Appendix G

The Holst Action and Ashtekar-Barbero Real Variables

G.1 The Holst Action

$$S(E,\omega) = \frac{1}{16\pi G}\int d^4x |E| E^a_I E^b_J P^{IJ}{}_{KL}\Omega_{ab}{}^{KL}(\omega) \tag{G.1}$$

where

$$P^{IJ}{}_{KL} = \delta^{[I}_K \delta^{J]}_L - \frac{1}{2\beta}\epsilon^{IJ}{}_{KL}. \tag{G.2}$$

G.1.1 3+1 Decomposition of the Holst Action

We relate the curvature of the Ashtekar-Barbero connection to the curvature of the spin connection,

$$\begin{aligned}
\mathcal{F}^i_{ab} &= 2\partial_{[a}(\Gamma^i_{b]} + \beta K^i_{b]}) - \epsilon^i{}_{jk}(\Gamma^j_a + \beta K^j_a)(\Gamma^k_b + \beta K^k_b) \\
&= 2\partial_{[a}(\Gamma^i_{b]} + \beta K^i_{b]}) - \frac{1}{2}\epsilon^i{}_{jk}\Big((\Gamma^j_a + \beta K^j_a)(\Gamma^k_b + \beta K^k_b) - a \leftrightarrow b\Big) \\
&= 2\partial_{[a}\Gamma^i_{b]} - \epsilon^i{}_{jk}\Gamma^j_{[a}\Gamma^k_{b]} - \beta\Big(2\partial_{[a}K^i_{b]} - \epsilon^i{}_{jk}(\Gamma^j_{[a}K^k_{b]} + K^j_{[a}\Gamma^k_{b]})\Big) - \beta^2\epsilon^i{}_{jk}K^j_{[a}K^k_{b]} \\
&= F^i_{ab} - \beta\Big(2\partial_{[a}K^i_{b]} - \epsilon^i{}_{jk}(\Gamma^j_{[a}K^k_{b]} - K^j_{[a}\Gamma^k_{b]})\Big) - \beta^2\epsilon^i{}_{jk}K^j_a K^k_b \\
&= F^i_{ab} - \beta(2\partial_{[a}K^i_{b]} - 2\epsilon^i{}_{jk}\Gamma^j_{[a}K^k_{b]} - \beta^2\epsilon^i{}_{jk}K^j_a K^k_b \\
&= F^i_{ab} - 2\beta\nabla_{[a}K^i_{b]} - \beta^2\epsilon^i{}_{jk}K^j_a K^k_b
\end{aligned} \tag{G.3}$$

G.1.2 The Diffeomorphism Constraint

For the diffeomorphism constraint we have (we introduce here a negative sign so that the constraint appears with a positve sign in the Hamiltonian, recall the basic form $\mathcal{L} = p\dot{q} - \mathcal{H}$)

$$\begin{aligned}
N^a C_a &= -\beta n_I N^a \big(\frac{\sqrt{q}E^b_J}{8\pi\beta G}\big) P^{IJ}{}_{KL} F_{ab}{}^{KL} \\
&= \beta N^a \tilde{E}^b_j P^{0j}{}_{kl} F_{ab}{}^{kl} \\
&= \beta N^a \tilde{E}^b_j \left(F_{ab}{}^{0j} - \frac{1}{2\beta}\epsilon^{0j}{}_{kl} F_{ab}{}^{kl} \right) \\
&= 2\beta N^a \tilde{E}^b_j \left(\partial_{[a}\omega_{b]}{}^{0j} + \omega_{[a}{}^{0k}\omega_{b]k}{}^{j} + \frac{1}{2\beta}\epsilon^j{}_{kl}(\partial_{[a}\omega_{b]}{}^{kl} + \omega_{[a}{}^{kL}\omega_{b]L}{}^{l}) \right) \\
&= 2\beta N^a \tilde{E}^b_j \left(\partial_{[a}K^j_{b]} + K^k_{[a}\omega_{b]k}{}^{j} + \frac{1}{2\beta}\big(2\partial_{[a}\Gamma_{b]}{}^{kl} + \epsilon^j{}_{kl}(\omega_{[a}{}^{k0}\omega_{b]0}{}^{l} + \omega_{[a}{}^{km}\omega_{b]m}{}^{l})\big) \right) \\
&= 2N^a \tilde{E}^b_j \left(\partial_{[a}A^j_{b]} - \beta\epsilon^j{}_{mk}\Gamma^m_{[a}K^k_{b]} + \frac{1}{2}\epsilon^j{}_{kl}(K^k_{[a}K^l_{b]} + \omega_{[a}{}^{km}\omega_{b]m}{}^{l}) \right)
\end{aligned} \tag{G.4}$$

