Appendix J

Details of Hawking's Calculation

J.1 Decomposition into Complete Basis

$$(\varphi_i, \varphi_j) = -\frac{1}{2} i \int_\Sigma (\varphi_i \nabla_\mu \varphi_j^* - \varphi_j^* \nabla_\mu \varphi_i) d\Sigma^\mu = \delta_{ij} \tag{J.1}$$

where $d\Sigma$ is an area element and Σ is a Cauchy surface.

J.2 Solution of Klein-Gordon Equation in Schwarzschild Spacetime

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0 \tag{J.2}$$

This becomes

$$\nabla_\mu (g^{\mu\nu} \partial_\nu \phi) = 0 \tag{J.3}$$

The covariant derivative of a vector field A^μ is given by

$$\nabla_\mu A^\mu = \partial_\mu A^\mu + \Gamma^\mu_{\nu\mu} A^\nu$$

As we have proven

$$\Gamma^\mu_{\nu\mu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g})$$

we have

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^\mu) \tag{J.4}$$

Thus the Klein-Gordon equation can be written

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{J.5}$$

$$(g_{\mu\nu}) = \begin{pmatrix} -(1-\frac{2M}{r}) & 0 & 0 & 0 \\ 0 & (1-\frac{2M}{r})^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{J.6}$$

g is given by

$$g = -r^4\sin^2\theta$$

$$\hat{L}^2 \equiv -\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}. \tag{J.7}$$

$$\left[-\frac{r}{r-2M}\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\} - \frac{1}{r^2}\hat{L}^2\right]\phi = 0. \tag{J.8}$$

Now write ϕ as

$$\phi = (Ae^{-i\omega t} + A^* e^{i\omega t})R(r)\Theta(\theta,\varphi) \tag{J.9}$$

$$\left[-\frac{r}{r-2M}(i\omega)^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\} - \frac{1}{r^2}\hat{L}^2\right]R(r)\Theta(\theta,\varphi) = 0. \tag{J.10}$$

dividing both sides

$$\frac{r^2}{R(r)}\left[-\frac{r}{r-2M}(i\omega)^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\}\right]R(r) = \frac{1}{\Theta(\theta,\varphi)}\hat{L}^2\Theta(\theta,\varphi). \tag{J.11}$$

By writing the constant as $l(l+1)$, we obtain

$$\hat{L}^2\Theta(\theta,\varphi) = l(l+1)\Theta(\theta,\varphi) \tag{J.12}$$

$$\left[\frac{r}{r-2M}\omega^2 + \frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(\frac{r-2M}{r}\right)\frac{\partial}{\partial r}\right\} - \frac{l(l+1)}{r^2}\right]R(r) = 0 \tag{J.13}$$

$\Theta(\theta, \varphi)$ can be expanded

$$\Theta(\theta, \varphi) = \sum_{l,m} a_{lm} Y_{lm}(\theta, \varphi) \tag{J.14}$$

where $Y_{lm}(\theta, \varphi)$ are the spherical harmonics.

$$R'(r_*) := rR(r) \tag{J.15}$$

where r_* is the tortise coordinate defined by

$$r_* := r + 2M \ln \left| \frac{1}{2M} - 1 \right| . \tag{J.16}$$

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial r_*}{\partial r} \frac{\partial}{\partial r_*} \\ &= \left(\frac{r}{r - 2M} \right) \frac{\partial}{\partial r_*} \end{aligned} \tag{J.17}$$

$$\begin{aligned} \frac{1}{r} \left(\frac{r}{r-2M} \right) \frac{\partial^2}{\partial r_*^2} R'(r_*) &= \frac{1}{r} \frac{\partial}{\partial r} \left[\left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right] (rR(r)) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left[\left(\frac{r-2M}{r} \right) R(r) + r \left(\frac{r-2M}{r} \right) \frac{\partial R(r)}{\partial r} \right] \\ &= \frac{2M}{r^3} + \frac{2}{r} \left(\frac{r-2M}{r} \right) \frac{\partial R(r)}{\partial r} + \frac{\partial}{\partial r} \left[\left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right] R(r) \\ &= \frac{2M}{r^3} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \left(\frac{r-2M}{r} \right) \frac{\partial}{\partial r} \right] R(r) \end{aligned} \tag{J.18}$$

