Chapter 2

Introduction to General Relativity and its Physical Observables

2.1 Introduction

What we learn about space-time geometry from observation We want to derive physical predictions that can be compared with observations. space-time properties do not have any meaning independent of observations. Space-time geometry, quite literally, has *no reality* independent of observations!

2.2 Special Relativity

We can imagine a frame, or coordinate system, as a collection of clocks (which give the time coordinate at each point) distributed along a set of rules (which give the spatial coordinates of the points):

Equally, we can consider another observer and who has constructed their own frame with their own set of rules and clocks. Let's suppose that we are at rest and that another observer is at constant speed v relative to us. The time coordinate t' that the observer assigns to a point in spacetime is the value that their clock at that point shows; the space coordinate x' they give to points in spacetime is the distance along their ruler of that point in spacetime.

Supposing that this observer is moving at speed v past us. The relation between these two coordinate systems is given by the Lorentz transformations, (??), and using these it is easy to find the coordinates in our frame of particular points in the moving observer's frame, as is shown in fig.(??)

The eather was supposed to provide a medium in which light waves could propagate

Maxwell's equations are among the laws of physics. These equations lead to the prediction that there should exist electromagnetic waves that move at a particular speed - the speed of light, c.

2.2.1 Relativity of Simultaneity

Consider the following thought experiment. A train is travelling along a track with velocity v relative to an observer A on the train platform. Inside the train is the observer B located at the centre of one of the carriages. There are two electrical trigger devices a carriage distance apart and an equal distance from A but on opposite sides. When the carriage containing B goes over the devices, as they do they momentarily activate two light sources located at the ends of the carriage as well as two light sources on the ground a carriage distance apart and an equal distance from A on oppisite sides (see fig 2.1). Observer A, being midway between the ground based light sources, upon receiving the flashes from them at the same instant will conclude that the two light sources switching on occurred *simultaneously*.

However, from A's perspective, B is travelling towards the light emanating from the light source located at the front of the carriage and away from the light emanating from the ligth source located at the rear of the carriage. By the constancy of the speed of light, A will observe B meet the light from the front source before the light from the rear source. Hence observer B concludes the front light source turned on before the rear light source.

Thus, spatially separated events that are simultaneous in one frame of reference are not simultaneous in another, moving relative to the first. This is called relativity of simultaneity.

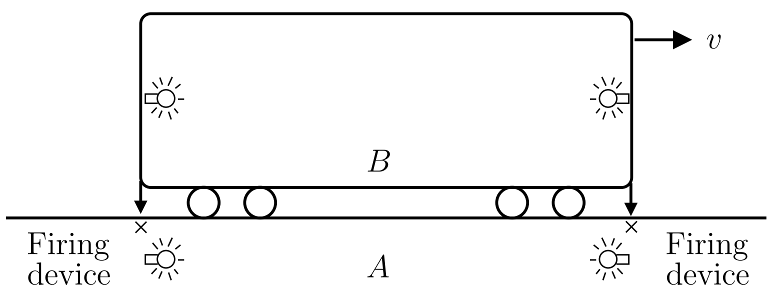

Figure 2.1: simultaneity.

2.2.2 Time Dilation

Consider a light source that directs a light pulse up toward a mirror a distance L above, the light pulse bounces off the mirror back down. It takes a time interval Δt_0 for the light pulse to make the "round trip" to the mirror and back down to the bottom mirror. The total distance is $2L$, so the time interval Δt_0 is

$$\Delta t_0 = \frac{2L}{c}. \tag{2.1}$$

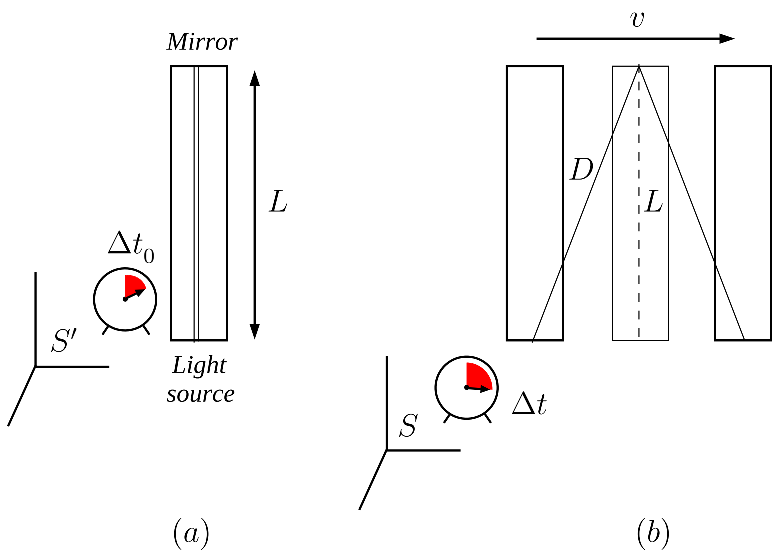

Figure 2.2: timedilF. (a) In system S' a light pulse is emmitted from a source at O' and is reflected back along the same line, and takes a time Δt_0 to perform a round trip. (b) Path of the same light pulse, as observed in the system S. The speed of the light pulse is the same as in system S', but the path is longer, and hence the moving clock takes a longer amount of time, Δt, to perform one tick.

The round-trip time measured by the observer seeing the system in motion at speed v is a different interval Δt. It will be greater than Δt_0, because the light pulse traces out a longer path, a total round trip distance of $2D$, and thus, with the same speed of light c, measures a longer time:

$$\Delta t = \frac{2D}{c} > \frac{2L}{c} = \Delta t_0, \tag{2.2}$$

By Pythagoras' theorem D is given by

$$D = \sqrt{L^2 + \left(\frac{v\Delta t}{2}\right)^2}. \tag{2.3}$$

and therefore we have

$$\Delta t = \frac{2D}{c} = \frac{2}{c}\sqrt{L^2 + \left(\frac{v\Delta t}{2}\right)^2}. \tag{2.4}$$

We wish to eliminate L from the equaion, using (2.1) we obtain,

$$\Delta t = \frac{2}{c}\sqrt{\left(\frac{c\Delta t_0}{2}\right)^2 + \left(\frac{v\Delta t}{2}\right)^2} \tag{2.5}$$

This becomes

$$(\Delta t)^2 = (\Delta t_0)^2 + \left(\frac{v}{c}\Delta t\right)^2 \tag{2.6}$$

and solving for Δt gives,

$$\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}. \tag{2.7}$$

The denominator is always less than unity, and so, as we have already noted, Δt is always larger than Δt_0. Think of a clock in the rest frame, the reading on that clock for the round trip will be Δt_0. When the round trip is measured by a clock in a frame moving with respect to this first clock, the time interval Δt that is recorded will be longer, and this observer thinks the first clock is running more slowly.

Quick calculation: the light pulse with sideways motion has a horizonal component v, and given the magnitude of the light pulse velocity vector must be equal to c, the vertical component must be $\sqrt{c^2 - v^2}$. As it is the vertical component that tells you how quickly the pulse goes up and down: $c\Delta t_0 = 2L = \sqrt{c^2 - v^2}\Delta t$ or $\Delta t = \Delta t_0/\sqrt{1 - v^2/c^2}$.

Proper time

Generally

$$\tau = \int_{t_0}^{t_1} \left(1 - \frac{v^2}{c^2}\right)^{1/2} dt \tag{2.8}$$

2.2.3 Length Contraction

The time taken for the light pulse to make the round trip from the source to the mirror and back is

$$\Delta t_0 = \frac{2l_0}{c} \tag{2.9}$$

$$c\Delta t_1 = l + v\Delta t_1,$$

so that

$$\Delta t_1 = \frac{l}{c-v}.$$

Similarly, one finds for the time Δt_2 for the return trip from the mirror to the source is

$$\Delta t_2 = \frac{l}{c+v}$$

The total time $\Delta t = \Delta t_1 + \Delta t_2$ for the round trip, as measured by O, is

$$\Delta t = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2l}{c(1-v^2/c^2)} \tag{2.10}$$

From the relation between Δt and Δt_0, (2.9) becomes

$$\Delta t\sqrt{1-v^2/c^2} = \frac{2l_0}{c}.$$

We obtain

$$l = l_0\sqrt{1-v^2/c^2}. \tag{2.11}$$

2.2.4 Lorentz Transformations

The question is when an event occurs at a point (x, y, z) at time t, as observed in a frame of reference S, what are the coordinates (x', y', z') and time t' of the event as observed in a second frame S' moving relative to S with constant velocity in the $x-$direction.

Pre-relativity we had the intuiative answer known as the **Galilean coordinate transformation**:

$$x = x' + vt, \quad y = y', \quad z = z'. \tag{2.12}$$

This together with $t = t'$. This transformaion conflicts with the principle of the constancy of the speed of light. Thus, the Galilean transformation needs to be modified. We now derive the modified equations.

As before, we assume that the origins coincide at $t = t' = 0$. Then in S the distance from O to O' is just vt.

The distance from O to P, as seen in S, is

$$x = vt + x'\sqrt{1 - v^2/c^2}. \tag{2.13}$$

Solving for x', we obtain

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \tag{2.14}$$

Now we note that the principal of relativity requires that the form of the transformation from S to S' be identical to that from S' to S, the only difference is a change in the sign of the relative velocity v. Thus from (2.13) it must be that

$$x' = -vt' + x\sqrt{1 - v^2/c^2}. \tag{2.15}$$

Equating (2.14) and (2.15) gives, after some rearangement, an equation between t' and t and x,

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}. \tag{2.16}$$

The lengths perpendicular to the direction of relative motion are unaffected, i.e. $y' = y$ and $z' = z$.

Collecting all the transformation equations, we have

$$
\begin{aligned}
x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \\
y' &= y, \\
z' &= z, \\
t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}.
\end{aligned} \tag{2.17}
$$

These equations are the **Lorentz transformation** formula, the relativistic generalisation of the Galilean transformation, (2.12).

Time dilation again

Here we derive the time dilation effect from the Lorentz transformation formula (2.16). See fig. 2.2. When the clock at rest with respect to the "stationary" system registers time t, the clock at rest with respect to the "moving" system is at position $x = vt$. We substitute $x = vt$ into (2.16) and obtain

$$
\begin{aligned}
t' &= \frac{t - v^2 t/c^2}{\sqrt{1 - v^2/c^2}} \\
&= t\frac{1 - v^2/c^2}{\sqrt{1 - v^2/c^2}} \\
&= t\sqrt{1 - v^2/c^2}
\end{aligned} \tag{2.18}
$$

which says the reading on the movig clock lags behind the reading on the clock at rest with respect to the "stationary" system S - again the observer in the "stationary" system concludes that the moving clock is runnining slow.

2.2.5 Velocities

Transformation of velocities

We can use the Lorentz transformation formula, (2.17), to derive relativistic velocity-transformation equations. Suppose we have a particle is in motion. In time dt it moves the distance $d\mathbf{r} = (dx, dy, dz)$ as measured in the frame S. We obtain the corresponding distance $d\mathbf{r}' = (dx', dy', dz')$ and time dt' in S' by taking differentials of equations (2.17)

Now dx/dt is the velocity-component u_1 in S, and dx'/dt' is the velocity-component in S', etc. In summary

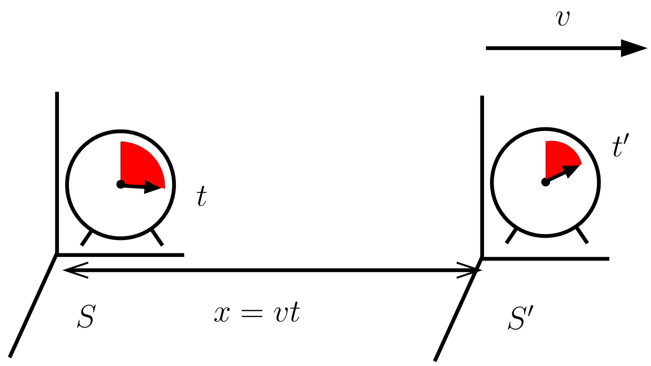

Figure 2.3: timedilLorz. Say we have a clock at rest with respect to the system S located at the origin of S. We also have a clock at rest with respect to the "moving" system S' located at the origin of S'. We assume that the origins coincide at an initial time $t = t' = 0$. Recall this assumption was made when deriving the Lorentz transformation equations.

$$(u_1, u_2, u_3) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt},\right)$$

and

$$(u_1', u_2', u_3') = \left(\frac{dx'}{dt'}, \frac{dy'}{dt'}, \frac{dz'}{dt'}\right).$$

Taking differentials of a Lorentz transformation

$$t' = \beta(t - vx/c^2), \qquad x' = \beta(x - vt), \qquad y' = y, \qquad z' = z,$$

we get

$$dt' = \beta(dt - vdx/c^2), \qquad dx' = \beta(dx - vdt), \qquad dy' = dy, \qquad dz' = dz \tag{2.19}$$

The transformation formula for the velocity component parallel to the direction of motion of the frame S' is then

$$\begin{aligned} \frac{dx'}{dt'} &= \frac{\beta(dx - vdt)}{\beta(dt - vdx/c^2)} \\ &= \frac{\frac{dx}{dt} - v}{1 - v\frac{dx}{dt}/c^2} \end{aligned}$$

or

$$u'_1 = \frac{u_1 - v}{1 - u_1 v/c^2}. \tag{2.20}$$

The transformation formula for the velocity components tranverse to the direction of motion of the frame S' are

$$\begin{aligned} \frac{dy'}{dt'} &= \frac{dy}{\beta(dt - v dx/c^2)} \\ &= \frac{\frac{dy}{dt}}{\beta\left[1 - v\frac{dx}{dt}/c^2\right]} \end{aligned}$$

or

$$u'_2 = \frac{u_2}{\beta(1 - u_1 v/c^2)} \tag{2.21}$$

and, obviously,

$$u'_3 = \frac{u_3}{\beta(1 - u_1 v/c^2)} \tag{2.22}$$

We see that the velocity componenets u_2 and u_2 transverse to the direction of motion of the frame S' are affected by the Lorentetz transformaiton.

Particle motion in the $x-$direction only

Consider the case where the particle only has motion in the $x-$direction, then

$$u'_1 = \frac{u_1 - v}{1 - v u_1/c^2}.$$

What if $u_1 = c$? We would obtain

$$u'_1 = \frac{c - v}{1 - v/c} = c.$$

This says that a particle moving with speed $u_1 = c$ relative to S also has speed $u'_1 = c$ relative to the to S', despite the relative motion of the two frames. Therefore equation (2.20) is consistent with the principle of the constancy of the speed of light which states the speed of light is the same in all inertial frames of reference.

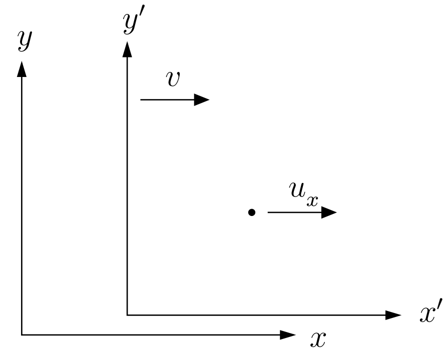

Figure 2.4: A particle moves with veolicty u_x, relative to frame S. We wish to know the velocity of the particle with respect to the frame S' which is moving with velocity v relative to frame S.

