

Appendix G

The Holst Action and Ashtekar-Barbero Real Variables

G.1 The Holst Action

$$S(E, \omega) = \frac{1}{16\pi G} \int d^4x |E| E_I^a E_J^b P^{IJ}{}_{KL} \Omega_{ab}{}^{KL}(\omega) \quad (\text{G.1})$$

where

$$P^{IJ}{}_{KL} = \delta_K^{[I} \delta_L^{J]} - \frac{1}{2\beta} \epsilon^{IJ}{}_{KL}. \quad (\text{G.2})$$

G.1.1 3+1 Decomposition of the Holst Action

We relate the curvature of the Ashtekar-Barbero connection to the curvature of the spin connection,

$$\begin{aligned} \mathcal{F}_{ab}^i &= 2\partial_{[a}(\Gamma_{b]}^i + \beta K_{b]}^i) - \epsilon^i{}_{jk}(\Gamma_a^j + \beta K_a^j)(\Gamma_b^k + \beta K_b^k) \\ &= 2\partial_{[a}(\Gamma_{b]}^i + \beta K_{b]}^i) - \frac{1}{2}\epsilon^i{}_{jk} \left((\Gamma_a^j + \beta K_a^j)(\Gamma_b^k + \beta K_b^k) - a \leftrightarrow b \right) \\ &= 2\partial_{[a}\Gamma_{b]}^i - \epsilon^i{}_{jk}\Gamma_{[a}^j\Gamma_{b]}^k - \beta \left(2\partial_{[a}K_{b]}^i - \epsilon^i{}_{jk}(\Gamma_{[a}^j K_{b]}^k + K_{[a}^j \Gamma_{b]}^k) \right) - \beta^2 \epsilon^i{}_{jk} K_{[a}^j K_{b]}^k \\ &= F_{ab}^i - \beta \left(2\partial_{[a}K_{b]}^i - \epsilon^i{}_{jk}(\Gamma_{[a}^j K_{b]}^k - K_{[a}^j \Gamma_{b]}^k) \right) - \beta^2 \epsilon^i{}_{jk} K_a^j K_b^k \\ &= F_{ab}^i - \beta(2\partial_{[a}K_{b]}^i - 2\epsilon^i{}_{jk}\Gamma_{[a}^j K_{b]}^k - \beta^2 \epsilon^i{}_{jk} K_a^j K_b^k) \\ &= F_{ab}^i - 2\beta \nabla_{[a} K_{b]}^i - \beta^2 \epsilon^i{}_{jk} K_a^j K_b^k \end{aligned} \quad (\text{G.3})$$

G.1.2 The Diffeomorphism Constraint

For the diffeomorphism constraint we have (we introduce here a negative sign so that the constraint appears with a positive sign in the Hamiltonian, recall the basic form $\mathcal{L} = p\dot{q} - \mathcal{H}$)

$$\begin{aligned}
N^a C_a &= -\beta n_I N^a \left(\frac{\sqrt{q} E_J^b}{8\pi\beta G} \right) P^{IJ} F_{ab}{}^{KL} \\
&= \beta N^a \tilde{E}_j^b P^{0j}{}_{kl} F_{ab}{}^{kl} \\
&= \beta N^a \tilde{E}_j^b \left(F_{ab}{}^{0j} - \frac{1}{2\beta} \epsilon^{0j}{}_{kl} F_{ab}{}^{kl} \right) \\
&= 2\beta N^a \tilde{E}_j^b \left(\partial_{[a} \omega_{b]}{}^{0j} + \omega_{[a}{}^{0k} \omega_{b]k}{}^j + \frac{1}{2\beta} \epsilon^j{}_{kl} (\partial_{[a} \omega_{b]}{}^{kl} + \omega_{[a}{}^{kL} \omega_{b]L}{}^l) \right) \\
&= 2\beta N^a \tilde{E}_j^b \left(\partial_{[a} K_{b]}^j + K_{[a}^k \omega_{b]k}{}^j + \frac{1}{2\beta} (2\partial_{[a} \Gamma_{b]}{}^{kl} + \epsilon^j{}_{kl} (\omega_{[a}{}^{k0} \omega_{b]0}{}^l + \omega_{[a}{}^{km} \omega_{b]m}{}^l)) \right) \\
&= 2N^a \tilde{E}_j^b \left(\partial_{[a} A_{b]}^j - \beta \epsilon^j{}_{mk} \Gamma_{[a}^m K_{b]}^k + \frac{1}{2} \epsilon^j{}_{kl} (K_{[a}^k K_{b]}^l + \omega_{[a}{}^{km} \omega_{b]m}{}^l) \right) \tag{G.4}
\end{aligned}$$

where we used $\epsilon^{0j}{}_{kl} = -\epsilon^j{}_{kl}$, $K_a^i = \omega_a{}^{0i}$ and $\omega_{ak}{}^j = \epsilon^j{}_{mk} \Gamma_a^m$. Also that $\omega_{[a}{}^{k0} \omega_{b]0}{}^l = \omega_{[a}{}^{0k} \omega_{b]}{}^{0l}$.

