Appendix E

ADM and First order Formalism of Einstein's Theory

covariant canonical quantization - one resolution - quantizing the space of solutions to the classical equations of motion.

well-posed initial value problem the classical solutions are

E.1 Intrinsic and Extrinsic Curvature

The extrinsic curvature is defined as the normal projection of the tensor $\nabla_a n_b$:

$$K_{ab} := q_a^c q_b^d \nabla_c n_d \tag{E.1}$$

On account of $\nabla_a(n_c n^c) = 0$ we have $n^c \nabla_a n_c = 0$

$$K_{ab} := q_a^c (g_b^d - n_b n^d) \nabla_c n_d = q_a^c \nabla_c n_b \tag{E.2}$$

The extrinsic curvature is symmetric:

$$K_{ab} = K_{ba}. \tag{E.3}$$

This follows from the definition of n^a, hypersurface orthogonal $n^a = -\partial_b \tau$,

$$\nabla_a n_b := \nabla_a(-\partial_b \tau) = -\partial_a \partial_b \tau + \Gamma_{ab}^c \partial_c \tau = -\underbrace{\partial_b \partial_a}\tau + \underbrace{\Gamma_{ba}^c}\partial_c \tau = \nabla_b(-\partial_a \tau) = \nabla_b n_a. \tag{E.4}$$

E.2 ADM Metric Formulation

In general the "flow of time" is not in the direction of the time-like normal n_a, and it is convenient to decompose t_a into components perpendicular

$$t^a = Nn^a + N^a. \tag{E.5}$$

The quantities N and N^a are the lapse and shift functions respectively and determine the projections of the time evolution vector field t^a along the direction perpendicular and tangent to the spacial slice.

The line element in these variables reads

$$ds^2 = (N^2 - N_a N^b)dt^2 - 2N_a dx^x dt + q_{ab} dx^a dx^b \tag{E.6}$$

We have

$$\sqrt{-\det g} = N\sqrt{q}.$$

Gauss' Equation

The curvature tensor in the spatial slice is defined by

$$\mathrm{R}_{abc}{}^d u_d := 2\mathcal{D}_{[a}\mathcal{D}_{b]}u_c \tag{E.7}$$

where $\mathcal{D}$ is the covariant derivative of the three metric. We wish to connect this with the four-dimensional Riemann tensor $R_{abc}{}^d$

$$\begin{aligned}
\mathrm{R}_{abc}{}^d u_d &= (\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)u_c \\
&= \mathcal{D}_a(q^p{}_b q^q{}_c \nabla_p u_q) - \mathcal{D}_b(q^p{}_a q^q{}_c \nabla_p u_q) \\
&= q^p{}_b q^q{}_c(\mathcal{D}_a \nabla_p u_q) - q^p{}_a q^q{}_c(\mathcal{D}_b \nabla_p u_q) \qquad \mathcal{D}_c q_{ab} := 0, \\
&= 2q^r{}_{[a} q^s{}_{b]} q^t{}_c \nabla_r (q^p{}_s q^q{}_t \nabla_p u_q)
\end{aligned} \tag{E.8}$$

This leads to the relation (exercise)

$$\mathrm{R}_{abc}{}^d + K_{ca}K_b{}^d - K_{cb}K_a{}^d = q^p{}_a q^q{}_b q^r{}_c q^d{}_s R_{pqr}{}^s, \tag{E.9}$$

known as **Gauss' equation**.

Evolution Equations

The change in time of the three-metric q_{ab} is given by the extrinsic curvature

$$K_{ab} = q_a{}^m q_b{}^n \nabla_m n_n = \frac{1}{2}\mathcal{L}_{\vec{n}} q_{ab}, \tag{E.10}$$

where $\mathcal{L}_{\vec{n}}$ is the Lie derivative in the direction of n_a.

$$\begin{aligned}
\mathcal{L}_{\vec{n}} q_{ab} &= n^c \nabla_c q_{ab} + q_{ac}\nabla_b n^c + q_{cb}\nabla_a n^c \\
&= n^c \nabla_c (g_{ab} + n_a n_b) + q_{ac}\nabla_b n^c + q_{cb}\nabla_a n^c \\
&= n^c (n_a \nabla_c n_b + n_b \nabla_c n_a) + (\nabla_a n_b + \nabla_b n_a) \\
&= (\delta_a^c + n^c n_a)\nabla_c n_b + (\delta_b^c + n^c n_b)\nabla_c n_a \\
&= q_a^c \nabla_c n_b + q_b^c \nabla_c n_a \\
&= 2K_{ab}.
\end{aligned} \tag{E.11}$$

$$2G_{ab} n^a n^b = R + K^2 - K^{ab} K_{ab}, \tag{E.12}$$

where K is the trace of K_{ab}, $K = K^{ab} q_{ab}$.

using the definition of ${}^4R_{abc}{}^d$,

$$2\nabla_{[a}\nabla_{b]} n_c = {}^4R_{abc}{}^d n_d \tag{E.13}$$

$$D_a K^a{}_b - D_b K^a{}_a = R_{cd} n^d q^c{}_b \tag{E.14}$$

$$\begin{aligned}
R_{\alpha\beta} u^\alpha u^\beta &= g^{\alpha\gamma} R_{\alpha\beta\gamma\delta} u^\alpha u^\beta u^\beta \nabla_\alpha \nabla_\beta u^\alpha - u^\beta \nabla_\beta \nabla_\alpha \\
&= \nabla_\alpha (u^\beta \nabla_\beta u^\alpha) - (\nabla_\alpha u^\beta)(\nabla_\beta u^\alpha) - \nabla_\beta (u^\beta \nabla_\alpha u^\alpha) + (\nabla_\beta u^\beta)^2 \\
&= \nabla_i (K u^i + a^i) - K_{\alpha\beta} K^{\alpha\beta} + K_\alpha^\alpha K_\beta^\beta
\end{aligned} \tag{E.15}$$

$$R = -R g_{\alpha\beta} u^\alpha u^\beta = 2(G_{\alpha\beta} - R_{\alpha\beta}) u^\alpha u^\beta, \tag{E.16}$$

and the identity

$$2G_{\alpha\beta} u^\alpha u^\beta = K_\alpha^\alpha K_\beta^\beta - K_{\alpha\beta} K^{\alpha\beta} + {}^{(3)}R \tag{E.17}$$

contracting

$$2\,{}^4\nabla_{[a}\,{}^4\nabla_{b]}k_c = \,{}^4R_{abc}{}^{d}k_d \tag{E.18}$$

on a and c and using the 3-metric to project b into Σ leads to a vector constraint

$$G_{ab}n^a q^b{}_m \equiv \nabla^a(K_{am} - Kq_{am}). \tag{E.19}$$

Assuming that the 4-metric g_{ab} satisfies the vacuum Einstein's equation,

$$G_{ab} = 0. \tag{E.20}$$

these constraint equations become

$$C := R + K^2 - K^{ab}K_{ab} \approx 0, \tag{E.21}$$
$$C_m := \nabla^a(K_{am} - Kq_{am}) \approx 0. \tag{E.22}$$

Gauss-Codacci relations

$$v_{p|q} := \nabla_b v_a e^a_p e^b_q \tag{E.23}$$

$$\begin{aligned} e^a_p e^b_q \nabla_a v_b &= e^a_q \nabla_a (v_b e^b_p) - v_b e^a_q \nabla_a e^b_p \\ &= \partial_a v_p - v^b e^a_q \nabla_a e_{pb} \\ &= \partial_a v_p - e^a_q e^c_r v^r \nabla_a e_{pc} \\ &= \partial_q v_p - \Gamma_{rpq} \end{aligned} \tag{E.24}$$

$$v_{p|q} = \partial_q v_p - \Gamma^q_{pq} v_q \tag{E.25}$$

The remaining six equations are the evolution equations for q_{ab} and K_{ab} which can be written in the form

$$\begin{aligned} \mathcal{L}_t q_{ab} &= 2NK_{ab} + \mathcal{L}_{\vec{N}} q_{ab} \\ \mathcal{L}_t K_{ab} &= N q_a{}^m q_b{}^n R_{mn} - NR_{ab} + 2NK_a{}^m K_{mb} - NKK_{ab} + D_a D_b N + \mathcal{L}_{\vec{N}} K_{ab} \end{aligned} \tag{E.26}$$

$$\begin{aligned}
\mathcal{L}_{\vec{N}} q_{ab} &= N^c \mathcal{D}_c q_{ab} + q_{ac} \mathcal{D}_b N^c + q_{cb} \mathcal{D}_a N^c \\
&= 2\mathcal{D}_{(a} N_{b)} \qquad \text{(E.27)}
\end{aligned}$$

so that

$$K_{ab} = \frac{1}{2N}(\dot{q}_{ab} - 2\mathcal{D}_{(a} N_{b)}) \qquad \text{(E.28)}$$

E.2.1 The Action

Using this the action can be written in terms of these variables:

$$S[N, \vec{N}, q] = \int dt \int d^3x \sqrt{q} N \left(K_{ab} K^{ab} - K^2 + R[q] \right) \qquad \text{(E.29)}$$

where $K^2 = K_a{}^a$. That is, in terms of the main variables, the Lagrangian density reads

$$\begin{aligned}
\mathcal{L}[N, \vec{N}, q] &= \sqrt{q} N \left(K_{ab} K^{ab} - K^2 \right) + \sqrt{q} N R[q] \\
&= \sqrt{q} N \left(q^{ac} q^{bd} K_{ab} K_{cd} - q^{ab} q^{cd} K_{ab} K_{cd} \right) + \sqrt{q} N R[q] \\
&= \sqrt{q} N (q^{ac} q^{bd} - q^{ab} q^{cd}) K_{ab} K_{cd} + \sqrt{q} N R[q] \\
&= \frac{\sqrt{q}(q^{ac} q^{bd} - q^{ab} q^{cd})(\dot{q}_{ab} - 2\mathcal{D}_{(a} N_{b)})(\dot{q}_{cd} - 2\mathcal{D}_{(c} N_{d)})}{4N} + \sqrt{q} N R[q] \qquad \text{(E.30)}
\end{aligned}$$

E.3 The Hamiltonian Formulation

The above form makes the hamiltonian analysis easy. The canonical momentum of the lapse and shift functions vanish because $\dot{N}$ and $\dot{N}_a$ do not appear in the action. The canonical momentum of the three metric is

$$\begin{aligned}
\pi^{ab} &= \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}} = \frac{\partial}{\partial \dot{q}_{ab}} \frac{\sqrt{q}(q^{a'c'} q^{b'd'} - q^{a'b'} q^{c'd'})(\dot{q}_{a'b'} - 2\mathcal{D}_{(a'} N_{b')})(\dot{q}_{c'd'} - 2\mathcal{D}_{(c'} N_{d')})}{4N} \\
&= \frac{\sqrt{q}(q^{ac'} q^{bd'} - q^{ab} q^{c'd'}) K_{c'd'}}{2} + \frac{\sqrt{q}(q^{a'a} q^{b'b} - q^{a'b'} q^{ab}) K_{a'b'}}{2} \\
&= \frac{\sqrt{q}}{2} \left[(q^{ac} q^{bd} - q^{ab} q^{cd}) + (q^{ca} q^{db} - q^{cd} q^{ab}) \right] K_{cd} \\
&= \sqrt{q}(q^{ac} q^{bd} - q^{ab} q^{cd}) K_{cd} \\
&= \sqrt{q} G^{abcd} K_{cd} \\
&= \sqrt{q}(K^{ab} - K q^{ab}) \qquad \text{(E.31)}
\end{aligned}$$

$$
\begin{aligned}
\pi^{ab} &= \sqrt{q}G^{abcd}K_{cd} \\
&= \sqrt{q}(K^{ab} - Kq^{ab})
\end{aligned} \tag{E.32}
$$

and the action written in Hamiltonian form reads

$$
\begin{aligned}
S[N,\vec{N},q,\pi] &= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} - \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd}\dot{q}_{ab} \\
&\qquad + \sqrt{q}N(q^{ac}q^{bd} - q^{ab}q^{cd})K_{ab}K_{cd} + \sqrt{q}NR[q]\Big) \\
&= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} + \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})(\dot{q}_{ab} - NK_{ab})K_{cd} + \sqrt{q}NR[q]\Big) \\
&= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} + \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})(2\mathcal{D}_{(a}N_{b)} + NK_{ab})K_{cd} + \sqrt{q}NR[q]\Big) \\
&= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} - N\Big\{\sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{ab}K_{cd} - \sqrt{q}R[q]\Big\} \\
&\qquad + \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})2\mathcal{D}_{(a}N_{b)}K_{cd}\Big) \\
&= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} - N\Big\{\sqrt{q}(K_{ab}K^{ab} - K^2) - \sqrt{q}R[q]\Big\} \\
&\qquad - 2N_{(a}\mathcal{D}_{b)}\sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd}\Big) \\
&= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} - N\Big\{\sqrt{q}(K_{ab}K^{ab} - K^2) - \sqrt{q}R[q]\Big\} \\
&\qquad - 2N_{(a}\mathcal{D}_{b)}\sqrt{q}(K^{ab} - q^{ab}K)\Big) \\
&= \int dt \int d^3x \Big(\pi^{ab}\dot{q}_{ab} - N\sqrt{q}\{K_{ab}K^{ab} - K^2 - R[q]\} - 2N_b\{\mathcal{D}_a\pi^{ab}\}\Big)
\end{aligned} \tag{E.33}
$$

so that

$$
S[N,\vec{N},q,\pi] = \int dt \int d^3x \left(\pi^{ab}\dot{q}_{ab} - NC(\pi,q) - 2N^aC_a(\pi,q)\right) \tag{E.34}
$$

where

$$
C = G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{q}R[q] \tag{E.35}
$$

is the scalar constraint, or Hamiltonian constraint, and

$$
C^b = \mathcal{D}_a\pi^{ab} \tag{E.36}
$$

is the vector, or diffeomorphism constraint. Here

$$G_{abcd} = \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}) \tag{E.37}$$

which is called the DeWitt super metric.

$$\begin{aligned}
G_{abcd}\pi^{ab}\pi^{cd} &= \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})\sqrt{q}(K^{ab} - q^{ab}K)\sqrt{q}(K^{cd} - q^{cd}K) \\
&= \frac{\sqrt{q}}{2}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})(K^{ab}K^{cd} - q^{ab}K^{cd}K - q^{cd}K^{ab}K + q^{ab}q^{cd}K^2) \\
&= \frac{\sqrt{q}}{2}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd})(K^{ab}K^{cd} - 2q^{ab}K^{cd}K + q^{ab}q^{cd}K^2) \\
&= \frac{\sqrt{q}}{2}\Big((2K^{ab}K_{ab} - K^2) - 2(2K - 3K)K + (3 + 3 - 9)K^2\Big) \\
&= \sqrt{q}\left(K^{ab}K_{ab} - K^2\right).
\end{aligned} \tag{E.38}$$

The constraints C and C^a must vanish because of the variation of the lapse and shift functions.

