Appendix I

Quantum Field Theory

I.1 Elementary Quantum Mechanics

I.1.1 Path Integrals and Functional Integrals

$$t_2 - t_1 \to -iT. \tag{I.1}$$

$$1 = \sum_n^\infty |n><n|, \qquad H|n>=|n>E_n, \tag{I.2}$$

$$\begin{aligned} <x_1|e^{-HT}|x_2> &= \sum_n <x_1|e^{-HT}|n><n|x_2> \\ &= \sum_n e^{-E_nT} <x_1|n><n|x_2> \\ &= \sum_n e^{-E_nT}\psi_n(x_1)\psi_n(x_2)^* \end{aligned} \tag{I.3}$$

$$\psi_0(x) = \lim_{T\to\infty} W[x,0,T] \tag{I.4}$$

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t)\; e^{i\int_{-\infty}^{t_1}\mathcal{L}(x,\dot{x})} \tag{I.5}$$

$$1 = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t)\; e^{i\int_{-\infty}^{\infty}\mathcal{L}(x,\dot{x})} \tag{I.6}$$

$$S[q(t)] \;=\; S[q_0(t)] + \frac{1}{2}\int dt'_1 dt'_2 \frac{\delta^2 S[q(t)]}{\delta q(t'_1)\delta q(t'_2)}\Big|_{q=q_0} (q(t'_1) - q_0(t'_1))(q(t'_2) - q_0(t'_2)) + \dots \tag{I.7}$$

$$q_{cl}(t) = \frac{1}{\sin\omega(t_1 - t_2)}\big(q_2 \sin\omega(t - t_1) - q_1 \sin\omega(t - t_2)\big) \tag{I.8}$$

which leads to the action

$$S_{cl}(t) = \frac{m\omega}{2\sin\omega(t_1 - t_2)}\big((q_1^2 + q_2)\cos\omega(t - t_1) - 2q_1 q_2\big) \tag{I.9}$$

I.1.2 Semi-Classical Limit

from elementary quantum mechanics, the phase of a semi-classical state determines where the state is peaked in the conjugate variables.

we expect semiclassical histories to dominate the path integral. These must be close to the equations of motion, fitting the boundary data.

$$\begin{aligned} \frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V &= 0 \\ \frac{\partial (a^2)}{\partial t} + \nabla \cdot \left(\frac{a^2}{m}\nabla S\right) &= 0. \end{aligned} \tag{I.10}$$

The Hamilton-Jacobi equation has solutions of the form

$$S(x,t) = W(x) - Et \tag{I.11}$$

provided that

$$\frac{|\nabla W|^2}{2m} + V = E. \tag{I.12}$$

The corresponding semi-classical wave function

$$\psi = a(x)\exp\left(\frac{iW(x)}{\hbar}\right)\exp\left(-i\frac{Et}{\hbar}\right) \tag{I.13}$$

I.1.3 Second Quantization

For many calculations, second quantized notation is much easier to use. For the normal-mode problem we can define the destruction operators for each mode exactly as for the simple harmonic oscillator

$$a_n = \sqrt{\frac{\omega_n}{2\hbar}} + i\frac{P_n}{\sqrt{2\hbar\omega_n}} \tag{I.14}$$

The Hamiltonian becomes

$$H = \sum_n \hbar\omega \left(a_n^\dagger a_n + \frac{1}{2} \right). \tag{I.15}$$

$$\begin{aligned} X_n &= \sqrt{\frac{\hbar}{2\omega_n}}(a_n^\dagger + a_n) \\ P_n &= \sqrt{\hbar\omega_n}(a_n^\dagger - a_n) \end{aligned} \tag{I.16}$$

$$\begin{aligned} x_n &= \sum_n \psi_k^{(n)} \sqrt{\frac{\hbar}{2\omega_n m_k}}(a_n^\dagger + a_n) \\ p_n &= \sqrt{\frac{\hbar\omega m_k}{2}}(a_n^\dagger - a_n) \end{aligned} \tag{I.17}$$

The eigenstates of energies,

$$\begin{aligned} E_{n_1,n_2,\ldots,n_N} &= \sum_{j=1}^{N} \hbar\omega_j \Big(n_j + \frac{1}{2} \Big), \\ |n_1, n_2, \ldots, n_N > &= \prod_{j=1}^{N} (a_j^\dagger)^{n_j} |0, 0, \ldots, 0 > \equiv \frac{a_j(^\dagger)^{n_j}}{\sqrt{n_j!}} \end{aligned} \tag{I.18}$$

$\boxed{\text{Gaussian integration}}$

I.1.4 N Real variables

$$Z(j) = \int \prod_{i=1}^{N} dx_i \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_{i=1}^{N} j_i x_i \right) \tag{I.19}$$

where the matrix A_{ij} is symmetric and strictly positive.A more compact notation is to represent the column vectors $(x_i...x_N)$ and $(j_1...j_N)$ as x and j, then the row vectors would be x^T and j^T where T=transpose. We then have:

$$\sum_{i,j=1}^{N} x_i A_{ij} x_j = x^T A x, \qquad \sum_{i=1}^{N} j_i x_i = j^T x \tag{I.20}$$

Change variables to x' given by

$$x = x' + A^{-1} j, \tag{I.21}$$

(the matrix A^{-1} exists because A is assumed positive) then

$$-\frac{1}{2} x^T A X + J^T x = -\frac{1}{2} x'^T A x' + \frac{1}{2} j^T A^{-1} j, \tag{I.22}$$

The integral then becomes

$$Z(j) = \exp(\frac{1}{2} j^T A^{-1} j) Z(0) \tag{I.23}$$

In many cases equation () is all one needs (for example in calculating correlation functions, from which Z(0) cancels). Z(0) reads

$$Z(0) = \int \prod_{i=1}^{N} dx'_i \exp(-\frac{1}{2} x^T A x), \tag{I.24}$$

Let R be an orthogonal transformation ($RR^T = I$) diagonalising A,

$$A = \mathbf{R}^T \mathbf{D} \mathbf{R}, \qquad \mathbf{D} = \begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ 0 & & \ddots & \\ & & & d_N \end{pmatrix}, \qquad d_i > 0 \qquad \forall i. \tag{I.25}$$

Make the following change of variables with unit Jacobian:

$$x' = \mathbf{R} x \qquad (det \mathbf{R} = 1), \tag{I.26}$$

$$\int \prod_{i=1}^{N} dx_i \exp(\frac{1}{2} x^T \mathbf{A} x) = \int \prod_{i=1}^{N} dx'_i \exp(\frac{1}{2} x'^T \mathbf{D} x') \tag{I.27}$$

The last integral is the product of N independent gaussian integrals, and is given by

$$(2\pi)^{N/2}\prod_{i=1}^{N}(d_i)^{-1/2} = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{I.28}$$

$$Z(0) = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{I.29}$$

$$\int\prod_{i=1}^{N} dx_i \exp(-x^T\mathbf{A}x + j^T x+) = \frac{\pi^{N/2}}{det\mathbf{A}^{1/2}}\exp((1/2)j^T A^{-1}j), \tag{I.30}$$