where we used $\epsilon^{0j}{}_{kl} = -\epsilon^j{}_{kl}$, $K^i_a = \omega_a{}^{0i}$ and $\omega_{ak}{}^{j} = \epsilon^j{}_{mk}\Gamma^m_a$. Also that $\omega_{[a}{}^{k0}\omega_{b]0}{}^{l} = \omega_{[a}{}^{0k}\omega_{b]}{}^{0l}$.

The term $\omega_{[a}{}^{km}\omega_{b]m}{}^{l}$ becomes,

$$\begin{aligned}
\epsilon^j{}_{kl}\omega_a{}^{km}\omega_{bm}{}^{l} &= \epsilon^j{}_{kl}(\epsilon^{km}{}_n\Gamma^n_a)(\epsilon_m{}^l{}_p\Gamma^p_b) \\
&= \epsilon^j{}_{kl}\epsilon_n{}^{km}\epsilon^l{}_{pm}\Gamma^n_a\Gamma^p_b \\
&= \epsilon^j{}_{kl}(\delta^l_n\delta^k_p - \eta_{np}\eta^{kl})\Gamma^n_a\Gamma^p_b \\
&= \epsilon^j{}_{kl}\Gamma^l_a\Gamma^k_b \\
&= -\epsilon^j{}_{kl}\Gamma^k_{[a}\Gamma^l_{b]}
\end{aligned} \tag{G.5}$$

Continuing with the calculation for the spatial diffeomorphism constraint,

$$
\begin{aligned}
N^a C_a &= N^a \tilde{E}^b_j \left(2\partial_{[a} A^j_{b]} - \beta \epsilon^j{}_{kl} \Gamma^k_{[a} K^l_{b]} - \frac{1}{2} \epsilon^j{}_{kl} \big(\Gamma^k_{[a} \Gamma^l_{b]} - K^k_{[a} K^l_{b]} \big) \right) \\
&= N^a \tilde{E}^b_j \left(2\partial_{[a} A^j_{b]} - \beta \frac{1}{2} \epsilon^j{}_{kl} \big(\Gamma^k_{[a} K^l_{b]} + K^k_{[a} \Gamma^l_{b]} \big) - \frac{1}{2} \epsilon^j{}_{kl} \big(\Gamma^k_{[a} \Gamma^l_{b]} - K^k_{[a} K^l_{b]} \big) \right) \\
&= N^a \tilde{E}^b_j \left(2\partial_{[a} A^i_{b]} - \frac{1}{2} \epsilon^j{}_{kl} (\Gamma^k_{[a} \Gamma^l_{b]} + \beta \Gamma^k_{[a} K^l_{b]} + \beta K^k_{[a} \Gamma^l_{b]} - K^k_{[a} K^l_{b]}) \right) \\
&= N^a \tilde{E}^b_j \left(2\partial_{[a} A^i_{b]} - \frac{1}{2} \epsilon^j{}_{kl} (\Gamma^k_{[a} \Gamma^l_{b]} + \beta \Gamma^k_{[a} K^l_{b]} + \beta K^k_{[a} \Gamma^l_{b]} + \beta^2 K^k_{[a} K^l_{b]}) \right. \\
&\quad \left. + (1+\beta^2) \epsilon^j{}_{kl} K^k_a K^l_b \right) \\
&= N^a \tilde{E}^b_j \left(2\partial_{[a} A^i_{b]} - \epsilon^j{}_{kl} (\Gamma^k_a + \beta K^k_a) (\Gamma^l_b + \beta K^l_b) + (1+\beta^2) \epsilon^j{}_{kl} K^k_a K^l_b \right) \\
&= N^a \tilde{E}^b_j \left(\mathcal{F}^j_{ab} + (1+\beta^2) \epsilon^j{}_{kl} K^k_a K^l_b \right) \qquad \text{(G.6)}
\end{aligned}
$$