E.4 Stuff

A similar calculation shows that the inverse spatial metric is given by

$$\gamma^{ab} = \delta^a_b + n^a n_b \tag{E.39}$$

E.5 The Cauchy Problem

The data (Σ, q_{ab}, K_{ab})

$$g_{\alpha\beta}(Q) = g_{\alpha\beta}(P) + g_{\alpha\beta,\,0}(P)x^0 + \sum_{n=2}^{\infty}\frac{1}{n!}\partial_0^n g_{\alpha\beta}|_P(x^0)^n. \tag{E.40}$$

$$R_{00} = -\frac{1}{2}g^{ab}g_{ab,00} + M_{00} = 0, \tag{E.41}$$

$$R_{0a} = \frac{1}{2}g^{0b}g_{ab,00} + M_{0a} = 0, \tag{E.42}$$

$$R_{ab} = -\frac{1}{2}g^{00}g_{ab,00} + M_{ab} = 0, \tag{E.43}$$

• under-determination The system does not contain $g_{0\alpha,00}$

• The system has represents ten equations in six unknowns $g_{ab,00}$; hence we have a problem of over-determination. compatibility requirements for the initial data.

$$R_{ab} = 0, \quad G^{\ 0}_{\alpha} = 0. \tag{E.44}$$

The first six equations are evolution equations for $g_{ab,00}$ and the last four equations are constraint equations which the initial data must satisfy

Einstein's equations

$$G_{munu} = 0$$

the equations $G_{0mu} = 0$ serve as constraints on the initial data for Einstein's equations, while the remaining equations describe time evolution. I.e., only for certain choices of a metric and its first time derivative at $t = 0$ can we get a solution of Einstein's equations. In fact, G_{0a} can be calculated knowing only the metric and its first time derivative at $t = 0$, and the equations saying they are zero are the constraints that this data must satisfy to get a solution of Einstein's equations.

In classical general relativity, G_{0i} not only gives one of Einstein's equations, namely $G_{0i} = 0$, it also "generates diffeomorphisms" of the 3-dimensional manifold S representing space. Just as in classical mechanics, observables give rise to one-parameter families of symmetries. For example, momentum gives rise to spatial translations, while energy (aka the Hamiltonian) gives rise to time translations. We say that the observable "generates" the one-parameter family of symmetries. This is (roughly) what is meant by saying that G_{0i} generates diffeomorphisms of S. Similarly, G_{00} generates diffeomorphisms of the spacetime $R \times S$ corresponding to time evolution.

The constraints corresponding to G_{0i} where $i = 1, 2, 3$ impose invariance under spatial active diffeomorphisms. Invariance under temporal active diffeomorphisms corresponds to G_{00} the constraint.

E.6 Gravitational Hamiltonian

$$S = \frac{1}{16\pi G} \int_V d^4x \sqrt{-g} g^{ab} R_{ab} \tag{E.45}$$

$$\delta S = \int_V d^4x \left((\delta\sqrt{-g}) g^{ab} R_{ab} + \sqrt{-g}(\delta g^{ab}) R_{ab} + \sqrt{-g} g^{ab} \delta R_{ab} \right) \tag{E.46}$$

$\delta\sqrt{-g} = \sqrt{-g} g_{ab} \delta g^{ab}/2$ and $\delta g^{ab} = -g_{ac} \delta g^{cd} g_{db}$

$$\delta S = \int_V d^4x \sqrt{-g} \left(R_{ab} - \frac{1}{2} g^{ab} R \right) \delta g^{ab} + \int_V d^4x \sqrt{-g} g^{ab} \delta R_{ab} \tag{E.47}$$

Last term of (E.47)

$$\begin{aligned}\int_V d^4x\sqrt{-g}g^{ab}\delta R_{ab} &= \int_V \sqrt{-g}d^4x\nabla_c(g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^a_{ab}) \\ &= \int_{\partial V} d\Sigma_c(g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^a_{ab}) \\ &= \int_{\partial V} |h|^{1/2}d^3x\epsilon n_c(g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^a_{ab}) \end{aligned} \tag{E.48}$$

variation of the metric connection

$$\delta\Gamma^c_{ab}|_{\partial V} = \frac{1}{2}g^{cd}(\delta_{da,b} + \delta g_{db,a} - g_{ab,d}) \tag{E.49}$$

$$\frac{1}{2}g^{ac}g^{bd}\delta g_{db,a}\bigg|_{\partial V} = \frac{1}{2}g^{ab}g^{cd}\delta g_{da,b}\bigg|_{\partial V} \tag{E.50}$$

where we have simply swapped around the dummy indices to make comparison with (E.49) easier.

$$g^{ab}\delta\Gamma^c_{ab}\bigg|_{\partial V} \tag{E.51}$$

$$\begin{aligned} n^c\delta &= n^c(\epsilon n^a n^b + h^{ab})(\delta g_{cb,a} - g_{ab,c}) \\ &= n^c h^{ab}(\delta g_{cb,a} - \delta g_{ab,c} \end{aligned} \tag{E.52}$$

$$n^c\delta = -h^{ab}\delta_{ab,c}n^c. \tag{E.53}$$

E.6.1 Boundary Term

$$\begin{aligned} K &= \nabla_a n^a \\ &= (\epsilon n^a n^b + h^{ab})\nabla_b n_a \quad \text{it follows from } \nabla_b(n^a n_a) = 0 \text{ that } n^a\nabla_b n_a = 0 \\ &= h^{ab}\nabla_b n_a \\ &= h^{ab}(\partial_b n_a - \Gamma^c_{ab}n_c) \end{aligned} \tag{E.54}$$

As the spacetime metric g^{ab} is taken to be fixed at the boundary the variation of h^{ab} is zero there. the variation is

$$\begin{aligned}\delta K &= -h^{ab}\delta\Gamma^{c}_{ab}n_c \\ &= -\frac{1}{2}h^{ab}[\delta(\partial_b g_{da}) + \delta(\partial_a g_{db}) - \delta(\partial_d g_{ab})]n^d \\ &= \frac{1}{2}h^{ab}\delta(\partial_d g_{ab})n^d \end{aligned} \tag{E.55}$$

where we have used that the tangental derivatives of δg_{ab} vanish on $\partial\mathcal{M}$ (i.e. $\delta(h^{cd}\partial_d g_{ab}) = 0$). The variation in boundary term

$$\oint_{\partial\mathcal{M}} \sqrt{|h|}d^3x\epsilon K \tag{E.56}$$

is

$$\delta S_B = \oint_{\partial\mathcal{M}} \sqrt{|h|}d^3x\epsilon h^{ab}\delta(\partial_c g_{ab})n^c \tag{E.57}$$

$$S = \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{|g|}d^4xR + \oint_{\partial\mathcal{M}} \sqrt{h}d^3x\epsilon(K - K_0). \tag{E.58}$$

E.6.2 Constraint Algebra

Classically, the constraints satisfy the following algebra (neglecting boundaryterms)

$$\{G(N), G(M)\} = -G([N, M]) \tag{E.59}$$
$$\{D(\vec{N}), G(M)\} = -G(\mathcal{L}_{\vec{N}}M) \tag{E.60}$$
$$\{D(\vec{N}), D(\vec{M})\} = -D([\vec{N}, \vec{M}]) \tag{E.61}$$
$$\{G(N), H(M)\} = 0 \tag{E.62}$$
$$\{D(\vec{N}), H(M)\} = -H(\mathcal{L}_{\vec{N}}M) \tag{E.63}$$
$$\{H(N), H(M)\} = \tag{E.64}$$

$$\begin{aligned}\frac{\delta C(N)}{\delta q_{ab}} &= -\frac{1}{2}N\tilde{C}(q,\tilde{p})q^{ab} + 2Nq^{-1/2}(\tilde{p}^{ac}\tilde{p}^{b}{}_{c} - \frac{1}{2}\tilde{p}\tilde{p}^{ab}) \\ &\quad + Nq^{1/2}(\mathcal{R}^{ab} - \mathcal{R}q^{ab}) - q^{1/2}(D^aD^bN - q^{ab}D^cD_cN)\end{aligned} \tag{E.65}$$

$$\frac{\delta C(N)}{\delta\tilde{p}^{ab}} = 2Nq^{-1/2}(\tilde{p}_{ab} - \frac{1}{2}\tilde{p}q_{ab}) \tag{E.66}$$

$$X_f = \int_\Sigma \frac{\delta f}{\delta \tilde{p}^{ab}} \frac{\delta}{\delta q_{ab}} - \frac{\delta f}{\delta \tilde{p}^{ab}} \frac{\delta}{\delta q_{ab}} \tag{E.67}$$

$$\{f, g\} = \int_\Sigma \frac{\delta f}{\delta \tilde{p}^{ab}} \frac{\delta g}{\delta q_{ab}} - \frac{\delta f}{\delta \tilde{p}^{ab}} \frac{\delta g}{\delta q_{ab}} \tag{E.68}$$

$$\begin{aligned} q_{ab} &\mapsto q_{ab} + \epsilon \frac{\delta f}{\delta \tilde{p}^{ab}} + \mathcal{O}(\epsilon^2) \\ \tilde{p}^{ab} &\mapsto \tilde{p}^{ab} - \epsilon \frac{\delta f}{\delta q_{ab}} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{E.69}$$

If f is $C(\vec{N})$ then we should have (at least on the constraint surface)

$$\frac{\delta C(\vec{N})}{\delta q_{ab}} = -\mathcal{L}_{\vec{N}} \tilde{p}^{ab} \quad \text{and} \quad \frac{\delta C(\vec{N})}{\delta \tilde{p}^{ab}} = -\mathcal{L}_{\vec{N}} q_{ab} \tag{E.70}$$

$$\begin{aligned} q_{ab} &\mapsto q_{ab} + \epsilon \mathcal{L}_{\vec{N}} q_{ab} + \mathcal{O}(\epsilon^2) \\ \tilde{p}^{ab} &\mapsto \tilde{p}^{ab} + \epsilon \mathcal{L}_{\vec{N}} \tilde{p}^{ab} + \mathcal{O}(\epsilon^2) \end{aligned} \tag{E.71}$$

$$2D_{(a}N_{b)} = \mathcal{L}_{\vec{N}} q_{ab} \tag{E.72}$$

We will need

$$\delta q = q q^{ab} \delta q_{ab} \qquad \delta \mathcal{R}_{ab} q^{ab} = D_a v^a \ \text{ for } \ v^a = -D^a(q^{bc}\delta q_{bc}) + D^b(q^{ac}\delta q_{bc})$$

$$\delta(q^{ac} q_{cb}) = \delta(\delta^a_b) = 0 \quad \Rightarrow \quad \delta q^{ab} = -q^{ac} q^{bd} \delta q_{cd}$$

$$\tilde{p}_{ab} = \tilde{p}^{cd} q_{ac} q_{bd}, \qquad \tilde{p} = \tilde{p}^{ab} q_{ab}$$

$$\delta \tilde{p}_{ab} = q_{ac} q_{bd} \delta \tilde{p}^{cd} + 2\tilde{p}^{cd} q_{bd} \delta q_{ac}, \qquad \delta \tilde{p} = q_{ab} \delta \tilde{p}^{ab} + \tilde{p}^{ab} \delta q_{ab}$$

Variation of $C(N)$ under independent variation of q_{ab} ($\delta \tilde{p}^{ab} = 0$) is

$$\begin{aligned}
\delta C(N) &= \delta \int_\Sigma N\Big(- q^{1/2}\mathcal{R} + q^{-1/2}(\tilde{p}^{ab}\tilde{p}_{ab} - \frac{1}{2}\tilde{p}^2)\Big) \\
&= \int_\Sigma N\Big(- \delta q^{1/2}\mathcal{R} - q^{1/2}\delta\mathcal{R} + \delta q^{-1/2}(\tilde{p}^{ab}\tilde{p}_{ab} - \frac{1}{2}\tilde{p}^2) + q^{1/2}(\tilde{p}^{ab}\delta\tilde{p}_{ab} - \tilde{p}\delta\tilde{p})\Big) \\
&= -\int_\Sigma N\Big(\frac{1}{2}q^{1/2}q^{ab}\delta q_{ab}\mathcal{R} + q^{1/2}\delta\mathcal{R} + \frac{1}{2}q^{-1/2}q^{ab}\delta q_{ab}(\tilde{p}^{ab}\tilde{p}_{ab} + \frac{1}{2}\tilde{p}^2) \\
&\qquad -2q^{-1/2}(\tilde{p}^{ac}\tilde{p}^b{}_c - \frac{1}{2}\tilde{p}q_{cd})\delta q_{ab}\Big)
\end{aligned} \tag{E.73}$$

$$\begin{aligned}
-\int_\Sigma Nq^{1/2}\delta\mathcal{R} &= -\int_\Sigma Nq^{1/2}\delta(\mathcal{R}_{ab}q^{ab}) \\
&= -\int_\Sigma Nq^{1/2}\big(\delta\mathcal{R}_{ab}q^{ab} + \mathcal{R}_{ab}\delta q^{ab}\big) \\
&= \int_\Sigma Nq^{1/2}\Big(D_a\big[D^a(q^{bc}\delta q_{bc}) - D^b(q^{ac}\delta q_{bc})\big] + \mathcal{R}_{ab}q^{ac}q^{bd}\delta q_{cd}\Big)
\end{aligned} \tag{E.74}$$

Variation of $C(N)$ under independent variation of $\tilde{p}^{ab}$ ($\delta q_{ab} = 0$) is

$$\begin{aligned}
\delta C(N) &= \delta \int_\Sigma N\Big(- q^{1/2}\mathcal{R} + q^{-1/2}(\tilde{p}^{ab}\tilde{p}_{ab} - \frac{1}{2}\tilde{p}^2)\Big) \\
&= \delta \int_\Sigma Nq^{-1/2}(\tilde{p}^{ab}\tilde{p}^{cd}q_{ac}q_{bd} - \frac{1}{2}\tilde{p}^2) \\
&= \int_\Sigma Nq^{-1/2}\big(2\tilde{p}^{cd}q_{ac}q_{bd}\delta\tilde{p}^{ab} - \tilde{p}\delta(\tilde{p}^{ab}q_{ab})\big) \\
&= \int_\Sigma 2Nq^{-1/2}(\tilde{p}_{ab} - \frac{1}{2}\tilde{p}q_{ab})\delta\tilde{p}^{ab}
\end{aligned} \tag{E.75}$$

$$f(M) = \int_\Sigma M^{a\ldots b}{}_{c\ldots d}\tilde{f}_{a\ldots b}{}^{c\ldots d}(q,\tilde{p}) \tag{E.76}$$

$$\begin{aligned}
\{C(\vec{N}), f(M)\} &= \int_\Sigma -\mathcal{L}_{\vec{N}}\tilde{p}^{ab}(\frac{\delta f(M)}{\delta\tilde{p}^{ab}}) - \mathcal{L}_{\vec{N}}q_{ab}(\frac{\delta f(M)}{\delta q_{ab}}) \\
&= \int_\Sigma M^{a\ldots b}{}_{c\ldots d}\left\{-\mathcal{L}_{\vec{N}}\tilde{p}^{ab}(\frac{\partial\tilde{f}_{a\ldots b}{}^{c\ldots d}}{\partial\tilde{p}^{ab}}) - \mathcal{L}_{\vec{N}}q_{ab}(\frac{\partial\tilde{f}_{a\ldots b}{}^{c\ldots d}}{\partial q_{ab}})\right\} \\
&= -\int_\Sigma M^{a\ldots b}{}_{c\ldots d}\,\mathcal{L}_{\vec{N}}\tilde{f}_{a\ldots b}{}^{c\ldots d}(q,\tilde{p}) \\
&= \int_\Sigma \big(\mathcal{L}_{\vec{N}}M^{a\ldots b}{}_{c\ldots d}\big)\,\tilde{f}_{a\ldots b}{}^{c\ldots d}(q,\tilde{p}) \\
&= f(\mathcal{L}_{\vec{N}}M)
\end{aligned} \tag{E.77}$$

$$\{C(\vec{N}), C(M)\} = C(\mathcal{L}_{\vec{N}}\vec{M}) = C([\vec{N}, \vec{M}]) \tag{E.78}$$

$$\{C(\vec{N}), C(M)\} = C(\mathcal{L}_{\vec{N}}M) \tag{E.79}$$

we need only to include the last term of (E.65).