The term $\omega_{[a}{}^{km} \omega_{b]m}{}^l$ becomes,

$$\begin{aligned}
\epsilon^j{}_{kl} \omega_a{}^{km} \omega_{bm}{}^l &= \epsilon^j{}_{kl} (\epsilon^{km}{}_n \Gamma_a^n) (\epsilon_m{}^l{}_p \Gamma_b^p) \\
&= \epsilon^j{}_{kl} \epsilon_n{}^{km} \epsilon^l{}_{pm} \Gamma_a^n \Gamma_b^p \\
&= \epsilon^j{}_{kl} (\delta_n^l \delta_p^k - \eta_{mp} \eta^{kl}) \Gamma_a^n \Gamma_b^p \\
&= \epsilon^j{}_{kl} \Gamma_a^l \Gamma_b^k \\
&= -\epsilon^j{}_{kl} \Gamma_{[a}^k \Gamma_{b]}^l \tag{G.5}
\end{aligned}$$

Continuing with the calculation for the spatial diffeomorphism constraint,

$$\begin{aligned}
N^a C_a &= N^a \tilde{E}_j^b \left(2\partial_{[a} A_{b]}^j - \beta \epsilon_{kl}^j \Gamma_{[a}^k K_{b]}^l - \frac{1}{2} \epsilon_{kl}^j (\Gamma_{[a}^k \Gamma_{b]}^l - K_{[a}^k K_{b]}^l) \right) \\
&= N^a \tilde{E}_j^b \left(2\partial_{[a} A_{b]}^j - \beta \frac{1}{2} \epsilon_{kl}^j (\Gamma_{[a}^k K_{b]}^l + K_{[a}^k \Gamma_{b]}^l) - \frac{1}{2} \epsilon_{kl}^j (\Gamma_{[a}^k \Gamma_{b]}^l - K_{[a}^k K_{b]}^l) \right) \\
&= N^a \tilde{E}_j^b \left(2\partial_{[a} A_{b]}^i - \frac{1}{2} \epsilon_{kl}^j (\Gamma_{[a}^k \Gamma_{b]}^l + \beta \Gamma_{[a}^k K_{b]}^l + \beta K_{[a}^k \Gamma_{b]}^l - K_{[a}^k K_{b]}^l) \right) \\
&= N^a \tilde{E}_j^b \left(2\partial_{[a} A_{b]}^i - \frac{1}{2} \epsilon_{kl}^j (\Gamma_{[a}^k \Gamma_{b]}^l + \beta \Gamma_{[a}^k K_{b]}^l + \beta K_{[a}^k \Gamma_{b]}^l + \beta^2 K_{[a}^k K_{b]}^l) \right. \\
&\quad \left. + (1 + \beta^2) \epsilon_{kl}^j K_a^k K_b^l \right) \\
&= N^a \tilde{E}_j^b \left(2\partial_{[a} A_{b]}^i - \epsilon_{kl}^j (\Gamma_a^k + \beta K_a^k) (\Gamma_b^l + \beta K_b^l) + (1 + \beta^2) \epsilon_{kl}^j K_a^k K_b^l \right) \\
&= N^a \tilde{E}_j^b \left(\mathcal{F}_{ab}^j + (1 + \beta^2) \epsilon_{kl}^j K_a^k K_b^l \right) \tag{G.6}
\end{aligned}$$

where in the second line we used $\epsilon_{kl}^j \Gamma_{[a}^k K_{b]}^l = -\epsilon_{kl}^j \Gamma_{[b}^k K_{a]}^l = -\epsilon_{kl}^j K_{[a}^k \Gamma_{b]}^l = \epsilon_{kl}^j K_a^k \Gamma_b^l$.