Under these transformations, equation () becomes

$$\frac{1}{r} \left[\frac{r}{r-2M} \omega^2 - \left(\frac{r}{r-2M} \right) \frac{\partial^2}{\partial r_*^2} - \frac{2M}{r^3} + \frac{1}{r^2} l(l+1) \right] R'(r_*) = 0 \tag{J.19}$$

Dividing by . we obtain

$$\frac{\partial^2}{\partial r_*^2} R'(r_*) + \left[\omega^2 - \frac{1}{r^2} \left\{ \frac{2M}{r} + l(l+1) \right\} \left(1 - \frac{2M}{r} \right) \right] R'(r_*) = 0 \tag{J.20}$$

taking $r \to \infty$

$$\frac{\partial^2}{\partial r_*^2} R'(r_*) + \omega^2 R'(r_*) = 0 \tag{J.21}$$

The solution for this equation is given by

$$R'(r_*) = C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*} \tag{J.22}$$

$$R(r) = \frac{1}{r}(C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*}). \tag{J.23}$$

Now write ϕ as

$$\begin{aligned} \phi &= (Ae^{-i\omega t} + A^* e^{i\omega t}) R(r) \Theta(\theta, \varphi) \\ &= (Ae^{-i\omega t} + A^* e^{i\omega t}) \frac{1}{r}(C_{\omega l} e^{-i\omega r_*} + C_{\omega l}^* e^{i\omega r_*}) \Theta(\theta, \varphi) \\ &= \sum \end{aligned} \tag{J.24}$$

Use affine parameters

$$\begin{aligned} v &:= t + r_* \\ u &:= t - r_* \end{aligned} \tag{J.25}$$

where v and u are respectively called the advanced and retarded time.

J.3 Bogoliubov Coefficients

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi} F_{\omega'}} \int_{-\infty}^{\infty} dv\ e^{-i\omega' v} p_\omega \tag{J.26}$$

$$\beta_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi} F_{\omega'}} \int_{-\infty}^{\infty} dv\ e^{i\omega' v} p_\omega. \tag{J.27}$$

where the partial wave is

$$p_\omega \sim \begin{cases} 0, & v > v_0 \\ \frac{P_\omega^-}{r\sqrt{2\pi\omega}} \exp\left[-i4M\omega \ln\left(\frac{v_0 - v}{c}\right)\right], & v \le v_0 \end{cases} \tag{J.28}$$

$$\alpha_{\omega\omega'} = \frac{r\sqrt{\omega'}}{\sqrt{2\pi}F_{\omega'}}\int_{-\infty}^{v_0} dv \frac{P_\omega(r)}{r\sqrt{2\pi\omega}} \exp\left[-i\frac{\omega}{\kappa}\ln\left(\frac{v_0 - v}{c}\right)\right] e^{-i\omega' v} \tag{J.29}$$

near the horizon $r = 2M$

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\left(\frac{P_\omega(2M)}{F_{\omega'}(2M)}\right) c^{i4M\omega}\sqrt{\frac{\omega'}{\omega}}\int_{-\infty}^{v_0} dv\ (v_0 - v)^{-i4M\omega} e^{-i\omega' v} \tag{J.30}$$

substituting $x = v_0 - v$, we obtain

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\left(\frac{P_\omega(2M)}{F_{\omega'}(2M)}\right) c^{i4M\omega}\sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0}\int_0^\infty dx\ x^{-i4M\omega} e^{-(-i\omega')x} \tag{J.31}$$

Recall

$$\Gamma(z) = \int_0^\infty du\ u^{z-1} e^{-u}$$

make the substitution $u = ts$

$$\Gamma(\epsilon) = t^\epsilon \int_0^\infty ds\ s^{\epsilon-1} e^{-ts}$$

Set $\epsilon = 1 - i4M\omega$ and $t = -i\omega'$ we obtain the final formula

$$\alpha_{\omega\omega'} = \frac{1}{2\pi}\left(\frac{P_\omega(2M)}{F_{\omega'}(2M)}\right) c^{i4M\omega}\sqrt{\frac{\omega'}{\omega}} e^{-i\omega' v_0}\ \Gamma(1 - i4M\omega)(-i\omega')^{1-i4M} \tag{J.32}$$