$$\begin{aligned}
\{C(N), C(M)\} &= \int_\Sigma \frac{\delta C(N)}{\delta q_{ab}} \frac{\delta C(M)}{\delta \tilde{p}^{ab}} - (N \leftrightarrow M) \\
&= \int_\Sigma \big(- q^{1/2}(D^a D^b N - q^{ab} D^c D_c N)\big)\big(2Mq^{-1/2}(\tilde{p}_{ab} - \frac{1}{2}\tilde{p} q_{ab})\big) - (N \leftrightarrow M) \\
&= \int_\Sigma -2M(D^a D^b N - q^{ab} D^c D_c N)(\tilde{p}_{ab} - \frac{1}{2}\tilde{p} q_{ab}) - (N \leftrightarrow M) \\
&= 2\int_\Sigma \big[(\partial^b N) D^a (M(\tilde{p}_{ab} - \frac{1}{2}\tilde{p} q_{ab}))\big] + \frac{1}{2}\partial_c N \partial^c M - (N \leftrightarrow M) \\
&= 2\int_\Sigma (\partial^b N) M D^a (\tilde{p}_{ab} - \frac{1}{2}\tilde{p} q_{ab}) - (N \leftrightarrow M) \\
&= 2\int_\Sigma (M\partial^b N - N\partial^b M)(q_{ac} q_{bd} - \frac{1}{2} q_{ab} q_{cd}) D^a \tilde{p}^{cd} \\
&= \int_\Sigma -2(N\partial^a M - M\partial^a N)(q_{ab} D_c \tilde{p}^{bc} - \frac{1}{2} q_{ab} D_c \tilde{p}) \\
&= \int_\Sigma -2(N\partial^a M - M\partial^a N) q_{ab} D_c \tilde{p}^{bc} \\
&= C(\vec{K}),
\end{aligned} \tag{E.80}$$

(some terms have been neglected because they were symmetric in M and N), where $K^a := (N\partial^a M - M\partial^a N) = q^{ab}(N\partial_b M - M\partial_b N)$.

Repeat this proof for tetrad first order formulation in appendix

E.7 First Order Formulation of Einstein Equations

E.8 Palatini Method in the Connection Formulation

The connection dynamics perspective to suggests re-interpreting gravity as a theory where the metric becomes a derived variable, with a frame field e and (Lorentz) connection ω the sole primary dynamical variables - the so-called Palatini Tetrad Formalism.

E.8.1 Method I

The connection ∇_a action on V_{aI} is

$$\nabla_a V_{bI} = \partial_a V_{bI} - \Gamma^c_{ab} V_{cI} + \Gamma_{aI}{}^J V_{bJ} \tag{E.81}$$

Obviously if we calculate $(\nabla_a \nabla_b - \nabla_b \nabla_a)V_c$ we retrieve the usual formula for the curvature,

$$2\nabla_{[a}\nabla_{b]} V_c = R_{abc}{}^d V_d \tag{E.82}$$

We want to calculate the commutator on V_I. First we get

$$\begin{aligned} \nabla_a \nabla_b V_I &= \nabla_a(\partial_b V_I + \Gamma_{bI}{}^J V_J) \\ &= \partial_a(\partial_b V_I + \Gamma_{bI}{}^J V_J) - \Gamma^c_{ab}(\partial_c V_I + \Gamma_{cI}{}^J V_J) + \Gamma_{aI}{}^K(\partial_b V_K + \Gamma_{bK}{}^J V_J) \end{aligned} \tag{E.83}$$

As Γ^c_{ab} is symmetric in a and b it will not contribute to the commutator and we get,

$$\begin{aligned} (\nabla_a \nabla_b - \nabla_b \nabla_a)V_I &= (\partial_a\partial_b - \partial_b\partial_a)V_I + \partial_a(\Gamma_{bI}{}^J V_J) - \partial_b(\Gamma_{aI}{}^J V_J) \\ &\quad \Gamma_{aI}{}^K \partial_b V_K - \Gamma_{bI}{}^K \partial_a V_K + \Gamma_{aI}{}^K \Gamma_{bK}{}^J V_J - \Gamma_{bI}{}^K \Gamma_{aK}{}^J \\ &= (\partial_a \Gamma_{bI}{}^J - \partial_a \Gamma_{bI}{}^J + \Gamma_{aI}{}^K \Gamma_{bK}{}^J - \Gamma_{bI}{}^K \Gamma_{aK}{}^J) V_J \end{aligned} \tag{E.84}$$

This defines the curvature $R_{abI}{}^J$ via

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)V_I = R_{abI}{}^J V_J. \tag{E.85}$$

Writing $V_c = e^I_c V_I$ and inserting it into (E.82)

$$\begin{aligned} R_{abc}{}^d V_d &= 2\nabla_{[a}\nabla_{b]} V_c \\ &= 2\nabla_{[a}\nabla_{b]}(e^I_c V_I) \\ &= 2e^I_c \nabla_{[a}\nabla_{b]} V_I \\ &= e^I_c R_{abI}{}^J e^d_J V_d \end{aligned} \tag{E.86}$$

where we have use that $\nabla_a e^I_b = 0$. Since the above is true for all V_d, we obtain:

$$R_{abc}{}^d = R_{abI}{}^J e^I_c e^d_J \tag{E.87}$$

The Ricci scalar is given by $g^{ab}R_{ab} = g^{ab}R_{acb}{}^{c}$. Using the previous expression we find

$$R_{acb}{}^{c} = R_{acI}{}^{J} e^{I}_{b} e^{c}_{J}. \tag{E.88}$$

Contracting over the two remaining spacetime indices then allows us to write the Ricci sclar in terms of the curvature of the spin connection and tetrads,

$$R = g^{ab}R_{acI}{}^{J} e^{I}_{b} e^{c}_{J} = R_{ab}{}^{IJ} e^{a}_{I} e^{b}_{J}. \tag{E.89}$$

The torsion tensor

$$T^{a} = \partial e^{a}_{I} + \omega^{J}_{b}{}_{I} e^{b}_{I} = 0 \tag{E.90}$$

The Einstein equations

$$\epsilon_{abcd}(e^{a}_{I} R^{bc}_{JK} + \Lambda e^{a}_{I} e^{b}_{J} e^{c}_{K}) = 0. \tag{E.91}$$

$$\epsilon_{abcd}(R^{cd}_{JK} + \Lambda e^{c}_{J} e^{d}_{K}) = 0. \tag{E.92}$$

implying

$$R^{ab}_{IJ} + \Lambda e^{a}_{[I} e^{b}_{J]} = 0. \tag{E.93}$$

Variations

$$\begin{aligned} \delta\eta_{IJ} &= \delta(e^{b}_{I} e_{bJ}) && \text{(E.94)} \\ &= \delta e^{b}_{I} e_{bJ} + \delta e_{bJ} e^{b}_{I} = 0 && \text{(E.95)} \end{aligned}$$

and contract this by e^{aJ} (at this point we assume the metric to be non-degenerate) and note $e_{bJ}e^{aJ} = \delta^{a}_{b}$ we arrive at

$$\delta e^{a}_{I} = -e^{b}_{I}\ \delta e^{b}_{J}\ e^{aJ}. \tag{E.96}$$

The variation of the curvature is

$$\delta R_{ab}{}^{IJ} = \partial_{[a} \delta\omega_{b]}{}^{IJ} + \delta\omega_{[a}{}^{IK} \omega_{b]K}{}^{J} + \omega_{[a}{}^{IK} \delta\omega_{b]K}{}^{J} \tag{E.97}$$

$$\delta\omega_b{}^{IJ} = \omega_b{}^{IJ} - \tilde{\omega}_b{}^{IJ} \tag{E.98}$$

$$\begin{aligned} \delta\omega'_b{}^{IJ} &= \omega'_b{}^{IJ} - \tilde{\omega'}_b{}^{IJ} \\ &= \left(O^I{}_K O^J{}_L \omega_b{}^{KL} - \partial_b O^{IJ}\right) - \left(O^I{}_K O^J{}_L \tilde{\omega}_b{}^{KL} - \partial_b O^{IJ}\right) \\ &= O^I{}_K O^J{}_L \delta\omega_b{}^{KL} \end{aligned} \tag{E.99}$$

$$\mathcal{D}_a T_b{}^{IJ} = \partial_a T_b{}^{IJ} + T_b{}^{IK}\omega_a{}^J{}_K + \omega_a{}^{IK} T_b{}^J{}_K + \Gamma^c_{ab} T_c{}^{IJ} \tag{E.100}$$

$$\mathcal{D}_{[a} T_{b]}{}^{IJ} = \partial_{[a} T_{b]}{}^{IJ} + T_{[a}{}^{IK}\omega_{b]}{}^J{}_K + \omega_{[a}{}^{IK} T_{b]}{}^J{}_K \tag{E.101}$$

with the last term missing because $\Gamma^c_{[ab]} = 0$.

$$\delta R_{ab}{}^{IJ} = \mathcal{D}_{[a}\delta\omega_{b]}{}^{IJ} \tag{E.102}$$

$$\delta(\mathcal{D}_{[a} e^I_{b]}) = \partial_{[a}\delta e^I_{b]} + \delta\omega^I_{[aJ} e^J_{b]} + \omega^{IJ}_{[a}\delta e_{bJ]} \tag{E.103}$$

$$\delta e_{bJ} = - \tag{E.104}$$

E.8.2 Method II

We introduce an arbitrary covariant derivative via

$$\mathcal{D}_a V_I = \partial_a V_I + \omega_{aI}{}^J V_J. \tag{E.105}$$

Where $\omega_{aI}{}^J$ is a Lorentz connection (the derivative annihilates the Minkowski metric η_{IJ}). We define a curvature via

$$\Omega_{abI}{}^J V_J = (\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a) V_I$$

and by a calculation similar to (E.84) we obtain

$$\Omega_{ab}{}^{IJ} = 2\partial_{[a}\omega_{b]}{}^{IJ} + \omega_{aI}{}^K\omega_{bK}{}^J - \omega_{bI}{}^K\omega_{aK}{}^J$$

The Ricci scalar of this curvature can be expressed as $e^a_I e^b_J \Omega_{ab}{}^{IJ}$. The action can be written

$$S_{EH} = \int d^4x \; e \; e_I^a e_J^b \Omega_{ab}{}^{IJ} \tag{E.106}$$

We introduce a connection compatible with the tetrad via $\nabla_a e_I^b = 0$. The difference between these two connections when applied to a tensor with purely internal indices is

$$C_{aI}{}^J V_J = (\mathcal{D}_a - \nabla_a) V_I \tag{E.107}$$

We would expect ∇_a to also annihilate the Minkowski metric $\eta_{IJ} = e_{bI} e_J^b$ and therefore,

$$\begin{aligned} 0 &= (\mathcal{D}_a - \nabla_a)\eta_{IJ} \\ &= C_{aI}{}^K \eta_{KJ} + C_{aJ}{}^K \eta_{IK} \\ &= C_{aIJ} + C_{aJI}. \end{aligned} \tag{E.108}$$

Implying $C_{aIJ} = C_{a[IJ]}$. The derivative defined by (E.105) only knows how to act on internal indices. However, we find it convenient to consider a torsion-free extension to spacetime indices. All calculations will be independent of this choice of extention. Applying $\mathcal{D}_a$ twice on V_I,

$$\begin{aligned} \mathcal{D}_a \mathcal{D}_b V_I &= \mathcal{D}_a(\nabla_b V_I + C_{bI}{}^J V_J) \\ &= \nabla_a(\nabla_b V_I + C_{bI}{}^J V_J) + C_{a_I}{}^K(\nabla_b V_K + C_{bK}^J V_J) + \overline{\Gamma}_{ab}^c(\nabla_c V_I + C_{cI}{}^J V_J) \end{aligned} \tag{E.109}$$

where $\overline{\Gamma}_{ab}^c$ is unimportant, we need only note that it is symmetric in a and b as it is torsion-free. Then

$$\begin{aligned} \Omega_{abI}{}^J V_J &= (\mathcal{D}_a \mathcal{D}_b - \mathcal{D}_b \mathcal{D}_a) V_I \\ &= (\nabla_a \nabla_b - \nabla_b \nabla_a) V_I + \nabla_a(C_{bI}{}^J V_J) - \nabla_b(C_{aI}{}^J V_J) \\ &\quad + C_{a_I}{}^K \nabla_b V_K - C_{b_I}{}^K \nabla_a V_K + C_{a_I}{}^K C_{bK}^J V_J - C_{b_I}{}^K C_{aK}^J V_J \\ &= R_{abI}{}^J V_J + (\nabla_a C_{bI}{}^J - \nabla_b C_{aI}{}^J + C_{a_I}{}^K C_{bK}^J - C_{b_I}{}^K C_{aK}^J) V_J + \end{aligned} \tag{E.110}$$

Hence

$$\Omega_{ab}{}^{IJ} - R_{ab}{}^{IJ} = 2\nabla_{[a} C_{b]}{}^{IJ} + 2C_{[a}{}^{IK} C_{b]K}{}^J \tag{E.111}$$

As

$$\mathcal{D}_a V_I = \partial_a V_I + \omega_{aI}{}^J V_J = \nabla_a V_I + C_{aI}{}^J V_J$$

the variation with respect to $\omega_{aI}{}^J$ (keeping the tetrad fixed) is the same as the variation of the resulting action with respect to $C_{aI}{}^J$. Substituting (E.111) into the action (E.106) gives

$$S_{EH} = \int d^4x \; e \; e_I^a e_J^b (R_{ab}{}^{IJ} + 2\nabla_{[a} C_{b]}{}^{IJ} + 2C_{[a}{}^{IK} C_{b]K}{}^J) \tag{E.112}$$