G.1.3 The Hamiltonian Constraint

$$\begin{aligned}
C &= -4\pi G \frac{N}{\sqrt{q}} \left(\frac{\sqrt{q} E_I^a}{8\pi\beta G} \right) \left(\frac{\sqrt{q} E_J^b}{8\pi\beta G} \right) P^{IJ}{}_{KL} F_{ab}^{KL} \\
&= -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} P^{ij}{}_{KL} F_{ab}{}^{KL} \\
&= -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \left(F_{ab}^{ij} - \frac{1}{2\beta} \epsilon^{ij}{}_{KL} F_{ab}{}^{KL} \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \left(F_{ab}^{ij} - \frac{1}{2\beta} (\epsilon^{ij}{}_{k0} F_{ab}{}^{k0} + \epsilon^{ij}{}_{0k} F_{ab}{}^{0k}) \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \left(2\partial_{[a} \omega_{b]}^{ij} + 2\omega_{[a}{}^{iK} \omega_{b]}{}^{Lj} \eta_{KL} + \frac{2}{\beta} \epsilon^{ij}{}_{k} (\partial_{[a} \omega_{b]}{}^{k0} + \omega_{[a}{}^{kl} \omega_{b]}{}^{l0}) \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \left(F_{ab}^{ij} + 2K_{[a}^i K_{b]}^j - \frac{2}{\beta} \epsilon^{ij}{}_{k} (\partial_{[a} K_{b]}^k + \omega_{[a}{}^{kl} K_{b]}^l) \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}{}_{k} \left(F_{ab}^k + \epsilon_{mn}^k K_{[a}^m K_{b]}^n - \frac{2}{\beta} \nabla_{[a} K_{b]}^k \right) \tag{G.7}
\end{aligned}$$

Substituting in (G.3) in this we obtain,

$$\begin{aligned}
C &= -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}_k \left(\mathcal{F}_{ab}^k + (1 + \beta^2) \epsilon^k_{mn} K_a^m K_b^n + 2 \frac{\beta^2 + 1}{\beta} \nabla_{[a} K_{b]}^k \right) \\
&= -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}_k \left(\mathcal{F}_{ab}^k + (1 + \beta^2) \epsilon^k_{mn} K_a^m K_b^n \right) - 4\pi G\beta^2 \left(\frac{\sqrt{q} E_i^a}{8\pi\beta G} \right) \frac{\tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}_k 2 \frac{\beta^2 + 1}{\beta} \nabla_{[a} K_{b]}^k \\
&= -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}_k \left(\mathcal{F}_{ab}^k + (1 + \beta^2) \epsilon^k_{mn} K_a^m K_b^n \right) - (\beta^2 + 1) \epsilon^{ij}_k E_i^a \tilde{E}_j^b \nabla_{[a} K_{b]}^k \quad (\text{G.8})
\end{aligned}$$

In a constrained Hamiltonian system, a dynamical quantity is called first class constraint if its Poisson bracket with all other constraints vanishes on the constraint surface (the surface implicitly defined by the simultaneous vanishing of all the constraints). A second class constraint is one that is not first class.

G.1.4 Addition Constraints

The Hamiltonian is then a sum over constraints,

$$\mathcal{H} = [A_a^i, \tilde{E}_j^b] = \int d^4x \left(\Lambda^i G_i + (1 + \beta) \omega_t^{0j} S_j + NC + N^a C_a \right) \quad (\text{G.9})$$

$$S_j = -\epsilon_{jm}{}^n K_b^m \tilde{E}_n^b$$

$$\begin{aligned}
\Gamma_{a(i} E_{j)}^b &= \delta_{ij} \frac{1}{2} \frac{1}{\sqrt{\det q}} \epsilon^{abc} e_{al} \partial_b e_c^l - \frac{\epsilon^{abc}}{2\sqrt{\det q}} (e_{ai} \partial_b e_{cj} + e_{aj} \partial_b e_{ci}) \\
&= -\frac{\epsilon^{abc}}{2\sqrt{\det q}} (e_{ai} \partial_b e_{cj} + e_{aj} \partial_b e_{ci} - \delta_{ij} e_{al} \partial_b e_c^l) \quad (\text{G.10})
\end{aligned}$$

Recall $e_I^a = E_I^a + n^a n_I$ and in the temporal gauge $n_i = 0$ so

$$\Gamma_{a(i} E_{j)}^b = -\frac{\epsilon^{abc}}{2\sqrt{\det q}} (E_{ai} \partial_b E_{cj} + E_{aj} \partial_b E_{ci} - \delta_{ij} E_{al} \partial_b E_c^l) \quad (\text{G.11})$$

