Appendix H

Basic Functional Analysis

H.1 Finite Hilbert Space.

For a finite-dimensional space every Hermitian or unitary operator can be described completely by its eigenvalues and eigenvectors.

For a finite-dimensional space the eigenvectors span the whole space.

This comprises the so-called spectral theorem.

H.1.1 The Hamilton-Cayley Theorem.

Let $\mathbf{A}$ be a square $N \times N$ matrix representing in some basis the operator A , and let λ be a parameter. The equation

$$\varphi(\lambda) := \det(\lambda\mathbf{E} - \mathbf{A}) = 0 \tag{H.1}$$

is called the characteristic equation of the operator A(or of the matrix $\mathbf{A}$). It is evident that $\psi(\lambda)$ is a polynomial of the Nth degree in λ with numerical coefficients the leading (that of λ^N being equal to 1

$$\varphi(\lambda) = \varphi_0 + \varphi_1\lambda + \varphi_2\lambda^2 + \cdots + \varphi_{N-1}\lambda^{N-1} + \varphi_N\lambda^N \tag{H.2}$$

The form of the characteristic (and thus the numerical values of the coefficients, ψ_i) does not depend on the choice of basis, since the determinant of a matrix, in our case, the matrix $\det(\lambda\mathbf{E} - \mathbf{A})$, is a scalar.

Replacing λ by the operator A we get the operator

$$A(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + \cdots + A_{N-1}\lambda^{N-1} + A_N\lambda^N \tag{H.3}$$

H.1.2 Projection Operators

In chapter 3 we saw that a projection operator is equivalent to operators that are self-adjoint and satify $P^2 = P$.

We say that a subspace $\mathcal{M}$ **reduces** a linear operator T if $T\psi$ is in $\mathcal{M}$ for every $\psi \in \mathcal{M}$ and $T\phi$ is in $\mathcal{M}^\perp$ for every ϕ in $\mathcal{M}^\perp$. Let P be the projection operator onto $\mathcal{M}$. We say that a subspace $\mathcal{M}$ is **invariant** under an operator if $T\psi$ is in $\mathcal{M}$ for every vector in $\mathcal{M}$.

Theorem H.1.1 *Let P be the projection operator onto the subspace $\mathcal{M}$. The following statements are equivalent:*

i) $\mathcal{M}$ reduces T;

ii) $PT = TP$

Proof: First we show i) implies ii). For any vector

$$\psi = \psi_{\mathcal{M}} + \psi_{\mathcal{M}^\perp}$$

we have

$$T\psi = T\psi_{\mathcal{M}} + T\psi_{\mathcal{M}^\perp}.$$

If $\mathcal{M}$ reduces T, then $T\psi_{\mathcal{M}}$ is in $\mathcal{M}$ and $T\psi_{\mathcal{M}^\perp}$ is in $\mathcal{M}^\perp$. Therefore

$$\begin{aligned} PT\psi &= PT(\psi_{\mathcal{M}} + \psi_{\mathcal{M}^\perp}) = PT\psi_{\mathcal{M}} \\ &= T\psi_{\mathcal{M}} = TP\psi_{\mathcal{M}} \\ &= TP\psi. \end{aligned} \tag{H.4}$$

Thus i) implies ii). We now show the converse. If ψ is in $\mathcal{M}$ and $PT = TP$, then $P\psi = \psi$ and

$$T\psi = TP\psi = PT\psi$$

so $T\psi$ is in $\mathcal{M}$. It is easy to see that $PT = TP$ and $(1-P)T = T(1-P)$ are equivalent. If ϕ is in $\mathcal{M}^\perp$ and $(1-P)T = T(1-P)$, then

$$T\phi = T(1-P)\phi = (1-P)T\phi$$

so $T\phi$ is in $\mathcal{M}^{\perp}$.

□

Theorem H.1.2 *If a subspace is invariant under T and $T^{\dagger}$ then it reduces T.*

Proof: Let $\mathcal{M}$ be a subspace that is invariant under T and $T^{\dagger}$ and let P be the projection operator onto $\mathcal{M}$. Then $TP\psi$ is in $\mathcal{M}$, so

$$TP\psi = PTP\psi$$

for every vector ψ, or

$$TP = PTP.$$

Because $\mathcal{M}$ is also invariant under $T^{\dagger}$, we have

$$T^{\dagger}P = PT^{\dagger}P.$$

Taking the adjoint, we have

$$PT = PTP.$$

Threfore

$$PT = TP.$$

This implies that $\mathcal{M}$ reduces T by theorem H.1.1.

□

Theorem H.1.3 *If P and Q are the projections on closed linear subspaces M and N of H, then $M \perp N$ if and only if $PQ = 0$ or equivalently $QP = 0$.*

Proof: First note that $PQ = 0$ is equivalent to $QP = 0$ through taking adjoints. If $M \perp N$, so that $N \subseteq M^{\perp}$, as $Q\psi$ is in N for every x we have $PQ\psi = 0$, or $PQ = 0$. Conversely, if $PQ = 0$ then for every ψ in N we have $P\psi = PQ\psi = 0$, so $N \subseteq M^{\perp}$ and $M \perp N$.