The first term does not involve $C_a{}^{IJ}$. The second term is a total derivative. From the last term we have from varying with respect to $C_{aI}{}^J$,

$$\begin{aligned} 0 &= \frac{\delta}{\delta C_{aI}{}^J} \int d^4x \; e \; 2 e^{M[c} e_N^{b]} C_{cM}{}^K C_{bK}{}^N \\ &= e 2 e^{M[c} e_N^{b]} \big(\delta_c^a \delta_M^I \delta_J^K C_{bK}{}^N + C_{cM}{}^K \delta_b^a \delta_K^I \delta_J^N \big) \\ &= e 2 (e^{I[a} e_N^{b]} C_{bJ}{}^N + e^{M[b} e_J^{a]} C_{bM}{}^I) \end{aligned} \tag{E.113}$$

or

$$e_I^{[a} e_K^{b]} C_{bJ}{}^K + e^{K[b} e_J^{a]} C_{bKI} = 0$$

or

$$C_{bI}{}^K e_K^{[a} e_J^{b]} + C_{bJ}{}^K e_I^{[a} e_K^{b]} = 0. \tag{E.114}$$

where we have used $C_{bKI} = -C_{bIK}$. This can be writtem more compactly as

$$e_M^{[a} e_N^{b]} \delta_{[I}^M \delta_{J]}^K C_{bK}{}^N = 0. \tag{E.115}$$

$(e_M^{[a} e_N^{b]} \delta_{[I}^M \delta_{J]}^K)$ is non-degenerate and so $C_{aI}{}^K = 0$. We will show (following gr-qc/9303032) that (E.114) implies $C_{aI}{}^J = 0$. First we define the spacetime tensor field by

$$S_{abc} := C_{aIJ} e_b^I e_c^J. \tag{E.116}$$

Then the condition $C_{aIJ} = C_{a[IJ]}$ is equivalent to $S_{abc} = S_{a[bc]}$. Now contract (E.114) with $e_a^I e_c^J$,

$$
\begin{aligned}
0 = \left(C_{bI}{}^K e_K^{[a} e_J^{b]} + C_{bJ}{}^K e_I^{[a} e_K^{b]}\right) e_a^I e_c^J &= \frac{1}{2} C_{bI}{}^K (e_K^a e_J^b - e_K^b e_J^a) e_a^I e_c^J \\
&\quad + \frac{1}{2} C_{bJ}{}^K (e_I^a e_K^b - e_I^b e_K^a) e_a^I e_c^J \\
&= \frac{1}{2} C_{bI}{}^K (\delta_K^I \delta_c^b - e_K^b \delta_J^I e_c^J) \\
&\quad + \frac{1}{2} C_{bJ}{}^K (\delta_I^I e_K^b e_c^J - \delta_K^I e_I^b e_c^J) \\
&= \frac{1}{2} (C_{cI}{}^I - C_{bJ}{}^K e_K^b e_c^J) \\
&\quad + \frac{1}{2} (4 C_{bJ}{}^K e_K^b e_c^J - C_{bJ}{}^I e_I^b e_c^J) \\
&= C_{bJ}{}^I e_c^J e_I^b \qquad \text{(E.117)}
\end{aligned}
$$

where we used $C_{aI}{}^I = C_{aIJ}\eta^{IJ} = C_{a[IJ]}\eta^{IJ} = 0$. As $S_{ab}{}^c = C_{aI}{}^J e_b^I e_J^c$, we have $S_{bc}{}^b = 0$. We write it as

$$(C_{bI}{}^J e_J^b) e_c^I = 0,$$

and as e_a^I are invertible this implies

$$C_{bI}{}^J e_J^b = 0.$$

Thus the terms $C_{bI}{}^K e_K^b e_J^a$ and $C_{bJ}{}^K e_I^a e_K^b$ of (E.114) both vanish and (E.114) reduces to

$$C_{bI}{}^K e_K^a e_J^b - C_{bJ}{}^K e_I^b e_K^a = 0. \qquad \text{(E.118)}$$

If we now contract (E.118) with $e_c^I e_d^J$, we get

$$
\begin{aligned}
0 = (C_{bI}{}^K e_K^a e_J^b - C_{bJ}{}^K e_I^b e_K^a) e_c^I e_d^J &= C_{bI}{}^K e_K^a e_c^I \delta_d^b - C_{bJ}{}^K \delta_c^b e_K^a e_d^J \\
&= C_{dI}{}^K e_c^I e_K^a - C_{cJ}{}^K e_d^J e_K^a
\end{aligned}
$$

or

$$S_{cd}{}^a = S_{(cd)}{}^a. \qquad \text{(E.119)}$$

Since we have $S_{abc} = S_{a[bc]}$ and $S_{abc} = S_{(ab)c}$, we can successively interchange the first two and then last two indices with appropriate sign change each time to obtain,

$$
\begin{aligned}
S_{abc} &= S_{bac} \\
&= -S_{bca} \\
&= -S_{cba} \\
&= S_{cab} \\
&= S_{acb} \\
&= -S_{abc}
\end{aligned}
$$

Implying $S_{abc} = 0$, or

$$
C_{aIJ} e^I_b e^J_c = 0,
$$

and since the e^I_a are invertible, we get $C_{aIJ} = 0$. This is the desired result.

This tells us that ∇ coincides with D when acting on objects with only internal indices.

We now consider varying with respect to e^I_b gives. We will need the formula for the variance of a determinant,

$$
\delta det(a) = det(a)(a^{-1})_{ji} \delta a_{ij}.
$$

This implies

$$
\delta e = e e^I_a \delta e^a_I.
$$

$$
\begin{aligned}
\frac{\delta S_{EH}}{\delta e^b_J} &= \int d^4x \frac{\delta}{\delta e^b_J}(e e^c_M e^d_N)\; \Omega_{cd}{}^{MN} \\
&= \int d^4x e(\frac{\delta}{\delta e^b_J} e^c_N e^d_M) \Omega_{cd}{}^{MN} + e^c_M e^d_N (\frac{\delta}{\delta e^b_J} e) \Omega_{cd}{}^{MN} \\
&= e[\delta^c_b \delta^J_N e^d_M + e^c_N \delta^d_b \delta^J_M] \Omega_{cd}{}^{MN} + e e^c_M e^d_N e^J_b \Omega_{cd}{}^{MN} \\
&= e[e^d_M \Omega_{bd}{}^{MJ} + e^c_N \Omega_{cb}{}^{JN}] + e \Omega_{cd}{}^{MN} e^c_M e^d_N e^J_b \\
&= -e(2 e^c_I \Omega_{cb}{}^{IJ} - \Omega_{cd}{}^{MN} e^c_M e^d_N e^J_b)
\end{aligned} \tag{E.120}
$$

We have

$$
e^c_I \Omega_{cb}{}^{IJ} - \frac{1}{2} \Omega_{cd}{}^{MN} e^c_M e^d_N e^J_b = 0 \tag{E.121}
$$

Substituting $\Omega_{ab}{}^{IJ}$ for $R_{ab}{}^{IJ}$

$$e^c_I R_{cb}{}^{IJ} - \frac{1}{2} R_{cd}{}^{MN} e^c_M e^d_N e^J_b = 0 \tag{E.122}$$

Multiplying (E.122) by e_{Ja} and using (E.88) with $R_{ab} = R_{ba}$ tells us that the Einstein tensor $G_{ab} := R_{ab} - \frac{1}{2} R g_{ab}$ of the metric defined by tetrads, $g_{ab} := e^I_a e^J_b \eta_{IJ}$, vanishes.

E.9 Inclusion of Matter

Rovelli [23]

However, GR is much more than just the theory of a specific physical force. Indeed, GR is a theory of space and time. It has modified in depth our understanding of space and time are, radically changing the Newtonian picture. This modification of the basic physical picture of the world does not refer to the gravitational interaction alone. Rather, it affects ***any*** *physical theory. Indeed, GR has taught us that the action of* ***all*** *physical systems must be generally covariant, not just the action of the gravitational field. Thus, GR is a theory with a universal reach, whose implications involve the redefinition of our description of the whole of fundamental physics...*

Particles and Fluids

E.9.1 Yang-Mills

Natural inclusion connection form of Einstein's equations

$$\mathcal{L}_{YM} := \frac{1}{2} (\, ^4\sigma) g^{ac} g^{bd} \mathrm{tr} \, ^4F_{ab} \, ^4F_{cd} \tag{E.123}$$

Each link is labelled by a spin j_l and an irreducible representation of G_{YM}.

E.9.2 Klein-Gordan - Scalar Matter Field

$$\mathcal{L}_{KG} = 4\pi(\, ^4\sigma)(g^{ab} \partial_a \phi \partial_b \phi + V(\phi^2)) \tag{E.124}$$

gravity coupled to to a scalar field ϕ with potential $V(\phi)$ with conjugate momentum π.

The coupled generalized Palatini action is

$$S[e^\beta_K, \omega_\alpha, \phi] = S_p[e^\beta_K, \omega_\alpha] + S_{KG}[e^\beta_K, \phi], \tag{E.125}$$

where

$$
\begin{aligned}
S_p[e_K^\beta, \omega_\alpha^{IJ}] &= \frac{1}{2\kappa}\int_{\mathcal{M}} d^4x(e)e_I^\alpha e_J^\beta(\Omega_{\alpha\beta}{}^{IJ} + \frac{1}{2\gamma}\epsilon^{IJ}{}_{KL}\Omega_{\alpha\beta}{}^{KL}), \\
S_{KG}[e_K^\beta, \phi] &= -\alpha_{\mathcal{M}}\int_{\mathcal{M}} d^4x(e)\eta^{IJ}\, e_I^\alpha e_J^\beta(\partial_\alpha\phi)\partial_\beta\phi,
\end{aligned} \tag{E.126}
$$

here e_K^β and ω_α^{IJ} are respectively the tetrad and Lorentz connection on $\mathcal{M}$, the real number γ, κ and $\alpha_{\mathcal{M}}$ are respectively the Barbero-Immirzi parameter, the gravitational constant and the coupling constant.

After 3+1 decomposition and Legendre transformation, similar to the case in Palatini formalism, we obtain the total Hamiltonian of the coupling system on the 3-manifold as:

$$
\mathcal{H}_{tot} = \int_\Sigma (\Lambda^i G_i + N^a \mathcal{V}_a + NC), \tag{E.127}
$$

where Λ^i, N^a and N are Lagrange multipliers, and the Gaussian, diffeomorphism and Hamiltonian constraints are expressed respectively as:

$$
G_i = D_a E_i^a = \partial E_i^a + \epsilon_{ij}{}^k A_a^j E_k^a = 0, \tag{E.128}
$$

$$
\mathcal{V}_a = E_i^b F_{ab}^i + \pi\partial_a\phi = 0, \tag{E.129}
$$

and

$$
\mathcal{H} = \frac{1}{l_p^2}\epsilon^{ijk}E_i^a E_j^b\left(F_{abk} + \frac{l_p^2 V(\phi)}{3}\epsilon_{abc}E_k^c\right) + \frac{1}{2}\pi^2 + \frac{1}{2}E^{ai}E_i^b(\partial_a\phi)(\partial_b\phi) = 0 \tag{E.130}
$$

E.9.3 Fermionic Matter

E.9.4 In the Language of Differential Geometry

$$
e^I(x) = e_a^I dx^a \tag{E.131}
$$

The spin connection

$$
\omega^I{}_J(x) = \omega^I{}_{aJ}(x)dx^a \tag{E.132}
$$

$$
R^I{}_J = R^I{}_{J\,ab}dx^a \wedge dx^b \tag{E.133}
$$

$$\mathcal{S}[e^I,\omega^{IJ}] = \frac{1}{4\kappa}\int_{\mathcal{M}} \epsilon_{IJKL}(e^I \wedge e^J \wedge R[\omega]^{KL} + \frac{\Lambda}{6} e^I \wedge e^J \wedge e^K \wedge e^L \wedge) \tag{E.134}$$

$$S(\omega, e) = \int_{\mathcal{M}} tr(e \wedge e \wedge F). \tag{E.135}$$

$$\delta F = d_\omega \delta\omega, \tag{E.136}$$

$$\begin{aligned} \delta S &= \int_{\mathcal{M}} \delta tr(e \wedge e \wedge F) \\ &= \int_{\mathcal{M}} tr(2\delta e \wedge e \wedge F + e \wedge e \wedge \delta F), \end{aligned} \tag{E.137}$$

using (E.136) we obtain

$$\delta S = \int_{\mathcal{M}} tr(2\delta e \wedge e \wedge F + e \wedge e \wedge d_\omega \delta\omega). \tag{E.138}$$

Integrating by parts,

$$\delta S = 2\int_{\mathcal{M}} tr(\delta e \wedge e \wedge F - e \wedge d_\omega e \wedge \delta\omega). \tag{E.139}$$

$$\begin{aligned} e \wedge F &= 0 \quad \text{var. of } e \\ e \wedge d_a e &= 0 \quad \text{var. of } \omega. \end{aligned} \tag{E.140}$$

The condition $e \wedge d_\omega e = 0$ implies $d_\omega e = 0$. This equation implies Γ is torsion-free, hence equal to the Levi-Civita connection of g. We then use this in $e \wedge F = 0$ and obtain the metric vacuum Einstein equation.

E.10 Self-dual Connection Formulation

E.10.1 Self-dual Curvature

We will need the totally antisymmetry tensor or Levi-Civita symbol, ϵ_{IJKL}. Recall that this is equal to either +1 or -1 depending on whether $IJKL$ is either an even or odd permutation of 0123, respectively, and zero if any two indices take the same value. The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}.

Given any anti-symmetric tensor T^{IJ}, we define its dual as

$$*T^{IJ} = \frac{1}{2}\epsilon_{KL}{}^{IJ}T^{\ KL} \tag{E.141}$$

The self-dual part of any tensor T^{IJ} is defined as

$$^{+}T^{IJ} := \frac{1}{2}\Big(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}\Big) \tag{E.142}$$

with the anti-self-dual part defined as

$$^{-}T^{IJ} := \frac{1}{2}\Big(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}\Big) \tag{E.143}$$

(the appearance of the imaginary unit i is related to the Minkowski signature as we will see below).

Tensor decomposition

Now given any anti-symmetric tensor T^{IJ}, we can decompose it as

$$T^{IJ} = \frac{1}{2}(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}) + \frac{1}{2}(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}) = {}^{+}T^{IJ} + {}^{-}T^{IJ} \tag{E.144}$$

where $^{+}T^{IJ}$ and $^{-}T^{IJ}$ are the self-dual and anti-self-dual parts of T^{IJ} respectively. Define the projector onto (anti-)self-dual part of any tensor as

$$P^{(\pm)} = \frac{1}{2}(1 \mp i*). \tag{E.145}$$

The meaning of these projectors can be made explicit. Let us concentrate of P^{+},

$$(P^{+}T)^{IJ} = (\frac{1}{2}(1 - i*)T)^{IJ} = \frac{1}{2}(\delta^{I}{}_{K}\delta^{J}{}_{L} - i\frac{1}{2}\epsilon_{KL}{}^{IJ})T^{KL} = \frac{1}{2}(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}T^{KL}) = {}^{+}T^{IJ}. \tag{E.146}$$

Then

$^{\pm}T^{IJ} = (P^{(\pm)}T)^{IJ}$.