where in the second line we used $\epsilon^j{}_{kl} \Gamma^k_{[a} K^l_{b]} = -\epsilon^j{}_{kl} \Gamma^k_{[b} K^l_{a]} = -\epsilon^j{}_{kl} K^l_{[a} \Gamma^k_{b]} = \epsilon^j{}_{kl} K^k_{[a} \Gamma^l_{b]}$.

G.1.3 The Hamiltonian Constraint

$$
\begin{aligned}
C &= -4\pi G \frac{N}{\sqrt{q}} \big(\frac{\sqrt{q} E^a_I}{8\pi\beta G} \big) \big(\frac{\sqrt{q} E^b_J}{8\pi\beta G} \big) P^{IJ}{}_{KL} F^{KL}_{ab} \\
&= -4\pi G \beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} P^{ij}{}_{KL} F_{ab}{}^{KL} \\
&= -4\pi G \beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} \left(F^{ij}_{ab} - \frac{1}{2\beta} \epsilon^{ij}{}_{KL} F_{ab}{}^{KL} \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} \left(F^{ij}_{ab} - \frac{1}{2\beta} (\epsilon^{ij}{}_{k0} F_{ab}{}^{k0} + \epsilon^{ij}{}_{0k} F_{ab}{}^{0k}) \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} \left(2\partial_{[a} \omega_{b]}{}^{ij} + 2\omega_{[a}{}^{iK} \omega_{b]}{}^{Lj} \eta_{KL} + \frac{2}{\beta} \epsilon^{ij}{}_k (\partial_{[a} \omega_{b]}{}^{k0} + \omega_{[a}{}^{kl} \omega_{b]l}{}^0) \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} \left(F^{ij}_{ab} + 2K^i_{[a} K^j_{b]} - \frac{2}{\beta} \epsilon^{ij}{}_k (\partial_{[a} K^k_{b]} + \omega_{[a}{}^{kl} K_{b]l}) \right) \\
&= -4\pi G \beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} \epsilon^{ij}{}_k \left(F^k_{ab} + \epsilon^k{}_{mn} K^m_{[a} K^n_{b]} - \frac{2}{\beta} \nabla_{[a} K^k_{b]} \right) \qquad \text{(G.7)}
\end{aligned}
$$

Substituting in (G.3) in this we obtain,

$$\begin{aligned}
C &= -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}{}_k \left(\mathcal{F}_{ab}^k + (1+\beta^2)\epsilon^k{}_{mn} K_a^m K_b^n + 2\frac{\beta^2+1}{\beta} \nabla_{[a} K_{b]}^k \right) \\
&= -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}{}_k \left(\mathcal{F}_{ab}^k + (1+\beta^2)\epsilon^k{}_{mn} K_a^m K_b^n \right) - \ 4\pi G\beta^2 \left(\frac{\sqrt{q} E_i^a}{8\pi\beta G}\right) \frac{\tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}{}_k 2\frac{\beta^2+1}{\beta} \nabla_{[a} K_{b]}^k \\
&= -4\pi G\beta^2 \frac{\tilde{E}_i^a \tilde{E}_j^b}{\sqrt{q}} \epsilon^{ij}{}_k \left(\mathcal{F}_{ab}^k + (1+\beta^2)\epsilon^k{}_{mn} K_a^m K_b^n \right) - \ (\beta^2+1)\epsilon^{ij}{}_k E_i^a \tilde{E}_j^b \nabla_{[a} K_{b]}^k
\end{aligned} \tag{G.8}$$

In a constrained Hamiltonian system, a dynamical quanity is called first class constraint if its Poisson bracket with all other constraints vanishes on the constraint surface (the surface implicitly defined by the simultaneous vanishing of all the constraints). A second class constraint is one that is not first class.