Addition of velocities

Say in the S' frame a particle moves at velocity u'_x. We wish to know what velocity the particle moves at in the S frame that sees the first frame at velocity v (see fig ()). As there is no fundamental distinction between the two frames S and S', the expression for u_x in terms of u'_x must have the sameform as (2.20), with u_x and u'_x interchanged and with the sign of v reversed, giving

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2}. \tag{2.23}$$

This formula can also be derived by rearranging solving (2.20) for u_x.

If we put $u'_x = c$ (in 2.23) we obtain

$$u_x = \frac{c + v}{1 + v/c} = c.$$

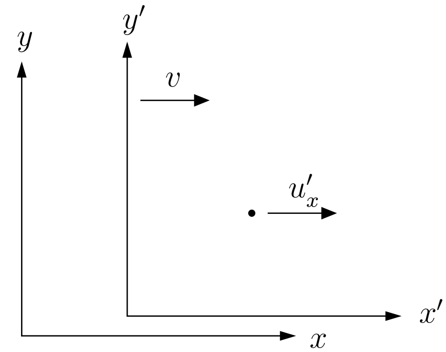

Figure 2.5: A particle moves with veolicty u'_x, relative to frame S', which is itself moving with velocity v relative to frame S. We wish to know the velocity of the particle with respect to the frame S.

2.2.6 Acceleration

2.2.7 The Relativistic Doppler Effect

Consider a source of light with wavelength λ_0 in its rest frame S. What wavelength will an observer S' moving at velocity u relative to S see the wavelength to be? First let us consider the non-relativistic effect. Say tw successive pulses are sent. The second will have to travel an extra distance

$$\Delta x = u_r dt'$$

Seeing as the pulse travels at the speed c, the second pulse arrives at extra time $\Delta x/c = u_r dt'/c$, so that they arrive with time difference

$$\Delta t = dt' + u_r dt'/c,$$

giving

$$\Delta t/dt' = 1 + u_r/c.$$

Now from the fundamental relations

$$\lambda_0 = cdt' \quad \text{and} \quad \lambda = c\Delta t$$

we obtain the classical Doppler formula

$$\lambda/\lambda_0 = 1 + u_r/c. \tag{2.24}$$

We now turn to the relativistic case. Because of time dialation the above is modified such that

$$\Delta t = \frac{dt'}{\sqrt{1 - v^2/c^2}} + u_r \frac{dt'/c}{\sqrt{1 - v^2/c^2}},$$

and so the special relativistic Doppler formula is

$$\lambda/\lambda_0 = \frac{1 + u_r/c}{\sqrt{1 - v^2/c^2}}. \tag{2.25}$$

If the velocity of the source is purely radial, then $u_r = v$ and the above equation becomes

$$\lambda/\lambda_0 = \sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{2.26}$$

2.2.8 Relativistic Momentum

If we look at a collision in one inertial frame of reference S and find that momentum is conserved. The we use the Lorentz transformation to obtain velocities in a second inertial system S'. If we use the Newtonian defintion of momentum, this is not conserved in the second system.

In order for momentum conservation in collisions to hold in all interial frames, the definition of momentum must be generalized.

$$\mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m_0 \mathbf{v}. \tag{2.27}$$

2.2.9 Relativistic Mass and Energy

The rest mass of a particle is the mass of the particle as maasured in the instantaneous frame of that particle. If a particle is maoving with respect to an observer O with velocity v then O measures the mass of the particle as

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} = \gamma m_0. \tag{2.28}$$

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right) = m_0 + \frac{1}{2} m_0 \frac{v^2}{c^2} \tag{2.29}$$

rest mass energy plus Newtonian kinetic energy of the particle:

$$mc^2 = m_0 c^2 + \frac{1}{2} m_0 v^2. \tag{2.30}$$

2.2.10 The Twin Paradox

for a complete resolution of this problem special relativity needs to be extended to include accelaration.

2.2.11 Lorentz group

Recall the Lorentz transformations

$$\begin{aligned} x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z, \\ t' &= \gamma(t - vx/c^2). \end{aligned} \tag{2.31}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{2.32}$$

This can be written in matrix form,

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v/c^2 & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}. \tag{2.33}$$

Here we shall demonstrate that Lorentz transformations form a group. The axioms of a group are:

1. Closure: If Λ_1 and Λ_2 are transformations then their composition is also a transformation of the group.

2. Associativity: $(\Lambda_1\Lambda_2)\Lambda_3 = \Lambda_1(\Lambda_2\Lambda_3)$.

3. Identity: There is an element of the group Λ_e of the group such that for any element Λ of the group

$$\Lambda\Lambda_e = \Lambda_e\Lambda = \Lambda.$$

4. Inverse: For any element Λ of the group there is an element Λ^{-1} such that

$$\Lambda\Lambda^{-1} = \Lambda_e$$

and

$$\Lambda^{-1}\Lambda = \Lambda_e.$$

We want consider the composition of two Lorentz transformations. Recall the formula for addition of velocities,

$$v_2' = \frac{v_1 + v_2}{1 + v_1 v_2/c^2}. \tag{2.34}$$

We have

$$\begin{aligned}
\gamma_2' &:= \frac{1}{\sqrt{1 - v_2'^2/c^2}} \\
&= \frac{1}{\sqrt{\left[1 - \left(\frac{v_1+v_2}{1+v_1v_2/c^2}\right)^2/c^2\right]}} \\
&= \frac{1 + v_1v_2/c^2}{\sqrt{\left[(1 + v_1v_2/c^2)^2 - (v_1 + v_2)^2/c^2\right]}} \\
&= \frac{1 + v_1v_2/c^2}{\sqrt{[(1 + v_1^2/c^2)(1 + v_2^2/c^2)]}} \\
&= \gamma_1\gamma_2(1 + v_1v_2/c^2)
\end{aligned} \tag{2.35}$$

and

$$
\begin{aligned}
\gamma_2' v_2' &= \gamma_1 \gamma_2 (1 + v_1 v_2/c^2) \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \\
&= \gamma_1 \gamma_2 (v_1 + v_2) \qquad (2.36)
\end{aligned}
$$

Employing these results, we can write,

$$
\begin{aligned}
\begin{pmatrix} x'' \\ y'' \\ z'' \\ t'' \end{pmatrix} &= \begin{pmatrix} \gamma_2 & 0 & 0 & -\gamma_2 v_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2 v_2/c^2 & 0 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & 0 & 0 & -\gamma_1 v_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1 v_1/c^2 & 0 & 0 & \gamma_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \\
&= \begin{pmatrix} \gamma_1 \gamma_2 (1 + v_1 v_2/c^2) & 0 & 0 & -\gamma_1 \gamma_2 (v_1 + v_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_1 \gamma_2 (v_1 + v_2)/c^2 & 0 & 0 & \gamma_1 \gamma_2 (1 + v_1 v_2/c^2) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \\
&= \begin{pmatrix} \gamma_2' & 0 & 0 & -\gamma_2' v_2' \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma_2' v_2'/c^2 & 0 & 0 & \gamma_2' \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \qquad (2.37)
\end{aligned}
$$

and so composition also gives a Lorentz transformation and is consistent.

We have:

Closure property follows from the fact that composition of two Lorentz transformations is also a Lorentz transformation.

The identity element is given by $v = 0$, if which case the matrix is the identity matrix

$$
\Lambda_e = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

The inverse element is given by $v_2 = -v_1$ as by (2.34) $v_2' = 0$

We check associativity via associativity of velocity composition. Set

$$
v_2' = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} \qquad (2.38)
$$

and

$$v_3' = \frac{v_2 + v_3}{1 + v_2 v_3/c^2}. \tag{2.39}$$

First we have

$$\begin{aligned}
\frac{v_2' + v_3}{1 + v_2' v_3/c^2} &= \frac{\frac{v_1+v_2}{1+v_1v_2/c^2} + v_3}{1 + \frac{v_1+v_2}{1+v_1v_2/c^2} v_3/c^2} \\
&= \frac{v_1 + v_2 + v_3 + v_1 v_2 v_3/c^2}{1 + v_1 v_2/c^2 + (v_1 + v_2) v_3/c^2}
\end{aligned} \tag{2.40}$$

and then

$$\begin{aligned}
\frac{v_1 + v_3'}{1 + v_1 v_3'/c^2} &= \frac{v_1 + \frac{v_2+v_3}{1+v_2v_3/c^2}}{1 + v_1 \frac{v_2+v_3}{1+v_2v_3/c^2}/c^2} \\
&= \frac{v_1 + v_2 + v_3 + v_1 v_2 v_3/c^2}{1 + v_2 v_3/c^2 + v_1 (v_2 + v_3)/c^2}
\end{aligned} \tag{2.41}$$

and so associativity holds.

2.3 The Principles of General Relativity

2.3.1 The Principle of Equivalence

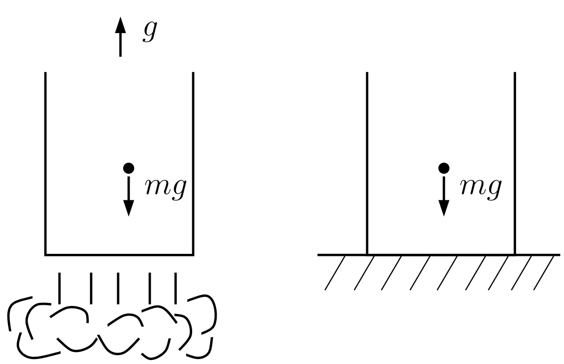

Figure 2.6: rocketEarth. .

The principle of equivalence states that it is impossible to distinguish the effect of gravity from acceleration and the absence of gravity from free fall.

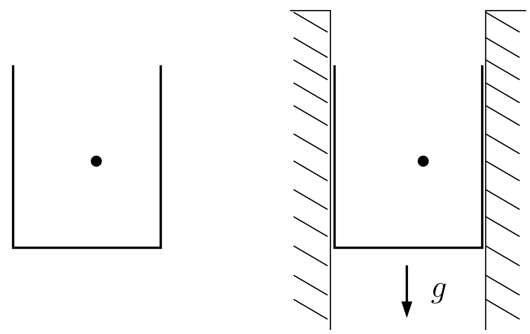

Figure 2.7: rocketEarth2. .

2.3.2 The Gravitation Red-shift: Warping Time

The principle of equivalence tells us that clock rates are affected by gravity

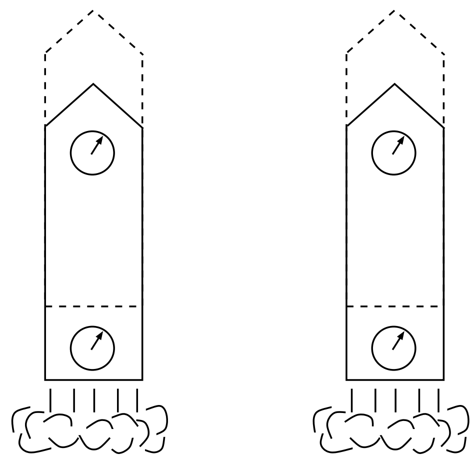

Figure 2.8: rocket. The clock at the top seems to run faster than the one on the bottom.

The time it takes light to travel down is to first order in H/c where H is the height of the rocket. In this time the bottom of the rocket has acquired a small aditional velocity, $v = gH/c$. The frequency is shifted

$$\omega = \omega_0 \left(1 + \frac{gH}{c^2}\right). \tag{2.42}$$

$$(\text{rate at the reciever}) \;=\; (\text{rate of emission}) \left(1 + \frac{gH}{c^2}\right) \tag{2.43}$$

where H is the Height of the emitter above the reciever.

the clock is in a perfectly legitimate frame of reference and works normally.

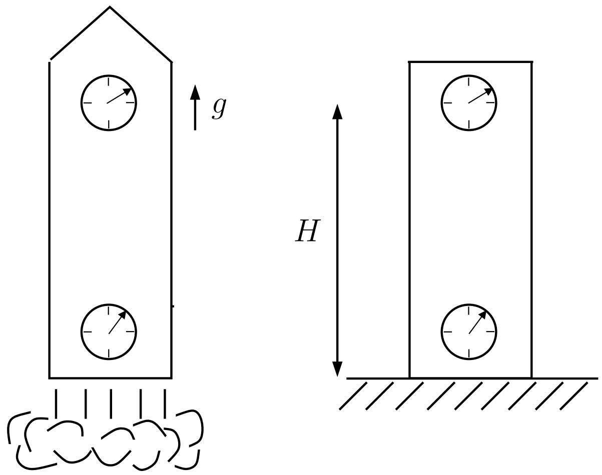

Figure 2.9: rocketaccel. The clock at the top seems to run faster than the one on the bottom.

2.3.3 The Curvature of Spacetime

The path of the light ray as observed in the accelerated frame of reference.

according to the principle of equivalence, the geometry along which light rays propagate in a rocket not flat but curved space, the spatial curvature depending upon the local value of the gravitation acceleration g.

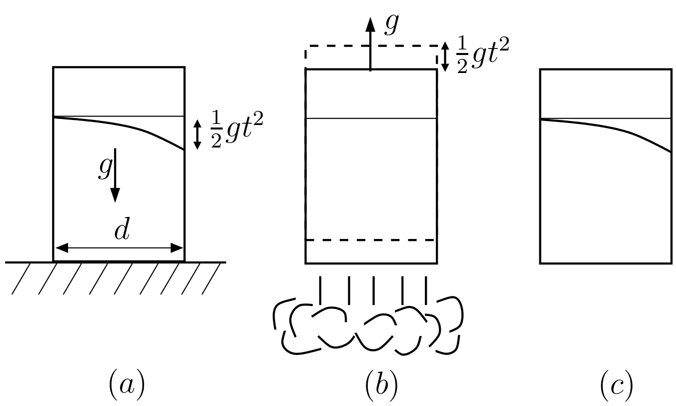

Figure 2.10: lightdeflec.

According to Maxwell's theory, lighht rays move in strisghht lines, tracing out the geometry of space.

However, light rays are bent by gravititational fields, which, in turn, respond to the pressence to matter - the only conclusion is that matter affects the gemetry of space.

Is then the geometry of space the same as the gravitational field? Almost. Consider a straight line in space. Two particles can travel along iiit, but one travels at a uniform speed, while the other is constantly acelerating. In spacetime they are travelling along different paths. The particle with constant speed travels on a straight line, in spacetime as well as space. athe accelerating particle travels on a curved path in spacetime.

Hence the geometry of spacetime can distinguish a particle at constant speed from one that is accelerating.

But the equivalence principle tells us that the effects of gravity cannot be distinguished, over small distances, frome the effects of acceleration. Hence by telling which trajectories are accelerated and which are not, the geometry of spacetime describes the effects of gravity. The geometry of spacetime is therefore the graviational field.

We demonstrate explicitly in the next section that the gravitational field is spacetime geometry, not just space geometry.