The Lie bracket

An important object is the Lie bracket defined by

$$[F,G]^{IJ} := F^{IK}G_K^{\ J} - G^{IK}F_K^{\ J}, \tag{E.147}$$

it appears in the curvature tensor (see the last two terms of ()), it also defines the algebraic structure. We have the results (proved below):

$$P^{(\pm)}[F,G]^{IJ} = [P^{(\pm)}F,G]^{IJ} = [F,P^{(\pm)}G]^{IJ} = [P^{(\pm)}F,P^{(\pm)}G]^{IJ} \tag{E.148}$$

and

$$[F,G] = [P^+F,P^+G] + [P^-F,P^-G]. \tag{E.149}$$

That is the Lie bracket, which defines an algebra, decomposes into two separate independent parts.

Identities for the totally anti-symmetric tensor

The internal indices of ϵ_{IJKL} are raised with the Minkowski metric η^{IJ}. Since η_{IJ} has signature $(-,+,+,+)$, it follows that

$$\epsilon^{IJKL} = -\epsilon_{IJKL}.$$

to see this consider,

$$\begin{aligned} \epsilon^{0123} &= \eta^{0I}\eta^{1J}\eta^{2K}\eta^{3L}\epsilon_{IJKL} \\ &= (-1)(+1)(+1)(+1)\epsilon_{0123} = -\epsilon_{0123}. \end{aligned}$$

We then have the identities,

$$\epsilon^{IJKO}\epsilon_{LMNO} = -6\delta^I_{[L}\delta^J_M\delta^K_{N]} \tag{E.150}$$

$$\begin{aligned} \epsilon^{IJMN}\epsilon_{KLMN} &= -4\delta^I_{[K}\delta^J_{L]} \\ &= -2(\delta^I_K\delta^J_L - \delta^I_L\delta^J_K). \end{aligned} \tag{E.151}$$

Definition of self-dual tensor

Therefore the square of the duality operator is minus the identity,

$$* * T^{IJ} = \frac{1}{4}\epsilon_{KL}{}^{IJ}\epsilon_{MN}{}^{KL}T^{\ MN} = -T^{IJ} \tag{E.152}$$

The minus sign here is due to the minus sign in (E.151), which is in turn due to the Minkowski signature. Had we used Euclidean signature, i.e. $(+,+,+,+)$, instead there would have been a positive sign. We define S^{IJ} to be self-dual if and only if

$$*S^{IJ} = iS^{IJ}. \tag{E.153}$$

(with Euclidean signature the self-duality condition would have been $*S^{IJ} = S^{IJ}$). Say S^{IJ} is self-dual, write it as a real and imaginary part,

$$S^{IJ} = \frac{1}{2}U^{IJ} + i\frac{1}{2}V^{IJ}.$$

Write the self-dual condition in terms of U and V,

$$*(U^{IJ} + iV^{IJ}) = \frac{1}{2}\epsilon_{KL}{}^{IJ}(U^{KL} + iV^{KL}) = i(U^{IJ} + iV^{IJ}).$$

Equating real parts we read off

$$V^{IJ} = -\frac{1}{2}\epsilon_{KL}{}^{IJ}U^{KL}$$

and so

$$S^{IJ} = \frac{1}{2}(U^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{IJ}U^{KL}) \tag{E.154}$$

where U^{IJ} is the real part of $2T^{IJ}$. Now given any tensor T^{IJ}, we can decompose it as

$$\begin{aligned} T^{IJ} &= \frac{1}{2}(T^{IJ} - \frac{i}{2}\epsilon_{KL}{}^{\ IJ}T^{IJ}) + \frac{1}{2}(T^{IJ} + \frac{i}{2}\epsilon_{KL}{}^{\ IJ}T^{IJ}) \\ &= {}^{+}T^{IJ} + {}^{-}T^{IJ} \end{aligned} \tag{E.155}$$

since $*({}^{+}T^{IJ}) = i\ {}^{+}T^{IJ}$ and $*({}^{-}T^{IJ}) = -i\ {}^{-}T^{IJ}$, ${}^{+}T^{IJ}$ and ${}^{-}T^{IJ}$ are the self-dual and anti-self-dual parts of T^{IJ}.

Important lengthy calculation

The following lengthy calculation is important as all the other important formula can easily be derived from it. From the definition of the Lie bracket and with the use of (E.150) we have

$$\begin{aligned}
*[F, *G]^{IJ} &= \frac{1}{2}\epsilon_{MN}{}^{IJ}(F^{MK}(*G)_K{}^N - (*G)^{MK}F_K{}^N) \\
&= \frac{1}{2}\epsilon_{MN}{}^{IJ}(F^{MK}\frac{1}{2}\epsilon_{OPK}{}^N G^{OP} - \frac{1}{2}\epsilon_{OP}{}^{MK}G^{OP}F_K{}^N) \\
&= \frac{1}{4}(\epsilon_{MN}{}^{IJ}\epsilon_{OP}{}^{KN} + \epsilon_{NM}{}^{IJ}\epsilon_{OP}{}^{NK})F^M{}_K G^{OP} \\
&= \frac{1}{2}\epsilon_{MN}{}^{IJ}\epsilon_{OP}{}^{KN}F^M{}_K G^{OP} \\
&= \frac{1}{2}\epsilon^{MIJN}\epsilon_{OPKN}F_M{}^K G^{OP} \\
&= -\frac{1}{2}\epsilon^{KIJN}\epsilon_{OPMN}F^M{}_K G^{OP} \\
&= \frac{1}{2}(\delta^K_O\delta^I_P\delta^J_M + \delta^K_M\delta^I_O\delta^J_P + \delta^K_P\delta^I_M\delta^J_O \\
&\quad - \delta^K_P\delta^I_O\delta^J_M - \delta^K_M\delta^I_P\delta^J_O - \delta^K_O\delta^I_M\delta^J_P)F^M{}_K G^{OP} \\
&= \frac{1}{2}(F^J{}_K G^{KI} + F^K{}_K G^{IJ} + F^I{}_K G^{JK} \\
&\quad - F^J{}_K G^{IK} - F^K{}_K G^{JI} - F^I{}_K G^{KJ}) \\
&= -F^{IK}G_K{}^J + G^{IK}F_K{}^J \\
&= -[F, G]^{IJ}
\end{aligned} \tag{E.156}$$

That gives the formula

$$*[F, *G]^{IJ} = -[F, G]^{IJ}. \tag{E.157}$$

from which everything else is then easy to derive.

Derivation of important results

First consider

$$\begin{aligned}
*[*F, G]^{IJ} &= - * [G, *F]^{IJ} \\
&= [G, F]^{IJ} = -[F, G]^{IJ}.
\end{aligned}$$

where in the first step we have used the anti-symmetry of the Lie bracket to swap $*F$ and G, in the second step we used (E.157) and in the last step we used the anti-symmetry of the Lie bracket again. Now using this we obtain

$$
\begin{aligned}
*[F,G]^{IJ} &= *(-*[*F,G]^{IJ}) \\
&= -**[*F,G]^{IJ} \\
&= [*F,G]^{IJ}.
\end{aligned}
$$

where we used $** = -1$ in the third step. Similarly we have

$$*[F,G]^{IJ} = [F,*G]^{IJ}$$

Now if we took $*[F,G]^{IJ} = [*F,G]^{IJ}$ and simply replaced G with $*G$ we would get $*[F,*G]^{IJ} = [*F,*G]^{IJ}$. Combining $-[F,G]^{IJ} = *[F,*G]^{IJ}$ (E.157) and $*[F,*G]^{IJ} = [*F,*G]^{IJ}$ we obtain

$$-[F,G]^{IJ} = [*F,*G]^{IJ}.$$

Summarising, we have

$$
\begin{aligned}
&*[F,*G]^{IJ} = -[F,G]^{IJ} = *[*F,G]^{IJ} && \text{(E.158)} \\
&*[F,G]^{IJ} = [*F,G]^{IJ} = [F,*G]^{IJ} && \text{(E.159)} \\
&[*F,*G]^{IJ} = -[F,G]^{IJ} && \text{(E.160)}
\end{aligned}
$$

Then

$$
\begin{aligned}
(P^{(\pm)}[F,G])^{IJ} &= \frac{1}{2}([F,G]^{IJ} \mp i*[F,G]^{IJ}) \\
&= \frac{1}{2}([F,G]^{IJ} + [\mp i*F,G]^{IJ}) \\
&= [P^{(\pm)}F,G]^{IJ} \qquad \text{(E.161)}
\end{aligned}
$$

Similarly we have $(P^{(\pm)}[F,G])^{IJ} = [F,P^{(\pm)}G]^{IJ}$. Now consider $[P^+F,P^-G]^{IJ}$,

$$
\begin{aligned}
[P^+F,P^-G]^{IJ} &= \frac{1}{4}[(1-i*)F,(1+i*)G]^{IJ} \\
&= \frac{1}{4}[F,G]^{IJ} - \frac{1}{4}i[*F,G]^{IJ} + \frac{1}{4}i[F,*G]^{IJ} + \frac{1}{4}[*F,*G]^{IJ} \\
&= \frac{1}{4}[F,G]^{IJ} - \frac{1}{4}i[*F,G]^{IJ} + \frac{1}{4}i[*F,G]^{IJ} - \frac{1}{4}[F,G]^{IJ} \\
&= 0. \qquad \text{(E.162)}
\end{aligned}
$$

Similarly $[P^-F,P^+G]^{IJ} = 0$. This implies

$$[P^{(\pm)}F,G]^{IJ} = [P^{(\pm)}F,P^{(\pm)}G + P^{(\mp)}G]^{IJ} = [P^{(\pm)}F,P^{(\pm)}G]^{IJ}.$$

Summary of main results

Altogether we have,

$$P^{(\pm)}[F,G]^{IJ} = [P^{(\pm)}F,G]^{IJ} = [F,P^{(\pm)}G]^{IJ} = [P^{(\pm)}F,P^{(\pm)}G]^{IJ}. \tag{E.163}$$

We then have that any bracket splits as

$$\begin{aligned} [F,G]^{IJ} &= [P^+F + P^-F, P^+G + P^-F]^{IJ} \\ &= [P^+F, P^+G]^{IJ} + [P^-F, P^-G]^{IJ}. \end{aligned} \tag{E.164}$$

into a part that depends only on self-dual Lorentzian tensors and is itself the self-dual part of $[F,G]^{IJ}$ by (E.163), and a part that depndends only on anti-self-dual Lorentzian tensors and is the anit-self-dual part of $[F,G]^{IJ}$ again by (E.163).

We can write

$$so(1,3)_{\mathbb{C}} = so(1,3)_{\mathbb{C}}^{+} + so(1,3)_{\mathbb{C}}^{-} \tag{E.165}$$

where $so(1,3)_{\mathbb{C}}^{\pm}$ contains only the self-dual (anti-self-dual) elements of $so(1,3)_{\mathbb{C}}$.

Self-dual curvature

Instead of considering the connection ${\omega_a}^{IJ}$ we will consider its self-dual part, ${}^+{A_a}^{IJ}$, with respect to the internal indices, that is,

$$i\,{}^+{A_a}^{IJ} = \frac{1}{2}{\epsilon_{MN}}^{IJ}\,{}^+{A_a}^{MN}$$

${}^+{A_a}^{IJ}$ is related to ${\omega_a}^{IJ}$ by

$${}^+{A_a}^{IJ} = \frac{1}{2}{\omega_a}^{IJ} - \frac{i}{4}{\epsilon_{MN}}^{IJ}{\omega_a}^{MN} \tag{E.166}$$

Define ${F_{ab}}^{JK}$ as the curvature of the self-dual connection,

$${F_{ab}}^{IJ} = \nabla_a\,{}^+{A_b}^{IJ} - \nabla_b\,{}^+{A_a}^{IJ} + {}^+{A_a}^{IK}\,{}^+{A_{bK}}^{J} - {}^+{A_b}^{IK}\,{}^+{A_{aK}}^{J} \tag{E.167}$$

We use the above results to show this corresponds to the self-dual part of the curvature of the usual connection:

$$
\begin{aligned}
F_{ab}{}^{IJ} &= \nabla_a \, {}^+A_b{}^{IJ} - \nabla_b \, {}^+A_a{}^{IJ} + \, {}^+A_a{}^{IK} \, {}^+A_{bK}{}^J - \, {}^+A_b{}^{IK} \, {}^+A_{aK}{}^J \\
&= (\nabla_a (P^+ \omega_b)^{IJ} - \nabla_b (P^+ \omega_a)^{IJ}) + [P^+ \omega_a, P^+ \omega_b]^{IJ} \\
&= (P^+ 2 \nabla_{[a} \omega_{b]})^{IJ} + (P^+ [\omega_a, \omega_b])^{IJ} \\
&= (P^+ \Omega_{ab})^{IJ}
\end{aligned} \tag{E.168}
$$

Which was the desired result. Thus we have,

$$
\begin{aligned}
F_{ab}{}^{IJ} &= \frac{1}{2} \Big(\Omega_{ab}{}^{IJ} - \frac{i}{2} \epsilon_{MN}{}^{IJ} \Omega_{ab}{}^{MN} \Big) \\
&= \nabla_{[a} \omega_{b]}{}^{IJ} + \omega_{[a}{}^{IK} \omega_{b]K}{}^J - \frac{i}{2} \epsilon_{MN}{}^{IJ} (\nabla_{[a} \omega_{b]}{}^{MN} + \omega_{[a}{}^{MK} \omega_{b]K}{}^N).
\end{aligned} \tag{E.169}
$$

Using

$$
\begin{aligned}
2 \nabla_{[a} \omega_{b]}{}^{IJ} + [\omega_a, \omega_b]^{IJ} &= (P^+ 2 \nabla_{[a} \omega_{b]})^{IJ} + (P^- 2 \nabla_{[a} \omega_{b]})^{IJ} + [P^+ \omega_a + P^- \omega_a, P^+ \omega_b + P^- \omega_b]^{IJ} \\
&= \Big((P^+ 2 \nabla_{[a} \omega_{b]})^{IJ} + [P^+ \omega_a, P^+ \omega_b]^{IJ} \Big) + \Big((P^- 2 \nabla_{[a} \omega_{b]})^{IJ} + [P^- \omega_a, P^- \omega_b]^{IJ} \Big).
\end{aligned} \tag{E.170}
$$

we see that the Palatini curvature decomposes as,

$$
\Omega_{ab}{}^{IJ}[\omega] = \Omega_{ab}{}^{IJ}[{}^+A] + \Omega_{ab}{}^{IJ}[{}^-A] \tag{E.171}
$$

and therefore the Palatini action can be written in terms of a self-dual and anti-self-dual part which depend respectively only on the self-dual and anti-self-dual connections.

E.10.2 Self-dual Action

In the self-dual formulation of general relativity, the variables are a self-dual Lorentz connection.

The other variable is a complex tetrad. The action in the self-dual formulation is built using the curvature of the self-dual Lorentz connection.