I.1.5 Complex variables

Gaussian integration

First consider the case of a single complex variable z=x+iy with integral

$$I = \int d^2ze^{-z^*Az+z^*j+zj^*} = \int dxdye^{-A(x^2+y^2)+2j_1x+2j_2y} \tag{I.31}$$

where $j = j_1 + ij_2$ and $A = a_1 + ia_2$, with $a_1 > 0$. Then I follows immediately:

$$I = \frac{\pi}{A}a^{j^*A^{-1}j} \tag{I.32}$$

Next, consider the case of N complex variables z_i,

$$I = \int\prod_{i=1}^{N} d^2z_i e^{-z^\dagger\mathbf{A}z+z^\dagger j+j^\dagger z}, \tag{I.33}$$

where transposes have been replaced by Hermitian conjugates. Assume that $\mathbf{A}$ can be diagonalized by a unitary transformation U,

$$A = U^\dagger DU, \tag{I.34}$$

where D is a diagonal matrix with elements d_i whose real parts are positive. Write

$$U = R + iS, \tag{I.35}$$

where R and S are real matrices; the relation $U^\dagger U = I$ entails

$$RR^T + SS^T = I, \qquad RS^T - SR^T = 0. \tag{I.36}$$

The transformation $z' = Uz$ amounts to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R & -S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{I.37}$$

and the matrix which transforms (x,y) into (x', y') is orthogonal so that the jacobian of the transformation is 1, which leads to the end result

$$\int \prod_{i=1}^{N} e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z} = \frac{\pi^N}{det\mathbf{A}} e^{j^\dagger \mathbf{A^{-1}} j} \tag{I.38}$$

I.2 Grassmann Integration

Grassmann Algebra

We consider a set of anticommutating Grassmann variables $\{\zeta_i\}_{i=1,\ldots,n}$, with complex linear coefficients, where n is the dimension of the algebra. The decisive relation defining the structure of the algebra is the anticommutation relation

$$\zeta_i \zeta_j + \zeta_j \zeta_i = 0 \tag{I.39}$$

for all i and j. As a particular consequence of this condition the square and all higher powers of a variable vanish,

$$\zeta_i^2 = 0 \tag{I.40}$$

The Grassmann algebra generate a Grassmann algebra of functions which have the form

$$f(\zeta) = f^{(0)} + \sum_i f_i^{(1)} + \sum_{i_1 < i_2} f_{i_1 i_2}^{(2)} \zeta_{i_1} \zeta_{i_2} + \ldots + f^{(n)} \zeta_{i_1} \zeta_{i_2} \ldots \zeta_{i_n} \tag{I.41}$$

where the coefficients $f^{(k)}$ are ordinary complex numbers.

On this algebra we will need to define the derivative. We first consider a simple Grassmann algebra of order $n = 2$ with the variables ζ_1 and ζ_2.

$$f(\zeta_1, \zeta_2) = f^{(0)} + f_1^{(1)}\zeta_1 + f_2^{(1)}\zeta_2 + f^{(2)}\zeta_1\zeta_2$$

$$\frac{\partial f}{\partial \zeta_1} = f_1^{(1)} + f^{(2)}\zeta_2, \qquad \frac{\partial f}{\partial \zeta_2} = f_2^{(1)} - f^{(2)}\zeta_1. \tag{I.42}$$

Note the minus sign in the last equation of (I.42),

$$\frac{\partial}{\partial \zeta_j}\zeta_1\zeta_2 = \delta_{j1}\zeta_2 - \delta_{j2}\zeta_1.$$

The general rule for differentiation of a product is given by

$$\frac{\partial}{\partial \zeta_j}\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m} = \delta_{ji_1}\zeta_{i_2}...\zeta_{i_m} - \delta_{ji_2}\zeta_{i_1}\zeta_{i_3}...\zeta_{i_m} + ... + (-1)^{m-1}\delta_{ji_2}\zeta_{i_1}\zeta_{i_2}...\zeta_{i_{m-1}} \tag{I.43}$$

The respective factor ζ_{i_k} is anticommuted to the left until the derivative operator can be directly applied. We may prove the following properties of the derivatives

$$\frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial \zeta_j}\frac{\partial}{\partial \zeta_i} = 0 \tag{I.44}$$

$$\frac{\partial}{\partial \zeta_i}\zeta_j + \zeta_j\frac{\partial}{\partial \zeta_i} = 0 \tag{I.45}$$

Grassmann integration

An attempt to to introduce an indefinite integral as the inverse of differentiation is bound to fail. This illustrated by the fact that according to (I.44) the second derivative of any Grassmann function vanishes, so that the inverse operation does not exist, for if there was an inverse to $\frac{\partial^2 F}{\partial \zeta^2}$ it should give

$$\int d\zeta \frac{\partial^2 F}{\partial \zeta^2} = \frac{\partial F}{\partial \zeta}$$

However as

$$0 = \int d\zeta 0 = \int d\zeta \frac{\partial^2 F}{\partial \zeta^2}$$

this would imply we always have

$$\frac{\partial F}{\partial \zeta} = 0$$

which is not true in general.

We must be content with some formal definition. One way to arrive at it is to require that integration be translationally invariant. For an arbitrary function $g(\zeta) = g_1 + g_2\zeta$ we have

$$\begin{aligned} \int d\zeta g(\zeta + \eta) &= \int d\zeta \left[g_1 + g_2(\zeta + \eta)\right] = \int d\zeta \left[g_1 + g_2\zeta\right] + \int d\zeta g_2\eta \\ &= \int d\zeta g(\zeta) + \left[\int d\zeta 1\right] g_2\eta = \int d\zeta g(\zeta) \end{aligned} \tag{I.46}$$

The translational invariance requires the integral of 1 is zero. The following postulates uniquely fix the value of any integral.

$$\int d\zeta 1 = 0, \tag{I.47}$$

$$\int d\zeta\zeta = 1. \tag{I.48}$$

Eq. (I.47) comes from the condition of translational invariance. The sole non-vanishing integral $\int d\zeta\zeta$ arbitrarily is assigned the value 1. This is a convenient normalisation condition.

We see that integration is equivalent to differentiation. Generalising integration rules to higher dimensions straightforward

$$\int d\zeta_i 1 = 0, \tag{I.49}$$

$$\int d\zeta_i\zeta_j = \delta_{ij}. \tag{I.50}$$

Note that the differentials $d\zeta_i$ must anticommute with all other Grassmann variables as integration is equivalent to differentiation. In order to obtain analog results of conventional integration we introduce complex Grassmann variables. Let us start with two disjoint sets of Grassmann variables $\zeta_1^*, ..., \zeta_n^*$ and $\zeta_1, ..., \zeta_n$, which are all mutually anticommutating

$$\{\zeta_i, \zeta_j\} = \{\zeta_i^*, \zeta_j^*\} = \{\zeta_i, \zeta_j^*\} = 0 \tag{I.51}$$

The two sets are related, using complex cunjugation, according to

$$
\begin{aligned}
(\zeta_i)^* &= \zeta_i^*, \\
(\zeta_i^*)^* &= -\zeta_i, \\
(\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m})^* &= \zeta_{i_m}^*...\zeta_{i_2}^*\zeta_{i_1}^* \\
(\lambda\zeta_i)^* &= \lambda^*\zeta_i^*
\end{aligned} \tag{I.52}
$$

where λ is a complex number.

In order to develop functional integral formalism for Grassmann fields we need to solve *Gaussian integrals*.

$$
\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp\left\{ - \sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l \right\} \tag{I.53}
$$

To simplify the notation, let us write this as

$$
I = \int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} \tag{I.54}
$$

The calculation in principle is very simply because grassmann functions can at worst be linear in each variable, causing the series expansion of the exponential function to terminate. On the other hand , according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes.