2.3.4 Curvature in a Weak Uniform Gravitation field

From a geometric point of view geodesics are the straightest paths between spacetime events. They are paths of maximum proper time. These are the paths of test bodies in free-fall.

$$\int_{t_1}^{t_2} d\tau \tag{2.44}$$

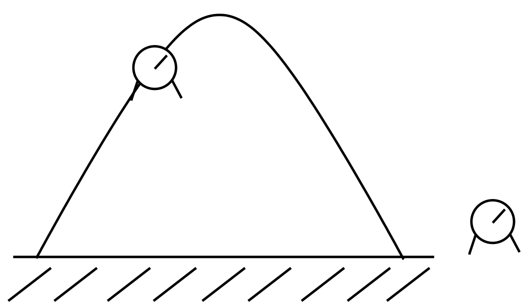

Figure 2.11: WeakGrav. geodesic.

This excess rate of the moving clock is

$$\frac{\omega_0 g H}{c^2} \tag{2.45}$$

Special relativistic effect is

$$\omega = \omega_0 \sqrt{(1 - v^2/c^2)} \tag{2.46}$$

For speeds much less than c, this is

$$\omega = \omega_0 (1 - v^2/2c^2) \tag{2.47}$$

If we measure a time dt on a fixed clock, the moving clock will register the time

$$dt \left[1 + \left(\frac{gH}{c^2} - \frac{v^2}{2c^2} \right) \right] \tag{2.48}$$

The total time excess over the trajectory is the integral of the extra term with respect to time

$$\frac{1}{c^2} \int_{t_1}^{t_2} \left(gH - \frac{v^2}{2} \right) dt \tag{2.49}$$

this is supposed to be a maximum.

$$\int_{t_1}^{t_2} \left(\frac{mv^2}{2} - m\phi \right) dt \tag{2.50}$$

the principle of least action Newton's law for an object in any potential.

2.3.5 The Principle of General Relativity

2.3.6 Background Independent Theories

A *background indepedent* theory is a physical theory defined on a base manifold $\mathcal{M}$ endowed with no extra structure, like geometry. I a theory does include any such geometric structure, it is *background independent*

In a background independent theory, there is no kinematics prior to and independent of the dynamics of the theory.

In any such background dependent theory, there is kinematics that is prior to and independent of the dynamics if the theory.

2.3.7 Einstein's Hole Argument

The above considerations may lead one to ask the following question: say we had two distinct spacetimes, i.e. the metric functions are not related through a coordinate transformation, could the difference between them still possibly be completely gauge?, i.e. is it possible that they are physically equivalent? We will explore this question.

As already mentioned, a point of a bare manifold is not distinguished from any other point. The theory of General Relativity is not based on any pre-existing geometrical structure, and as such there is no way to identify points of two manifolds. Now, suppose we have two different manifolds $\mathcal{M}$ and $\mathcal{M}'$ and that for both the metric and the matter distributionis are know everywhere outside of some hole in the manifold (see fig (2.3.7)).

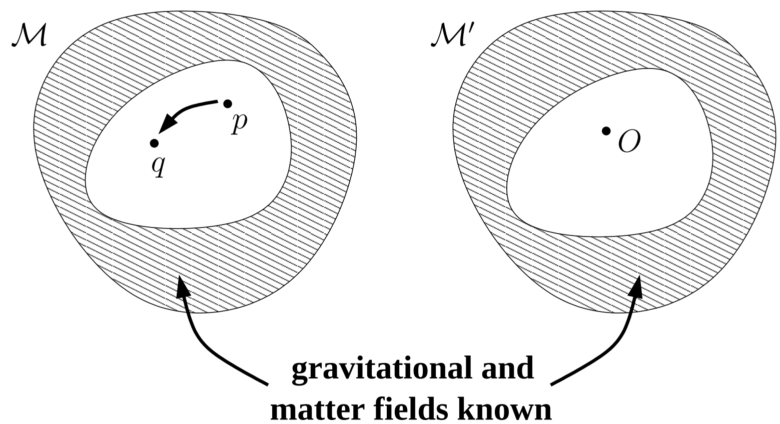

Figure 2.12: Hole. Einstein's hole argument.

First of all, let us suppose we have fixed a coordinate system x^a in the manifold $\mathcal{M}$: any point $p \in \mathcal{M}$ is labeled by $x^a(p)$. Define the point identification map as carrying the backgroound coordinate over $\mathcal{M}'$:

$$\begin{aligned} &\psi : \mathcal{M} \to \mathcal{M}' \\ &p \mapsto O = \psi(p) \quad \text{with} \quad x^a(p) \equiv x^a(O) \end{aligned} \tag{2.51}$$

O is the point on the manifold $\mathcal{M}'$ corresponding to p through the diffeomorphism ψ; ψ assigns the same coordinate labels between related points.

If there were a preferred coordinate system on both these spacetimes, we could accomplish this identification using this preferred coordinate system. However, there is no such coordinate system due to general covariance: there should be no preferred coordinate system with a physical role to play, which is equivalent to the condition that the theory isn't based on any pre-existing geometric structure.

A change in map ψ, keeping the coordinates on $\mathcal{M}$ fixed, is a guage transformation. We could as well use a different gauge φ and think of O as the point of $\mathcal{M}'$ corresponding to a different point q in the manifold $\mathcal{M}$, with coordinates $x^a(q)$:

$$
\begin{aligned}
&\varphi : \mathcal{M} \to \mathcal{M}' \\
&q \mapsto O = \varphi(q) = \psi(p) \quad \text{with} \quad x^a(q) \neq x^a(O)
\end{aligned}
\tag{2.52}
$$

The two different ways of mapping $\mathcal{M}'$ through the coordinate system of $\mathcal{M}$ suggest a one-to-one correspondence between different points in the manifold $\mathcal{M}$: the composition of maps

$$
\begin{aligned}
&\Phi : \mathcal{M} \to \mathcal{M}' \to \mathcal{M} \\
&p \mapsto q = \Phi(p) = \varphi^{-1}(\psi(p))
\end{aligned}
\tag{2.53}
$$

is a gauge transformation which does not change the coordinate label system but moves the points on the manifold, and then evaluate the coordinates of the new points: $\overline{x}^a(q) = \Phi^a(x^b(p))$, that is an *active diffeomorphism* on the original spacetime.

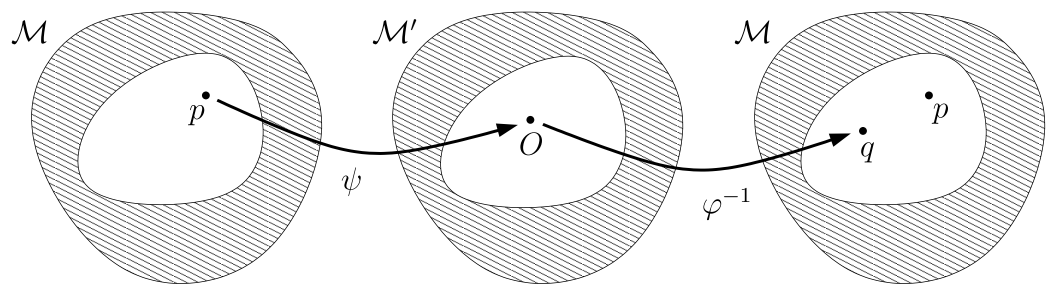

Figure 2.13: Hole3. Einstein's hole argument.$\Phi : \mathcal{M} \to \mathcal{M}' \to \mathcal{M}$

Stuff from cosmological perturbation theory in a matter dominated..

General Relativity is invariant under diffeomorphisms; diffeomorphisms are coordinate transformations in some sence and choosing the coordinate system means fixing the chart between open subsets of $\mathcal{M}$ and open subsets of $\mathbb{R}^{n+1}$. This invariance under diffeomorphisms reflects the redundancy in the description of the metric components g_{ab} and can be seen in the indetermination of E.E. system; it is also known as gauge freedom. In other words....

In what follows we will then refer to gauge (or gauge choice) as a coordinate choice or more loosely to a family of coordinates choices, and a gauge tranfromations as equivalent to a coordinate transformation.

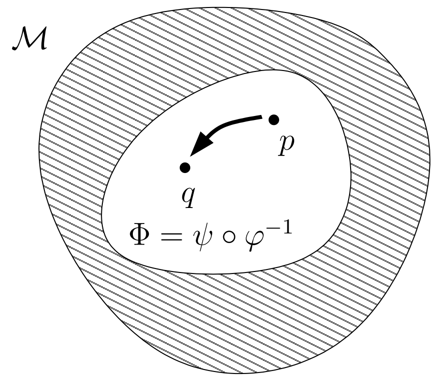

Figure 2.14: Hole4. Einstein's hole argument. A gauge transformation which does not change the coordinate label system but moves the points on the manifold, and then evaluate the coordinates of the new point

2.4 Observables

We recall what we learned in the first chapter: one introduces fields, electromagnetic, gravitational, etc fields over the space-time manifold, collectively denoted φ. Physical theories are still defined over space-time, but which are invariant under active diffeomorphisms $\phi : \mathcal{M} \to \mathcal{M}$ of the space-time manifold $\mathcal{M}$ into itself. The diffeomorphism group $Diff_{\mathcal{M}}$

$$S[\varphi, X_n] = S[\phi(\varphi), \phi(X_n)] \qquad \text{for all } \phi \in Diff_{\mathcal{M}} \tag{2.54}$$

What we learn about space-time geometry from observation We want to derive physical predictions that can be compared with observations. space-time properties do not have any meaning independent of observations. Space-time geometry, quite literally, has *no reality* independent of observations!

Any quantity whose definition is dependent on a coordinate system cannot be an observable in GR. For example consider a surface which is defined by a set of points in some coordinate system, the area of the surface is given by

$$A = \int_\Sigma dS$$

This quantity may be is invariant under coordinate transformations, however, it is not invariant under an active diffeomorphisms because under such a transformation the surface stays where it is while the metric gets dragged across the manifold, the new metric imposes a different spacetime geometry and as so assigns a different area to the surface.

The area of a surface defined by a physical object, such as a table, is an observable. Under an active transformation the surface gets dragged across together with the metric.

There are several open difficulties connected with the treatment of the notion of time in general covariant quantum theories, and it is important to distinguish carefully between them.

General covariant theories can be formulated in the lagrangian language in terms of evolution in a non-physical, fictitious coordinate time. The coordinate time (as well as the spatial coordinates) can in principle be discarded from the formulation of the theory without loss of physical content, because results of real gravitational experiments are always expressed in coordinate-free form. Let us generically denote the fields of the theory as $f_A(\vec{x}, x^o), A = 1 \ldots N$. These include for instance metric field, matter fields, electromagnetic field, and so on, and are subject to equations of motion invariant under coordinate transformations. Given a solution of the equations of motion

$$f_A = f_A(\vec{x}, x^o),$$

we cannot compare directly the quantities $f_A(\vec{x}, x^o)$ with experimental data. Results of experiments, in fact, are expressed in terms of physical distances and physical time intervals, which are functions of the various fields (including of course the metric field) independent from the coordinates $\vec{x}, x^o$. We have to compute coordinate independent quantities out of the quantities $f_A(\vec{x}, x^o)$, and compare these with the experimental data.4

The strategy employed in experimental gravitation, is to use concrete physical objects as clocks and as spatial references. Clocks and other reference system objects are concrete physical objects also in non generally covariant theories; what is new in general covariant theories is that these objects cannot be taken as independent from the dynamics of the system, as in non general covariant physics. They must be components of the system itself. Let these "reference system objects" be described by the variables $f_1 \ldots f_4$ in the theory. We are more concerned here with temporal determination than with space determination. Examples of physical clocks are: a laboratory clock (the rate of which depends by the local gravitational field), the pulsar's pulses, or an arbitrary combination of solar system variables, these variables are employed as independent variables with respect to which the physical evolution of any other

2.5 Tensor Calculus

2.5.1 Tensors

Contravariant tensors

Consider the differential distance vector $d\mathbf{r}$. The components of this in the $x-$coordinate system are dx^a, in another coordinate syste, the $x'-$coordinate system they are dx'^a. They are related by

$$dx'^a = \frac{\partial x'^a}{\partial x^b} dx^b \tag{2.55}$$

Any set of quantities X^a transforming this way, namely,

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b, \tag{2.56}$$

is called a contravariant vector. This is also called a contravariant tensor of rank 1. There are higher rank contravariant tensors, for example a contravariant tensor of rank 2 is a quantity which transforms as

$$X'^{ab} = \frac{\partial x'^a}{\partial x^c} \frac{\partial x'^b}{\partial x^d} X^{cd}, \tag{2.57}$$

Covariant tensors

We begin again by considering the transformation property of a prototype quantity. Let

$$\phi = \phi(x^a) \tag{2.58}$$

be a real-valued function on the manifold. Now x^a can be thought of as a function of x'^b, the above equation can be written

$$\phi = \phi(x^a(x')). \tag{2.59}$$

Differentiating this with respect to x'^b and using the function of a function rule, we obtain

$$\frac{\partial \phi}{\partial x'^b} = \frac{\partial \phi}{\partial x^c} \frac{\partial x^c}{\partial x'^b} \tag{2.60}$$

This is the prototype equation we seeked. Any quantity X_a that transforms according to

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b \tag{2.61}$$

is called a covector or covariant tensor of rank one. Similarly, we can define a covariant tensor of rank 2 by the transformation law

$$X'_a = \frac{\partial x^b}{\partial x'^a} X_b \tag{2.62}$$

2.5.2 Covariant Derivative

The derivative of a vector is not as straightforward beacause the basis vectors ε_i are in general not constant.

$$\mathbf{X} = X^i \varepsilon_i$$

Direct differentiation yields

$$\frac{\partial \mathbf{X}}{\partial x^j} = \frac{\partial X^i}{\partial x^j} \varepsilon_i + V^i \frac{\partial \varepsilon_i}{\partial x^j} \tag{2.63}$$

Now

$$\frac{\partial \varepsilon_i}{\partial x^j}$$

will be some linear combination of the ε_i withe the coefficient depending on i and j. We write

$$\frac{\partial \varepsilon_a}{\partial x^b} = \Gamma^d_{ab} \varepsilon_d \tag{2.64}$$

Contracting with ε^c, we have

$$\Gamma^c_{ab} = \varepsilon^c \cdot \frac{\partial \varepsilon_a}{\partial x^b} \tag{2.65}$$

This is the Christoffel connection. With these (2.63) can be written

$$\frac{\partial \mathbf{X}}{\partial x^b} = \frac{\partial X^a}{\partial x^b}\varepsilon_a + V^a\Gamma^c_{ab}\varepsilon_c \tag{2.66}$$

which can be written, interchanging dummy indicies, to

$$\frac{\partial \mathbf{X}}{\partial x^b} = \left(\frac{\partial X^a}{\partial x^b} + V^c\Gamma^a_{cb}\right)\varepsilon_a \tag{2.67}$$

The quantity in the brackets is the so-called covaraiant derivative, $\nabla_b X^a$. We have

$$\nabla_b X^a = \frac{\partial X^a}{\partial x^b} + \Gamma^a_{cb}V^c \tag{2.68}$$

Next we define the covariant derivative of a scalar to be just its ordinary derivative

$$\nabla_b \phi = \partial_b \phi. \tag{2.69}$$

If we demand that covariant derivatiation satisfy the product rule then we find

$$\nabla_b X_a = \frac{\partial X_a}{\partial x^b} - \Gamma^c_{ab}V_c \tag{2.70}$$

The expression for the general tensor is

$$\nabla_c T^{a\cdots}_{b\cdots} = \partial_c T^{a\cdots}_{b\cdots} + \Gamma^a_{dc}T^{d\cdots}_{b\cdots} + \cdots - \Gamma^d_{bc}T^{a\cdots}_{d\cdots} - \ldots \tag{2.71}$$

2.5.3 The Metric Connection

2.5.4 Curvature Tensor

The vector X defines a curve through the point p via parallel transport, The vector Y defines another curve through p. We can form and attempted parallelagram.

$$\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$$

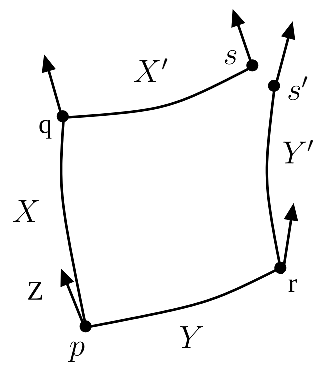

Figure 2.15: We display the geometric interpretation of the curvature tensor. Carry a third vector Z, by parallel transport from p to s via q, comparing this with transporting this from p to s' via r we find a discrepancy between the two vectors given in terms of the curvature tensor components $R_{abc}{}^{d}$ by the formula $\epsilon^2 X^a Y^b Z^c R_{abc}{}^{d}$.