As in the Palatini formulism, one can use the tetrad to define a metric g on M by

$$
g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta
$$

However, because the tetrad is complex, the metric is now complex.

The self-dual action can be written

$$S_{EH} = \int d^4x\ e\ e_I^a e_J^b F_{ab}{}^{IJ} \tag{E.172}$$

We introduce a connection compatible with the tetrad via $\nabla_a e_I^b = 0$. We will like to proceed analogously as with the Palatini case and show that one of the equations of motion implies that ${}^+\mathcal{D}_a$ is the self-dual part of the unique, torsion-free covariant derivative ∇_a compatible with e_a^I.

If $\Gamma_{aI}{}^J$ denotes the Chrristoffel symbol of ∇_a, we define the self-dual part ${}^+\nabla_a$ of ∇_a by

$${}^+\nabla_a v_I := \partial_a v_I + {}^+\Gamma_{aI}{}^J v_J, \tag{E.173}$$

where ${}^+\Gamma_{aI}{}^J$ is the self-dual part of $\Gamma_{aI}{}^J$. The difference between these two connections when applied to a tensor with purely internal indices is

$${}^+C_{aI}{}^J V_J = ({}^+\mathcal{D}_a - {}^+\nabla_a) V_I. \tag{E.174}$$

Note that ${}^+C_{aI}{}^J = A_{aI}{}^J - {}^+\Gamma_{aI}{}^J$ is indeed the self-dual part of $C_{aI}{}^J$.

$$F_{ab}{}^{IJ} - {}^+R_{ab}{}^{IJ} = 2\nabla_{[a}\,{}^+C_{b]}{}^{IJ} + 2\,{}^+C_{[a}{}^{IK}\,{}^+C_{b]K}{}^J \tag{E.175}$$

The self-dual action is written as

$$S_{EH} = \int d^4x\ e\ e_I^a e_J^b ({}^+R_{ab}{}^{IJ} + 2\nabla_{[a}\,{}^+C_{b]}{}^{IJ} + 2\,{}^+C_{[a}{}^{IK}\,{}^+C_{b]K}{}^J) \tag{E.176}$$

Variation with respect to ${}^+C_{b]}{}^{IJ}$ produces

$${}^+C_{bI}{}^K e_K^{[a} e_J^{b]} + {}^+C_{bJ}{}^K e_I^{[a} e_K^{b]} = 0. \tag{E.177}$$

where we have used ${}^+C_{bKI} = -\,{}^+C_{bIK}$. This can be writtem more compactly as

$$e_M^{[a} e_N^{b]} \delta_{[I}^M \delta_{J]}^K\,{}^+C_{bK}{}^N = 0. \tag{E.178}$$

The variation with respect to the tetrad goes along the similarly except $\Omega_{ab}{}^{IJ}$ replaced everywhere by $F_{ab}{}^{IJ}$.

E.11 Ashtekar's Canonical Formalism

GR can be expressed in terms of a complex field $A_a^i(x)$ and a 3d real field $\tilde{E}_i^a(x)$, defined on a three-dimensional space Σ without boundaries, satisfying the reality condition

$$A_a^i - \bar{A}_a^i = \Sigma_a^i[E] \tag{E.179}$$

where Σ is defined in appendix A. The theory is defined by the Hamiltonian system

$$\tilde{E}_i^a(x) = \frac{\delta S}{\delta A_a^i(x)} \tag{E.180}$$

This indicates that the quantity $\tilde{E}_i^a(x)$ is the momentum conjugate to $A_a^i(x)$. In Maxwell and Yang-Mills theories, the momentum conjugate to the three dimensional connection A is called the electric field.

The Lagrainian and Ashtekar Lagrangian give the same equations of motion from the varaiation of the spin connection, and once this equation is solved and substituted into, the two Lagrangians differ only by a term that vanishes due to the Bianchi identity.

E.12 Generators of Symmetry Transformations

$$\{E_i^a(x), G(y)\} = \epsilon^{ijk}\Lambda_j \tilde{E}_k^a \tag{E.181}$$

This Poisson bracket generates an infintesmal roatation on the internal space index of E_i^a. $R(\delta\theta)^{ij} = \delta^{ij} + \epsilon^{ijk}\Lambda_j$.

$$\{A_a^i(x), G(y)\} = \mathcal{D}_a\Lambda_j, \tag{E.182}$$

The theory contains the "vector" constraint ()

$$V(\vec{N}) = \int_\Sigma d^3 N^a E^{bi} F_{ab}^i \tag{E.183}$$

which when combined with the Gauss constraint, gives the statial diffeomorphism constraint

$$\begin{aligned} D(\vec{N}) = V(\vec{N}) - G(A_a^i N^a) &= \int_\Sigma d^3x[N^a E^{bi} F_{ab}^i = \\ &= \int_\Sigma d^3x[E^{bi}\partial_a A_b^i - \partial_b(E^{bi}A_a^i)] \end{aligned} \tag{E.184}$$

The diffeomorphism constraint generates the in intesimal change in the fields along the diffeomorphism flow. When applied to a Wilson loop we dont need to include the Gauss gauge term $G(A_a^i N^a)$ and there is no difference between the constraints $V(\vec{N})$ and $D(\vec{N})$.

$$\begin{aligned} \{A_a^i, D(\vec{N})\} &= \mathcal{L}_{\vec{N}} A_a^i, \\ \{E^{ai}, D(\vec{N})\} &= \mathcal{L}_{\vec{N}} E^{ai} \end{aligned} \tag{E.185}$$

The Hamiltonian constraint generates

$$\{A_a^i, H(N)\} = 2\beta N \epsilon^{ijk} E^{bi} F_{ab}^k, \tag{E.186}$$

$$\{E^{ai}, H(N)\} = -2\beta N D_b (N E^{aj} E^{bk}). \tag{E.187}$$

E.12.1 The Gauss-law Constraint Generates Gauge Transformations

$$\widehat{E_i(x)}\Psi = -i\frac{\delta\Psi}{\delta A^i(x)}. \tag{E.188}$$

$$|\psi\rangle \to |\psi\rangle + i\epsilon \left(\int d^3x \Lambda \nabla^i \widehat{E_i(x)} \right) |\psi\rangle \tag{E.189}$$

$$\Psi[A] \to \Psi[A] + \epsilon \int d^3x \Lambda \nabla^i \frac{\delta\Psi}{\delta A^i(x)}. \tag{E.190}$$

Integrating by parts, this gives us

$$\Psi[A] \to \Psi[A] + \epsilon \int d^3x (\nabla^i \Lambda) \frac{\delta\Psi}{\delta A^i(x)}. \tag{E.191}$$

But this is just the first terms of an infinite-dimensional Taylor expansion, and we have

$$\Psi[A] \to \Psi[A + \epsilon \nabla^i \Lambda] \tag{E.192}$$

which is a gauge transformation. The constraint operators are the generators of the gauge group. It follows that any physical state, i.e. any state satisfying

$$\nabla^i \widehat{E_i(x)} |\psi\rangle = 0, \tag{E.193}$$

must be gauge invariant.

E.12.2 Incorperating Matter in the Quantum Theory

fermions at open ends of loops - natural gauge invariant objects.

Lattice Gauge Theories

E.13 Toy Model: Free Particle described using Half-Complex Coordinates.

E.13.1 Complex Variables and Reality Conditions

Canonical transformation

$$P\dot{Q} - \tilde{\mathcal{H}}(Q,P) = p\dot{q} - \mathcal{H}(q,p) - \frac{\partial F}{\partial t}. \tag{E.194}$$

$$\begin{aligned} \frac{\partial \tilde{\mathcal{H}}}{\partial P} &= \dot{Q} \\ \frac{\partial \tilde{\mathcal{H}}}{\partial Q} &= -\dot{P} \end{aligned} \tag{E.195}$$

$$P = -\frac{\partial F_1}{\partial Q}, \quad Q = q, \tag{E.196}$$

$$F_1(q,Q) = -qQ + ipQ \tag{E.197}$$

$Q = q$ $P = z = q - ip$

$$z = q - ip. \tag{E.198}$$

In terms of these variables the Hamiltonian reads

$$\mathcal{H}_0(x,z) = -\frac{1}{2m}(x-z)^2. \tag{E.199}$$

Consider z as a configuration variable

To find the solutions corresponding to real x and p, we have to impose the condition

$$z + \bar{z} = 2x \tag{E.200}$$

which x is the real part of z

Equation (E.200) is called the *reality condition*.

E.13.2 Quantization in Compex Coordinates.

A complex cononical transformation $x \rightarrow x, \;\; p \rightarrow x - ip$

If we know what x and p are then we know what x and z are and conversely if we know what x and z are we know what x and p are; so they are equivalent descriptions.

a free particle described in the coordinates

The Shrödinger equation in complex coordinates is

$$i\hbar\frac{\partial\psi(z,t)}{\partial t} = \mathcal{H}_0\left(\hbar\frac{\partial}{\partial z}, z\right)\psi(z,t) = -\frac{1}{2m}\left(\hbar\frac{\partial}{\partial z} - z\right)^2\psi(z,t). \tag{E.201}$$

$$z + z^\dagger = 2\hbar\frac{\partial}{\partial z}. \tag{E.202}$$

Now, equation (E.202) only makes sense after we have specified a the scalar product, because the adjoint of an operator is defined in terms of the the scalar product.

$$(\psi, \phi) = \int d\bar{z}dz f(z,\bar{z})\overline{\psi(z)}\phi(z). \tag{E.203}$$

We will know what the inner product if we can specify f.

$$(z + \bar{z})f(z,\bar{z}) = -2\hbar\frac{\partial}{\partial z}f(z,\bar{z}) \tag{E.204}$$

This gives

$$f(z,\bar{z}) = e^{-\frac{(z+\bar{z})^2}{4\hbar}} \tag{E.205}$$

E.14 The Holst Action

$$S(E,\omega) = \frac{1}{16\pi G}\int d^4x |E| E^a_I E^b_J P^{IJ}{}_{KL}\Omega_{ab}{}^{KL}(\omega) \tag{E.206}$$

where

$$P^{IJ}{}_{KL} = \delta^{[I}_K \delta^{J]}_L - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL}. \tag{E.207}$$

E.14.1 3+1 Decomposition of the Holst Action

We relate the curvature of the Ashtekar-Barbero connection to the curvature of the spin connection,

$$\begin{aligned}
\mathcal{F}^i_{ab} &= 2\partial_{[a}(\Gamma^i_{b]} + \beta K^i_{b]}) - \epsilon^i{}_{jk}(\Gamma^j_a + \beta K^j_a)(\Gamma^k_b + \beta K^k_b) \\
&= 2\partial_{[a}(\Gamma^i_{b]} + \beta K^i_{b]}) - \frac{1}{2}\epsilon^i{}_{jk}\Big((\Gamma^j_a + \beta K^j_a)(\Gamma^k_b + \beta K^k_b) - a \leftrightarrow b\Big) \\
&= 2\partial_{[a}\Gamma^i_{b]} - \epsilon^i{}_{jk}\Gamma^j_{[a}\Gamma^k_{b]} - \beta\Big(2\partial_{[a}K^i_{b]} - \epsilon^i{}_{jk}(\Gamma^j_{[a}K^k_{b]} + K^j_{[a}\Gamma^k_{b]})\Big) - \beta^2\epsilon^i{}_{jk}K^j_{[a}K^k_{b]} \\
&= F^i_{ab} - \beta\Big(2\partial_{[a}K^i_{b]} - \epsilon^i{}_{jk}(\Gamma^j_{[a}K^k_{b]} - K^j_{[a}\Gamma^k_{b]})\Big) - \beta^2\epsilon^i{}_{jk}K^j_aK^k_b \\
&= F^i_{ab} - \beta(2\partial_{[a}K^i_{b]} - 2\epsilon^i{}_{jk}\Gamma^j_{[a}K^k_{b]} - \beta^2\epsilon^i{}_{jk}K^j_aK^k_b \\
&= F^i_{ab} - 2\beta\nabla_{[a}K^i_{b]} - \beta^2\epsilon^i{}_{jk}K^j_aK^k_b
\end{aligned} \tag{E.208}$$

E.14.2 The Diffeomorphism Constraint

For the diffeomorphism constraint we have (we introduce here a negative sign so that the constraint appears with a positve sign in the Hamiltonian, recall the basic form $\mathcal{L} = p\dot{q} - \mathcal{H}$)

$$\begin{aligned}
N^aC_a &= -\beta n_I N^a\Big(\frac{\sqrt{q}E^b_J}{8\pi\beta G}\Big)P^{IJ}{}_{KL}F_{ab}{}^{KL} \\
&= \beta N^a\tilde{E}^b_j P^{0j}{}_{kl}F_{ab}{}^{kl} \\
&= \beta N^a\tilde{E}^b_j\left(F_{ab}{}^{0j} - \frac{1}{2\beta}\epsilon^{0j}{}_{kl}F_{ab}{}^{kl}\right) \\
&= 2\beta N^a\tilde{E}^b_j\left(\partial_{[a}\omega_{b]}{}^{0j} + \omega_{[a}{}^{0k}\omega_{b]k}{}^j + \frac{1}{2\beta}\epsilon^j{}_{kl}(\partial_{[a}\omega_{b]}{}^{kl} + \omega_{[a}{}^{kL}\omega_{b]L}{}^l)\right) \\
&= 2\beta N^a\tilde{E}^b_j\left(\partial_{[a}K^j_{b]} + K^k_{[a}\omega_{b]k}{}^j + \frac{1}{2\beta}\big(2\partial_{[a}\Gamma_{b]}{}^{kl} + \epsilon^j{}_{kl}(\omega_{[a}{}^{k0}\omega_{b]0}{}^l + \omega_{[a}{}^{km}\omega_{b]m}{}^l)\big)\right) \\
&= 2N^a\tilde{E}^b_j\left(\partial_{[a}A^j_{b]} - \beta\epsilon^j{}_{mk}\Gamma^m_{[a}K^k_{b]} + \frac{1}{2}\epsilon^j{}_{kl}(K^k_{[a}K^l_{b]} + \omega_{[a}{}^{km}\omega_{b]m}{}^l)\right)
\end{aligned} \tag{E.209}$$

where we used $\epsilon^{0j}{}_{kl} = -\epsilon^{j}{}_{kl}$, $K^i_a = \omega_a{}^{0i}$ and $\omega_{ak}{}^{j} = \epsilon^{j}{}_{mk}\Gamma^m_a$. Also that $\omega_{[a}{}^{k0}\omega_{b]0}{}^{l} = \omega_{[a}{}^{0k}\omega_{b]}{}^{0l}$.

