx 0. \tag{D.562}$$

$$\begin{aligned}
-\{f, f'\} + \{f, f'\} \ &\equiv\ \{f, C_\mu\} K^{\mu\nu} \{C_\nu, f'\} \\
&\equiv\ \{f, C_j\} K^{1j,1k} \{C_k, f'\} + \{f, C_j\} K^{1j,2k} \{T_k, f'\} + \\
&\quad\ \{f, C_j\} K^{2j,1k} \{C_k, f'\} + \{f, T_j\} K^{2j,2k} \{T_k, f'\}
\end{aligned} \tag{D.563}$$

Now substitute the weak equalities (D.562)

$$\begin{aligned}
-\{f, f'\} + \{f, f'\} \ &\approx\ \sum_{m,n} \{f, C_j\} (A^{-1})_{mj} \{T_m, T_n\} (A^{-1})_{nk} \{C_k, f'\} - \{f, C_j\} (A^{-1})_{kj} \{T_k, f'\} + \\
&\quad\ + \{f, T_j\} (A^{-1})_{jk} \{C_k, f'\} + 0 \\
&=\ -\sum_{m,n} (X_m \cdot f) \{T_m, T_n\} (X_n \cdot f') + (X_k \cdot f) \{T_k, f'\} - (X_k \cdot f') \{T_k, f\}
\end{aligned} \tag{D.564}$$

Suppose

$$X_j \prod_l (X_l)_l^n[\{f,f'\}^* - \{f,f'\}]$$

$$\simeq \quad X_j \sum_{k_1}^{n} \cdots \sum_{k_n}^{n} \tag{D.565}$$

we will use the notation

$$(X_l \cdot f)_{\{k^j\}} := \begin{cases} X_1^{k_1} \ldots X_l^{k_l+1} \ldots X_j^{k_j+1} \ldots X_n^{k_n} \cdot f & l \neq j \\ X_1^{k_1} \ldots X_j^{k_j+2} \ldots X_n^{k_n} \cdot f & l = j \end{cases} \tag{D.566}$$

$$f'_{\{n^j-k\}} := (X_1)^{n_1-k_1} \ldots (X_j)^{n_j+1-k_1} \ldots (X_1)^{n_n-k_n} \cdot f' \tag{D.567}$$

$$\begin{aligned} & X_j \cdot \sum_{k_1}^{n_1} \cdots \sum_{k_n}^{n_1} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} n_n \\ k_n \end{pmatrix} \sum_l (X_1^{k_1} \ldots X_l^{k_l+1} \ldots X_n^{k_n}) \cdot f\{T_l, X_1^{n_1-k_1} \ldots X_n^{n_n-k_n} \cdot f'\} \\ \approx \quad & \sum_{k_1}^{n_1} \cdots \sum_{k_n}^{n_1} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} n_n \\ k_n \end{pmatrix} \\ & -\sum_l [(X_l \cdot f)_{\{k^j\}}\{T_l, f'_{\{n-k\}}\} + (X_l \cdot f)_{\{k\}}\{T_l, f'_{\{n^j-k\}}\} + (X_l \cdot f)_{\{k\}}\{X_j \cdot T_l, f'_{\{n-k\}}\}] \end{aligned} \tag{D.568}$$

Similarly for $X_j \cdot \sum_l (X_l \cdot f')_{\{n-k\}}\{T_l, f_{\{k\}}\}$

$$\begin{aligned} & X_j \cdot \sum_{l,m} (X_l \cdot f)_{\{k\}}(X_m \cdot f)_{\{n-k\}}\{T_l, T_m\} \\ \approx \quad & \sum_{l,m} [(X_l \cdot f)_{\{k^j\}}(X_m \cdot f)_{\{n-k\}}\{T_l, T_m\} + (X_l \cdot f)_{\{k\}}(X_m \cdot f)_{\{n^j-k\}}\{T_l, T_m\} \\ + \quad & (X_l \cdot f)_{\{k\}}(X_m \cdot f)_{\{n-k\}}(\{X_l \cdot T_l, T_m\} + \{T_l, X_l \cdot T_m\})]] \end{aligned} \tag{D.569}$$