2.6 Space-Time Measurements

2.6.1 Measurements of Time Intervals and Space Distances

theory of measurements - what can be directly measured has physical reality

in particular proper-time readings, angles, frequencies and energy fluxes?? a consistent formulism for a theory of measurement. a conceptual (and practical) need to establish a relation between measurements and the geometry (up to active diffeomorphisms) of space-time. no details of experiments but to the relations between the observable quantities and geometrical terms like curvature components, spacial distances and proper recession velocities.

how are geometric concept of spatial distance is related to the observer's proper time when the curvature does not vanish

connect the spacial distance.... ??

$$L(p,\gamma) := \left(2\Omega(S_{A_0})\right)^{1/2} = |(\sigma_p - \sigma_{A_0})||\xi^2|^{1/2}_{A_0} \tag{2.72}$$

This is the mathematical construction. Such a quantity has physical meaning when applied to the experimental set up described at the beginning, thus we wish to derive this value from physical quantities that we measure.

Details

$$\Omega(p',p) = \frac{1}{2}(s_1 - s_0)\int_{s_0}^{s_1} g_{ab} X^a X^b ds \tag{2.73}$$

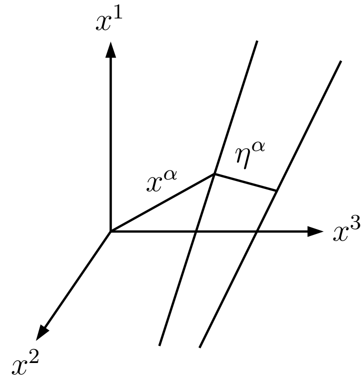

Figure 2.16: geodesic deviation.

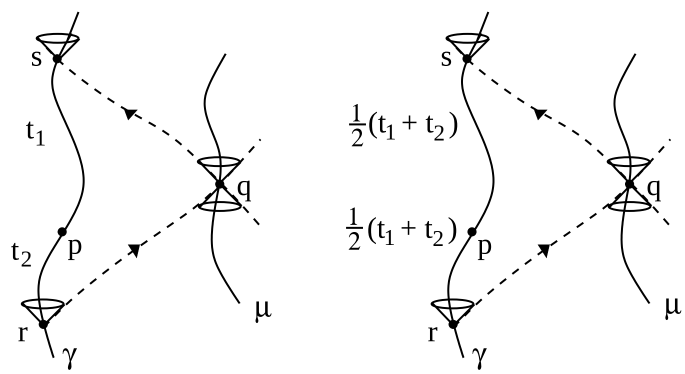

Figure 2.17: clock time.

taken along γ with $X^a = d$, has a value independent of the particular special parameter chosen. If, as we shall suppose, the points p' and p determine a unique geodesic passing through them, then Ω is a function of these two points. As a function of the eight variables x and x we shall call it the world-function of space-time.

$$\left.\frac{d\Omega}{ds}\right|_{q_0} = 0 \tag{2.74}$$

$$\begin{aligned} \frac{d\Omega}{dt} &= \frac{\Omega\big(\gamma_t(s_0) + \dot{\gamma}_t(s_0)dt, \gamma_t(s_1) + \dot{\gamma}_t(s_1)dt\big) - \Omega(\gamma_t(s_0), \gamma_t(s_1))}{dt} \\ &= \frac{\partial\Omega}{\partial x^{a_0}}\,\frac{d\gamma_t}{dt}(s_0) + \frac{\partial\Omega}{\partial x^{a_1}}\,\frac{d\gamma_t}{dt}(s_1) \end{aligned} \tag{2.75}$$

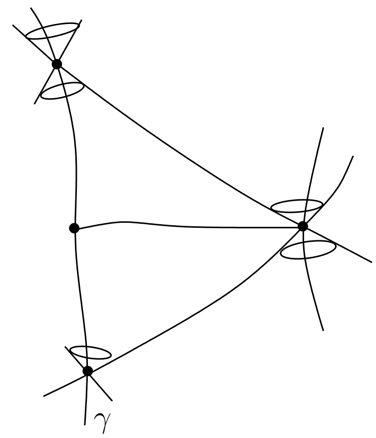

Figure 2.18: measLocation.

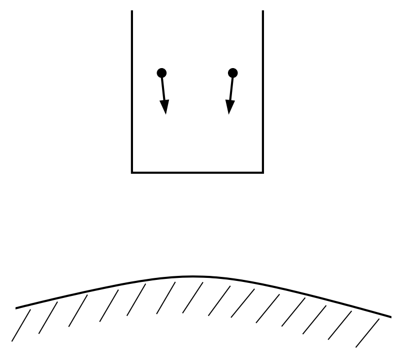

Figure 2.19: tidalforceF. .

2.6.2 Measurement Tools

Clocks and Rulers

Relativistic Doppler Effect

In special relativity, the Doppler effect is shown by

$$\frac{\Delta\tau_B}{\Delta\tau_A} = \frac{1 - \frac{v_{AB}}{c} \cdot \frac{k}{|k|}}{\sqrt{1 - \frac{v_{AB}^2}{c^2}}} \tag{2.76}$$

where τ_A and τ_B are the proper periods of a light signal emitted by a source A and received by B respectively, v_{AB} is the relative velocity from A to B, k is the wave vector in flat space from A to B.

A frequency shift can be thought of as induced by both gravity and by proper motion of the source relative to the observer.

2.6.3 Geodesis Deviation

[6]

$$\frac{d^2(Y^a u_a)}{ds^2} = \mathcal{K}^a{}_b Y^a u_a \tag{2.77}$$

where we have put $\mathcal{K}^a{}_b = R^a{}_{bcd} X^b X^c$. There is a standard method to obtain an iterative equation for such differential equations - the method of Green's functions. We write

$$Y^a(s')u_a = (\text{solution for } \mathcal{K}^a{}_b = 0) + \int_{s_0}^{s_1} G(s,s')(\mathcal{K}^a{}_b Y^b(s)u_a)ds \tag{2.78}$$

where $G(s,s')$ satisfies

$$\frac{d^2 G(s,s')}{ds^2} = \delta(s,s') \tag{2.79}$$

It is easy to see how (2.78) corresponds to the differential equation (2.77),

$$\begin{aligned} \frac{d^2}{ds'^2} Y^a(s')u_a &= \int_{s_0}^{s_1} \frac{d^2}{ds'^2} G(s,s')(\mathcal{K}^a{}_b Y^b(s)u_a)ds \\ &= \int_{s_0}^{s_1} \delta(s-s')(\mathcal{K}^a{}_b Y^b(s)u_a)ds \\ &= \mathcal{K}^a{}_b Y^a(s')u_a \end{aligned} \tag{2.80}$$

The zeroth order solution (i.e. for $\mathcal{K}^a{}_b = 0$) is easy

$$\frac{d^2 Y^a(s')u_a}{ds'^2} = 0 \quad \Rightarrow \quad Y^a(s')u_a = A + Bs'$$

boundary conditions $Y^{b'}(s=s_0)u_{b'} = Y^{a_0}(\Gamma_{a_0}{}^{b'} u_{b'})$ and $Y^{b'}(s=s_1) = Y^{a_1}(\Gamma_{a_0}{}^{b'} u_{b'})$

$$Y^{b'}(s')u_{b'} = \frac{(s_1 - s')}{s_1 - s_0}\Gamma_{a_0}{}^{b'} Y^{a_0} u_{b'} + \frac{(s'-s_0)}{s_1 - s_0}\Gamma_{a_1}{}^{b'} Y^{a_1} u_{b'} \tag{2.81}$$

It is easily verified that the Green's function should be

$$G(s,s') = \begin{cases} \alpha(s-s_0)(s_1-s') & s \le s' \\ \alpha(s'-s_0)(s_1-s) & s \ge s' \end{cases} . \tag{2.82}$$

Taking the derivative gives a step function:

$$\frac{dG}{ds}(s,s') = \begin{cases} \alpha(s_1-s') & s \le s' \\ -\alpha(s'-s_0) & s \ge s' \end{cases} . \tag{2.83}$$

The derivative of this is proportional to the delta function:

$$\frac{d^2G}{ds^2}(s,s') = A\delta(s-s'). \tag{2.84}$$

We choose α so that $A = 1$. We find α by integrating this last equation over s

$$\begin{aligned} A &= \int_{s_0}^{s_1} \delta(s-s')ds = \int_{s_0}^{s_1} \frac{d^2G(s,s')}{ds^2} ds \\ &= \left[\frac{dG}{ds}(s,s')\right]_{s_0}^{s_1} = \alpha(s_1-s') + \alpha(s'-s_0) = \alpha(s_1-s_0) \end{aligned}$$

So that $\alpha = (s_1-s_0)^{-1}$.

$$\begin{aligned} \int_{s_0}^{s_1} G\frac{d^2(Y^a u_a)}{ds^2} ds &= \int_{s_0}^{s_1} G\mathcal{K}^a{}_b Y^a u_a \\ &= \left[G\frac{d}{ds}(Y^a u_a)\right]_{s_0}^{s_1} - \int_{s_0}^{s_1} \frac{dG}{ds}\frac{d}{ds}(Y^a u_a)ds \end{aligned} \tag{2.85}$$

$$\begin{aligned} &- \alpha\int_{s_0}^{s'} (s_1-s')\frac{d}{ds}(u_a Y^a)ds + \alpha\int_{s'}^{s_1} (s'-s_0)\frac{d}{ds}(u_a Y^a)ds \\ &= -\alpha(s_1-s')[u_{a'}Y^{a'} - u_{a_0}Y^{a_0}] + \alpha(s'-s_0)[u_{a_1}Y^{a_1} - u_{a'}Y^{a'}] \\ &= \int_{s_0}^{s_1} G\mathcal{K}^a{}_b Y^a u_a ds. \end{aligned} \tag{2.86}$$

$$Y^{b'}(s') = \alpha(s_1-s')\Gamma_{a_0}{}^{b'}Y^{a_0} + \alpha(s'-s_0)\Gamma_{a_1}{}^{b'}Y^{a_1} - \int_{s_0}^{s_1} G(s,s')\mathcal{K}^a{}_c Y^c \Gamma_a{}^{b'} ds \tag{2.87}$$

2.6.4 World Function

$$\frac{DV^a}{Ds} := X^b \nabla_b V^a, \qquad \frac{DV^a}{Dt} := Y^b \nabla_b V^a \tag{2.88}$$

$$\frac{DX^a}{Ds} = 0, \tag{2.89}$$

$$\frac{DY^a}{Dt} = \frac{DX^a}{Ds}. \tag{2.90}$$

Introduce the world function

$$\Omega(p_0, p_1) = \frac{1}{2}(s_1 - s_0)^2 X^a X_a \tag{2.91}$$

where X denotes the tangent vector to the (unique) geodesic connecting p_0 to p_1.

As we move along κ_0, κ_1 X changes - we consider X as a function of t.

$$\frac{d}{dt}\Omega(\gamma_t(s_0), \gamma_t(s_1)) = (s_1 - s_2)^2 X_b \frac{D}{dt} X^b \tag{2.92}$$

$$\begin{aligned}
\frac{d\Omega}{dt} &= \frac{\Omega\big(\gamma_t(s_0) + \dot{\gamma}_t(s_0)dt, \gamma_t(s_1) + \dot{\gamma}_t(s_1)dt\big) - \Omega(\gamma_t(s_0), \gamma_t(s_1))}{dt} \\
&= \frac{\partial \Omega}{\partial x^{a_0}} \frac{d\gamma_t}{dt}(s_0) + \frac{\partial \Omega}{\partial x^{a_1}} \frac{d\gamma_t}{dt}(s_1) \\
&= \frac{\partial \Omega}{\partial x^{a_0}} Y^{a_0} + \frac{\partial \Omega}{\partial x^{a_1}} Y^{a_1}
\end{aligned} \tag{2.93}$$

From (5.24)

$$X_a \frac{DY^a}{Dt} = X_a \frac{DX^a}{Ds} \tag{2.94}$$

Because of the antisymmetry of the indices of the curvature tensor,

$$\left(\frac{D^2 Y}{Ds^2}\right)^a X_a = R^a{}_{bcd} X^b X^c Y^d X_a. \tag{2.95}$$

$$X_a \frac{D^2 Y^a}{Ds^2} = 0 \tag{2.96}$$

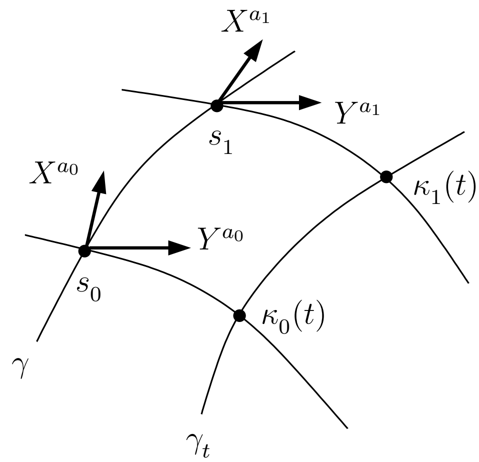

Figure 2.20: worldfunc1.

so that

$$\frac{d^2}{ds^2}(X_a Y^a) = \frac{d}{ds}\Big(Y^a \underbrace{\frac{DX^a}{Ds}}_{=0} + X_a \frac{DY^a}{Ds}\Big)$$
$$= X_a \frac{D^2 Y^a}{Ds^2} = 0 \tag{2.97}$$

which means

$$X_a \frac{DY^a}{Ds} = \frac{d}{ds}(X_a Y^a) = a = const. \tag{2.98}$$

If we integrate along $\gamma(t)$ from s_0 to s_1, we get :

$$a = -(s_0 - s_1)^{-1}(X_{a_0} Y^{a_0} - X_{a_0} Y^{a_0}) \tag{2.99}$$

$$\frac{d\Omega}{dt} = \Omega_{a_0} Y^{a_0} + \Omega_{a_1} Y^{a_1} = (s_0 - s_1)[X_{a_1} Y^{a_1} - X_{a_0} Y^{a_0}] \tag{2.100}$$