Obviously

$$\mathcal{D}_b \tilde{E}_i^b = \partial_b \tilde{E}_i^b - \epsilon_{ij}{}^k (\Gamma_b^j + \beta K_b^j) \tilde{E}_k^b = 0 \quad \epsilon_{ij}{}^k K_b^j \tilde{E}_k^b = 0$$

imply

$$\partial_b \tilde{E}_i^b - \epsilon_{ij}{}^k \Gamma_b^j \tilde{E}_k^b = 0$$

Contracting this the totally anti-symmetric tensor

$$\begin{aligned}
0 = \epsilon^i{}_{jk}(\partial_b \tilde{E}_i^b - \epsilon_{ij'}{}^{k'} \Gamma_b^{j'} \tilde{E}_{k'}) &= \epsilon^i{}_{jk} \partial_b \tilde{E}_i^b - \epsilon^i{}_{jk} \epsilon_{ij'}{}^{k'} \Gamma_b^{j'} \tilde{E}_{k'}^b \\
&= \epsilon^i{}_{jk} \partial_b \tilde{E}_i^b - (\delta_{j'}^j \delta_{k'}^k - \delta_{j'}^k \delta_{k'}^j) \Gamma_b^{j'} \tilde{E}_{k'}^b \\
&= \epsilon^i{}_{jk} \partial_b \tilde{E}_i^b - (\Gamma_b^j \tilde{E}_k^b - \Gamma_b^k \tilde{E}_j^b) \tag{G.12}
\end{aligned}$$

$$|\det(\tilde{E})| = |\det(E)|^2$$

$$E_i^a = |\det(\tilde{E})|^{-1/2} \tilde{E}_i^a$$

$$\begin{aligned}
\Gamma_{a[i} E_{j]}^a &= \frac{1}{2} |\det(E)|^{-1} \epsilon_{ijk} \partial_a (|\det(E)| E_k^a) \\
&= \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + \frac{E_k^a}{|\det(E)|} \partial_a |\det(E)|) \\
&= \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^l \partial_a E_l^c) \tag{G.13}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{ai} E_j^a &= \Gamma_{a(i} E_{j)}^a + \Gamma_{a[i} E_{j]}^a \\
&= -\frac{\epsilon^{abc}}{2\sqrt{\det q}} (E_{ai} \partial_b E_{cj} + E_{aj} \partial_b E_{ci} - \delta_{ij} E_{an} \partial_b E_c^n) + \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^l \partial_a E_l^c) \\
&= -\frac{\epsilon^{klm}}{2} E_k^a E_l^b E_m^c (E_{ai} \partial_b E_{cj} + E_{aj} \partial_b E_{ci} - \delta_{ij} E_{an} \partial_b E_c^n) + \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^n \partial_a E_n^c) \\
&= -\frac{1}{2} (\epsilon^{ilm} E_l^b E_m^c \partial_b E_{cj} + \epsilon^{jlm} E_l^b E_m^c \partial_b E_{ci} - \epsilon^{klm} \delta_{ij} E_l^b E_m^c \partial_b E_c^k) + \frac{1}{2} \epsilon_{ijk} (\partial_a E_k^a + E_k^a E_c^n \partial_a E_n^c) \\
&= \frac{1}{2} E_l^b (\epsilon^{ilk} E_{aj} \partial_b E_k^a + \epsilon^{jlk} E_{ai} \partial_b E_k^a + \epsilon^{mlk} \delta_{ij} E_k^c \partial_b E_c^m) + \frac{1}{2} \epsilon_{ijl} (\partial_b E_l^b + E_l^b E_c^n \partial_b E_n^c) \\
&= \\
&= \\
&= \frac{1}{2} \epsilon^{ikl} E_l^b (E_{a,b}^k - E_{b,a}^k) E_j^a + \frac{1}{2} \epsilon^{ikl} E_l^b E_k^c E_{c,b}^j \\
&= \frac{1}{2} \epsilon^{ikl} E_l^b (E_{a,b}^k - E_{b,a}^k + E_k^c E_a^m E_{c,b}^m) E_j^a \tag{G.14}
\end{aligned}$$