Putting it all together we have

$$\begin{aligned} = \quad & \sum_{\{k\}} \sum_{m=1}^{n} \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m+1} \cdots \sum_{k_n=1}^{n_n} \begin{pmatrix} n_1 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} n_n \\ k_n \end{pmatrix} \\ & \sum_l [(X_1^{k_1} \ldots X_l^{k_l+1} \cdot \ldots X_j^{k_j+1} \cdot \ldots X_n^{k_n}) \cdot f' + \\ & (X_1^{k_1} \ldots X_l^{k_l+1} \ldots X_n^{k_n}) \cdot f\{T_l, X_1^{n_1-k_1} \ldots X_1^{n_j+1-k_1} \ldots X_1^{n_n-k_n} \cdot f'\} \end{aligned} \tag{D.570}$$

$$
\begin{aligned}
&= \sum_{\{k\}} \sum_{m=1}^{n} \sum_{k_1=1}^{n_1} \cdots \sum_{k_m=1}^{n_m+1} \cdots \sum_{k_n=1}^{n_n} \binom{n_1}{k_1} \cdots \binom{n_n}{k_n} \\
&\quad [\{f_{\{k\}}, f'_{\{n-k\}}\} - \sum_j (X_j \cdot f)_{\{k\}} \{T_j, f'_{\{n-k\}}\} + \\
&+ \sum_j (X_j \cdot f')_{\{n-k\}} \{T_j, f_{\{k\}}\} + \sum_{j,m} (X_j \cdot f)_{\{k\}} (X_m \cdot f')_{\{n-k\}} \{T_j, T_m\}] \qquad \text{(D.571)}
\end{aligned}
$$

hence it remains to show

$$
\begin{aligned}
0 \;\approx\; & \sum \prod_l \binom{n_l}{k_l} \\
& [-\sum_l (X_l \cdot f)_{\{k\}} \{X_j \cdot T_l, f'_{\{n-k\}} + \sum_l (X_l \cdot f')_{\{k\}} \qquad \text{(D.572)}
\end{aligned}
$$

We have

$$
\begin{aligned}
X_j \cdot T_l &= \sum_m \{A_{jm} C_m, T_l\} \\
&= \sum_m A_{jm} \{C_m, T_l\} + C_m \{A_{mk}, T_l\} \\
&= \sum_m [A_{jm} (A^{-1})_{ml} + C_m \{A_{mk}, T_l\} \\
&= \delta_{jl} + \sum_m C_m \{(A^{-1})_{jm}, T_l\} =: \delta_{jl} + \sum_m C_m B_{jlm} \qquad \text{(D.573)}
\end{aligned}
$$

hence

$$
\{X_j \cdot T_l, g\} \approx \sum_{m,n} B_{jlm} A_{mn} (X_n \cdot g) =: \sum_m D_{jln} (X_n \cdot g) \qquad \text{(D.574)}
$$

using (D.573) and (D.574)

$$
\{X_j \cdot T_l, T_m\} + \{T_l, X_j \cdot T_m\} \approx \sum_n (B_{jln} A_{nm} - B_{jmn} A_{nl}) = D_{jlm} - D_{jml}. \qquad \text{(D.575)}
$$

We now simplify the right hand side of (D.572)

$$
\begin{aligned}
& \binom{n_l}{k_l} \times \\
& \sum_{l,m} D_{jlm}[-(X_l \cdot f)_{\{k\}}(X_m \cdot f'_{\{n-k\}}) + (X_l \cdot f'_{\{k\}}(X_m \cdot f)_{\{n-k\}}) \\
+ \quad & [D_{jlm} - D_{jml}](X_l \cdot f)_{\{k\}}(X_m \cdot f'_{\{n-k\}})] \\
& \sum_{l,m} D_{jlm} \prod_i (X_i)^{n_i}[[- \qquad \text{(D.576)}
\end{aligned}
$$

Worked example: Homomorphism of F^{τ}_{T}.

Establish the properties (a) of (D.344) and (b) of (D.345).

(a) The first is easy,

$$
\begin{aligned}
F^{\tau}_{f+f',T} &= \sum_{\{k\}} Y_{\{k\}}(f_{\{k\}} + f'_{\{k\}}) \\
&= \sum_{\{k\}} Y_{\{k\}} f_{\{k\}} + \sum_{\{k\}} Y_{\{k\}} f'_{\{k\}} \\
&= F^{\tau}_{f,T} + F^{\tau}_{f',T}. \qquad \text{(D.577)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
F^{\tau}_{f,T} F^{\tau}_{f',T} &= \sum_{\{k\},\{l\}} Y_{\{k\}} Y_{\{l\}} f_{\{k\}} f'_{\{l\}} \\
&= \sum_{\{k\}} Y_{\{k\}} \sum_{k_1,\ldots,k_n} \prod_l \binom{n_l}{k_l} f_{\{k\}} f'_{\{n-k\}} \\
&\simeq \sum_{\{k\}} Y_{\{k\}} \prod_l (X_l)^{n_l}(ff') = F^{\tau}_{ff',T} \qquad \text{(D.578)}
\end{aligned}
$$