Let us consider the case where we have two pairs of variables. The exponential then reads

$$
e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!}(\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2 - \frac{1}{3!}(\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^3 + \dots \tag{I.55}
$$

Obviously this series terminates beyond second order, so we have

$$
e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!}(\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2. \tag{I.56}
$$

Let us consider the integration of the first two terms, obviously we have

$$
\begin{aligned}
\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; 1 &= 0 \\
\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b) &= 0
\end{aligned}
$$

as the number of variables integrated over is greater than the number of variables appearing in the integrand. For the case of two pairs of variables one effectively has

$$
\begin{aligned}
e^{-\zeta^* M\zeta} &\rightarrow \frac{1}{2!}(\zeta^* M\zeta)^2 \\
&= \frac{1}{2!}(\zeta_1^* M_{11}\zeta_1 + \zeta_1^* M_{12}\zeta_2 + \zeta_2^* M_{21}\zeta_1 + \zeta_2^* M_{22}\zeta_2)^2 \\
&= (M_{11}M_{22} - M_{12}M_{21})\zeta_1^*\zeta_1\zeta_2^*\zeta_2
\end{aligned}
\tag{I.57}
$$

where the last line follows from the anticommutating character of the Grassmann numbers. The integration of $\zeta_1^*\zeta_1\zeta_2^*\zeta_2$, gives unity, and so for this case

$$
\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = detM
\tag{I.58}
$$

One should suspect that this result holds in general. For the case of N pairs of variables, only the term of order $(\zeta^* M\zeta)^N$ survives in the expansion of the exponential and contributes to the integral:

$$
\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = \frac{(-1)^N}{N!}\int \prod_{k=1}^{N}(d\zeta_k^* d\zeta_k)(\zeta^* M\zeta)^N .
\tag{I.59}
$$

In view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than go through this combinatorial exercise we will follow the method given in (Brown QFT).(page83) which is presented in Appendix(B). The integral is some function $I(M)$ of the matrix M, let us derive a differential equation for this function. Since $\zeta^* M\zeta$ contains the product of two anti commutating variables and thus a commutating variable itself,

$$
\begin{aligned}
\delta_M(\zeta^* M\zeta)^N &= (\zeta^*(M+\delta M)\zeta)^N - (\zeta^* M\zeta)^N \\
&= n(\zeta^*\delta M\zeta)(\zeta^* M\zeta)^{n-1}
\end{aligned}
\tag{I.60}
$$

$$
\begin{aligned}
\delta I &= \int [d\zeta^* d\zeta]\ (e^{-\zeta^*(M+\delta M)\zeta} - e^{-\zeta^* M\zeta}) \\
&= \int [d\zeta^* d\zeta] \sum_{n=0} \frac{(-1)^n}{n!}[(\zeta^*(M+\delta M)\zeta)^n - (\zeta^* M\zeta)^n] \\
&= -\int [d\zeta^* d\zeta]\ (\zeta^*\delta M\zeta)\sum_{n=0}\frac{(-1)^{n-1}}{(n-1)!}(\zeta^* M\zeta)^{n-1} \\
&= -\int [d\zeta^* d\zeta]\ \zeta^*\delta M\zeta\ e^{-\zeta^* M\zeta}
\end{aligned}
\tag{I.61}
$$

Since $\zeta^* M\zeta$ commute, the derivative of $e^{-\zeta^* M\zeta}$ is given by

$$\begin{aligned}
\frac{\partial}{\partial \zeta_k^*} e^{-\zeta^* M\zeta} &= \frac{\partial}{\partial \zeta_k^*} \sum_{n=0} \frac{(-1)^n}{n!} (\zeta^* M\zeta)^n \\
&= \sum_{n=1} \frac{(-1)^n}{n!} \Bigg([\frac{\partial}{\partial \zeta_k^*} (\zeta^* M\zeta)](\zeta^* M\zeta)^{n-1} + \\
&\quad + (\zeta^* M\zeta)[\frac{\partial}{\partial \zeta_k^*} (\zeta^* M\zeta)](\zeta^* M\zeta)^{n-2} + \cdots + (\zeta^* M\zeta)^{n-1}[\frac{\partial}{\partial \zeta_k^*} (\zeta^* M\zeta)] \Bigg) \\
&= -\sum_{m=1} M_{km}\zeta_m \sum_{n=1} \frac{(-1)^{n-1}}{(n-1)!} (\zeta^* M\zeta)^{n-1} \\
&= -\sum_{m=1} M_{km}\zeta_m e^{-\zeta^* M\zeta}. \qquad \text{(I.62)}
\end{aligned}$$

Hence

$$\begin{aligned}
\delta I &= \int [d\zeta^* d\zeta] \zeta^* \delta M (M^{-1} M\zeta \, e^{-\zeta^* M\zeta}) \\
&= \int [d\zeta^* d\zeta] \zeta^* \delta M M^{-1} \frac{\partial}{\partial \zeta^*} e^{-\zeta^* M\zeta} \qquad \text{(I.63)}
\end{aligned}$$

From () we have

$$\frac{\partial}{\partial \zeta_k^*} (\zeta_m^* F) = \delta_{km} F - \zeta_m^* \frac{\partial}{\partial \zeta_k^*} F. \qquad \text{(I.64)}$$

Therefore,

$$\delta I = \int [d\zeta^* d\zeta] \sum_{k,m} (\delta M M^{-1})_{mk} \left\{ \delta_{km} e^{-\zeta^* M\zeta} - \frac{\partial}{\partial \zeta_k^*} (\zeta_m^* e^{-\zeta^* M\zeta}) \right\} \qquad \text{(I.65)}$$

Since the Grassmann integral of a derivative vanishes,

$$\delta I = (Tr \delta M M^{-1}) I, \qquad \text{(I.66)}$$

which gives

$$\delta \ln I = \frac{\delta I}{I} = Tr(\delta M M^{-1}), \qquad \text{(I.67)}$$

It turns out that

$$Tr(\delta M M^{-1}) = \delta(\ln\ detM) \tag{I.68}$$

which we digress to prove.

$$detM = \sum_n M_{kn} C_{nk} \tag{I.69}$$

Taking the derivative of this, noting that C_{mk} is independent of the element M_{km},

$$\frac{\partial}{\partial M_{km}} detM = C_{mk} \tag{I.70}$$

Now the well known formul for the inverse matrix M^{-1} is

$$(M^{-1})_{kl} = \frac{C_{kl}}{detM} \tag{I.71}$$

So that we have

$$\frac{\partial}{\partial M_{km}} detM = (M^{-1})_{km} detM \tag{I.72}$$

or

$$\frac{\partial}{\partial M_{km}} \ln detM = (M^{-1})_{km}. \tag{I.73}$$

Accodingly,

$$\begin{aligned} \delta \ln detM &= \sum_{k,m} \delta M_{km} \frac{\partial \ln detM}{\partial M_{km}} \\ &= \sum_{k,m} \delta M_{km} (M^{-1})_{mk} \\ &= Tr(\delta M M^{-1}) \\ &= Tr(M^{-1} \delta M). \end{aligned} \tag{I.74}$$

We have thus established (I.68). Therefore the equation (I.67) for $I(M)$ becomes

$$\delta \ln I = \delta(\ln\ detM), \tag{I.75}$$

with the solution

$$I(M) = Const.\, detM. \tag{I.76}$$

Treating the problem as a differential equation for $I(M)$, we set $M = 1$ in order to determine the proportionalitiy constant,

$$\begin{aligned} I(1) &= Const. = \left[\int d\zeta^* d\zeta e^{-\zeta^*\zeta}\right]^N \\ &= \prod_{k=1}^{N} \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^{N} \zeta_i^* \zeta_i)^N \\ &= \frac{1}{N!} \prod_{k=1}^{N} \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^{N} \zeta_i^* \zeta_i)^N \\ &= (-1)^N \int d\zeta_N^* d\zeta_N \dots d\zeta_1^* d\zeta_1 (\zeta_N^* \zeta_N \dots \zeta_1^* \zeta_1) \\ &= 1^N = 1. \end{aligned} \tag{I.77}$$

Hence the constant is unity and we do obtain the expected result:

$$I = \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) e^{-\zeta^* M \zeta} = detM \tag{I.78}$$

This should be compared to the ordinary integration where the corresponding integral gives $\det M^{-1}$.