G.1.4 Addition Constraints

The Hamiltonian is then a sum over constraints,

$$\mathcal{H} = [A_a^i, \tilde{E}_j^b] = \int d^4x \left(\Lambda^i G_i + (1+\beta)\omega_t^{0j} S_j + NC + N^a C_a \right) \tag{G.9}$$

$$S_j = -\epsilon_{jm}{}^n K_b^m \tilde{E}_n^b$$

$$\begin{aligned}
\Gamma_{a(i} E_{j)}^b &= \delta_{ij} \frac{1}{2} \frac{1}{\sqrt{detq}} \epsilon^{abc} e_{al} \partial_b e_c^l - \frac{\epsilon^{abc}}{2\sqrt{detq}} (e_{ai}\partial_b e_{cj} + e_{aj}\partial_b e_{ci}) \\
&= -\frac{\epsilon^{abc}}{2\sqrt{detq}} (e_{ai}\partial_b e_{cj} + e_{aj}\partial_b e_{ci} - \delta_{ij} e_{al}\partial_b e_c^l)
\end{aligned} \tag{G.10}$$

Recall $e_I^a = E_I^a + n^a n_I$ and in the temporal gauge $n_i = 0$ so

$$\Gamma_{a(i} E_{j)}^b = -\frac{\epsilon^{abc}}{2\sqrt{detq}} (E_{ai}\partial_b E_{cj} + E_{aj}\partial_b E_{ci} - \delta_{ij} E_{al}\partial_b E_c^l) \tag{G.11}$$

Obviously

$$\mathcal{D}_b \tilde{E}_i^b = \partial_b \tilde{E}_i^b - \epsilon_{ij}{}^k (\Gamma_b^j + \beta K_b^j) \tilde{E}_k^b = 0 \qquad \epsilon_{ij}{}^k K_b^j \tilde{E}_k^b = 0$$

imply

$$\partial_b \tilde{E}_i^b - \epsilon_{ij}{}^k \Gamma_b^j \tilde{E}_k^b = 0$$

Contracting this the totally anti-symmetric tensor

$$\begin{aligned}
0 = \epsilon^{i}{}_{jk}(\partial_b \tilde{E}^b_i - \epsilon_{ij'}{}^{k'}\Gamma^{j'}_b \tilde{E}^b_{k'}) &= \epsilon^{i}{}_{jk}\partial_b \tilde{E}^b_i - \epsilon^{i}{}_{jk}\epsilon_{ij'}{}^{k'}\Gamma^{j'}_b \tilde{E}^b_{k'} \\
&= \epsilon^{i}{}_{jk}\partial_b \tilde{E}^b_i - (\delta^j_{j'}\delta^k_{k'} - \delta^k_{j'}\delta^j_{k'})\Gamma^{j'}_b \tilde{E}^b_{k'} \\
&= \epsilon^{i}{}_{jk}\partial_b \tilde{E}^b_i - (\Gamma^j_b \tilde{E}^b_k - \Gamma^k_b \tilde{E}^b_j) \qquad \text{(G.12)}
\end{aligned}$$

$$|det(\tilde{E})| = |\det(E)|^2$$

$$E^a_i = |\det(\tilde{E})|^{-1/2}\tilde{E}^a_i$$

$$\begin{aligned}
\Gamma_{a[i}E^a_{j]} &= \frac{1}{2}|\det(E)|^{-1}\epsilon_{ijk}\partial_a(|\det(E)|E^a_k) \\
&= \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + \frac{E^a_k}{|\det(E)|}\partial_a|\det(E)|) \\
&= \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + E^a_k E^l_c \partial_a E^c_l) \qquad \text{(G.13)}
\end{aligned}$$