Worked example: Evolving Constants

$$
\begin{aligned}
F_{f,T}^{\tau+\tau_0} &= \sum_{n_1,n_2,\cdots=0}^{\infty} \frac{(\tau_1+\tau_1^0-T_1)^{n_1}}{n_1!}\cdots\frac{(\tau_n+\tau_n^0-T_n)^{n_n}}{n_n!} X_1^{n_1}\dots X_n^{n_n}\cdot f \\
&\approx \sum_{n_1,\dots,n_n=0}^{\infty}\left[\sum_{k_1=0}^{n_1}\frac{1}{n_1!}\begin{pmatrix} n_1 \\ k_1 \end{pmatrix}(\tau_1^0-T_1)^{k_1}\tau_1^{n_1-k_1}\right]\cdots\left[\sum_{k_n=0}^{n_n}\frac{1}{n_n!}\begin{pmatrix} n_n \\ k_n \end{pmatrix}(\tau_n^0-T_n)^{k_n}\tau_n^{n_n-k_2}\dots\right] \\
&\quad ; \qquad \times X_1^{k_1}\dots X_n^{k_n}\cdot f \\
&\approx \sum_{n_1,\dots,n_n=0}^{\infty}\sum_{k_1=0}^{n_1}\cdots\sum_{k_n=0}^{n_n}\frac{(\tau_1^0-T_1)^{k_1}}{k_1!}\cdots\frac{(\tau_n^0-T_n)^{k_n}}{k_n!}\frac{\tau_1^{n_1-k_1}}{(n_1-k_1)!}\cdots\frac{\tau_n^{n_n-k_n}}{(n_n-k_n)!}X_1^{k_1}\dots X_n^{k_n}\cdot f
\end{aligned}
\tag{D.579}
$$

Using (D.542)

$$
\begin{aligned}
F_{f,T}^{\tau+\tau_0} &\approx \sum_{k_1,\dots k_n,=0}^{\infty}\frac{(\tau_1^0-T_1)^{k_1}}{k_1!}\cdots\frac{(\tau_n^0-T_n)^{k_n}}{k_1!}X_1^{k_1}\dots X_n^{k_n}\left[\sum_{l_1,\dots l_n,=0}^{\infty}\frac{\tau_1^{k_1}}{l_1!}\cdots\frac{\tau_n^{k_n}}{l_n!}X_1^{n_1-k_1}\dots X_n^{n_n-k_n}\right]\cdot f \\
&= F_{\alpha'_\tau(f),T}^{\tau_0}
\end{aligned}
\tag{D.580}
$$

$$
\begin{aligned}
\alpha'_\tau(f) &:= \sum_{n=0}^{\infty}\frac{1}{m!}(\tau_1X_1+\dots\tau_nX_n)^m\cdot f \\
&= \sum_{n=0}^{\infty}\frac{1}{m!}\sum_{j_1,\dots j_n=0}^{n}\tau_{j_1}X_{j_1}\dots\tau_{j_n}X_{j_n}\cdot f
\end{aligned}
\tag{D.581}
$$

A simple and familiar combinatorial problem: there are $m!$ ways of ordering the n terms $\tau_{j_1}\dots\tau_{j_n}$, however, there are $k_r!$ ways to order k_r identical τ_r's - so

$$
\begin{aligned}
\alpha'_\tau(f) &= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{\sum_j k_j=n}\frac{m!}{k_1!\dots k_n!}\tau_{j_1}^{k_1}\dots\tau_{j_n}^{k_n}X_{j_1}\dots X_{j_n}\cdot f \\
&= \sum_{k_1,\dots k_n=0}^{\infty}\frac{\tau_{j_1}^{k_1}}{k_1!}\cdots\frac{\tau_{j_n}^{k_n}}{k_n!}(X_{j_1})^{k_1}\dots(X_{j_n})^{k_n}\cdot f
\end{aligned}
\tag{D.582}
$$

Worked example: Dirac brackets to Posion brackets for $f(q,p)$

$$
\begin{aligned}
\{f,f'\}^* &= \{f,f'\} - \{f,C_j\}\{C_j,C_k\}\{C_k,f'\} - \{f,C_j\}\{C_j,T_k\}\{T_k,f'\} - \\
&\quad \{f,T_j\}\{T_j,C_k\}\{C_k,f'\}\{f,T_j\}\{T_j,T_k\}\{T_k,f'\}
\end{aligned} \tag{D.583}
$$

$$
\{f(q,p),C_k\} = \{\frac{\partial f}{\partial q} \tag{D.584}
$$

$$
\begin{aligned}
\{f(q,p),f'(q,p)\}^* &= \{f(q,p),f'(q,p)\} - \{f(q,p),C_j\}\underbrace{\{C_j,C_k\}}_{\approx 0}\{C_k,f'(q,p)\} - \\
&\quad \{f(q,p),C_j\}\{C_j,T_k\}\underbrace{\{T_k,f'(q,p)\}}_{\approx 0} - \\
&\quad \underbrace{\{f(q,p),T_j\}}_{\approx 0}\{T_j,C_k\}\{C_k,f'(q,p)\} \\
&\quad \underbrace{\{f(q,p),T_j\}}_{\approx 0}\{T_j,T_k\}\underbrace{\{T_k,f'(q,p)\}}_{\approx 0}
\end{aligned} \tag{D.585}
$$

where we used the first class property in the first line.

