

Appendix J

Details of Hawking's Calculation

J.1 Decomposition into Complete Basis

$$(\varphi_i, \varphi_j) = -\frac{1}{2}i \int_{\Sigma} (\varphi_i \nabla_{\mu} \varphi_j^* - \varphi_j^* \nabla_{\mu} \varphi_i) d\Sigma^{\mu} = \delta_{ij} \quad (\text{J.1})$$

where $d\Sigma$ is an area element and Σ is a Cauchy surface.

J.2 Solution of Klein-Gordon Equation in Schwarzschild Spacetime

$$g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} \phi = 0 \quad (\text{J.2})$$

This becomes

$$\nabla_{\mu} (g^{\mu\nu} \partial_{\nu} \phi) = 0 \quad (\text{J.3})$$

The covariant derivative of a vector field A^{μ} is given by

$$\nabla_{\mu} A^{\mu} = \partial_{\mu} A^{\mu} + \Gamma_{\nu\mu}^{\mu} A^{\nu}$$

As we have proven

$$\Gamma_{\nu\mu}^{\mu} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g})$$

we have

$$\nabla_{\mu} A^{\mu} = \frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} A^{\mu}) \quad (\text{J.4})$$

Thus the Klein-Gordon equation can be written

$$\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \phi) = 0 \quad (\text{J.5})$$

$$(g_{\mu\nu}) = \begin{pmatrix} -(1 - \frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix} \quad (\text{J.6})$$

g is given by

$$g = -r^4 \sin^2 \theta$$

$$\hat{L}^2 \equiv -\frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \quad (\text{J.7})$$

$$\left[-\frac{r}{r-2M} \frac{\partial^2}{\partial t^2} + \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right\} - \frac{1}{r^2} \hat{L}^2 \right] \phi = 0. \quad (\text{J.8})$$

Now write ϕ as

$$\phi = (Ae^{-i\omega t} + A^*e^{i\omega t})R(r)\Theta(\theta, \varphi) \quad (\text{J.9})$$

$$\left[-\frac{r}{r-2M} (i\omega)^2 + \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right\} - \frac{1}{r^2} \hat{L}^2 \right] R(r)\Theta(\theta, \varphi) = 0. \quad (\text{J.10})$$

dividing both sides

$$\frac{r^2}{R(r)} \left[-\frac{r}{r-2M} (i\omega)^2 + \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right\} \right] R(r) = \frac{1}{\Theta(\theta, \varphi)} \hat{L}^2 \Theta(\theta, \varphi). \quad (\text{J.11})$$

By writing the constant as $l(l+1)$, we obtain

$$\hat{L}^2 \Theta(\theta, \varphi) = l(l+1) \Theta(\theta, \varphi) \quad (\text{J.12})$$

$$\left[\frac{r}{r-2M} \omega^2 + \frac{1}{r^2} \frac{\partial}{\partial r} \left\{ r^2 \left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right\} - \frac{l(l+1)}{r^2} \right] R(r) = 0 \quad (\text{J.13})$$

$\Theta(\theta, \varphi)$ can be expanded

$$\Theta(\theta, \varphi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi) \quad (\text{J.14})$$

where $Y_{lm}(\theta, \varphi)$ are the spherical harmonics.

$$R'(r_*) := rR(r) \quad (\text{J.15})$$

where r_* is the tortoise coordinate defined by

$$r_* := r + 2M \ln \left| \frac{1}{2M} - 1 \right|. \quad (\text{J.16})$$

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial r_*}{\partial r} \frac{\partial}{\partial r_*} \\ &= \left(\frac{r}{r-2M} \right) \frac{\partial}{\partial r_*} \end{aligned} \quad (\text{J.17})$$

$$\begin{aligned} \frac{1}{r} \left(\frac{r}{r-2M} \right) \frac{\partial^2}{\partial r_*^2} R'(r_*) &= \frac{1}{r} \frac{\partial}{\partial r} \left[\left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right] (rR(r)) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left[\left(\frac{r-2M}{r} \right) R(r) + r \left(\frac{r-2M}{r} \right) \frac{\partial R(r)}{\partial r} \right] \\ &= \frac{2M}{r^3} + \frac{2}{r} \left(\frac{r-2M}{r} \right) \frac{\partial R(r)}{\partial r} + \frac{\partial}{\partial r} \left[\left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right] R(r) \\ &= \frac{2M}{r^3} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right] R(r) \end{aligned} \quad (\text{J.18})$$