$$\begin{aligned}
\Gamma_{ai}E^a_j &= \Gamma_{a(i}E^a_{j)} + \Gamma_{a[i}E^a_{j]} \\
&= -\frac{\epsilon^{abc}}{2\sqrt{detq}}(E_{ai}\partial_b E_{cj} + E_{aj}\partial_b E_{ci} - \delta_{ij}E_{an}\partial_b E^n_c) + \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + E^a_k E^l_c \partial_a E^c_l) \\
&= -\frac{\epsilon^{klm}}{2}E^a_k E^b_l E^c_m(E_{ai}\partial_b E_{cj} + E_{aj}\partial_b E_{ci} - \delta_{ij}E_{an}\partial_b E^n_c) + \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + E^a_k E^n_c \partial_a E^c_n) \\
&= -\frac{1}{2}\Big(\epsilon^{ilm}E^b_l E^c_m \partial_b E_{cj} + \epsilon^{jlm}E^b_l E^c_m \partial_b E_{ci} - \epsilon^{klm}\delta_{ij}E^b_l E^c_m \partial_b E^k_c\Big) + \frac{1}{2}\epsilon_{ijk}(\partial_a E^a_k + E^a_k E^n_c \partial_a E^c_n) \\
&= \frac{1}{2}E^b_l\Big(\epsilon^{ilk}E_{aj}\partial_b E^a_k + \epsilon^{jlk}E_{ai}\partial_b E^a_k + \epsilon^{mlk}\delta_{ij}E^c_k \partial_b E^m_c\Big) + \frac{1}{2}\epsilon_{ijl}(\partial_b E^b_l + E^b_l E^n_c \partial_b E^c_n) \\
&= \\
&= \\
&= \frac{1}{2}\epsilon^{ikl}E^b_l(E^k_{a,b} - E^k_{b,a})E^a_j + \frac{1}{2}\epsilon^{ikl}E^b_l E^c_k E^j_{c,b} \\
&= \frac{1}{2}\epsilon^{ikl}E^b_l(E^k_{a,b} - E^k_{b,a} + E^c_k E^m_a E^m_{c,b})E^a_j \qquad \text{(G.14)}
\end{aligned}$$

G.1.5 Final Total Hamiltonian

$$\mathcal{H} = [A^i_a, \tilde{E}^b_j] = \int d^4x \left(\Lambda^i G_i + NC + N^a C_a\right) \qquad \text{(G.15)}$$

$$C_a = \tilde{E}^b_j \mathcal{F}^j_{ab} \qquad \text{(G.16)}$$

$$C = -4\pi G\beta^2 \frac{\tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} \left(\epsilon^{ij}{}_k \mathcal{F}^k_{ab} + 2(1+\beta^2) K^i_{[a} K^j_{b]} \right) \tag{G.17}$$

or

$$C = -4\pi G\beta^2 \left(\frac{\epsilon^{ij}{}_k \mathcal{F}^k_{ab} \tilde{E}^a_i \tilde{E}^b_j}{\sqrt{q}} + 2(1+\beta^2) \frac{\tilde{E}^a_{[i} \tilde{E}^b_{j]}}{\sqrt{q}} (A^i_a - \Gamma^i{}_a)(A^j_b - \Gamma^j{}_b) \right) \tag{G.18}$$

G.2 Inclusion of Matter

G.2.1 Yang-Mills

G.2.2 Klein-Gordan - Scalar Matter

G.2.3 Fermionic Matter

G.3 Biblioliographical notes

In this chapter I have relied on the following refferences:

Eric Poisson

Kenneth smith, Dynamic Singularity Excision in Numerical Relativity.