From (2.100) we can read off,

$$\Omega_{a_o} = -(s_0 - s_1) Y_{a_o}, \quad \Omega_{a_1} = (s_0 - s_1) Y_{a_1} \tag{2.101}$$

second derivative

$$
\begin{aligned}
\frac{d^2}{dt^2}\Omega &= \frac{D}{Dt}\big(\frac{\partial\Omega}{\partial x^{a_0}}Y^{a_0}+\frac{\partial\Omega}{\partial x^{a_1}}Y^{a_1}\big)\\
&= Y^{a_0}Y^{b_0}\nabla_{b_0}\Omega_{a_0}+\Omega_{a_0}\frac{D}{Dt}Y^{a_0}\\
&+ Y^{a_1}Y^{b_1}\nabla_{b_1}\Omega_{a_1}+\Omega_{a_1}\frac{D}{Dt}Y^{a_1}
\end{aligned} \tag{2.102}
$$

$$
\frac{d^2}{dt^2}\Omega = Y^{a_0}Y^{b_0}\Omega_{a_0b_0}+Y^{a_1}Y^{b_1}\Omega_{a_1b_1}+ \tag{2.103}
$$

now we need a solution for the derivative of the connecting vector field.

$$
\frac{D}{Dt}\Omega_{a_0} = -(s_0-s_1)\frac{D}{Dt}Y_{a_0} \tag{2.104}
$$

end up with

$$
\Omega_{a_0b_0} = g_{a_0b_0}+\alpha g_{a_0e_0}\int_{s_0}^{s_1}(s_1-s)^2\mathcal{K}^c{}_d\Gamma_c{}^{e_0}\Gamma_{b_0}{}^d\;ds+\mathcal{O}(|\mathrm{Riem}|^2) \tag{2.105}
$$

$$
\Omega_{a_0b_1} = -g_{a_0d_0}\Gamma_{c_1}{}^{d_0}+\alpha g_{a_0e_0}\int_{s_0}^{s_1}(s_1-s)(s-s_0)\mathcal{K}^c{}_d\Gamma_{b_1}{}^d\Gamma_c{}^{e_0}\;ds+\mathcal{O}(|\mathrm{Riem}|^2) \tag{2.106}
$$

$$
\Omega(s) = \Omega(s_{A_0})+\sum_{n=1}^{\infty}\frac{1}{n!}\frac{d^n\Omega}{ds^n}\bigg|_{s_{A_0}}(s-s_{A_0})^n. \tag{2.107}
$$

$$
\begin{aligned}
\frac{d\Omega}{ds} &= \Omega_i\dot\gamma^i\\
\frac{d^2\Omega}{ds^2} &= \Omega_{ij}\dot\gamma^i\dot\gamma^j+\Omega_i\dot\gamma^i a^i\\
\frac{d^3\Omega}{ds^3} &= \Omega_{ijk}\dot\gamma^i\dot\gamma^j\dot\gamma^k+\dot\gamma^i a^i
\end{aligned} \tag{2.108}
$$

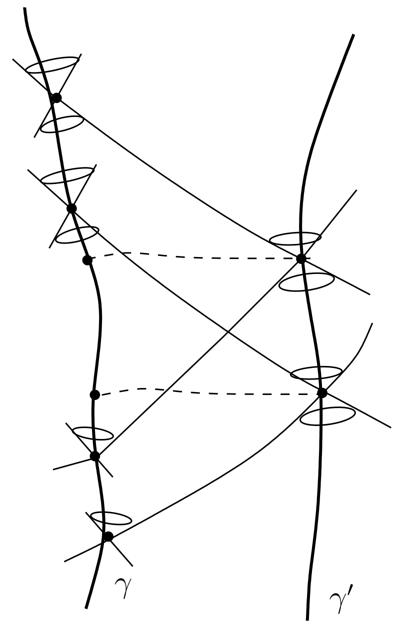

Figure 2.21: measVelocity.

2.6.5 Measurement of Relative Velocities

2.6.6 Derivation of Lorentz Transformation Formula in GR

2.7 Derivation of Vacuum Field Equations

2.7.1 The Newtonian Equation of Deviation

We examine two neigbouring test particles in free fall in a gravitational field in Newtonian theory in Euclidian space. The first particle's curve is denoted by

$$x^\alpha(t)$$

and its equation of motion is given in terms of the gravitational potential $\phi(x)$

$$\ddot{x}^\alpha(t) = -\partial^\alpha \phi(x^a(t)) \tag{2.109}$$

The second particle is located at $y^a(t)$ with

$$y^a(t) = x^a(t) + \eta^\alpha(t) \tag{2.110}$$

defining a small connecting vector $\eta^\alpha(t)$ between the curves of the two particles. The second particle's equation of motion is

$$\begin{aligned}\ddot{x}^\alpha + \ddot{\eta}^\alpha &= -\partial^\alpha \phi(x^a(t) + \eta^\alpha(t)) \\ &= -\partial^\alpha \phi(x^a(t)) - \eta^\beta \partial_\beta \partial^\alpha \phi(x^a(t))\end{aligned} \tag{2.111}$$

This implies the equation for the connecting vector

$$\ddot{\eta}^\alpha = -\eta^\beta \partial_\beta \partial^\alpha \phi. \tag{2.112}$$

This is called the Newtonian equation of deviation. Let

$$K^\alpha{}_\beta = K_\beta{}^\alpha = \partial^\alpha \partial_\beta \phi, \tag{2.113}$$

then the Newtonian equation of deviation can be written

$$\ddot{\eta}^\alpha + K^\alpha{}_\beta \eta^\beta = 0. \tag{2.114}$$

Laplace's equation can be expressed as

$$K^\alpha{}_\alpha = 0. \tag{2.115}$$

In the next sections we investigate the analogies between the Newtonian expression for the variation of the freely falling particles and the General Relativistic expression for the geodesic deviation. We then look at the vanishing of the trace of the analogy of $K^\alpha{}_\beta$ to find the vacuum field equations of General Relativity.

2.7.2 Equation of geodesic deviation

Write

$$x^a = x^a(\tau, \nu) \tag{2.116}$$

where τ is thhe proper time along the geodesic C_1 and ν paramaterises a curve connecdting the geodesic C_2. We define

$$v^a = \frac{dx^a}{d\tau} \tag{2.117}$$

and

$$\xi^a = \frac{dx^a}{d\nu}. \tag{2.118}$$

Then v^a is the tangent vector to the timelike geodesic at each point along C_1 and ξ^a is a connecting vector connecting the neighbouring curves.

Now

$$\begin{aligned} [v,\xi]^a &:= v^b \partial_b \xi^a - \xi^b \partial_b v^a \\ &= \frac{dx^a}{d\tau}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{d\nu}\right) - \frac{dx^a}{d\nu}\frac{\partial}{\partial x^b}\left(\frac{dx^a}{d\tau}\right) \\ &= \frac{d^2x^a}{d\tau d\nu} - \frac{d^2x^a}{d\nu d\tau} \\ &= 0. \end{aligned} \tag{2.119}$$

We can replace partial derivatives with covariant ones:

$$\begin{aligned} 0 &= v^b \partial_b \xi^a - \xi^b \partial_b v^a \\ &= v^b \partial_b (\xi^a + \Gamma^a_{bc}\xi^c) - \xi^b(\partial_b v^a + \Gamma^a_{bc} v^c) \\ &= v^b \nabla_b \xi^a - \xi^b \nabla_b v^c \end{aligned} \tag{2.120}$$

where we have used $\Gamma^a_{bc} = \Gamma^a_{cb}$. Let use the directional derivative notation: $\nabla_X \equiv X^b \nabla_b$. So we have from (2.120)

$$\nabla_v \xi^a = \nabla_\xi v^a \tag{2.121}$$

Applying the directional derivative ∇_v to both sides gives

$$\nabla_v \nabla_v \xi^a = \nabla_v \nabla_\xi v^a. \tag{2.122}$$

Consider the identity

$$\nabla_X(\nabla_Y Z^a) - \nabla_Y(\nabla_X Z^a) - \nabla_{[X,Y]} Z^a = R^a{}_{bcd} Z^b X^c Y^d \tag{2.123}$$

If we set $X^a = Z^a = v^a$ and $Y^a = \xi^a$, then then the second is $\nabla_\xi(\nabla_v v^a)$ vanishes because v^a is tangent to a geodesic,

$$\nabla_v v^a = v^b \nabla_v v^a = 0. \tag{2.124}$$

The third term, $[v, \xi]^b \nabla_b v^a$, vanishes by (2.119). Thus (2.123) becomes

$$\nabla_v \nabla_\xi v^a - R^a{}_{bcd} v^b v^c \xi^d = 0. \tag{2.125}$$

Substituting (2.122) into this we obtain

$$\nabla_v \nabla_v \xi^a - R^a{}_{bcd} v^b v^c \xi^d = 0. \tag{2.126}$$

By definition

$$\frac{D^2 \xi^a}{D\tau^2} \equiv \nabla_v \nabla_v \xi^a$$

and so, the geodesic deviation equation is

$$\frac{D^2 \xi^a}{D\tau^2} = R^a{}_{bcd} v^b v^c \xi^d. \tag{2.127}$$

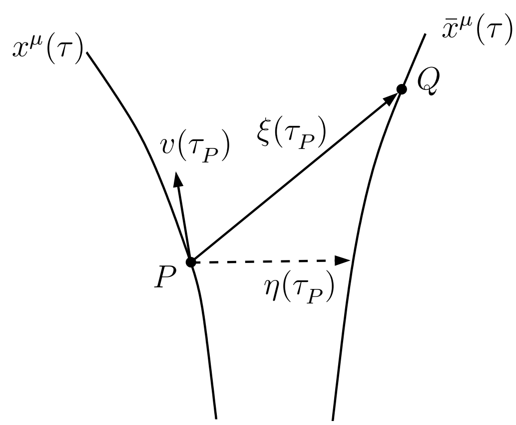

Figure 2.22: η is the orthogonal connecting vector.

We are only interested in the spatial part of ξ^a and its geodesic deviation equation. To this end we introduce the projection operator which acts on tensors and gives the spatial information orthogonal to the timelike vector v^a.

Projection operator

First note that since $d\tau^2 = g_{ab}dx^a dx^b$,

$$v^a v_a = g_{ab}v^a v^b = g_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau} = 1, \tag{2.128}$$

i.e. v^a is a unit tangent vector. The projection operator defined by

$$h^a{}_b := \delta^a{}_b - v^a v_b \tag{2.129}$$

it projects tensors into the three-space orthogonal to v^a

Obviously

$$h^a{}_b v^b = v^a - v^a v_b v^b = 0$$

If $w^a u_a = 0$ then

$$h^a{}_b w^b = w^a - v^a v_b w^b = w^a$$

and if $h^a{}_b w^b = w^a$ then

$$-v^a v_b w^b = w^a$$

and so

$$-v_b w^b = w^a v_a \quad \Rightarrow \quad w^a v_a = 0$$

$$\begin{aligned} h^a{}_b h^b{}_c &= (\delta^a{}_b - v^a v_b)(\delta^b{}_c - v^b v_c) \\ &= (\delta^a{}_c - 2v^a v_c + v^a v_b v^b v_c \\ &= \delta^a{}_c - v^a v_c = h^a{}_c \end{aligned} \tag{2.130}$$

$$h^a{}_a := \delta^a{}_a - v^a v_a = 4 - 1 = 3$$

Obviously

$$h_{ab} = h_{ba}$$

Orthogonal connecting vector and its equation of geodesic deviation

We thus define the orthogonal connecting vector η^a by

$$\eta^a := h^a{}_b \xi^b. \tag{2.131}$$

In the following we will use that

$$v_a \xi^b \nabla_b v^a = 0$$

which comes from

$$\begin{aligned} 0 &= \xi^b \nabla_b (1) \\ &= \xi^b \nabla_b (v^a v_a) \\ &= v_a \xi^b \nabla_b v^a + v^a \xi^b \nabla_b v_a \\ &= 2 v_a \xi^b \nabla_b v^a \end{aligned} \tag{2.132}$$

since the covariant derivative of 1 is zero. This is because we can always go to a frame in free fall and use cartesian coordinates in which the connection Γ^a_{bc} vanishes. As

$$\xi^a = \eta^a + v^a v_b \xi^b$$

we have

$$\begin{aligned} \frac{D\xi^a}{D\tau} &= v^c \nabla_c \xi^a \\ &= v^c \nabla_c (\eta^a + v^a v_b \xi^b) \\ &= v^c \nabla_c \eta^a + (v^c \nabla_c v^a) v_b \xi^b + v^a (v^c \nabla_c v_b) \xi^b + v^a v_b (v^c \nabla_c \xi^b) \\ &= v^c \nabla_c \eta^a + v^a v_b (\xi^c \nabla_c v^b) \\ &= \frac{D\eta^a}{D\tau}, \end{aligned} \tag{2.133}$$

where we used the geodesic equation $v^b \nabla_b v^a = 0$ and (2.132). We also have

$$R^a{}_{bcd} v^b v^c \xi^d = R^a{}_{bcd} v^b v^c (\eta^d + v^d v_e \xi^e) = R^a{}_{bcd} v^b v^c \eta^d \tag{2.134}$$

since $R^a{}_{bcd}$ is anti-symmetric in c and d.

So we have

$$\frac{D^2\eta^a}{D\tau^2} - R^a{}_{bcd}v^b v^c \eta^d = 0 \tag{2.135}$$

which is the same as (2.127) but with ξ^a replaced with η^a. However, this is still a four-vector equation whereas the Newtonian deviation equation is a three-vector equation.

2.7.3 The Newtonian Correspondence

At any point on the curve C_1, we introduce an orthogonal frame of three unit spacelike vectors

$$e_\alpha{}^a = (e_\mathbf{1}{}^a, e_\mathbf{2}{}^a, e_\mathbf{3}{}^a)$$

which are all orthogonal to v^a and where α is a bold label running from 1 to 3. We define

$$e_\alpha{}^a = v^a,$$

Combine these four vectors

$$e_\mathbf{i}{}^a \quad (\mathbf{i} = 0, 1, 2, 3).$$

They satisfy the orthonormaility relations

$$e_\mathbf{i}{}^a e_{\mathbf{j}a} = \eta_\mathbf{ij} \tag{2.136}$$

where $\eta_\mathbf{ij}$ is the Minkowski metric, that is,

$$\eta_\mathbf{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The four vectors are said to for a tetrad. We have introduced the frame notation for convience, but it turns out that frames posses a powerful formulism of their own, for example in spinor analysis. In particular they are used in the construction of the basic variables of loop quauntum gravity. Also it turn out they are essential when introducing Dirac spinor fields into general relativity.

Frame field - and the precise analogue with η^α

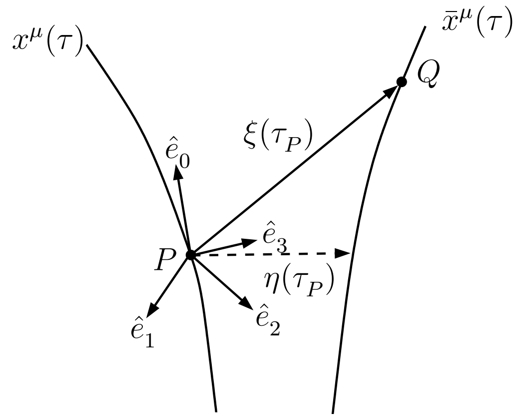

Figure 2.23: We find the spatial frame components η^α of the orthogonal connecting vector by projecting onto a spatial frame field. This is the precise analogue of the Newtonian connecting vector.