The term $\omega_{[a}{}^{km}\omega_{b]m}{}^{l}$ becomes,

$$\begin{aligned}
\epsilon^{j}{}_{kl}\omega_a{}^{km}\omega_{bm}{}^{l} &= \epsilon^{j}{}_{kl}(\epsilon^{km}{}_{n}\Gamma^n_a)(\epsilon_m{}^{l}{}_{p}\Gamma^p_b) \\
&= \epsilon^{j}{}_{kl}\epsilon_n{}^{km}\epsilon^{l}{}_{pm}\Gamma^n_a\Gamma^p_b \\
&= \epsilon^{j}{}_{kl}(\delta^l_n\delta^k_p - \eta_{np}\eta^{kl})\Gamma^n_a\Gamma^p_b \\
&= \epsilon^{j}{}_{kl}\Gamma^l_a\Gamma^k_b \\
&= -\epsilon^{j}{}_{kl}\Gamma^k_{[a}\Gamma^l_{b]}
\end{aligned} \tag{E.210}$$

Continuing with the calculation for the spatial diffeomorphism constraint,

$$\begin{aligned}
N^aC_a &= N^a\tilde{E}^b_j\left(2\partial_{[a}A^j_{b]} - \beta\epsilon^{j}{}_{kl}\Gamma^k_{[a}K^l_{b]} - \frac{1}{2}\epsilon^{j}{}_{kl}\big(\Gamma^k_{[a}\Gamma^l_{b]} - K^k_{[a}K^l_{b]}\big)\right) \\
&= N^a\tilde{E}^b_j\left(2\partial_{[a}A^j_{b]} - \beta\frac{1}{2}\epsilon^{j}{}_{kl}\big(\Gamma^k_{[a}K^l_{b]} + K^k_{[a}\Gamma^l_{b]}\big) - \frac{1}{2}\epsilon^{j}{}_{kl}\big(\Gamma^k_{[a}\Gamma^l_{b]} - K^k_{[a}K^l_{b]}\big)\right) \\
&= N^a\tilde{E}^b_j\left(2\partial_{[a}A^i_{b]} - \frac{1}{2}\epsilon^{j}{}_{kl}(\Gamma^k_{[a}\Gamma^l_{b]} + \beta\Gamma^k_{[a}K^l_{b]} + \beta K^k_{[a}\Gamma^l_{b]} - K^k_{[a}K^l_{b]})\right) \\
&= N^a\tilde{E}^b_j\bigg(2\partial_{[a}A^i_{b]} - \frac{1}{2}\epsilon^{j}{}_{kl}(\Gamma^k_{[a}\Gamma^l_{b]} + \beta\Gamma^k_{[a}K^l_{b]} + \beta K^k_{[a}\Gamma^l_{b]} + \beta^2K^k_{[a}K^l_{b]}) \\
&\quad + (1+\beta^2)\epsilon^{j}{}_{kl}K^k_aK^l_b\bigg) \\
&= N^a\tilde{E}^b_j\left(2\partial_{[a}A^i_{b]} - \epsilon^{j}{}_{kl}(\Gamma^k_a + \beta K^k_a)(\Gamma^l_b + \beta K^l_b) + (1+\beta^2)\epsilon^{j}{}_{kl}K^k_aK^l_b\right) \\
&= N^a\tilde{E}^b_j\left(\mathcal{F}^j_{ab} + (1+\beta^2)\epsilon^{j}{}_{kl}K^k_aK^l_b\right)
\end{aligned} \tag{E.211}$$

where in the second line we used $\epsilon^{j}{}_{kl}\Gamma^k_{[a}K^l_{b]} = -\epsilon^{j}{}_{kl}\Gamma^k_{[b}K^l_{a]} = -\epsilon^{j}{}_{kl}K^l_{[a}\Gamma^k_{b]} = \epsilon^{j}{}_{kl}K^k_{[a}\Gamma^l_{b]}$.

E.14.3 The Hamiltonian Constraint

$$
\begin{aligned}
C &= -4\pi G\frac{N}{\sqrt{q}}(\frac{\sqrt{q}E^a_I}{8\pi\beta G})(\frac{\sqrt{q}E^b_J}{8\pi\beta G})P^{IJ}{}_{KL}F^{KL}_{ab} \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}P^{ij}{}_{KL}F_{ab}{}^{KL} \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\left(F^{ij}_{ab}-\frac{1}{2\beta}\epsilon^{ij}{}_{KL}F_{ab}{}^{KL}\right) \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\left(F^{ij}_{ab}-\frac{1}{2\beta}(\epsilon^{ij}{}_{k0}F_{ab}{}^{k0}+\epsilon^{ij}{}_{0k}F_{ab}{}^{0k})\right) \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\left(2\partial_{[a}\omega_{b]}{}^{ij}+2\omega_{[a}{}^{iK}\omega_{b]}{}^{Lj}\eta_{KL}+\frac{2}{\beta}\epsilon^{ij}{}_k(\partial_{[a}\omega_{b]}{}^{k0}+\omega_{[a}{}^{kl}\omega_{b]l}{}^0)\right) \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\left(F^{ij}_{ab}+2K^i_{[a}K^j_{b]}-\frac{2}{\beta}\epsilon^{ij}{}_k(\partial_{[a}K^k_{b]}+\omega_{[a}{}^{kl}K_{b]l})\right) \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\epsilon^{ij}{}_k\left(F^k_{ab}+\epsilon^k{}_{mn}K^m_{[a}K^n_{b]}-\frac{2}{\beta}\nabla_{[a}K^k_{b]}\right) \qquad \text{(E.212)}
\end{aligned}
$$

Substituting in (E.208) in this we obtain,

$$
\begin{aligned}
C &= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\epsilon^{ij}{}_k\left(\mathcal{F}^k_{ab}+(1+\beta^2)\epsilon^k{}_{mn}K^m_aK^n_b+2\frac{\beta^2+1}{\beta}\nabla_{[a}K^k_{b]}\right) \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\epsilon^{ij}{}_k\left(\mathcal{F}^k_{ab}+(1+\beta^2)\epsilon^k{}_{mn}K^m_aK^n_b\right)-\ 4\pi G\beta^2(\frac{\sqrt{q}E^a_i}{8\pi\beta G})\frac{\tilde{E}^b_j}{\sqrt{q}}\epsilon^{ij}{}_k2\frac{\beta^2+1}{\beta}\nabla_{[a}K^k_{b]} \\
&= -4\pi G\beta^2\frac{\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{q}}\epsilon^{ij}{}_k\left(\mathcal{F}^k_{ab}+(1+\beta^2)\epsilon^k{}_{mn}K^m_aK^n_b\right)-\ (\beta^2+1)\epsilon^{ij}{}_kE^a_i\tilde{E}^b_j\nabla_{[a}K^k_{b]} \qquad \text{(E.213)}
\end{aligned}
$$

In a constrained Hamiltonian system, a dynamical quanity is called first class constraint if its Poisson bracket with all other constraints vanishes on the constraint surface (the surface implicitly defined by the simultaneous vanishing of all the constraints). A second class constraint is one that is not first class.

E.14.4 Addition Constraints

The Hamiltonian is then a sum over constraints,

$$\mathcal{H}=[A^i_a,\tilde{E}^b_j]=\int d^4x\left(\Lambda^iG_i+(1+\beta)\omega^{0j}_tS_j+NC+N^aC_a\right) \qquad \text{(E.214)}$$

$$S_j=-\epsilon_{jm}{}^nK^m_b\tilde{E}^b_n$$

$$\begin{aligned}
\Gamma_{a(i}E^b_{j)} &= \delta_{ij}\frac{1}{2}\frac{1}{\sqrt{detq}}\epsilon^{abc}e_{al}\partial_b e^l_c - \frac{\epsilon^{abc}}{2\sqrt{detq}}(e_{ai}\partial_b e_{cj} + e_{aj}\partial_b e_{ci}) \\
&= -\frac{\epsilon^{abc}}{2\sqrt{detq}}(e_{ai}\partial_b e_{cj} + e_{aj}\partial_b e_{ci} - \delta_{ij}e_{al}\partial_b e^l_c)
\end{aligned} \tag{E.215}$$

Recall $e^a_I = E^a_I + n^a n_I$ and in the temporal gauge $n_i = 0$ so

$$\Gamma_{a(i}E^b_{j)} = -\frac{\epsilon^{abc}}{2\sqrt{detq}}(E_{ai}\partial_b E_{cj} + E_{aj}\partial_b E_{ci} - \delta_{ij}E_{al}\partial_b E^l_c) \tag{E.216}$$

Obviously

$$\mathcal{D}_b\tilde{E}^b_i = \partial_b\tilde{E}^b_i - \epsilon_{ij}{}^k(\Gamma^j_b + \beta K^j_b)\tilde{E}^b_k = 0 \qquad \epsilon_{ij}{}^k K^j_b\tilde{E}^b_k = 0$$

imply

$$\partial_b\tilde{E}^b_i - \epsilon_{ij}{}^k\Gamma^j_b\tilde{E}^b_k = 0$$

Contracting this the totally anti-symmetric tensor

$$\begin{aligned}
0 = \epsilon^i{}_{jk}(\partial_b\tilde{E}^b_i - \epsilon_{ij'}{}^{k'}\Gamma^{j'}_b\tilde{E}^b_{k'}) &= \epsilon^i{}_{jk}\partial_b\tilde{E}^b_i - \epsilon^i{}_{jk}\epsilon_{ij'}{}^{k'}\Gamma^{j'}_b\tilde{E}^b_{k'} \\
&= \epsilon^i{}_{jk}\partial_b\tilde{E}^b_i - (\delta^j_{j'}\delta^k_{k'} - \delta^k_{j'}\delta^j_{k'})\Gamma^{j'}_b\tilde{E}^b_{k'} \\
&= \epsilon^i{}_{jk}\partial_b\tilde{E}^b_i - (\Gamma^j_b\tilde{E}^b_k - \Gamma^k_b\tilde{E}^b_j)
\end{aligned} \tag{E.217}$$

$$|det(\tilde{E})| = |\det(E)|^2$$

$$E^a_i = |\det(\tilde{E})|^{-1/2}\tilde{E}^a_i$$

$$\begin{aligned}
\Gamma_{a[i}E^a_{j]} &= \frac{1}{2}|\det(E)|^{-1}\epsilon_{ijk}\partial_a(|\det(E)|E^a_k) \\
&= \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + \frac{E^a_k}{|\det(E)|}\partial_a|\det(E)|) \\
&= \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + E^a_k E^l_c\partial_a E^c_l)
\end{aligned} \tag{E.218}$$

$$
\begin{aligned}
\Gamma_{ai} E_j^a &= \Gamma_{a(i} E_{j)}^a + \Gamma_{a[i} E_{j]}^a \\
&= -\frac{\epsilon^{abc}}{2\sqrt{detq}} (E_{ai} \partial_b E_{cj} + E_{aj} \partial_b E_{ci} - \delta_{ij} E_{an} \partial_b E_c^n) + \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^l \partial_a E_l^c) \\
&= -\frac{\epsilon^{klm}}{2} E_k^a E_l^b E_m^c (E_{ai} \partial_b E_{cj} + E_{aj} \partial_b E_{ci} - \delta_{ij} E_{an} \partial_b E_c^n) + \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^n \partial_a E_n^c) \\
&= -\frac{1}{2} \Big(\epsilon^{ilm} E_l^b E_m^c \partial_b E_{cj} + \epsilon^{jlm} E_l^b E_m^c \partial_b E_{ci} - \epsilon^{klm} \delta_{ij} E_l^b E_m^c \partial_b E_c^k \Big) + \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^n \partial_a E_n^c) \\
&= \frac{1}{2} E_l^b \Big(\epsilon^{ilk} E_{aj} \partial_b E_k^a + \epsilon^{jlk} E_{ai} \partial_b E_k^a + \epsilon^{mlk} \delta_{ij} E_k^c \partial_b E_c^m \Big) + \frac{1}{2} \epsilon_{ijl} (\partial_b E_l^b + E_l^b E_c^n \partial_b E_n^c) \\
&= \\
&= \\
&= \frac{1}{2} \epsilon^{ikl} E_l^b (E_{a,b}^k - E_{b,a}^k) E_j^a + \frac{1}{2} \epsilon^{ikl} E_l^b E_k^c E_{c,b}^j \\
&= \frac{1}{2} \epsilon^{ikl} E_l^b (E_{a,b}^k - E_{b,a}^k + E_k^c E_a^m E_{c,b}^m) E_j^a
\end{aligned} \tag{E.219}
$$

E.14.5 Final Total Hamiltonian

$$
\mathcal{H} = [A_a^i, \tilde{E}_j^b] = \int d^4x \left(\Lambda^i G_i + NC + N^a C_a \right) \tag{E.220}
$$

$$
C_a = \tilde{E}_j^b \mathcal{F}_{ab}^j \tag{E.221}
$$

$$
C = -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \left(\epsilon^{ij}{}_k \mathcal{F}_{ab}^k + 2(1+\beta^2) K_{[a}^i K_{b]}^j \right) \tag{E.222}
$$

or

$$
C = -4\pi G \beta^2 \left(\frac{\epsilon^{ij}{}_k \mathcal{F}_{ab}^k \tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} + 2(1+\beta^2) \frac{\tilde{E}_{[i}^a \tilde{E}_{j]}^b}{\sqrt{q}} (A_a^i - \Gamma^i{}_a)(A_b^j - \Gamma^j{}_b) \right) \tag{E.223}
$$

E.15 Biblioliographical notes

In this chapter I have relied on the following refferences:

Eric Poisson

Kenneth smith, Dynamic Singularity Excision in Numerical Relativity.