Robert Bartnik, Jim Isenberg, *The Constraint Equations*?? or should this be refferenced in

G.4 Worked Exercises and Details

Details (I1.1) Gauss gauge

Derive Eq's above

$$\mathcal{G}^i(x) = \mathcal{D}_a E^{ai}(x) = \partial_a E^{ai}(x) + \epsilon^i_{jk} A^j_a(x) E^{ak}(x) \tag{G.19}$$

$$\{E^a_i(x), G(y)\} = \int d^3y \{E^a_i(x), \Lambda^j(y) \mathcal{G}^j\} = \tag{G.20}$$

$$\int d^3y\{A^i_a(x),\Lambda^j(y)\mathcal{G}^j\} = \int d^3y\{A^i_a(x),\Lambda^j(y)\mathcal{D}_aE^{bj}(y)\} =$$
$$\int d^3z \int d^3y \frac{\delta A^i_a(x)}{\delta A^k_c(z)}\Lambda^j(y)\frac{\delta[\partial_a E^{aj}(y)+\epsilon^i_{mn}A^m_a(y)E^{an}(z)]}{\delta E^c_k(z)} \tag{G.21}$$

$$\int d^3xN^i(x)\frac{\delta\mathcal{D}_aE^a_i(x)}{\delta A^k_c(z)} = \int d^3xN^i(x)\epsilon^i_{jk}\frac{\delta A^j_a(x)E^{ak}(x)}{\delta A^k_c(z)}$$
$$= \int d^3xN^i(x)\epsilon^i_{jk}\delta^3(x-z)\delta^c_aE^{ak}(x) = \tag{G.22}$$

$$\int d^3yM^j(y)\frac{\delta\mathcal{D}_aE^a_i(y)}{\delta E^c_k(z)} = \int d^3yM^j(y)\frac{\delta[\partial_aE^a_j(y)+\epsilon^i_{jk}A^j_a(y)E^{ak}(y)]}{\delta E^c_k(z)}$$
$$= -\int d^3y[\partial_cM^j(y)-\epsilon^i_{jk}M^j(y)A^k_c(y)]\delta^3(y-z)\delta^a_c = -[\partial_aM^j(z)+\epsilon^i_{jk}M^j(z)A^k_c(z)] = \tag{G.23}$$

Details (I1.1) Spacial diff

$$V(\vec{N}) = \int_\Sigma d^3N^aE^{bi}F^i_{ab} \tag{G.24}$$

Details (I1.1) Ham

$$F^i_{ab} = \partial_aA^i_b - \partial_bA^i_a + A^j_aA^k_b\epsilon_{ijk} \tag{G.25}$$

Obviously

$$\{A^i_a, F^j_{bc}\} = 0 \tag{G.26}$$

as F^i_{ab} depends only on the connection field. So that the Poisson bracket is simple:

$$\{A^i_a, H(N)\} = \{A^i_a,\ \epsilon^{i'j'k'}F^{i'}_{bc}\tilde{E}^b_{j'}\tilde{E}^c_{i'}\} = \epsilon^{i'j'k'}F^{i'}_{bc}\{A^i_a,\tilde{E}^b_{j'}\tilde{E}^c_{i'}\} \tag{G.27}$$

Using the "product rule" for Poisson brackets this is

$$\begin{aligned}\{A^i_a, H(N)\} &= F^{i'}_{bc}\epsilon^{i'j'k'}\left(\{A^i_a, \tilde{E}^b_{j'}\}\tilde{E}^c_{i'} + \{A^i_a, \tilde{E}^c_{i'}\}\tilde{E}^b_{j'}\right) \\ &= 2F^{i'}_{bc}\epsilon^{i'j'k'}\{A^i_a, \tilde{E}^b_{j'}\}\tilde{E}^c_{i'} \\ &= 2\epsilon^{ijk}\tilde{E}^c_j F^k_{bc}\end{aligned} \tag{G.28}$$

where in the in the second step we used that the field-strength tensor is anti-symmetric in its spacetime indices. We have proved (F.46).