Grassmann generating Functional

It is not surprising that the Gaussian integral formula (I.78) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp -\zeta^* M \zeta = detM e^{-\frac{1}{2}\rho^T A^{-1} \rho}. \tag{I.79}$$

Here ρ is an n-component vector of Grassmann variables. Equation (I.79) is obtained by translating the integration variable, $\zeta' = \zeta + A^{-1}\rho$.

The construction of functional integration in section (4.1.2) did not make use of any special properties of the integration over field variables which might restrict the validity to ordinary c-numbers.

$$\boxed{\begin{array}{c}\int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[-\int d^dx'\, d^dx \bar{\chi}(x')A(x',x)\chi(x) + \int d^dx(\bar{\rho}(x)\chi(x) + \bar{\chi}(x)\rho(x))\right] \\ = detA \exp\left[\int d^dx'\, d^dx \bar{\rho}(x')A^{-1}(x',x)\rho(x)\right].\end{array}} \tag{I.80}$$

in which the measure is $\propto \prod_{\mathbf{r}} d\bar{\varphi}(r)d\varphi(r)$ and $Z(\rho = 0) = detA$. Note that to normalise the functional we divide by $detA$ as apposed to $det(A^{-1})$ in the bosonic case (**??**).

It is rather straightforward to extend the results of esction 4.1 to the fermionic case: The Grassmann functional derivative is defined

$$\frac{\delta G[\chi(y)]}{\delta\chi(x)} = \lim_{\Delta V_i \to 0} \frac{\partial G}{\partial \chi_i} \quad \text{where } \mathbf{x} \text{ is located in cell } \Delta V_i \tag{I.81}$$

The (2n)-point correlators

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \langle \chi(y_n), \ldots, \chi(y_1); \bar{\chi}(x_1), \ldots, \bar{\chi}(x_n) \rangle \tag{I.82}$$

can now be obtained by forming derivatives of the generating functional [1]

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \left. \frac{\delta^{2n} Z[\rho, \bar{\rho}]}{\delta\rho(x_n) \cdots \delta\rho(x_1)\delta\bar{\rho}(y_1) \cdots \delta\bar{\rho}(y_n)} \right|_{\rho=\bar{\rho}=0}. \tag{I.83}$$

I.3 Quantization on the Space of Classical Solutions

As we shall see, the Fock quantization is naturally constructed from solutions to the classical equations of motion and relies heavily on the linear structure of the space of solutions (The Klein-Gordon and Maxwell equations are linear). Thus, it can only be implemented for quantizing linear (free) field theories.

It is adapted so that it can be generalized to curved (globally-hyperbolic) space-times. Recall that a globally hyperbolic spacetime is one in which the entire history of the universe can be predicted from conditions at the instant of time represented by a hyper-surface Σ, that is, Σ is a Cauchy surface.

The Fock quantization is equivalent to the definition of a complex structure on the classical phase space.

The quantization of a linear field. Construction of the Fock representation.

The relations

[1] The order of the derivatives was chosen in such that we get agreement with the bosionic case. This is not a trivial matter as the Grassmann derivatives $\delta/\delta\rho(x)$ and $\delta/\delta\rho(x)$ anticommute with the field variables $\chi(x)$ and $\bar{\chi}(x)$. One can show, however, that there is an even number of commutations when we carry out the differentiations of (I.83) and write the result in the form (I.82).

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0 \tag{I.84}$$
$$\{q_\mu, q_\nu\} = \delta_{\mu\nu} \tag{I.85}$$

$$\Omega(y_1, y_2) = \sum_\mu (p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}) \tag{I.86}$$

The Hamilton function, H, on Γ takes the form

$$H(t; y) = \sum_{\mu\nu} \frac{1}{2} K_{\mu\nu} y^\mu y^\nu \tag{I.87}$$

where $y^\mu = (q_1, \ldots, q_n; p_1, \ldots, p_n)$, of a point y.

Hamilton's equation's of motion are

$$\frac{dy^\mu}{dt} = \sum_{\rho\nu} \Omega^{\mu\rho} K_{\rho\nu} y^\nu \tag{I.88}$$

Let $y_1(t), y_2(t)$ be two solutions of the equations of motion.

$$\begin{aligned} \frac{d}{dt}\Omega(y_1, y_2) &= \frac{d}{dt}\sum_\mu (p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}) \\ &= \sum_\mu \Big(\frac{dp_{1\mu}}{dt} q_{2\mu} + p_{1\mu} \frac{dq_{2\mu}}{dt} - \frac{dp_{2\mu}}{dt} q_{1\mu} - p_{2\mu} \frac{dq_{1\mu}}{dt} \Big) \\ &= \Big(\frac{\partial H_1}{\partial q_{1\mu}} q_{2\mu} - p_{1\mu} \frac{\partial H_2}{\partial p_{2\mu}} - \frac{\partial H_2}{\partial q_{2\mu}} q_{1\mu} + p_{2\mu} \frac{\partial H_1}{\partial p_{1\mu}} \Big) \\ &= 0 \end{aligned} \tag{I.89}$$

where $H_1 = \frac{1}{2} K_{\mu\nu} y_{1\mu} y_{1\nu}$ and $H_2 = \frac{1}{2} K_{\mu\nu} y_{2\mu} y_{2\nu}$. Thus for a linear dynamical system the symplectic product of two solutions is conseved (This is a consequence of more a general result see section ??). Thus the symplectic structure, $\Omega : \Gamma \times \Gamma \to \mathbb{R}$, on Γ gives rise to a natural symplectic structure (also denoted Ω) on the vector space of solutions, $\mathcal{B}$.

$$\begin{aligned} \{\Omega(y_1, y), \Omega(y_2, y)\} &= \{\sum_\mu (p_{1\mu} q_\mu - p_\mu q_{1\mu}), \sum_\nu (p_{2\nu} q_\nu - p_\nu q_{2\nu})\} \\ &= \sum_{\mu,\nu} (-p_{1\mu} q_{2\mu} \{q_\mu, p_\nu\} - p_{2\mu} q_{1\mu} \{p_\mu, q_\nu\}) \\ &= -\sum_\mu (p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\nu}) \\ &= -\Omega(y_1, y_2) \end{aligned} \tag{I.90}$$

We write

$$\{\Omega(y_1, \cdot\,), \Omega(y_2, \cdot\,)\} = -\Omega(y_1, y_2). \tag{I.91}$$

$$[\hat{\Omega}(y_1, \cdot\,), \hat{\Omega}(y_2, \cdot\,)] = -i\Omega(y_1, y_2)I. \tag{I.92}$$

for all $y_1, y_2 \in \Gamma$.