□

We say that two projections P and Q are orthogonal if $PQ = 0$.

Theorem H.1.4 *If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of H, then $P = P_1 + P_2 + \cdots + P_n$ is a projection if and only if the P_i's are pairwise orthognal ($P_i P_j = 0$ whenever $i \neq j$). In this case P is the projection onto*

$$M = M_1 + M_2 + \cdots + M_n.$$

Proof: Since P is self-adjoint, to prove it is a projection, we need only show it is idempotent, that is, $P^2 = P$. As the P_i's are pairwise orthogonal,

$$\begin{aligned} P^2 &= (P_1 + P_2 + \cdots + P_n)(P_1 + P_2 + \cdots + P_n) \\ &= P_1^2 + P_2^2 + \cdots + P_n^2 \\ &= P_1 + P_2 + \cdots + P_n = P. \end{aligned} \tag{H.5}$$

Therefore P is a projection. Conversely, assume that P is idempotent. Let ψ be a vector in the range of P_i then

$$\begin{aligned} \|\psi\|^2 &= \|P_i\psi\|^2 \leq \sum_{j=1}^{n} \|P_j\psi\|^2 \\ &= \sum_{j=1}^{n} (P_j\psi, \psi) = (P\psi, \psi) \\ &= \|P\psi\|^2 \leq \|\psi\|^2 \end{aligned}$$

Given we started with $\|\psi\|^2$ and ended with $\|\psi\|^2$, equality must hold throught, so

$$\sum_{j=1}^{n} \|P_j\psi\| = \|P_i\psi\|$$

and

$$\|P_j\psi\| = 0 \ \text{ for } \ j \neq i.$$

Therefore the range of P_i is contained in the null space of P_j, that is, $M_i \subseteq M_j^{\perp}$, for every $j \neq i$. So $M_i \perp M_j$ whenever $j \neq i$, and by the previous theorem we conclude that the P_i's are pairwise orthogonal. We now prove the final statement. Denote the range of P by $\mathrm{Ran}(P)$. First note that since $\|P\psi\| = \|\psi\|$ for every ψ in M_i, each M_i is contained in the range of P, and therefore

$$M \subseteq Ran(P). \tag{H.6}$$

Next, if ψ is in the range of P, then

$$\begin{aligned}\psi &= P\psi \\ &= P_1\psi + P_2\psi + \cdots P_n\psi\end{aligned}$$

is obviously in M, and so ψ is in M, hence

$$Ran(P) \subseteq M. \tag{H.7}$$

Comparing (H.6) and (H.7) implies $M = Ran(P)$.

□

H.1.3 Spectral Theorem for Finite Spaces

An operator on H is said to be **normal** if it commutes with its adjoint, that is, $NN^\dagger = N^\dagger N$. Self-adjoint and unitary operators are examples of normal operators. The spectral theorem states that for each normal operator N on H has a spectral resolution, that is, there exist distinct complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and non-zero pairwise orthogonal projections $P_1, P_2, \ldots, P_m$ such that $\sum_{i=1}^m P_i = I$, and

$$N = \sum_{i=1}^m \lambda_i P_i.$$

Before coming on to the spectral theorem, we prove results for normal operators.

Lemma H.1.5 *If T ia normal, then ψ is an eigenvector of T with eigenvalue λ if and only if ψ is an eigenvector of $T^\dagger$ with eigenvalue λ^*.*

Proof:

Firsrt we show that an operator T is normal if and only if $\|T^\dagger\psi\| = \|T\psi\|$ for every ψ. Obviously $\|T^\dagger\psi\| = \|T\psi\|$ is equivalent to $\|T^\dagger\psi\|^2 = \|T\psi\|^2$, that is, $(T^\dagger, \psi T^\dagger\psi) = (T\psi, T\psi)$. This in turn is equivalent to $(TT^\dagger\psi, \psi) = (T^\dagger T\psi, \psi)$. Finally this is eqivalent to $([TT^\dagger - T^\dagger T]\psi, \psi) = 0$.

Since T is normal, it is obvious that also is $T - \lambda I$ for any scalar λ,

$$(T - \lambda I)(T^\dagger - \lambda^* I) = TT^\dagger - \lambda^* T - \lambda T^\dagger + |\lambda|^2 I = (T^\dagger - \lambda^* I)(T - \lambda I).$$

Hence we have

$$\|T - \lambda I\| = \|T^\dagger - \lambda^* I\|$$

for all ψ, and the lemma follows at once.

$\square$

Lemma H.1.6 *If T ia normal, then each M_i reduces T.*

Proof:

It is obvious that each M_i is mapped onto itself under T. By theorem H.1.2 it suffices to show that each M_i is invariant under $T^\dagger$. But this follows from lemma H.1.5, for if ψ_i is a vector in M_i, so $T\psi_i = \lambda_i \psi_i$, then $T^\dagger \psi_i = \lambda_i^* \psi_i$ is also in M_i.