Treating $e_{\mathbf{i}}{}^a$ as a 4×4 matrix, we can define its inverse $e^{\mathbf{j}}{}_a$ by

$$e_{\mathbf{i}}{}^a e^{\mathbf{j}}{}_a = \delta^{\mathbf{j}}_{\mathbf{i}} \tag{2.137}$$

Multiply this by $e^{\mathbf{i}}{}_b$, we have

$$(e^{\mathbf{i}}{}_b e_{\mathbf{i}}{}^a) e^{\mathbf{j}}{}_a = e^{\mathbf{j}}{}_b$$

this implies

$$e^{\mathbf{i}}{}_b e_{\mathbf{i}}{}^a = \delta^a_b \tag{2.138}$$

We propagate the frame along C_1 by parallel propagation.

$$\frac{D}{D\tau}(e_{\mathbf{i}}{}^a) = 0 \tag{2.139}$$

We define spatial frame components of the orthogonal connecting vector η^α

$$\eta^\alpha = e^\alpha{}_a \eta^a \tag{2.140}$$

This is the precise analogue with η^α from the Newtonian equation of deviation. Note that

$$\eta^{\mathbf{0}} = e^{\mathbf{0}}{}_a \eta^a = e^{\mathbf{0}}{}_a h^a{}_b \xi^b = (v_a h^a{}_b)\xi^b = 0 \tag{2.141}$$

To find the spatial part of (2.135) we contract with $e^{\alpha}{}_a$, and then using parallel progation of the frame, we find

$$\frac{D^2\eta^\alpha}{D\tau^2} - R^a{}_{bcd} e^{\alpha}{}_a v^b v^c \eta^d = 0 \tag{2.142}$$

Now

$$\begin{aligned} \eta^d &= \delta^d_c \eta^c \\ &= e_{\mathbf{i}}{}^d e^{\mathbf{i}}{}_c \eta^c \\ &= e_{\mathbf{0}}{}^d e^{\mathbf{0}}{}_c \eta^c + e_\beta{}^d e^\beta{}_c \eta^c = e_\beta{}^d \eta^\beta \end{aligned} \tag{2.143}$$

Using this the spatial part of the equation of geodesic deviation becomes

$$\frac{D^2\eta^\alpha}{D\tau^2} + K_\beta{}^\alpha \eta^\beta = 0, \tag{2.144}$$

where

$$K^\alpha{}_\beta = -R^a{}_{bcd} e^\alpha{}_a v^b v^c e_\beta{}^d \tag{2.145}$$

We now have the analogue of the Newtonian deviation equation (2.114).

2.7.4 The Vacuum Field Equations

From the analogies between the Newtonian expression for the variation of the freely falling particles and the General Relativistic expression for the geodesic deviation, we investigate the vanishing of the trace (2.145), namely,

$$R^a{}_{bcd} e^\alpha{}_a v^b v^c e_\alpha{}^d = 0 \tag{2.146}$$

Let us introduce a special coordinate system in which

$$e_i{}^a = \delta_i{}^a \tag{2.147}$$

Then (2.146)

$$R^{\alpha}{}_{00\alpha} = 0.$$

Since the Riemann tensor is anti-symmetric in the last pair of indices

$$R^{0}{}_{000} = -R^{0}{}_{000} = 0.$$

Therefore

$$R^{a}{}_{00a} = 0.$$

Then

$$\begin{aligned} 0 &= R^{a}{}_{00a} \\ &= R^{a}{}_{bca}\delta^b_0\delta^c_0 \\ &= R^{a}{}_{bca}v^b v^c \\ &= -R^{a}{}_{bac}v^b v^c \\ &= -R_{bc}v^b v^c \end{aligned} \tag{2.148}$$

As $R_{bc}v^b v^c$ is a scalar if it vanishes in one coordinate system it must vanish in all coordinate systems. Moreover, since it vanishes for all observers (world lines passing through P), it vanises for all v^a at P. We prove this in the following. Now

$$R_{bc}v^b v^c = 0$$

for arbitrary timelike vector v^a (note that v^a need not be normalised). Let

$$v^a = u^a + \sum_{\alpha=1}^{3} \lambda^\alpha w^a_\alpha,$$

where $u^a u_a = 1$, $w^a_\alpha w_{\alpha a} = -1$, $u_a w^a_\alpha = 0$

$$v^a v_a = (u^a + \sum_{\alpha=1}^{3} \lambda^\alpha w^a_\alpha)(u_a + \sum_{\beta=1}^{3} \lambda^\beta w_{\beta a}) = u^a u_a - \sum_{\alpha=1}^{3} (\lambda^\alpha)^2 w^a_\alpha w_{\alpha a} = 1 - \sum_{\alpha=1}^{3} (\lambda^\alpha)^2$$

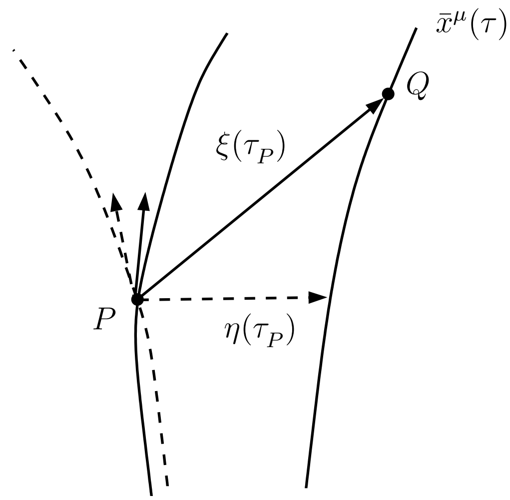

Figure 2.24: Different world line passing through P corresponds to different observer with different v^a.

we must have $0 \leq \sum_{\alpha=1}^{3}(\lambda^\alpha)^2 < 1$, λ^α arbitrary otherwise.

$$R_{ab}(u^a + \sum_{\alpha=1}^{3} \lambda^\alpha w_\alpha^a)(u^b + \sum_{\beta=1}^{3} \lambda^\beta w_\alpha^b) = 0$$

or

$$R_{ab}u^a u^b + 2\sum_{\alpha=1}^{3} \lambda^\alpha R_{ab}u^a w_\alpha^b + \sum_{\alpha,\beta=1}^{3} \lambda^\alpha \lambda^\beta R_{ab} w_\alpha^a w_\beta^b = 0 \tag{2.149}$$

Consider the special coordinate system in which $u^a = \delta_0^a$ and $w_\alpha^a = \delta_\alpha^a$

Choose $\lambda^\alpha = 0$ then

$$R_{ab}u^a u^b = R_{ab}\delta_0^a \delta_0^b = R_{00} = 0.$$

Differentiate with respect to λ^α and put $\lambda^\alpha = 0$

$$R_{ab}u^a w_\alpha^b = R_{ab}\delta_0^a \delta_\alpha^b = R_{0\alpha} = R_{\alpha 0} = 0 \tag{2.150}$$

Differentiate with respect to λ^α and λ^β then

$$R_{ab}w_\alpha^a w_\beta^c = R_{ab}\delta_\alpha^b \delta_\beta^b = R_{\alpha\beta} = 0$$

Since altogether $R_{ab} = 0$ holds in our special coordinate system it holds in all coordinate systems. So we have

$$[R_{ab}]_P = 0.$$

And finally since P is arbitrary, we find the vacuum field equation

$$R_{ab} = 0. \tag{2.151}$$

2.8 GPS Observables

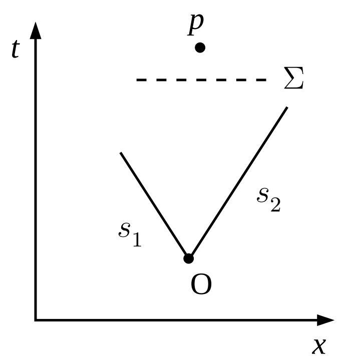

Figure 2.25: GPScoord. s_1 and s_2 are the GPS coordinates of the point p. Σ is a Cauchy surface with p in its future domain of dependence.

2.9 Measurement of an Area

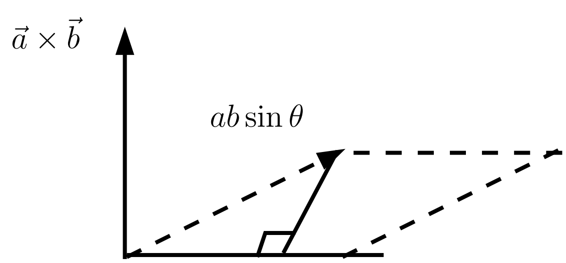

Figure 2.26: crossParea

$$\vec{n} = \frac{\vec{a} \times \vec{b}}{\|\vec{a} \times \vec{b}\|}$$

$$A = ab \sin\theta = \|\vec{a} \times \vec{b}\|$$

$$\begin{aligned}
dA &= \|dx\| \; \|dy\| \sin\theta \\
&= \sqrt{\|dx\|^2 \; \|dy\|^2 \cos^2\theta} \\
&= \sqrt{\|dx\|^2 \; \|dy\|^2 (1 - \sin^2\theta)} \\
&= \sqrt{\|dx\|^2 \; \|dy\|^2 - (dx \cdot dy)^2} \\
&=
\end{aligned} \tag{2.152}$$

Area in GR

$$n_a^T = \epsilon_{bcda} \frac{\partial x^b}{\partial \tau^1} \frac{\partial x^c}{\partial \tau^2} \frac{\partial x^d}{\partial \tau^3} \tag{2.153}$$

It does not depend on the metric. a normal to the hyper surface Σ,

$$n_a^T = \epsilon_{bcda} \frac{\partial x^b}{\partial \rho^1} \frac{\partial x^c}{\partial \rho^2} \frac{\partial x^d}{\partial \rho^3} \tag{2.154}$$

$$n_{ab} = \frac{1}{2} \epsilon_{abcd} \frac{\partial x^c}{\partial u} \frac{\partial x^d}{\partial v} \tag{2.155}$$

$$\begin{aligned}
A = A(S) &= \\
&= \int_S du dv \sqrt{\det\left(\frac{\partial x^a}{\partial u^i} \frac{\partial x^b}{\partial u^j} g_{ab} \right)}
\end{aligned} \tag{2.156}$$

where $u^i = (u, v)$ and the determinant is on the indices $i, j = 1, 2$.

2.10 Matter

2.10.1 Dust

Incoherent matter (dust)

Dust is the simplest kind of energy field: a field of non-interacting incohernt matter (force-free motion - moves under gravity alone). As no force is exterted on the particles, the particles extert no force themselves and hence dust is pressureless. Such a field is characterised by two quantities, the 4-velocity flow

$$u^a = \frac{dx^a}{d\tau}$$

where τ is the proper time along the world line of the dust particle and a scalar field

$$\rho_0(x)$$

descibing the proper density of the flow, that is, the density which would be measured by by an observer moving with the flow. Using these two characteristics of the matter field, the simplest second-rank tensor field we can construct is

$$T^{ab} = \rho_0(x) u^a(x) u^b(x) \tag{2.157}$$

and this turns out to be the energy-momentum tensor.

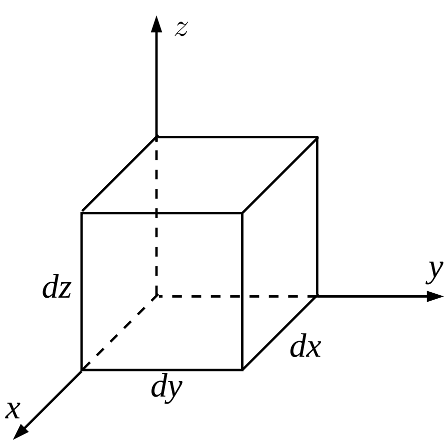

Figure 2.27: continuityEM Y and Y.

$$\begin{aligned} &= \left[\frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z)\right] dxdydz \\ &= \nabla \cdot (\rho \mathbf{u}) dxdydz \end{aligned} \tag{2.158}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{2.159}$$

T^{00} is the energy density.

cT^{0x} is the energy flow per unit area parallel to the x direction.

T^{ij} is the flow of momentum component x per unit area in the x direction

$$\begin{aligned} F_x &= \rho\frac{u_x(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) - u_x(x,y,z,t)}{\Delta t} \\ &= \rho\left[\frac{\Delta t}{\Delta t}\frac{\partial u_x}{\partial t} + \frac{\Delta x}{\Delta t}\frac{\partial u}{\partial x} + \frac{\Delta y}{\Delta t}\frac{\partial u}{\partial y} + \frac{\Delta z}{\Delta t}\frac{\partial u}{\partial z}\right] \\ &= \rho\left[\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}\right]. \end{aligned} \tag{2.160}$$

Euler equation of motion

$$\rho[\frac{\partial}{\partial t}(u_x\hat{\mathbf{i}} + u_y\hat{\mathbf{j}} + u_z\hat{\mathbf{k}}) + u_x\frac{\partial u_x}{\partial x}\hat{\mathbf{i}} + u_y\frac{\partial u_y}{\partial y}\hat{\mathbf{j}} + +u_z\frac{\partial u_z}{\partial z}\hat{\mathbf{k}}] = 0. \tag{2.161}$$

$$\rho[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}] = 0. \tag{2.162}$$

$$\nabla_b T^{ab} = 0. \tag{2.163}$$

The *energy-momentum tensor* for the matter field is

$\rho = \gamma^2\rho_0$

The component T^{00} may therefore be interpreted as the relativistic energy density of the matter field

$$T_{ab} = \rho\begin{pmatrix} 1 & u_x & u_y & u_z \\ u_x & u_x^2 & u_x u_y & u_x u_z \\ u_y & u_x u_y & u_y^2 & u_y u_z \\ u_z & u_x u_z & u_y u_z & u_z^2 \end{pmatrix} \tag{2.164}$$

$$T^{0\nu}{}_{,\nu} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho u_x}{\partial x} + \frac{\partial\rho u_y}{\partial y} + \frac{\partial\rho u_z}{\partial z} \tag{2.165}$$

classical equation of continuity

$$\frac{\partial\rho}{\partial t} + div\rho\mathbf{u} = 0. \tag{2.166}$$

This kinematic relation expresses quite generally the conservation of a quantity of material with density ρ moving with a velocity field $\mathbf{u}$. Here it expresses the conservation of matter in the sense of special relativty, which is the same as the conservation of energy.

$$\begin{aligned} T^{1\nu}{}_{,\nu} &= \frac{\partial(\rho u_x)}{\partial t} + \frac{\partial(\rho u_x^2)}{\partial x} + \frac{\partial(\rho u_x u_y)}{\partial y} + \frac{\partial(\rho u_x u_z)}{\partial z} \\ &= \rho \left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_y}{\partial y} + u_z \frac{\partial u_z}{\partial z} \right) \\ &\quad + u^2 \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z} \right). \end{aligned} \tag{2.167}$$

The second term zanishes by conservation of energy equation. The remainder gives

$$T^{1\nu}{}_{,\nu} = \rho \left(\frac{\partial u_x}{\partial t} + \mathbf{u} \cdot \nabla u_x \right) \tag{2.168}$$

Similarly, the other terms of the divergence may be included in

$$T^{i\nu}{}_{\nu} = \rho \left(\frac{\partial u^i}{\partial t} + \mathbf{u} \cdot \nabla u^i \right) \tag{2.169}$$

The RHS is famililiar from hydrodynamics - it describes the force free motion of a field of matter when set to zero.

$$\rho[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}] = 0. \tag{2.170}$$

We see that the requirement that the energy-momentum tensor have zero divergence in special relativity is eqivalent to conservation of energy and conservation of momentum in the matter field - hence the name energy-momentum tensor.