Under these transformations, equation () becomes

$$\frac{1}{r} \left[\frac{r}{r-2M} \omega^2 - \left(\frac{r}{r-2M} \right) \frac{\partial^2}{\partial r_*^2} - \frac{2M}{r^3} + \frac{1}{r^2} l(l+1) \right] R'(r_*) = 0 \quad (\text{J.19})$$

Dividing by . we obtain

$$\frac{\partial^2}{\partial r_*^2} R'(r_*) + \left[\omega^2 - \frac{1}{r^2} \left\{ \frac{2M}{r} + l(l+1) \right\} \left(1 - \frac{2M}{r} \right) \right] R'(r_*) = 0 \quad (\text{J.20})$$

taking $r \rightarrow \infty$

$$\frac{\partial^2}{\partial r_*^2} R'(r_*) + \omega^2 R'(r_*) = 0 \quad (\text{J.21})$$

The solution for this equation is given by

$$R'(r_*) = C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*} \quad (\text{J.22})$$

$$R(r) = \frac{1}{r} (C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*}). \quad (\text{J.23})$$

Now write ϕ as

$$\begin{aligned} \phi &= (Ae^{-i\omega t} + A^* e^{i\omega t}) R(r) \Theta(\theta, \varphi) \\ &= (Ae^{-i\omega t} + A^* e^{i\omega t}) \frac{1}{r} (C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*}) \Theta(\theta, \varphi) \\ &= \sum \end{aligned} \quad (\text{J.24})$$

Use affine parameters

$$\begin{aligned} v &:= t + r_* \\ u &:= t - r_* \end{aligned} \quad (\text{J.25})$$

where v and u are respectively called the advanced and retarded time.

J.3 Bogoliubov Coefficients

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dv e^{-i\omega'v} p_{\omega} \quad (\text{J.26})$$

$$\beta_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{\infty} dv e^{i\omega'v} p_{\omega}. \quad (\text{J.27})$$

where the partial wave is

$$p_{\omega} \sim \begin{cases} 0, & v > v_0 \\ \frac{P_{\omega}^-}{r\sqrt{2\pi\omega}} \exp[-i4M\omega \ln(\frac{v_0-v}{c})], & v \leq v_0 \end{cases} \quad (\text{J.28})$$

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}} \int_{-\infty}^{v_0} dv \frac{P_{\omega}(r)}{r\sqrt{2\pi\omega}} \exp \left[-i\frac{\omega}{\kappa} \ln \left(\frac{v_0 - v}{c} \right) \right] e^{-i\omega'v} \quad (\text{J.29})$$

near the horizon $r = 2M$

$$\alpha_{\omega\omega'} = \frac{1}{2\pi} \left(\frac{P_{\omega}(2M)}{F_{\omega'}(2M)} \right) e^{i4M\omega} \sqrt{\frac{\omega'}{\omega}} \int_{-\infty}^{v_0} dv (v_0 - v)^{-i4M\omega} e^{-i\omega'v} \quad (\text{J.30})$$

substituting $x = v_0 - v$, we obtain

$$\alpha_{\omega\omega'} = \frac{1}{2\pi} \left(\frac{P_{\omega}(2M)}{F_{\omega'}(2M)} \right) e^{i4M\omega} \sqrt{\frac{\omega'}{\omega}} e^{-i\omega'v_0} \int_0^{\infty} dx x^{-i4M\omega} e^{-(i\omega')x} \quad (\text{J.31})$$

Recall

$$\Gamma(z) = \int_0^{\infty} du u^{z-1} e^{-u}$$

make the substitution $u = ts$

$$\Gamma(\epsilon) = t^{\epsilon} \int_0^{\infty} ds s^{\epsilon-1} e^{-ts}$$

Set $\epsilon = 1 - i4M\omega$ and $t = -i\omega'$ we obtain the final formula

$$\alpha_{\omega\omega'} = \frac{1}{2\pi} \left(\frac{P_{\omega}(2M)}{F_{\omega'}(2M)} \right) e^{i4M\omega} \sqrt{\frac{\omega'}{\omega}} e^{-i\omega'v_0} \Gamma(1 - i4M\omega) (-i\omega')^{1-i4M} \quad (\text{J.32})$$