$$W(y) = \exp[i\Omega(y, \dot{\ })] \tag{I.93}$$

a unitary operator $\hat{W}(y)$ that satisfies

$$\hat{W}(y_1)\hat{W}(y_2) = \exp(i\Omega(y_1, y_2)/2)\hat{W}(y_1 + y_2) \tag{I.94}$$

and

$$\hat{W}^\dagger(y) = \hat{W}(-y) \tag{I.95}$$

I.3.1 Harmonic Oscillators

the annihilation operator

$$a = \sqrt{\frac{\omega}{2}}q + i\sqrt{\frac{1}{2\omega}}p \tag{I.96}$$

$$\begin{aligned} [a, a^\dagger] &= I && \text{(I.97)} \\ H &= \omega(a^\dagger a + \frac{1}{2}I) && \\ [H, a] &= -\omega a. && \text{(I.98)} \end{aligned}$$

the ground state, $|\Psi_0>$, of the hamonic oscillator satisifes

$$a|\Psi_0>= 0 \tag{I.99}$$

The $n-$th excited state, $|\Psi_n>$, is given by

$$|\Psi_n>= \sqrt{\frac{1}{n!}}(a^\dagger)^n|\Psi_0> \tag{I.100}$$

and satisfies

$$H|\Psi_n >= (n+\frac{1}{2})\omega|\Psi_n > \tag{I.101}$$

The time evolution operator $U_t = \exp(-iHt)$. In the Heisenberg representation, the annihilator operator satisfies

$$\frac{da_H}{dt} = i[H, a_H] = -i\omega a_H \tag{I.102}$$

where

$$a_H(t) \equiv U_t^{-1} a U_t. \tag{I.103}$$

Hence

$$a_H(t) = \exp(-i\omega)\ a \tag{I.104}$$

and

$$\begin{aligned} q_H(t) &= \sqrt{\frac{1}{2\omega}}(a_H + a_H^\dagger) \\ &= \sqrt{\frac{1}{2\omega}}(\exp(-i\omega t)\ a + \exp(i\omega t)\ a^\dagger) \end{aligned} \tag{I.105}$$

n Harmonic Oscillators

consider a collection of n decoupled time-independent harmonic oscillators of frequencies $\omega_1, \ldots, \omega_n$. The Hilbert space of the combined system, $\mathcal{F}$, is the tensor product of the Hilbert spaces of the individual systems,

$$\mathcal{F} = \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_n \equiv L^2(Q) \tag{I.106}$$

where Q is the configuration space of the collection of oscillators.

The operrators q_i, p_k extend to

$$\begin{aligned} \hat{q}_i\psi &= \psi_1 \otimes \cdots \otimes \psi_i \otimes \cdots \otimes \psi_n = \psi_1 \otimes \cdots \otimes q_i\psi_i \otimes \cdots \otimes \psi_n \\ \hat{p}_k\psi &= \psi_1 \otimes \cdots \otimes \psi_k \otimes \cdots \otimes \psi_n = \psi_1 \otimes \cdots \otimes -i\hbar\frac{\partial}{\partial q_i}\psi_k \otimes \cdots \otimes \psi_n \end{aligned} \tag{I.107}$$

and form an irreducible representation of the Weyl relations (I.94),(I.95).

the states obtained by applying the creation operators to Ψ_0 span $\mathcal{F}$.

I.3.2 'Fock Space' Quantization

Formulate an alternative construction in terms of the symplectic vector space $(\mathcal{B}, \Omega)$ of classical solutions to the equations of motion rather than in terms of the phase space, Γ.

First we complexify $\mathcal{B}$, that is form the $2n$-complex dimensional vector space $\mathcal{B}^{\mathbb{C}}$ comprising of elements

$$y(t) + i\tilde{y}(t) = (q(t) + i\tilde{q}(t), p(t) + i\tilde{p}(t)) \tag{I.108}$$

where $y(t), \tilde{y}(t) \in \mathcal{B}$. We extend Ω to $\mathcal{B}^{\mathbb{C}}$ by

$$\Omega(y_1, y_2) = \sum_{\mu} \big((p_{1\mu} + i\tilde{p}_{1\mu})(q_{2\mu} + i\tilde{q}_{2\mu}) - (p_{2\mu} + i\tilde{p}_{2\mu})(q_{1\mu} + i\tilde{q}_{1\mu})\big) \tag{I.109}$$

Consider the map

$$\begin{aligned} (y_1, y_2) &= -i\Omega(\overline{y}_1, y_2) \\ &= -i\sum_{\mu} \big((p_{1\mu} - i\tilde{p}_{1\mu})(q_{2\mu} + i\tilde{q}_{2\mu}) - (p_{2\mu} + i\tilde{p}_{2\mu})(q_{1\mu} - i\tilde{q}_{1\mu})\big) \end{aligned} \tag{I.110}$$

This has the properties of an inner product, except positive definitness:

$$\begin{aligned} (y_1, y_2 + y_3) &= -i\Omega(\overline{y}_1, y_2 + y_3) \\ &= -i\Omega(\overline{y}_1, y_2) - i\Omega(\overline{y}_1, y_3) \\ &= (y_1, y_2) + (y_1, y_3) \end{aligned} \tag{I.111}$$

$$\begin{aligned} (y_1, \alpha y_2) &= -i\Omega(y_1, \alpha y_2) \\ &= -i\alpha\Omega(\overline{y}_1, y_2) \\ &= \alpha(y_1, y_2) \end{aligned} \tag{I.112}$$

$$\begin{aligned} \overline{(y_1, y_2)} &= +i\overline{\Omega(y_1, y_2)} \\ &= -i\Omega(\overline{y}_2, y_1) \\ &= (y_2, y_1) \end{aligned} \tag{I.113}$$

However we do not have $(y, y) \geq 0$ and $(y, y) = 0$ does not imply $y = 0$:

$$q_i(t) = \alpha_i \exp(-i\omega_i t) + \beta_i \exp(i\omega_i t) \tag{I.114}$$

$p_i(t) = -i\omega_i(\alpha_i \exp(-i\omega_i t) - \beta_i \exp(i\omega_i t))$

$$\begin{aligned}
(y, y) &= -i\Omega(\overline{y}, y) \\
&= \sum_i -i\big(\overline{p_i} q_i - p_i \overline{q_i}\big) \\
&= \sum_i \omega_i \Big((\overline{\alpha_i} \exp(i\omega_i t) - \overline{\beta}_i \exp(-i\omega_i t))(\alpha_i \exp(-i\omega_i t) + \beta_i \exp(i\omega_i t)) + \\
&\quad + \big(\alpha_i \exp(-i\omega_i t) - \beta_i \exp(i\omega_i t))(\overline{\alpha}_i \exp(i\omega_i t) + \overline{\beta}_i \exp(-i\omega_i t)) \Big) \\
&= 2 \sum_i \omega_i (|\alpha_i|^2 - |\beta_i|^2)
\end{aligned} \tag{I.115}$$

Positive frequency solutions and a Hilbert space

solutions of motion which oscillate with purely positive frequency:

$$q_i(t) = \alpha_i \exp(-i\omega_i t). \tag{I.116}$$

Then (,) is positive definite,

$p_i(t) = -i\omega_i \alpha_i \exp(-i\omega_i t)$

$q_{1i}(t) = \alpha_i \exp(-i\omega_i t)$, $q_{2i}(t) = \alpha_i \exp(-i\omega_i (t + \tau_{i12}))$

$$(y_1, y_1) = 2 \sum_i \omega_i |\alpha_i|^2 \geq 0 \tag{I.117}$$

Therefore the $n-$dimensional subspace $\mathcal{B}^{\mathbb{C}}$ spanned by the complex positve frequency solutions of the classical equations of motion, which we denote $\mathcal{H}$, is an $n-$dimensional complex Hilbert space. Next we will see that we can introduce a representation of the Heisenberg position and momentum operators on $\mathcal{F}'$ associated with $\mathcal{H}$. The construction of the Hilbert space $\mathcal{F}'$ with these Heisenberg position and momentum operators define a quantum theory of the system of harmonic oscillators.