Robert Bartnik, Jim Isenberg, *The Constraint Equations*?? or should this be refferenced in

E.16 Worked Exercises and Details

Gauss-Codazi

Gauss' equation.

$$
\begin{aligned}
\mathrm{R}_{abc}{}^{d}u_d &= (\mathcal{D}_a\mathcal{D}_b - \mathcal{D}_b\mathcal{D}_a)u_c \\
&= 2q^r{}_{[a}q^s{}_{b]}q^t{}_c\nabla_r(\underbrace{q^p{}_s}_{I}\underbrace{q^q{}_t}_{II}\underbrace{\nabla_p u_q}_{III})
\end{aligned} \tag{E.224}
$$

tackle each term separately.

$$
\mathrm{R}_{abc}{}^{d}u_d = (I) + (II) + (III), \tag{E.225}
$$

Proof.

$$
\begin{aligned}
\text{(I)} &= 2q^r{}_{[a}q^s{}_{b]}\,(q^t{}_c q^q{}_t)\,(\nabla_r q^p{}_s)(\nabla_p u_q) \\
&= 2q^r{}_{[a}q^s{}_{b]}q^q{}_c\,\left(\nabla_r(\delta^p_s + n^p n_s)\right)(\nabla_p u_q) \quad \text{using } q^t{}_c q^q{}_t = q^t{}_q \\
&= 2q^r{}_{[a}q^s{}_{b]}q^q{}_c\,(\nabla_r n^p n_s)(\nabla_p u_q) \\
&= 2q^r{}_{[a}q^s{}_{b]}q^q{}_c\,(n^p\nabla_r n_s + n_s\nabla_r n^p)(\nabla_p u_q) \\
&= 0.
\end{aligned} \tag{E.226}
$$

We now move to the second piece of (E.224),

$$
\begin{aligned}
\text{(II)} &= 2q^r{}_{[a}q^s{}_{b]}q^t{}_c q^p{}_s\,(\nabla_r q^q{}_t)(\nabla_p u_q) \\
&= 2q^r{}_{[a}q^p{}_{b]}q^t{}_c\,\left(\nabla_r(\delta^q_t + n^q n_t)\right)(\nabla_p u_q) \quad \text{using } q^s{}_b q^p{}_s = q^t{}_c \\
&= 2q^r{}_{[a}q^p{}_{b]}q^t{}_c\,(\nabla_r n^q n_t))\,(\nabla_p u_q) \\
&= 2q^r{}_{[a}q^p{}_{b]}q^t{}_c(\nabla_r n_t)n^q(\nabla_p u_q),
\end{aligned} \tag{E.227}
$$

We will use that $u_a n^a = 0$ implies $u_a\nabla_b n^a = -n^a\nabla_b u_a$,

$$
\begin{aligned}
&= 2q^r{}_{[a}q^s{}_{b]}q^t{}_c(\nabla_r n_t)(\nabla_p n^q)u_q \\
&= 2q^r{}_{[a}q^s{}_{b]}q^t{}_c(\nabla_r n_t)(\nabla_p n^q)u_q \\
&= -2K_{c[a}K_{b]q}u^q \\
&= -(K_{ca}K_b{}^d - K_{cb}K_a{}^d)u_d.
\end{aligned} \tag{E.228}
$$

$$
\begin{aligned}
(\mathrm{III}) &= 2q^r{}_{[a}q^s{}_{b]}q^t{}_c q^p{}_s q^q{}_t\,(\nabla_r\nabla_p u_q)\\
&= 2q^r{}_{[a}q^s{}_{b]}q^q{}_c\,(\nabla_r\nabla_p u_q)\\
&= 2q^{[r}{}_a q^{s]}{}_b q^t{}_c(\nabla_{[r}\nabla_{p]}u_q)\\
&= 2q^r{}_a q^p{}_b q^q{}_c \mathrm{R}_{rpq}{}^s u_s\\
&= 2q^r{}_a q^p{}_b q^q{}_c q^d{}_s \mathrm{R}_{rpq}{}^s u_d. \qquad \text{(E.229)}
\end{aligned}
$$

Putting it all together

$$
\mathrm{R}_{abc}{}^d u_d = (0) - \Big((K_{ca}K_b{}^d - K_{cb}K_a{}^d)u_d\Big) + (q^r{}_a q^p{}_b q^q{}_c q^d{}_s \mathrm{R}_{rpq}{}^s u_d) \qquad \text{(E.230)}
$$

As u_d is arbitrary, this yields Gauss' equation (E.9).

□

Gauss-Codazi

Details Hamiltonian.

The determinate of the metric can be written $e = N\sqrt{q}$

$$
\begin{aligned}
ds^2 &= N^2dt^2 - q_{ab}(dx^a + N^a dt)(dx^b + N^b dt)\\
&= (N^2 - q_{ab}N^aN^b)dt^2 - q_{ab}N^b dx^a dt - q_{ab}N^a dt dx^b - q_{ab}dx^a dx^b \qquad \text{(E.231)}
\end{aligned}
$$

$$
g_{ab} = \begin{pmatrix} N^2 - N_aN^a & -q_{a1}N^a & -q_{a2}N^a & -q_{a3}N^a \\ -q_{1b}N^b & & & \\ -q_{2b}N^b & & -q_{ab} & \\ -q_{3b}N^b & & & \end{pmatrix} \qquad \text{(E.232)}
$$

as can be seen from

$$
\begin{aligned}
ds^2 &= (dt, dx^1, dx^2, dx^3) \begin{pmatrix} N^2 - N_a N^a & -q_{a1} N^a & -q_{a2} N^a & -q_{a3} N^a \\ -q_{1b} N^b & & & \\ -q_{2b} N^b & & -q_{ab} & \\ -q_{3b} N^b & & & \end{pmatrix} \begin{pmatrix} dt \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \\
&= (dt, dx^1, dx^2, dx^3) \begin{pmatrix} (N^2 - N_a N^a) dt - q_{ab} N^a dx^b \\ -q_{1b} N^b dt - q_{1b} dx^b \\ -q_{2b} N^a dt - q_{2b} dx^b \\ -q_{3b} N^b dt - q_{3b} dx^b \end{pmatrix} \\
&= (N^2 - N_a N^a) dt^2 - q_{ab} N^a dx^b dt - dx^b q_{ab} N^b dt - dx^a q_{ab} dx^b \qquad \text{(E.233)}
\end{aligned}
$$

We now evaluate the determinant (E.232)

$$
\begin{aligned}
\det g = (N^2 - N_a N^a) \det \begin{pmatrix} & & \\ & -q_{ab} & \\ & & \end{pmatrix} + q_{a1} N^a \det \begin{pmatrix} -q_{1b} N^b & -q_{12} & -q_{13} \\ -q_{2b} N^b & -q_{22} & -q_{23} \\ -q_{3b} N^b & -q_{32} & -q_{33} \end{pmatrix} + \\
-q_{a2} N^a \det \begin{pmatrix} -q_{1b} N^b & -q_{11} & -q_{13} \\ -q_{2b} N^b & -q_{21} & -q_{23} \\ -q_{3b} N^b & -q_{31} & -q_{33} \end{pmatrix} + q_{a3} N^a \det \begin{pmatrix} -q_{1b} N^b & -q_{11} & -q_{12} \\ -q_{2b} N^b & -q_{21} & -q_{22} \\ -q_{3b} N^b & -q_{31} & -q_{32} \end{pmatrix} \qquad \text{(E.234)}
\end{aligned}
$$

Consider the determinate in the second term on the RHS of (E.234)

$$
\det \begin{pmatrix} -q_{1b} N^b & -q_{12} & -q_{13} \\ -q_{2b} N^b & -q_{22} & -q_{23} \\ -q_{3b} N^b & -q_{32} & -q_{33} \end{pmatrix} = - \det \begin{pmatrix} q_{11} N^1 + q_{12} N^2 + q_{13} N^3 & q_{12} & q_{13} \\ q_{21} N^1 + q_{22} N^2 + q_{23} N^3 & q_{22} & q_{23} \\ q_{31} N^1 + q_{32} N^2 + q_{33} N^3 & q_{32} & q_{33} \end{pmatrix} \qquad \text{(E.235)}
$$

We use two properties of determinates () and () to write (E.235) as

$$
-N^1 \det \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} - N^2 \det \begin{pmatrix} q_{12} & q_{12} & q_{13} \\ q_{22} & q_{22} & q_{23} \\ q_{32} & q_{32} & q_{33} \end{pmatrix} - N^3 \det \begin{pmatrix} q_{13} & q_{12} & q_{13} \\ q_{23} & q_{22} & q_{23} \\ q_{33} & q_{32} & q_{33} \end{pmatrix} \qquad \text{(E.236)}
$$

since the determinant of a matrix with a repeated column is zero (see appendex A) the second and third terms of (E.236) vanish and we are left with

$$
-N^1 \det \begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix} = -N^1 \det q \qquad \text{(E.237)}
$$

The other terms in (E.236) are treated analogously and putting it all together (E.234) becomes

$$\begin{aligned}\det g &= -(N^2 - N_a N^a)\det q - q_{a1}N^a N^1 \det q - q_{a2}N^a N^2 \det q - q_{a3}N^a N^3 \det q \\ &= (-N^2 + N_a N^a - q_{ab}N^a N^b)\det q \qquad (E.238)\\ &= -N^2 \det q \qquad (E.239)\end{aligned}$$

Therefore

$$e = det(e_{ab}) = \sqrt{-g} = N\sqrt{q}. \qquad (E.240)$$

Details (I1.1) Gauss gauge

Derive Eq's above

$$\mathcal{G}^i(x) = \mathcal{D}_a E^{ai}(x) = \partial_a E^{ai}(x) + \epsilon^i_{jk} A^j_a(x) E^{ak}(x) \qquad (E.241)$$

$$\{E^a_i(x), G(y)\} = \int d^3y \{E^a_i(x), \Lambda^j(y)\mathcal{G}^j\} = \qquad (E.242)$$

$$\int d^3y \{A^i_a(x), \Lambda^j(y)\mathcal{G}^j\} = \int d^3y \{A^i_a(x), \Lambda^j(y)\mathcal{D}_a E^{bj}(y)\} = \\ \int d^3z \int d^3y \frac{\delta A^i_a(x)}{\delta A^k_c(z)} \Lambda^j(y) \frac{\delta[\partial_a E^{aj}(y) + \epsilon^i_{mn} A^m_a(y) E^{an}(z)]}{\delta E^c_k(z)} \qquad (E.243)$$

$$\int d^3x N^i(x) \frac{\delta \mathcal{D}_a E^a_i(x)}{\delta A^k_c(z)} = \int d^3x N^i(x) \epsilon^i_{jk} \frac{\delta A^j_a(x) E^{ak}(x)}{\delta A^k_c(z)} \\ = \int d^3x N^i(x) \epsilon^i_{jk} \delta^3(x-z) \delta^c_a E^{ak}(x) = \qquad (E.244)$$

$$\int d^3y M^j(y) \frac{\delta \mathcal{D}_a E^a_i(y)}{\delta E^c_k(z)} = \int d^3y M^j(y) \frac{\delta[\partial_a E^a_j(y) + \epsilon^i_{jk} A^j_a(y) E^{ak}(y)]}{\delta E^c_k(z)} \\ = -\int d^3y [\partial_c M^j(y) - \epsilon^i_{jk} M^j(y) A^k_c(y)] \delta^3(y-z) \delta^a_c = -[\partial_a M^j(z) + \epsilon^i_{jk} M^j(z) A^k_c(z)] = \qquad (E.245)$$

Details (I1.1) Spacial diff

$$V(\vec{N}) = \int_\Sigma d^3 N^a E^{bi} F^i_{ab} \tag{E.246}$$

Details (I1.1) Ham

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b \epsilon_{ijk} \tag{E.247}$$

Obviously

$$\{A^i_a, F^j_{bc}\} = 0 \tag{E.248}$$

as F^i_{ab} depends only on the connection field. So that the Poisson bracket is simple:

$$\{A^i_a, H(N)\} = \{A^i_a,\ \epsilon^{i'j'k'} F^{i'}_{bc} \tilde{E}^b_{j'} \tilde{E}^c_{i'}\} = \epsilon^{i'j'k'} F^{i'}_{bc} \{A^i_a, \tilde{E}^b_{j'} \tilde{E}^c_{i'}\} \tag{E.249}$$

Using the "product rule" for Poisson brackets this is

$$\begin{aligned} \{A^i_a, H(N)\} &= F^{i'}_{bc} \epsilon^{i'j'k'} \left(\{A^i_a, \tilde{E}^b_{j'}\} \tilde{E}^c_{i'} + \{A^i_a, \tilde{E}^c_{i'}\} \tilde{E}^b_{j'} \right) \\ &= 2 F^{i'}_{bc} \epsilon^{i'j'k'} \{A^i_a, \tilde{E}^b_{j'}\} \tilde{E}^c_{i'} \\ &= 2 \epsilon^{ijk} \tilde{E}^c_j F^k_{bc} \end{aligned} \tag{E.250}$$

where in the in the second step we used that the field-strength tensor is anti-symmetric in its spacetime indices. We have proved (E.186).

Details (I1.1) Ham

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b \epsilon_{ijk} \tag{E.251}$$

Obviously

$$\{A^i_a, F^j_{bc}\} = 0 \tag{E.252}$$

as F^i_{ab} depends only on the connection field. So that the Poisson bracket is simple:

$$\{A^i_a, H(N)\} = \{A^i_a,\ \epsilon^{i'j'k'} F^{i'}_{bc} \tilde{E}^b_{j'} \tilde{E}^c_{i'}\} = \epsilon^{i'j'k'} F^{i'}_{bc} \{A^i_a, \tilde{E}^b_{j'} \tilde{E}^c_{i'}\} \tag{E.253}$$

Using the "product rule" for Poisson brackets this is

$$\begin{aligned}
\{A_a^i, H(N)\} &= F_{bc}^{i'}\epsilon^{i'j'k'}\left(\{A_a^i, \tilde{E}_{j'}^b\}\tilde{E}_{i'}^c + \{A_a^i, \tilde{E}_{i'}^c\}\tilde{E}_{j'}^b\right) \\
&= 2F_{bc}^{i'}\epsilon^{i'j'k'}\{A_a^i, \tilde{E}_{j'}^b\}\tilde{E}_{i'}^c \\
&= 2\epsilon^{ijk}\tilde{E}_j^c F_{bc}^k
\end{aligned} \tag{E.254}$$

where in the in the second step we used that the field-strength tensor is anti-symmetric in its spacetime indices. We have proved (E.186).

Details (L2.2)

$$\{G(N_i), G(M_j)\} = \int\int d^3x d^3y N^i(x)M^j(y)\{\mathcal{G}^i(x), \mathcal{G}^j(y)\} =$$

$$= \int d^3z \int d^3x \int d^3y N^i(x)M^j(y)\left[\frac{\delta\mathcal{G}^i(x)}{\delta A_c^k(z)}\frac{\delta\mathcal{G}^j(y)}{\delta E_k^c(z)} - \frac{\delta\mathcal{G}^i(x)}{\delta E_k^c(z)}\frac{\delta\mathcal{G}^j(y)}{\delta A_c^k(z)}\right] \tag{E.255}$$

$$\int d^3x N^i(x)\frac{\delta\mathcal{G}^i(x)}{\delta A_c^k(z)} = N^i(z)\epsilon_{jk}^i E^{ck}(z) \tag{E.256}$$

$$\int d^3y M^j(y)\frac{\delta\mathcal{G}^j(y)}{\delta E_k^c(z)} = -\mathcal{D}_a M_k(z) \tag{E.257}$$

$$\{G(N_i), G(M_j)\} = \int d^3z\left[N^i(z)\epsilon_{jk}^i E^{ck}(z)\mathcal{D}_a M_k(z) - M^i(z)\epsilon_{jk}^i E^{ck}(z)\mathcal{D}_a N_k(z)\right] \tag{E.258}$$

Jacobi identity

$$\epsilon_{ijm}\epsilon_{mkn} + \epsilon_{jmk}\epsilon_{min} + \epsilon_{kim}\epsilon_{mjn} = 0 \tag{E.259}$$

$$\{G(N_i), G(M_j)\} = -G([N, M]) \tag{E.260}$$

Details

$$\{C_N, C_M\} = \{C_N, (\mathcal{D}_a A_b^i - \mathcal{D}_b A_a^i)\tilde{E}_j^a\tilde{E}_k^b\}\epsilon^{ijk} \tag{E.261}$$

$$\{C_N, C_M\} = \int d^3x (N\partial_a M - M\partial_a N)(\tilde{E}^{[a}\tilde{E}^{b]}[\tilde{E}^c, F_{ab}]$$
$$\int d^3x (N\partial_a M - M\partial_a N)(\tilde{E}^c \tilde{E} \tag{E.262}$$