Details (I1.1) Ham

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b \epsilon_{ijk} \tag{G.29}$$

Obviously

$$\{A^i_a, F^j_{bc}\} = 0 \tag{G.30}$$

as F^i_{ab} depends only on the connection field. So that the Poisson bracket is simple:

$$\{A^i_a, H(N)\} = \{A^i_a,\ \epsilon^{i'j'k'}F^{i'}_{bc}\tilde{E}^b_{j'}\tilde{E}^c_{i'}\} = \epsilon^{i'j'k'}F^{i'}_{bc}\{A^i_a, \tilde{E}^b_{j'}\tilde{E}^c_{i'}\} \tag{G.31}$$

Using the "product rule" for Poisson brackets this is

$$\begin{aligned}\{A^i_a, H(N)\} &= F^{i'}_{bc}\epsilon^{i'j'k'}\left(\{A^i_a, \tilde{E}^b_{j'}\}\tilde{E}^c_{i'} + \{A^i_a, \tilde{E}^c_{i'}\}\tilde{E}^b_{j'}\right) \\ &= 2F^{i'}_{bc}\epsilon^{i'j'k'}\{A^i_a, \tilde{E}^b_{j'}\}\tilde{E}^c_{i'} \\ &= 2\epsilon^{ijk}\tilde{E}^c_j F^k_{bc}\end{aligned} \tag{G.32}$$

where in the in the second step we used that the field-strength tensor is anti-symmetric in its spacetime indices. We have proved (F.46).

Details (L2.2)

$$\{G(N_i), G(M_j)\} = \int\int d^3x d^3y N^i(x) M^j(y) \{\mathcal{G}^i(x), \mathcal{G}^j(y)\} =$$

$$= \int d^3z \int d^3x \int d^3y N^i(x) M^j(y) \left[\frac{\delta\mathcal{G}^i(x)}{\delta A^k_c(z)}\frac{\delta\mathcal{G}^j(y)}{\delta E^c_k(z)} - \frac{\delta\mathcal{G}^i(x)}{\delta E^c_k(z)}\frac{\delta\mathcal{G}^j(y)}{\delta A^k_c(z)}\right] \tag{G.33}$$

$$\int d^3x N^i(x) \frac{\delta \mathcal{G}^i(x)}{\delta A_c^k(z)} = N^i(z)\epsilon^i_{jk} E^{ck}(z) \tag{G.34}$$

$$\int d^3y M^j(y) \frac{\delta \mathcal{G}^j(y)}{\delta E_k^c(z)} = -\mathcal{D}_a M_k(z) \tag{G.35}$$

$$\{G(N_i), G(M_j)\} = \int d^3z \left[N^i(z)\epsilon^i_{jk} E^{ck}(z)\mathcal{D}_a M_k(z) - M^i(z)\epsilon^i_{jk} E^{ck}(z)\mathcal{D}_a N_k(z) \right] \tag{G.36}$$

Jacobi identity

$$\epsilon_{ijm}\epsilon_{mkn} + \epsilon_{jmk}\epsilon_{min} + \epsilon_{kim}\epsilon_{mjn} = 0 \tag{G.37}$$

$$\{G(N_i), G(M_j)\} = -G([N, M]) \tag{G.38}$$

Details

$$\{C_N, C_M\} = \{C_N, (\mathcal{D}_a A^i_b - \mathcal{D}_b A^i_a)\tilde{E}^a_j \tilde{E}^b_k\}\epsilon^{ijk} \tag{G.39}$$

$$\begin{aligned} \{C_N, C_M\} = \int d^3x (N\partial_a M - M\partial_a N)(\tilde{E}^{[a}\tilde{E}^{b]}[\tilde{E}^c, F_{ab}] \\ \int d^3x (N\partial_a M - M\partial_a N)(\tilde{E}^c \tilde{E} \end{aligned} \tag{G.40}$$