Worked example: Reduced quant

$$
\alpha^\tau(Q^a) = \sum_{k_1,k_2,\cdots=0}^{\infty} \frac{\tau_{j_1}}{k_1!}\cdots\frac{\tau_{j_n}}{k_n!}F_{\prod_j X_j^{k_j}\cdot q^a,T} \tag{D.586}
$$

$$
\begin{aligned}
\sum_{\{k\}}\prod_j \frac{\tau^{k_j}}{k_j!}F_{\prod_j X_j^{k_j}\cdot q^a,T} &= \sum_{\{k\}}\prod_j \frac{\tau^{k_j}}{k_j!}\sum_{l_1,l_2,\cdots=0}^{\infty}\frac{(\tau_1-T_1)^{l_1}}{l_1!}\cdots\frac{(\tau_n-T_n)^{l_n}}{l_n!}X_1^{l_1}\ldots X_n^{l_n}\cdot(X_i^{j_1}\ldots X_n^{j_n}\cdot q^a) \\
&= \sum_{l_1,l_2,\cdots=0}^{\infty}\frac{(\tau_1-T_1)^{l_1}}{l_1!}\cdots\frac{(\tau_n-T_n)^{l_n}}{l_n!}X_1^{l_1}\ldots X_n^{l_n}\cdot(\sum_{\{k\}}\prod_j\frac{\tau^{k_j}}{k_j!}X_i^{j_1}\ldots X_n^{j_n}\cdot q^a) \\
&= \sum_{l_1,l_2,\cdots=0}^{\infty}\frac{(\tau_1-T_1)^{l_1}}{l_1!}\cdots\frac{(\tau_n-T_n)^{l_n}}{l_n!}X_1^{l_1}\ldots X_n^{l_n}\cdot\alpha'_\tau(q^a) \\
&= F_{\alpha'_\tau(q^a),T}
\end{aligned} \tag{D.587}
$$

$$
X_l\cdot T_j = \delta_{lj}, \qquad X_{l_q}\cdot X_{l_r} = X_{l_q}\delta_{lj} = 0. \tag{D.588}
$$

$$F^{\tau}_{T_j,T} \approx \sum_{l_1,l_2,\cdots=0}^{\infty} \frac{(\tau_1 - T_1)^{l_1}}{l_1!} \cdots \frac{(\tau_n - T_n)^{l_n}}{l_n!} X_1^{l_1} \cdot \ldots X_n^{l_n} \cdot T_j + \mathcal{O}(X^2) \cdot T_j X_1^{l_1} \cdot \ldots X_n^{l_n} \cdot T_j$$
$$= T_j + \sum_{m=1}^{n} \delta_{mj}(\tau_m - T_m)$$
$$= \tau_j \tag{D.589}$$

$$X_r \cdot C_j := A_{rl}\{C_l, C_j\} = A_{rl} f_{lj}\ {}^m C_m \tag{D.590}$$

$$F^{\tau}_{C_j,T} \tag{D.591}$$

Worked example: Reduced

$$\{H_j, F^0_{f,T}\} \approx F^0_{\{E_j,f\}^*,T} = F^0_{\{E_j,f\},T} = F^0_{\{\tilde{C}_j,f\}T}$$
$$= \sum_{\{k\}} \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!} \prod_l X_l^{k_l} \cdot \tilde{X}_j \cdot f$$
$$\approx \sum_{\{k\}} \prod_l \frac{(\tau_1 - T_l)^{k_l}}{k_l!} \tilde{X}_j \cdot \prod_l X_l^{k_l} \cdot f$$
$$\approx \tilde{X}_j \cdot F^0_{f,T} - \sum_{\{k\}} (\tilde{X}_j \cdot \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!}) \prod_l X_l^{k_l} \cdot f$$
$$\approx + \sum_{\{k\}} \prod_l \frac{(\tau_l - T_l)^{k_l}}{k_l!} X_j \cdot \prod_l X_l^{k_l} \cdot f$$
$$= (\frac{\partial}{\partial \tau_j})_{\tau=0}\ \alpha^{\tau}(F_T(f)) \tag{D.592}$$

$$\{H_j(\tau), F^{\tau}_{f,T}\} := \frac{\partial}{\partial \tau_j} \alpha^{\tau}(F_T(f)) \tag{D.593}$$

where we used $\{T_j, E_k\} = \{T_j, f\} = 0$