The Symmetric Fock space

This alternative construction denoted by $(\mathcal{F}', q_i', p_i', H')$

The symmetric Fock space, $\mathcal{F}_s(\mathcal{H})$, associated with $\mathcal{H}$ are defined by the direct sum of the complex numbers with all the symmetrized tensor products of $\mathcal{H}$

$$\mathcal{F}_s(\mathcal{H}) = \oplus_{n=0}^{\infty} (\otimes_s^n \mathcal{H}) \tag{I.118}$$

an element $\psi \in \otimes^n \mathcal{H}$ will be denoted as $\psi^{a_1 a_2 \dots a_n}$. The subspace $\psi \in \otimes_s^n \mathcal{H}$ consists of those elements satisfying

$$\psi^{a_1 a_2 \dots a_n} = \psi^{(a_1 a_2 \dots a_n)}. \tag{I.119}$$

We need to give the general defintion of the annihilation and creation operators on a symmetric Fock space. Let $\mathcal{H}$ be a Hilbert space with $\mathcal{F}_s(\mathcal{H})$ its associated symmetric Hilbert space. Each $\Psi \in \mathcal{F}_s(\mathcal{H})$ can be written as

$$\Psi = (\psi, \psi^{a_1}, \psi^{a_1 a_2}, \dots) \tag{I.120}$$

$$\begin{aligned} a(\overline{\xi})\Psi &= (\overline{\xi}_a \psi^a, \sqrt{2}\overline{\xi}_a \psi^{a a_1}, \sqrt{3}\overline{\xi}_a \psi^{a a_1 a_2}, \dots) \\ a^\dagger(\xi)\Psi &= (0, \psi \xi^a, \sqrt{2}\xi^{(a_1}\psi^{a_2)}, \sqrt{3}\xi^{(a_1}\psi^{a_2 a_3)}, \dots) \end{aligned} \tag{I.121}$$

$$\begin{aligned} a(\overline{\xi})a^\dagger(\eta)\Psi &= a(\overline{\xi})(0, \psi\eta^{a_1}, \sqrt{2}\eta^{(a_1}\psi^{a_2)}, \sqrt{3}\eta^{(a_1}\psi^{a_2 a_3)}, \dots) \\ &= (\psi \xi_{a_1}\eta^{a_1}, 2\overline{\xi}_{a_1}\eta^{(a_1}\psi^{a_2)}, 3\overline{\xi}_{a_1}\eta^{(a_1}\psi^{a_2 a_3)}) \end{aligned} \tag{I.122}$$

$$\begin{aligned} a^\dagger(\eta)a(\overline{\xi})\Psi &= a^\dagger(\eta)((\overline{\xi}_a\psi^a), \sqrt{2}\,\overline{\xi}_a\psi^{a a_1}, \sqrt{3}\,\overline{\xi}_a\psi^{a a_1 a_2}, \dots) \\ &= (0, \overline{\xi}_a \psi^{(a}\eta^{a_1)}, 2\overline{\xi}_a\psi^{(a a_1}\eta^{a_2)}, 3\overline{\xi}_a\psi^{(a a_1 a_2}\eta^{a_3)}, \dots). \end{aligned} \tag{I.123}$$

$[a(\overline{\xi}), a^\dagger(\eta)]\Psi = (\xi_a \eta^a \psi, \xi_a \eta^{(a}\psi^{a_1)}, \xi_a \eta^{(a}\psi^{a_1 a_2)}, \dots) = \overline{\xi}_a \eta^a \Psi$

$a(\overline{\xi})$ and $a^\dagger(\eta)$ satisfy the commutation relation,

$$[a(\overline{\xi}), a^\dagger(\eta)] = \overline{\xi}_a \eta^a I. \tag{I.124}$$

The state

$$|0> = (1, 0, 0, \dots) \tag{I.125}$$

represents the vacuum state. The vacuum state is uniquely characterized by the condition

$$a(\overline{\xi})|0> = 0. \tag{I.126}$$

Let

$$\xi_i \in \mathcal{H}$$

denote the complex positive frequency solution in which only the $i-$th harmonic oscillator is excited,

$$\xi_i(t) = (0; 0, \dots, q_i(t); p_i(t), \dots, 0; 0)$$

normalized so that

$$\|\xi_i\| = (\xi_i, \xi_i) = 1$$

that is we take $|\alpha_i|$ in (I.116) so that

$$2\omega_i|\alpha_i|^2 = 1.$$

Then $\{\xi_i\}$ forms an orthonormal basis of $\mathcal{H}$. A general element of $\mathcal{H}$ can be then be written as

$$\sum_{j=1}^{n} \beta_j \xi_j = (\beta_1\alpha_1 \exp(-i\omega_1 t), \dots, \beta_j\alpha_j \exp(-i\omega_j t), \dots, \beta_n\alpha_n \exp(-i\omega_n t)) \tag{I.127}$$

where β_i are arbitrary complex numbers.

Let a_i denote the annihilation operator on $\mathcal{F}_s(\mathcal{H})$ associated with $\overline{\xi}_i$.

take Heisenberg position and momentum operators on $\mathcal{F}' = \mathcal{F}_s(\mathcal{H})$ by

$$\begin{aligned} q'_{iH}(t) &= \xi_i(t)a_i + \overline{\xi}_i(t)a_i^\dagger & \text{(I.128)} \\ p'_{iH}(t) &= \frac{dq'_{iH}}{dt} & \text{(I.129)} \end{aligned}$$

$$p'_{iH} = -i\omega_i\alpha_i \exp(-i\omega_i t)$$

$$
\begin{aligned}
[q'_{iH}, p'_{iH}]\Psi &= [\xi_i(t)a_i + \overline{\xi}_i(t)a_i^\dagger, \frac{d\xi_i}{dt}(t)a_i + \frac{d\overline{\xi}_i}{dt}(t)a_i^\dagger]\Psi \\
&= [\xi_i(t)a_i, \frac{d\overline{\xi}_i}{dt}(t)a_i^\dagger]\Psi + [\overline{\xi}_i(t)a_i^\dagger, \frac{d\xi_i}{dt}(t)a_i]\Psi \\
&= (\xi_i \frac{d\overline{\xi}_i}{dt} - \overline{\xi}_i \frac{d\xi_i}{dt})\Psi \\
&= 2i\omega_i|\alpha_i|^2\Psi \\
&= i\Psi. \qquad \text{(I.130)}
\end{aligned}
$$

by normalization of ξ_i. For $i \neq j$

$$
[q'_{iH}, p'_{jH}] = (\xi_i \frac{d\overline{\xi}_j}{dt} - \overline{\xi}_i \frac{d\xi_j}{dt})\Psi = 0 \qquad \text{(I.131)}
$$