Perfect fluid

[42] *Dust as a Standard of Space and Time in Canonical Quantum Gravity*, [gr-qc/9409001].

Review basics of dust

Energy momentum tensor:

$$T^{ab} = \rho u^a u^b \tag{2.171}$$

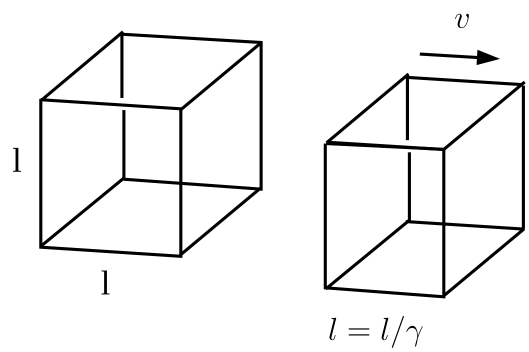

Figure 2.28: Lcfluid. The Lorentz contraction of a fluid element.

$$\partial_b T^{ab} = 0 \tag{2.172}$$

$$\frac{\partial(\rho \mathbf{u})}{\partial t} + \frac{\partial(\rho u_x \mathbf{u})}{\partial x} + \frac{\partial(\rho u_y \mathbf{u})}{\partial y} + \frac{\partial(\rho u_z \mathbf{u})}{\partial z} = 0 \tag{2.173}$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla(u) \right] = 0 \tag{2.174}$$

In classical fluid mechanics the equivalent of Newton's 2nd law equation of motion is called the Navier-Stokes equation. It is written as

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla(u) \right] = -\nabla p + \rho \vec{X} \tag{2.175}$$

where p is the pressure in the fluid and $\vec{X}$ is the external body force per unit mass. The dust equation of motion is just a special case of the Navier-Stokes equation where there is no pressure and no external forces acting on the body.

General relativistic dust

Making the simplest generalisation

$$\nabla_b T^{ab} = 0. \tag{2.176}$$

This equation in General relativity contains the geodesic equation of motion. We derive this. For dust we have

$$\nabla_b(\rho_0 u^a u^b) = 0. \tag{2.177}$$

Or

$$u^a \nabla_b(\rho_0 u^b) + \rho_0 u^b (\nabla_b u^a) = 0. \tag{2.178}$$

Contracting this equation with u_a gives

$$u_a u^a \nabla_b(\rho_0 u^b) + \rho_0 u^b u_a (\nabla_b u^a) = 0. \tag{2.179}$$

Since we are using proper time the 4-velocity is normalised such that

$$u_a u^a = c^2. \tag{2.180}$$

implying

$$u_a(\nabla_b u^a) = 0. \tag{2.181}$$

Substituting this into (2.179) implies

$$\nabla_b(\rho_0 u^b) = 0. \tag{2.182}$$

This equation substituted back into (2.178) we obtainn

$$\rho_0 u^b (\nabla_b u^a) = 0. \tag{2.183}$$

This particles of dust obey the geodesic equation.

2.10.2 Perfect Fluid

there is the

i) four velocity $u^a = dx^a/d\tau$

ii) a proper density field $\rho_0 = \rho_0(x)$

iii) and a scalar pressure field $p = p(x)$

$$T^{ab} = \rho_0 u^a u^b + S^{ab} \tag{2.184}$$

We work in the classical limit with low fluid velocities and low pressure, this means neglecting terms of order

$$u^2 \quad \text{and} \quad pu$$

We furthermore assume that the pressure is sufficiently small so that the elestic energy density of the fluid is small compared to the energy due to matter density. With these assumptons we can write the conservation of energy completely in terms of the proper matter density ρ_0:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_x}{\partial x} + \frac{\partial \rho u_y}{\partial y} + \frac{\partial \rho u_z}{\partial z} = 0. \tag{2.185}$$

The conservation of energy equation is acheived using just the divergence of $\rho_0 u^0 u^b$ alone so we take $S^{0i} = S^{i0} = 0$.

The Navier-Stokes equation from fluid dynamics reads

$$\rho \left(\frac{\partial u^i}{\partial t} + \mathbf{u} \cdot \nabla u^i \right) = -\frac{\partial p}{\partial x^i} \tag{2.186}$$

The RHS is the accelaration experienced by an observer moving with the fluid, the volume element accelerated by the pressure-force density $-\partial p / \partial x^i$.

$$T^{i\nu}{}_{\nu} = \rho \left(\frac{\partial u^i}{\partial t} + \mathbf{u} \cdot \nabla u^i \right) + S^{ij}{}_{j} \tag{2.187}$$

Implying

$$S^{ij}{}_{j} = \frac{\partial p}{\partial x^i}. \tag{2.188}$$

Something satisfying this is easily seen to be

$$S^{ij} = p \delta^{ij} \tag{2.189}$$

or reexpressed

$$S^{ij} = p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.190}$$

The extension of S^{ij} to a 4×4 matrix is

$$S^{\mu\nu} = p \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.191}$$

The conservation of energy and the equations of motion have therefore been combined into a single matrix equation

$$T^{\mu\nu}{}_{,\nu} = 0 \tag{2.192}$$

where $T^{\mu\nu}$ is explicitly

$$T^{\mu\nu} = \rho_0 \begin{pmatrix} 1 & u_x & u_y & u_z \\ u_x & 0 & 0 & 0 \\ u_y & 0 & 0 & 0 \\ u_z & 0 & 0 & 0 \end{pmatrix} + p \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.193}$$

We wish to write this in a Lorentz covariant form - extend it to a tensor. We argue that S^{ab} must be of the form

$$S^{\mu\nu} = p(\lambda u^\mu u^\nu + \mu g^{\mu\nu}) \tag{2.194}$$

where λ and μ are costants. Neglecting terms u^2 and pu leads to

$$S^{\mu\nu} = p \left[\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \mu \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \right] \tag{2.195}$$

This must reduce to (2.191), so we must choose $\lambda = 1$ and $\mu = -1$. Thus

$$S^{\mu\nu} = p(u^\mu u^\nu - g^{\mu\nu}) \tag{2.196}$$

and the energy-momentum tensor of a perfect fluid is

$$T^{\mu\nu} = (\rho_0 + p)u^\mu u^\nu - pg^{\mu\nu}. \tag{2.197}$$

2.10.3 Maxwell's Equations

Maxwell's Equations are

$$\begin{aligned} \nabla \cdot \vec{E} = \rho \qquad & \nabla \cdot \vec{B} = 0 \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \qquad & \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{j} \end{aligned} \tag{2.198}$$

where $\vec{E}$ is th electric field, $\vec{B}$ is the magnetic field, ρ is the charge density, and $\vec{j}$ is the current density.

It is more convenient to work in terms of potentials. With the 4-vector potential

$$A^a = (\phi, \vec{A}) \tag{2.199}$$

the electric and magnetic field can be written

$$\vec{E} = -\nabla\phi - \frac{\partial A}{\partial t}, \qquad \vec{B} = \nabla \times \vec{A} \tag{2.200}$$

The electromagnetic field strength tensor is defined as

$$F^{ab} = \partial^a A^b - \partial^b A^a \tag{2.201}$$

which turns out to be the matrix given by

$$F^{ab} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.202}$$

The field equation is

$$\partial^a F^{bc} = 0, \tag{2.203}$$

which along with the Bianchi identity

$$\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0, \tag{2.204}$$

constitute Maxwell's equations.

In Minkoski spacetime, the electromagnetic field is described by the lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{ab} F^{ab}. \tag{2.205}$$

2.10.4 Scalar Field

consider a scalar field ϕ in Minkowski spacetime, whose lagrangian is

$$\mathcal{L} = \frac{1}{2}(\eta^{ab}\partial_a\phi\partial_b\phi - m^2\phi^2) \tag{2.206}$$

The corresponding field equation is the so called Klein-Gordon equation:

$$\partial_a\partial^a\phi - m^2\phi = 0. \tag{2.207}$$

The Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{ab}(x)\partial_a\phi(x)\partial_b\phi(x) - m^2\phi^2(x)) \tag{2.208}$$

$$g^{ab}\nabla_a\nabla_b\phi + m^2\phi = 0 \tag{2.209}$$

Using the identity

$$\partial_a\sqrt{-g} = \frac{\sqrt{-g}}{2} g^{cd}\partial_a g_{cd} \tag{2.210}$$

the field equation is

$$\Box\phi(x) + m^2\phi(x) = 0 \tag{2.211}$$

where

$$\Box = \nabla_a \partial^a = \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} g^{ca} \partial_c) \tag{2.212}$$

2.10.5 Yang-Mills

$$\mathcal{L} = -\frac{1}{4} \sqrt{-g} g^{ac} g^{bd} F^I_{ab} F^I_{cd} \tag{2.213}$$

2.10.6 Fermionic Matter

In Minkowski spacetime, Dirac's field lagrangian is

$$\mathcal{L}_D = \frac{ic\hbar}{2} (\overline{\psi} \gamma^a \partial_a \psi - \partial_a \overline{\psi} \gamma^a \psi) - mc^2 \overline{\psi} \psi. \tag{2.214}$$

The corresponding field equation is the Dirac equation

$$i\hbar \gamma^a \partial_a \psi - mc\psi = 0. \tag{2.215}$$

There is no finite dimensional spinor representation of the group $GL(4)$. However, as Dirac discovered, there is a spinorial representations of the Lorentz group. Spinors can then be defined at every point on the curved manifold only if they transform within that flat tangent space. Since the Lorentz group acts on the tangent space indices, we can define spinors on the tangent space.

Dirac matrices are contracted onto tetrads:

$$\gamma^I e^a_I(x) = \gamma^a(x) \tag{2.216}$$

$$\{\gamma^a(x), \gamma^b(x)\} = 2g^{ab}(x) \tag{2.217}$$

we are free to perform a different Lorentz transformation on each tangent space We must introduce annother gauge field, called the spin connection.Examine how the derivative of ψ transforms under $\psi \to S(\Lambda(x))\psi$,

$$\begin{aligned} \partial_a S(\Lambda)\psi &= S(\Lambda)\partial_a \psi + \partial_a S(\Lambda))\psi \\ &= S(\Lambda)(\partial_a + S^{-1}(\partial_a S)\psi \end{aligned} \tag{2.218}$$

$$\mathcal{D}_a\psi = (\partial_a + \omega_a)\psi \tag{2.219}$$

$\mathcal{D}_a\psi$ should tranform in the same way as ψ, provided that ω_a tranforms as

$$\omega_a \to S\psi S^{-1} - (\partial_a S)S^{-1}. \tag{2.220}$$

The connection is expanded in terms of the gauge group generators, the spinor representation of the Lorentz group, $\sigma_{IJ} = [\gamma_I, \gamma_J]/2$.

$$\omega_a = \omega_a^{IJ}\hat{\sigma}_{IJ} \tag{2.221}$$

We have a spinor $\psi(x)$ that is defined to be a scalar under coordinate transformations and an ordinary spinor under flat tangent space Lorentz transformations:

$$\mathcal{D}_a\psi = \partial_a\psi + \omega_a^{IJ}\hat{\sigma}_{IJ}\psi \tag{2.222}$$

The generally covariant Dirac equation is therefore given by

$$(i\gamma^a\mathcal{D}_a - m)\psi(x) = 0 \tag{2.223}$$

This is the Euler-Lagrange equation of the Lagraingian:

$$\mathcal{L}_D = \frac{1}{2\kappa^2}e(x)\overline{\psi}(x)(i\gamma^a\mathcal{D}_a - m)\psi(x) \tag{2.224}$$

where $e = \det(e_I^a) = \sqrt{-g}$. We are working with the connection of a two component spinor and we wish to

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} = c\begin{pmatrix} 0 & \hat{\sigma} \\ \hat{\sigma} & 0 \end{pmatrix}\cdot\hat{p}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{2.225}$$

or

$$\begin{aligned} \frac{\partial\varphi}{\partial t} &= c\hat{\sigma}\cdot\hat{p}\chi + mc^2\varphi \\ \frac{\partial\chi}{\partial t} &= c\hat{\sigma}\cdot\hat{p}\varphi - mc^2\chi \end{aligned} \tag{2.226}$$

we choose a different representation of the γ matrices from the standard set (). The four matrices

$$\gamma^0 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \hat{\sigma}^i \\ -\hat{\sigma}^i & 0 \end{pmatrix} \tag{2.227}$$

satisfy

$$\gamma^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad C = \begin{pmatrix} -\epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \quad \mathcal{C} = \begin{pmatrix} 0 & \epsilon \\ -\epsilon & 0 \end{pmatrix} \tag{2.228}$$

from the Lagrangian

$$\mathcal{L}_D = \hat{\sigma}^a_{AA'}\overline{\xi}^{A'}\mathcal{D}_a\xi^A - (\mathcal{D}_a\overline{\eta}^A)\eta^{A'} - \frac{im}{\sqrt{2}}(\overline{\eta}_A\xi^A - \overline{\xi}^{A'}\eta_{A'}) \tag{2.229}$$

The Dirac equation can be written in terms of the right- and left-handed components as

$$i\gamma^a\partial_a\psi_R = \psi_L, \quad i\gamma^a\partial_a\psi_L = \psi_R. \tag{2.230}$$

In the case of massless particles, these two equations decouple

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a}\cdot\sigma\right) \tag{2.231}$$

$$\Psi \to \Psi' = \hat{U}\Psi(x) = \exp\left(i\mathbf{a}(x)\cdot\hat{\mathbf{T}}\right)\Psi(x) \tag{2.232}$$

$$\tau\cdot a \to \tau\cdot a' = \exp\left(\frac{i}{2}\mathbf{a}\cdot\sigma\right)(\tau\cdot a)\exp\left(\frac{i}{2}\mathbf{a}\cdot\sigma\right) \tag{2.233}$$

effects a rotation of the vector field r around the axis $\mathbf{n} = \mathbf{r}/a$ by an angle $|a|$.