$\square$

We are now in a position to prove the spectral theorem.

Theorem H.1.7 *Let T be an arbitrary operator on H. By ... we know that the distinct eigenvalues of T form a non-empty finite set of complex numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be these eigenvalues, let $M_1, M_2, \ldots, M_n$ be their corresponding eigenspaces, and let $P_1, P_2, \ldots, P_n$ be the projections onto these eigenspaces. The following statements are equivalent:*

i) The M_i's are pairwise orthogonal and span H.

ii) The P_i's orthogonal, $I = \sum_i^m P_i$ and $T = \sum_i^m \lambda_i P_i$.

iii) T is normal.

Proof:

First we prove that i) implies ii). By i) every vector ψ in H can be expressed uniquely in the form

$$\psi = \psi_1 + \psi_2 + \cdots + \psi_m, \tag{H.8}$$

where ψ_i is in M_i for each i and the ψ_i's are pairwise orthogonal. It follows that

$$\begin{aligned} T\psi &= T\psi_1 + T\psi_2 + \cdots + T\psi_m \\ &= \lambda_1\psi_1 + \lambda_2\psi_2 + \cdots + \lambda_m\psi_m. \end{aligned} \tag{H.9}$$

By theorem (H.1.3) the M_i's being pairwise orthognal is equivalent to the projection operators P_i being pairwise orthogonal.

$$
\begin{aligned}
Ix = x &= x_1 + x_2 + \cdots + x_m \\
&= P_1 x + P_2 x + \cdots + P_m x \\
&= (P_1 + P_2 + \cdots + P_m)x \qquad \text{(H.10)}
\end{aligned}
$$

for every x in $\mathcal{H}$, so

$$
I = P_1 + P_2 + \cdots + P_m. \qquad \text{(H.11)}
$$

Equation (H.9)

$$
\begin{aligned}
T\psi &= \lambda_1 \psi_1 + \lambda_2 \psi_2 + \cdots + \lambda_m \psi_m \\
&= \lambda_1 P_1 \psi + \lambda_2 P_2 \psi + \cdots + \lambda_m P_m \psi \\
&= (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)\psi. \qquad \text{(H.12)}
\end{aligned}
$$

for every ψ, therefore

$$
T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m. \qquad \text{(H.13)}
$$

We now show that ii) implies iii). By (H.13) we have

$$
T^\dagger = \lambda_1^* P_1 + \lambda_2^* P_2 + \cdots + \lambda_m^* P_m,
$$

and by using that the P_i's are pairwise orthogonal we obtain

$$
\begin{aligned}
TT^\dagger &= (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\lambda_1^* P_1 + \lambda_2^* P_2 + \cdots + \lambda_m^* P_m) \\
&= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m \qquad \text{(H.14)}
\end{aligned}
$$

and, similarly,

$$
T^\dagger T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m.
$$

so that

$$
TT^\dagger = T^\dagger T.
$$

Lastly we show that iii) implies i), completing the proof that all conditions of the theorem imply each other. First we show that if T is normal, then the M_i's are pairwise orthogonal. Let ψ_i and ψ_j be vectors in M_i and M_j for $i \neq j$, so that $T\psi_i = \lambda_i \psi_i$ and $T\psi_j = \lambda_j \psi_j$. We have

$$\begin{aligned} \lambda_i(\psi_i, \psi_j) &= (\lambda_i^* \psi_i, \psi_j) \\ &= (T^\dagger \psi_i, \psi_j) \\ &= (\psi_i, T\psi_j) \\ &= (\psi_i, \lambda_j \psi_j) = \lambda_j(\psi_i, \psi_j) \end{aligned} \tag{H.15}$$

where we used lemma H.1.5 in going from the frist line to second line, and since $\lambda_i \neq \lambda_j$, it is obvious that $(\psi_i, \psi_j) = 0$, which is the desired result.

We now prove that the M_i's span H when T is normal. By theorem H.1.3, the fact that the M_i's are pairwise orthogonal implies the P_i's are pairwise orthogonal. In turn by theorem H.1.4 we have that $M = M_1 + M_2 + \cdots + M_m$ is a closed linear subspace of H, and that its associated projection is

$$P = P_1 + P_2 + \cdots + P_m.$$

Since by lemma H.1.6 each M_i reduces T, from theorem H.1.1 we have that $TP_i = P_i T$ for each P_i. It follows from this that $TP = PT$, so, from H.1.1 again, M also reduces T, and consequently $M^\perp$ is invariant under T. Say $M^\perp \neq \{\emptyset\}$. All the eigenvectors of T reside in M, the restriction of T to $M^\perp$ represents an operator on the space $M^\perp$ but one which has no eigenvectors, and hence no eigenvalues. It follows from the fact that every operator must have at least one eigenvalue that we have a contradiction. We conclude that $M^\perp = \{\emptyset\}$ and hence the M_i's span H.

□

The expression

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m. \tag{H.16}$$

for T, when it exists, is called the **spectral resolution** of T. The spectral theorem in particular proves that if T is normal, then it has the spectral resolution (H.16).