as . So we have

$$
[q'_{iH}, p'_{jH}] = i\delta_{ij}. \qquad \text{(I.132)}
$$

$$
(\psi_1, \psi_2)_2 = (-i)^2 \Omega_{\mu\nu}\Omega_{\rho\sigma}\psi^{\mu\rho}\psi^{\nu\sigma} \qquad \text{(I.133)}
$$

$$
\begin{aligned}
\xi_i(t) &= (0;0,\ldots,q_i(t);p_i(t),0;0,,\ldots,0;0) \\
\xi_j(t) &= (0;0,\ldots,0;0,q_j(t);p_j(t),\ldots,0;0) \qquad \text{(I.134)}
\end{aligned}
$$

an orthonormal basis

$$
\psi^{ab} = \sum_{i,j} \beta_i^{(1)}\beta_j^{(2)}\xi^a_{(i}\xi^b_{j)}
$$

their norm is

$$
\begin{aligned}
\|\xi_{(i}\xi_{j)}\|^2 &= (\xi_{(i}\xi_{j)}, \xi_{(i}\xi_{j)}) \\
&= (-i)^2\Omega_{ac}\Omega_{bd}\ \xi^a_{(i}\xi^b_{j)}\ \xi^c_{(i}\xi^d_{j)} \\
&= (-i)^2\frac{1}{(2!)^2}\Omega_{ac}\Omega_{bd}\ (\xi^a_i\xi^b_j + \xi^a_j\xi^b_i)(\xi^c_i\xi^d_j + \xi^c_j\xi^d_i) \\
&= (\xi_i, \xi_i)(\xi_j, \xi_j) \\
&= 1. \qquad \text{(I.135)}
\end{aligned}
$$

The inner product of the symmetric Fock space is

$$(\Psi_1, \Psi_2)_{\mathcal{F}} = \sum_{n=0}^{\infty} n!(\psi_1, \psi_2)_n \tag{I.136}$$

Fock space vectors are normed

$$\|\Psi_n\|_{\mathcal{F}}^2 = n!(\psi_n, \psi_n) \tag{I.137}$$

The Fock norm is

$$\|\Psi\|_{\mathcal{F}}^2 = \sum_{n=0}^{\infty} n!(\psi_n, \psi_n) \tag{I.138}$$

Both constructions provide irreducible representations of the Weyl relations - there are no subspaces other than $\{0\}$ and the full Hilbert space which are invariant under all the operators $\hat{W}(y)$. The unitary equivalence follows from the Stone-von Neumann theorem.

The unitary equivalence

Write $a_i^\dagger := a^\dagger(\xi_i)$

$U : \mathcal{F} \to \mathcal{F}'$ given by

$$\begin{aligned}
U\Psi_0 &= |0> = (1, 0, 0, \dots) \\
\\
U\Psi_{1i} &= a_i^\dagger|0> \\
&= (0, \xi_i, 0, 0, \dots) \\
\\
U\Psi_{ij} &= \frac{1}{2!}a_j^\dagger a_i^\dagger|0> \\
&= \frac{1}{2!}a_j^\dagger(0, \xi_i, 0, 0, \dots) \\
&= \frac{1}{\sqrt{2!}}(0, 0, \xi_{(i}\xi_{j)}, 0, \dots) \\
U\Psi_{2ij} &= \frac{1}{3!}a_j^\dagger(a_i^\dagger)^2|0> \\
&= \frac{1}{\sqrt{3!}}(0, 0, \xi_{(i}\xi_i\xi_{j)}, 0, \dots) \\
&\dots
\end{aligned} \tag{I.139}$$

Unitary as

$$\begin{aligned}(U^\dagger\Psi_{ni}, U\Psi_{ni})_{\mathcal{F}} &= (<0|\frac{(a_i)^n}{n!}, \frac{(a_i^\dagger)^n}{n!}|0>)_{\mathcal{F}} \\ &= (\xi_{(i}\xi_i\dots\xi_{i)}, \xi_{(i}\xi_i\dots\xi_{i)}) \\ &= 1 \\ &= <\Psi_{ni}, \Psi_{ni}>\end{aligned} \tag{I.140}$$

$$q'_{iH} = Uq_iU^\dagger \qquad p'_{iH} = Up_iU^\dagger \tag{I.141}$$

$$\begin{aligned}<0|a_i^n\mathcal{O}(t)a_i^{\dagger m}|0> &= <\Psi_n|U^\dagger(U\mathcal{O}(t)U^\dagger)U|\Psi_m> \\ &= \end{aligned} \tag{I.142}$$

Hilbert spaces

The Hilbert space $\overline{\mathcal{H}}$ is the complex conjugate of $\mathcal{H}$ with $\{\overline{e}_1, \dots, \overline{e}_j, \dots\}$ an orthonormal basis.

We shall introduce the abstract index notation for the Hilbert spaces since it is most convenient way of describing the Fock space. Given a space $\mathcal{H}$, we can construct the spaces $\overline{\mathcal{H}}$, the complex conjugate space; $\mathcal{H}^*$, the dual space; and $\overline{\mathcal{H}}^*$ the dual to the complex conjugate.

By using Riesz lemma, we may identify $\overline{\mathcal{H}}$ with $\mathcal{H}^*$ and $\mathcal{H}$ with $\overline{\mathcal{H}}^*$.

Unitary transformations

$$\begin{aligned}K_1 &: \mathcal{B}_\mu^{\mathbb{C}} \to \mathcal{H}_1 & K_2 &: \mathcal{B}_\mu^{\mathbb{C}} \to \mathcal{H}_2 \\ \overline{K}_1 &: \mathcal{B}_\mu^{\mathbb{C}} \to \overline{\mathcal{H}}_1 & \overline{K}_2 &: \mathcal{B}_\mu^{\mathbb{C}} \to \overline{\mathcal{H}}_2\end{aligned} \tag{I.143}$$

$K_1 + \overline{K}_1 = I$ and $K_2 + \overline{K}_2 = I$ and for $\psi \in \mathcal{H}_1$, $K_1\psi = \psi$ and $\overline{K}_1\psi = \emptyset$, e.t.c.

$$\begin{aligned}A &: \mathcal{H}_2 \to \mathcal{H}_1 & C &: \mathcal{H}_1 \to \mathcal{H}_2 \\ B &: \mathcal{H}_2 \to \overline{\mathcal{H}}_1 & D &: \mathcal{H}_1 \to \overline{\mathcal{H}}_2\end{aligned} \tag{I.144}$$

$$U\hat{\Omega}_1(\psi,\ \cdot\ )U^{-1} = \hat{\Omega}_2(\psi,\ \cdot\ ) \tag{I.145}$$

$$U[ia_1(\overline{K}_1\psi) - ia_1^\dagger(K_1\psi)]U^{-1} = ia_2(\overline{K}_2\psi) - ia_2^\dagger(K_2\psi) \tag{I.146}$$

where a_1, $a_1^\dagger$ and a_2, $a_2^\dagger$ are the annihilation and creation operators on $\mathcal{F}_s(\mathcal{H}_1)$ and $\mathcal{F}_s(\mathcal{H}_2)$, respectively.