G.1.5 Final Total Hamiltonian

$$\mathcal{H} = [A_a^i, \tilde{E}_j^b] = \int d^4x (\Lambda^i G_i + NC + N^a C_a) \tag{G.15}$$

$$C_a = \tilde{E}_j^b \mathcal{F}_{ab}^j \tag{G.16}$$

$$C = -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \left(\epsilon^{ij} \mathcal{F}_{ab}^k + 2(1 + \beta^2) K_{[a}^i K_{b]}^j \right) \quad (\text{G.17})$$

or

$$C = -4\pi G\beta^2 \left(\frac{\epsilon^{ij} \mathcal{F}_{ab}^k \tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} + 2(1 + \beta^2) \frac{\tilde{E}_{[i}^a \tilde{E}_{j]}^b}{\sqrt{q}} (A_a^i - \Gamma_a^i) (A_b^j - \Gamma_b^j) \right) \quad (\text{G.18})$$

G.2 Inclusion of Matter

G.2.1 Yang-Mills

G.2.2 Klein-Gordan - Scalar Matter

G.2.3 Fermionic Matter

G.3 Bibliographical notes

In this chapter I have relied on the following references:

Eric Poisson

Kenneth smith, Dynamic Singularity Excision in Numerical Relativity.

Robert Bartnik, Jim Isenberg, *The Constraint Equations??* or should this be referenced in

G.4 Worked Exercises and Details

Details (I1.1) Gauss gauge

Derive Eq's above

$$\mathcal{G}^i(x) = \mathcal{D}_a E^{ai}(x) = \partial_a E^{ai}(x) + \epsilon_{jk}^i A_a^j(x) E^{ak}(x) \quad (\text{G.19})$$

$$\{E_i^a(x), G(y)\} = \int d^3y \{E_i^a(x), \Lambda^j(y) \mathcal{G}^j\} = \quad (\text{G.20})$$

$$\int d^3y \{A_a^i(x), \Lambda^j(y) \mathcal{G}^j\} = \int d^3y \{A_a^i(x), \Lambda^j(y) \mathcal{D}_a E^{bj}(y)\} =$$

$$\int d^3z \int d^3y \frac{\delta A_a^i(x)}{\delta A_c^k(z)} \Lambda^j(y) \frac{\delta [\partial_a E^{aj}(y) + \epsilon_{mn}^i A_a^m(y) E^{an}(z)]}{\delta E_k^c(z)} \quad (\text{G.21})$$

$$\int d^3x N^i(x) \frac{\delta \mathcal{D}_a E_i^a(x)}{\delta A_c^k(z)} = \int d^3x N^i(x) \epsilon_{jk}^i \frac{\delta A_a^j(x) E^{ak}(x)}{\delta A_c^k(z)}$$

$$= \int d^3x N^i(x) \epsilon_{jk}^i \delta^3(x-z) \delta_a^c E^{ak}(x) = \quad (\text{G.22})$$

$$\int d^3y M^j(y) \frac{\delta \mathcal{D}_a E_i^a(y)}{\delta E_k^c(z)} = \int d^3y M^j(y) \frac{\delta [\partial_a E_j^a(y) + \epsilon_{jk}^i A_a^j(y) E^{ak}(y)]}{\delta E_k^c(z)}$$

$$= - \int d^3y [\partial_c M^j(y) - \epsilon_{jk}^i M^j(y) A_c^k(y)] \delta^3(y-z) \delta_c^a = - [\partial_a M^j(z) + \epsilon_{jk}^i M^j(z) A_c^k(z)] = \quad (\text{G.23})$$

Details (I1.1) Spacial diff

$$V(\vec{N}) = \int_{\Sigma} d^3N^a E^{bi} F_{ab}^i \quad (\text{G.24})$$

Details (I1.1) Ham

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + A_a^j A_b^k \epsilon_{ijk} \quad (\text{G.25})$$

Obviously

$$\{A_a^i, F_{bc}^j\} = 0 \quad (\text{G.26})$$

as F_{ab}^i depends only on the connection field. So that the Poisson bracket is simple:

$$\{A_a^i, H(N)\} = \{A_a^i, \epsilon^{i'j'k'} F_{bc}^{i'} \tilde{E}_{j'}^b \tilde{E}_{k'}^c\} = \epsilon^{i'j'k'} F_{bc}^{i'} \{A_a^i, \tilde{E}_{j'}^b \tilde{E}_{k'}^c\} \quad (\text{G.27})$$

Using the “product rule” for Poisson brackets this is

$$\begin{aligned}
\{A_a^i, H(N)\} &= F_{bc}^{i'j'k'} \left(\{A_a^i, \tilde{E}_{j'}^b\} \tilde{E}_{i'}^c + \{A_a^i, \tilde{E}_{i'}^c\} \tilde{E}_{j'}^b \right) \\
&= 2F_{bc}^{i'j'k'} \{A_a^i, \tilde{E}_{j'}^b\} \tilde{E}_{i'}^c \\
&= 2\epsilon^{ijk} \tilde{E}_j^c F_{bc}^k
\end{aligned} \tag{G.28}$$

where in the in the second step we used that the field-strength tensor is anti-symmetric in its spacetime indices. We have proved (F.46).

Details (I1.1) Ham

$$F_{ab}^i = \partial_a A_b^i - \partial_b A_a^i + A_a^j A_b^k \epsilon_{ijk} \tag{G.29}$$

Obviously

$$\{A_a^i, F_{bc}^j\} = 0 \tag{G.30}$$

as F_{ab}^i depends only on the connection field. So that the Poisson bracket is simple:

$$\{A_a^i, H(N)\} = \{A_a^i, \epsilon^{i'j'k'} F_{bc}^{i'} \tilde{E}_{j'}^b \tilde{E}_{i'}^c\} = \epsilon^{i'j'k'} F_{bc}^{i'} \{A_a^i, \tilde{E}_{j'}^b \tilde{E}_{i'}^c\} \tag{G.31}$$

Using the “product rule” for Poisson brackets this is

$$\begin{aligned}
\{A_a^i, H(N)\} &= F_{bc}^{i'j'k'} \left(\{A_a^i, \tilde{E}_{j'}^b\} \tilde{E}_{i'}^c + \{A_a^i, \tilde{E}_{i'}^c\} \tilde{E}_{j'}^b \right) \\
&= 2F_{bc}^{i'j'k'} \{A_a^i, \tilde{E}_{j'}^b\} \tilde{E}_{i'}^c \\
&= 2\epsilon^{ijk} \tilde{E}_j^c F_{bc}^k
\end{aligned} \tag{G.32}$$

where in the in the second step we used that the field-strength tensor is anti-symmetric in its spacetime indices. We have proved (F.46).

Details (L2.2)

$$\begin{aligned}
\{G(N_i), G(M_j)\} &= \int \int d^3x d^3y N^i(x) M^j(y) \{\mathcal{G}^i(x), \mathcal{G}^j(y)\} = \\
&= \int d^3z \int d^3x \int d^3y N^i(x) M^j(y) \left[\frac{\delta \mathcal{G}^i(x)}{\delta A_c^k(z)} \frac{\delta \mathcal{G}^j(y)}{\delta E_k^c(z)} - \frac{\delta \mathcal{G}^i(x)}{\delta E_k^c(z)} \frac{\delta \mathcal{G}^j(y)}{\delta A_c^k(z)} \right]
\end{aligned} \tag{G.33}$$

$$\int d^3x N^i(x) \frac{\delta \mathcal{G}^i(x)}{\delta A_c^k(z)} = N^i(z) \epsilon_{jk}^i E^{ck}(z) \quad (\text{G.34})$$

$$\int d^3y M^j(y) \frac{\delta \mathcal{G}^j(y)}{\delta E_k^c(z)} = -\mathcal{D}_a M_k(z) \quad (\text{G.35})$$

$$\{G(N_i), G(M_j)\} = \int d^3z \left[N^i(z) \epsilon_{jk}^i E^{ck}(z) \mathcal{D}_a M_k(z) - M^i(z) \epsilon_{jk}^i E^{ck}(z) \mathcal{D}_a N_k(z) \right] \quad (\text{G.36})$$

Jacobi identity

$$\epsilon_{ijm} \epsilon_{mkn} + \epsilon_{jmk} \epsilon_{min} + \epsilon_{kim} \epsilon_{mjn} = 0 \quad (\text{G.37})$$

$$\{G(N_i), G(M_j)\} = -G([N, M]) \quad (\text{G.38})$$

Details

$$\{C_N, C_M\} = \{C_N, (\mathcal{D}_a A_b^i - \mathcal{D}_b A_a^i) \tilde{E}_j^a \tilde{E}_k^b\} \epsilon^{ijk} \quad (\text{G.39})$$

$$\begin{aligned} \{C_N, C_M\} &= \int d^3x (N \partial_a M - M \partial_a N) (\tilde{E}^{[a} \tilde{E}^{b]}) [\tilde{E}^c, F_{ab}] \\ &\quad \int d^3x (N \partial_a M - M \partial_a N) (\tilde{E}^c \tilde{E}^d) \end{aligned} \quad (\text{G.40})$$