choosing $\psi \in \overline{\mathcal{H}}_1$ writing $\chi = \overline{\psi}$ $(\chi \in \mathcal{H}_1)$

$$U a_1(\overline{\chi}) U^{-1} = a_2(\overline{C\chi}) - a_2^\dagger(\overline{D\chi}) \tag{I.147}$$

$$\begin{aligned}
(\psi, \chi)_{\mathcal{H}_2} &= -i\Omega(\overline{\psi}, \chi) \\
&= -i\Omega(\overline{K_1\psi + \overline{K}_1\psi}, K_1\chi + \overline{K}_1\chi) \\
&= -i\Omega(\overline{A\psi + B\psi}, A\chi + B\chi) \\
&= -i\Omega(\overline{A\psi}, A\chi) - i\Omega(\overline{B\psi}, B\chi) \\
&= (A\psi, A\chi)_{\mathcal{H}_1} - (B\psi, B\chi)_{\overline{\mathcal{H}}_1}
\end{aligned} \tag{I.148}$$

$$A^\dagger A - B^\dagger B = I \tag{I.149}$$

$$A^\dagger \overline{B} = B^\dagger \overline{A} \tag{I.150}$$

similarly

$$C^\dagger C - D^\dagger D = I \tag{I.151}$$

$$C^\dagger \overline{D} = D^\dagger \overline{C} \tag{I.152}$$

$$A^\dagger = C \tag{I.153}$$

$$\overline{B}^\dagger = -D. \tag{I.154}$$

We can ask how the Fock quantization compares with the standard Schrodinger

representation we are used to in ordinary quantum mechanics. Recall that in this case, quantum states are given by complex-valued functions on con?guration space $\psi(q_i)$. There is however, a unitarily equivalent representation where the wave functions are (analytic) functions on phase space $\varphi(z_j = q_j - ip_j)$. This is the so called Bargmann representation of quantum mechanics. This is not usually done in ordinary quantum mechanics, but we could in fact construct a Fock space for the harmonic oscillator, where the 'particles' would be quanta of energy [??]. In this case the basis is given by the $|n>$ kets, corresponding to the eigenstates of the Hamiltonian. The most natural representation for this construction, in terms of wave-functions is the one given by Bargmann.

I.3.3 The Fock Representation of Field Theory

Fock space vectors are $\Psi_n(g)$, normed by

$$\|\Psi_n\|^2 = n!(g_n, g_n)_{L^2} \tag{I.155}$$

Together they span the Fock space

$$\mathcal{H} = \{\Psi : \Psi = (\Psi_0, \Psi_1, \dots, \Psi_n, \dots)\} \tag{I.156}$$

with norm

$$\|\Psi\|^2_{\mathcal{H}} = \sum_{n=0}^{\infty} n!(g_n, g_n)_{L^2}. \tag{I.157}$$

Creation and Annilihilation Operators

$$\begin{aligned} (a(f)g_n)(x_1, \dots, x_{n-1}) &= \int dx_n f(x_n) g_n(x_1, \dots, x_n) \\ a(f)\Psi_0 &= 0. \end{aligned} \tag{I.158}$$

There are creation operators which add a particle with wavefunction f

$$\begin{aligned} (a^\dagger(f)\Psi_n)(x_1, \dots, x_{n+1}) &= \frac{1}{(n+1)!} \sum_\sigma f(x_{\sigma(1)}) g_n(x_{\sigma(2)}, \dots, x_{\sigma(n)}) \\ &= . \end{aligned} \tag{I.159}$$

find the commutation relations

$$\begin{aligned} [a(f), a(g)] &= [a^\dagger(f), a^\dagger(g)] = 0 \\ \left[a(f), a^\dagger(g)\right] &= (f, g). \end{aligned} \tag{I.160}$$

Coherent States

$$e(f) = \exp(-\frac{1}{2}|f|^2)e^{a^\dagger(f)}\Omega \tag{I.161}$$

coherent states are eigenstates of the annihilation opereatros:

$$a(g)e(f) = (g,f)\cdot e(f) \tag{I.162}$$

$$\begin{aligned} e^{\varphi(f)} &= e^{\frac{1}{2}(f,f)}e^{a^*(f)}e^{a(f)} = e^{\frac{1}{2}(f,f)} : e^{\varphi(f)} : \\ &= (\Omega, e^{i\varphi(f)}\Omega) = e^{-\frac{1}{2}(f,f)}. \end{aligned} \tag{I.163}$$

Summary

Fock space

$$\{\mathcal{H}, \Omega, \varphi\} \tag{I.164}$$

is a Hilbert space $\mathcal{H}$ with field operators φ and vector Ω "vacuum" or "ground state" such that

$$C(f) = \left(\Omega, e^{i\varphi(f)}\Omega\right) = e^{-\frac{1}{2}(f,f)} \tag{I.165}$$

I.3.4 The Fock Representation of a Free Scalar Field

The algebra $\mathcal{S}$ of fundamental observables to be quantized, consists of suitable linear functionals on Γ.

(i) Linearity: for $f \in C_0^\infty$ $f \mapsto \phi(f)$ is linear.

(ii) ϕ solves the Klein-Gordon equation (in a distributional sense)

(iii) Hermiticty: $\phi(f) = \phi(f)^*$

(iv) Weyl relations: canonical commutation relations

$$[\phi(f), \phi(h)] = E(f \otimes h)\mathbb{I}.$$

where E is the causal propagator, i.e. the difference between the advanced and the retarded fundamental solutions the Klein-Gordon equation.

I.3.5 The Fock Representation of the Maxwell Field

The Hilbert space is given by $\mathcal{H}_F = L^2(\mathcal{S}', d\mu_F)$, where $\mathcal{S}'$ is the appropriate space of tempered distributions on Σ and μ_F the Gaussian measure.

The algebra $\mathcal{S}$ of fundamental observables to be quantized, consists of suitable linear functionals on Γ.

The next step is to construct the so called one-particle Hilbert space $\mathcal{H}_0$ from the space Γ. As mentioned before, the one particle Hilbert space $\mathcal{H}_0$ receives this name since it can be interpreted as the Hilbert space of a one particle relativistic system (in the electro-magnetic case, the photon). The one-particle space is constructed by defining a complex structure on Γ compatible with the naturally defined symplectic structure thereon, in order to define a Hermitian inner product on Γ. The completion with respect to this inner product will be the one-particle Hilbert space $\mathcal{H}_0$.

From the Hilbert space $\mathcal{H}_0$ one constructs its symmetric (since we are considering Bose fields) Fock space $\mathcal{F}_s(\mathcal{H})$, the Hilbert space of the theory. The final step is to represent the algebra $\mathcal{S}$ of observables in the Fock space as suitable combinations of (naturally defined) creation and annihilation operators.

I.3.6 Quantum Field Theory on Curved Spacetime - The Basics

Cauchy-surface independent symplectic form

$$\begin{aligned}\omega_{\mathcal{M}}(\phi_1, \phi_2) &:= \int_\Sigma (\pi_1\phi_2 - \pi_2\phi_1) d\mu_g^{(S)} \\ &= \int_\Sigma (\phi_2 n^a \nabla_a \phi_1 - \phi_1 n^a \nabla_a \phi_2) d\mu_g^{(S)} \end{aligned} \tag{I.166}$$

Σ being an arbitrary Cauchy surface with unit surface normal future-pointing vector n and $d\mu_g^{(S)}$ the measure induced on Σ by the metric g.

the smeared Heisenberg operator $\hat{\phi}(f) : \mathcal{F}_s(\mathcal{H}) \to \mathcal{F}_s(\mathcal{H})$ by

$$\hat{\phi}(f) = ia(\overline{K(Ef))}) - ia^\dagger(K(Ef)) \tag{I.167}$$