2.10.7 Energy-Momentum Tensor

$$T^{\mu}{}_{\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\nu\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\partial_\nu\overline{\psi} - \delta^\mu_\nu\mathcal{L} \tag{2.234}$$

from which is the energy density is

$$
\begin{aligned}
T^0{}_0 &= -\psi^\dagger i\hbar\hat{\alpha}\cdot\nabla c\psi + m_0 c^2\overline{\psi}\psi \\
&= \psi^\dagger(\hat{\alpha}\cdot\hat{p} + \hat{\beta}m_0c^2)\psi
\end{aligned} \tag{2.235}
$$

2.11 Action Principle

Scalar actions are invariant under active diffeomorphisms

The Einstein-Hilbert action is the most elementry variational principle from which field equtions for general relativity can be derived. However, the Einstein-Hilbert action is appropriate only when the underlying spacetime manifold topology, $\mathcal{V}$, is closed, i.e., a manifold which is both compact and without boundary. The Einstein-Hilbert action is (including the cosmological constant)

$$
S_{EH} = \frac{1}{16\pi}\int_{\mathcal{V}}(R-2\Lambda)\sqrt{-g}dx^4, \tag{2.236}
$$

To compute its variation, we need only the identity:

$$
\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu} \tag{2.237}
$$

and the Palatini identity

$$
\delta R_{\alpha\beta} \equiv \nabla_\mu(\delta\Gamma^\mu_{\alpha\beta}) - \nabla_\beta(\delta\Gamma^\mu_{\alpha\mu}). \tag{2.238}
$$

We derive the Palatini identity. Contracting the Riemann tensor

$$
R^\rho{}_{\alpha\mu\beta} = \partial_\mu\Gamma^\rho_{\beta\alpha} - \partial_\beta\Gamma^\rho_{\mu\alpha} + \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\beta\alpha} - \Gamma^\rho_{\beta\sigma}\Gamma^\sigma_{\mu\alpha}
$$

over ρ and μ gives the Ricci tensor

$$
R_{\alpha\beta} = R^\rho{}_{\alpha\rho\beta} = \partial_\rho\Gamma^\rho_{\beta\alpha} - \partial_\beta\Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\beta\alpha} - \Gamma^\rho_{\beta\sigma}\Gamma^\sigma_{\rho\alpha}.
$$

We then carry out a variation in the connection which results in a change in the Ricci tensor

$$
\begin{aligned}
\delta R_{\alpha\beta} &= \partial_\rho \delta\Gamma^\rho_{\beta\alpha} - \partial_\beta \delta\Gamma^\rho_{\rho\alpha} + \delta\Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\beta\alpha} + \Gamma^\rho_{\rho\sigma}\delta\Gamma^\sigma_{\beta\alpha} - \delta\Gamma^\rho_{\beta\sigma}\Gamma^\sigma_{\rho\alpha} - \Gamma^\rho_{\beta\sigma}\delta\Gamma^\sigma_{\rho\alpha} \\
&= \partial_\rho \delta\Gamma^\rho_{\beta\alpha} - \partial_\beta \delta\Gamma^\rho_{\rho\alpha} + \delta(\Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\beta\alpha}) - \delta(\Gamma^\rho_{\beta\sigma}\Gamma^\sigma_{\rho\alpha}). \qquad (2.239)
\end{aligned}
$$

Now since $\delta\Gamma^\rho_{\beta\alpha}$ is the difference between two connections, it is a tensor and we can calculate its covariant derivative

$$
\nabla_\lambda(\delta\Gamma^\rho_{\beta\alpha}) = \partial_\lambda(\delta\Gamma^\rho_{\beta\alpha}) + \Gamma^\rho_{\sigma\lambda}\delta\Gamma^\sigma_{\alpha\beta} - \Gamma^\sigma_{\alpha\lambda}\delta\Gamma^\rho_{\sigma\beta} - \Gamma^\sigma_{\beta\lambda}\delta\Gamma^\rho_{\alpha\sigma}
$$

From which we see

$$
\begin{aligned}
\nabla_\rho(\delta\Gamma^\rho_{\beta\alpha}) &= \partial_\rho(\delta\Gamma^\rho_{\beta\alpha}) + \Gamma^\rho_{\sigma\rho}\delta\Gamma^\sigma_{\alpha\beta} - \Gamma^\sigma_{\alpha\rho}\delta\Gamma^\rho_{\sigma\beta} - \Gamma^\sigma_{\beta\rho}\delta\Gamma^\rho_{\alpha\sigma} \\
&= \partial_\rho(\delta\Gamma^\rho_{\beta\alpha}) + \Gamma^\rho_{\sigma\rho}\delta\Gamma^\sigma_{\alpha\beta} - \delta(\Gamma^\sigma_{\alpha\rho}\Gamma^\rho_{\sigma\beta}) \qquad (2.240)
\end{aligned}
$$

and using $\nabla_\beta X_\alpha = \partial_\beta X_\alpha - \Gamma^\sigma_{\beta\alpha}X_\sigma$ on $\delta\Gamma^\rho_{\rho\alpha}$ gives

$$
\nabla_\beta(\delta\Gamma^\rho_{\rho\alpha}) = \partial_\beta(\delta\Gamma^\rho_{\rho\alpha}) - \Gamma^\sigma_{\alpha\beta}\delta\Gamma^\rho_{\sigma\rho} \qquad (2.241)
$$

Then subtracting (2.241) from (2.240) we gives

$$
\begin{aligned}
\nabla_\rho(\delta\Gamma^\rho_{\beta\alpha}) - \nabla_\beta(\delta\Gamma^\rho_{\rho\alpha}) &= \partial_\rho(\delta\Gamma^\rho_{\beta\alpha}) - \partial_\beta(\delta\Gamma^\rho_{\rho\alpha}) + \Gamma^\rho_{\sigma\rho}\delta\Gamma^\sigma_{\alpha\beta} - \delta(\Gamma^\sigma_{\alpha\rho}\Gamma^\rho_{\sigma\beta}) + \Gamma^\sigma_{\alpha\beta}\delta\Gamma^\rho_{\sigma\rho} \\
&= \partial_\rho(\delta\Gamma^\rho_{\beta\alpha}) - \partial_\beta(\delta\Gamma^\rho_{\rho\alpha}) + \delta(\Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\alpha\beta}) - \delta(\Gamma^\sigma_{\alpha\rho}\Gamma^\rho_{\sigma\beta}) \\
&= \delta R_{\alpha\beta}, \qquad (2.242)
\end{aligned}
$$

establishing the Palatini identity.

We can use these results to carry out the Einstein-Hilbert action

$$
\begin{aligned}
S_{EH} &= \frac{1}{16\pi}\int_V (R - 2\Lambda)\sqrt{-g}dx^4 \\
&= \frac{1}{16\pi}\int_V \left(g^{\alpha\beta}R_{\alpha\beta} - 2\Lambda\right)\sqrt{-g}d^4x \qquad (2.243)
\end{aligned}
$$

with the use of Leibniz for products,

$$
\begin{aligned}
\delta S_{EH} &= \int_V \delta\left(g^{\alpha\beta}R_{\alpha\beta} - 2\Lambda\right)\sqrt{-g}dx^4 \\
&= \int_V \left(R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + g^{\alpha\beta}R_{\alpha\beta}\delta\sqrt{-g} - 2\Lambda\delta\sqrt{-g} + \sqrt{-g}g^{\alpha\beta}\delta R_{\alpha\beta}\right)dx^4 \\
&= \int_V \left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{-g}dx^4 + \int_V g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}dx^4. \quad (2.244)
\end{aligned}
$$

The first term gives us what we need for the left-hand side of the Einstein field equations (with cosmological constant). We must account for the second term. By the Palatini idenity

$$
g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\mu \delta V^\mu, \qquad \delta V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\beta_{\alpha\beta} \tag{2.245}
$$

As such the term $\int_V g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}dx^4$ converts into a boundary term. If the spacetime is closed this term is irrelevant. This is discussed in the next section.

2.11.1 GHY boundary term

In the event that the manifold has a boundary $\partial\mathcal{V}$, the action should be supplemented by a boundary term so that the variational principle is well defined. The appropriate action is

$$
S_{EH} + S_{GHY} = \int_{\mathcal{V}} dx^4\sqrt{-g}(R - 2\Lambda) + \frac{1}{8\pi}\int_{\partial\mathcal{V}} d^3y\epsilon\sqrt{|h|}K, \tag{2.246}
$$

where S_{EH} is the usual Einstein-Hilbert action, S_{GHY} is the Gibbons-Hawking-York boundary term, h_{ab} is the inuced metric on the boundary, h is the determinant, K is the trace of the second fundamental form, ϵ is equal to $+1$ where $\partial\mathcal{V}$ is timelike and -1 where $\partial\mathcal{V}$ is spacelike, and y^a are the coordinates intrinsic to the boundary $\partial\mathcal{V}$. Varing this action with respect to the metric $g_{\alpha\beta}$, subject to the condition

$$
\delta g_{\alpha\beta}\big|_{\partial\mathcal{V}} = 0, \tag{2.247}
$$

gives the Einstein equations; the addition of the boundary term means that in performing the variation, the geometry intrinsic to the boundary encoded in the induced metric h_{ab} is held fixed.

That boundary term needed in the gravitation case is due to the fact that R, the gravitatioal Lagrangian density, contains second derivatives of the metric tensor. This non-typical feature of field theories, which are usually formulated in terms of Lagrangians that involve first derivatives of fields varied over.

The GHY term is essential. When one passes to the Hamiltonian formulism, it is necessary to include the GHY term in order to reproduce the correct Arnowitt-Deser-Minner (ADM enery). The term is required to ensue the old path integrl (a la Hawking et al) for quantum gravity has the correct composition properties. When calculating black hole entropy using the Euclidean semiclassical approach, the entire contribution comes from the GHY term. This term has been has had more recent applications in loop quantum gravity in calculating transition amplitudes and background-independent scattering amplitudes.

In order to have a finite value for the action, we may have to subtract off a surface term for flat spacetime:

$$S_{EH}+S_{GHY,0}=\frac{1}{16\pi}\int_V\sqrt{-g}(R-2\Lambda)dx^4+\frac{1}{8\pi}\int_{\partial V}d^3y\epsilon\sqrt{|h|}Kd^3y-\frac{1}{8\pi}\int_{\partial V}d^3y\epsilon\sqrt{|h|}K_0d^3y \tag{2.248}$$

or

$$S_{EH}+S_{GHY,0}=\frac{1}{16\pi}\int_V\sqrt{-g}(R-2\Lambda)dx^4+\frac{1}{8\pi}\int_{\partial V}d^3y\epsilon\sqrt{|h|}(K-K_0), \tag{2.249}$$

where K_0 is the extrinsic curvature of the boundary imbedded in flat space. As $\sqrt{h}$ is invariant under variations of $g_{\alpha\beta}$ this additional term does not effect the field equtions, as such that this is referred to as the non-dynamical term.

2.11.2 Introduction to hypersurfaces

Defining hypersurfaces

In a four-dimensional spacetime manifold, a hypersurface is a three-dimensional submanifold that can be either timelike, spacelike, or null.

A particular hyper-surface Σ can be selected either by imposing a constraint on the coordinates

$$f(x^\alpha)=0, \tag{2.250}$$

or by giving parametric equations,

$$x^\alpha=x^\alpha(y^a), \tag{2.251}$$

where y^a $(a = 1, 2, 3)$ are coordinates intrinsic to the hyper-surfaace.

For example, a two-sphere in three-dimensional Euclidean space can be described either by

$$f(x^\alpha) = x^2 + y^2 + z^2 - r^2 = 0,$$

where r is the radius of the sphere, or by

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad \textit{and}\ z = r\cos\theta,$$

where θ and ϕ are intrinsic coordinates.

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where θ and ϕ are intrinsic coordinates.

Hypersurface orthogonal vector fields

We start with the family of hyper-surfaces given by

$$f(x^\alpha) = C \tag{2.252}$$

where different members of the family correspond to different values of the constant C. Consider two neighbouring points P and Q with coordinates x^α and $x^\alpha + dx^\alpha$, respectively, lying in the same hyper-surface. We then have to first order

$$C = f(x^\alpha + dx^\alpha) = f(x^\alpha) + \frac{\partial f}{\partial x^\alpha}dx^\alpha. \tag{2.253}$$

Subtracting off $C = f(x^\alpha)$ from this equation gives

$$\frac{\partial f}{\partial x^\alpha}dx^\alpha = 0 \tag{2.254}$$

at P. This implies that $f_{,\alpha}$ is normal to the hyper-surface. A unit normal n_α can be introduced in the case where the hyper-surface is not null. This is defined by

$$n^\alpha n_\alpha \equiv \epsilon = \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike,} \\ +1 & \text{if } \Sigma \text{ is timelike,} \end{cases}, \tag{2.255}$$

and we require that n^α point in the direction of increasing $f : n^\alpha f_{,\alpha} > 0$. It can then easily be checked that n_α is given by

$$n_\alpha = \frac{\epsilon f_{,\alpha}}{|g^{\alpha\beta} f_{,\alpha} f_{,\beta}|^{\frac{1}{2}}} \tag{2.256}$$

if the hypersurface either spacelike or timelke.

Induced and transverse metric and transverse metric

The three vectors

$$e^\alpha_a = \left(\frac{\partial x^\alpha}{\partial y^a}\right)_{\partial\mathcal{M}} \qquad a = 1, 2, 3 \tag{2.257}$$

are tangential to the hyper-surface.

The induced metric is the three-tensor h_{ab} defined by

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b. \tag{2.258}$$

This acts as a metric tensor on the hyper-surface in the y^a coordinates. For displacements confined to the hyper-surface (so that $x^\alpha = x^\alpha(y^a)$)

$$\begin{aligned} ds^2 &= g_{\alpha\beta}dx^\alpha dx^\beta \\ &= g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial y^a}dy^a\right)\left(\frac{\partial x^\beta}{\partial y^b}dy^b\right) \\ &= \left(g_{\alpha\beta}e^\alpha_a e^\beta_b\right)dy^a dy^b \\ &= h_{ab}dy^a dy^b. \end{aligned} \tag{2.259}$$

Because the three vectors $e_1^\alpha, e_2^\alpha, e_3^\alpha$ are tangential to the hyper-surface,

$$n_\alpha e_a^\alpha = 0 \tag{2.260}$$

where n_α is the unit vector ($n_\alpha n^\alpha = \pm 1$) normal to the hyper-surface.

We introduce what is called the transverse metric

$$h_{\alpha\beta} = g_{\alpha\beta} - \epsilon n_\alpha n_\beta. \tag{2.261}$$

It isolates the part of the metric that is transverse to the normal n^α.

Is is easily seen that this four-tensor

$$h^\alpha{}_\beta = \delta^\alpha{}_\beta - \epsilon n^\alpha n_\beta \tag{2.262}$$

projects out the part of a four-vector transverse to the normal n^α as

$$h^\alpha{}_\beta n^\beta = (\delta^\alpha{}_\beta - \epsilon n^\alpha n_\beta) n^\beta = (n^\alpha - \epsilon^2 n^\alpha) = 0 \quad \text{and if} \quad w^\alpha n_\alpha = 0 \quad \text{then} \quad h^\alpha{}_\beta w^\beta = w^\beta. \tag{2.263}$$

We have

$$h_{ab} = h_{\alpha\beta} e_a^\alpha e_b^\beta. \tag{2.264}$$

If we define h^{ab} to be the inverse of h_{ab}, it is easy to check

$$h^{\alpha\beta} = h^{ab} e_a^\alpha e_b^\beta \tag{2.265}$$

where

$$h^{\alpha\beta} = g^{\alpha\beta} - \epsilon n^\alpha n^\beta. \tag{2.266}$$

Note that variation subject to the condition

$$\delta g_{\alpha\beta}\big|_{\partial\mathcal{M}} = 0, \tag{2.267}$$

implies that $h_{ab} = g_{\alpha\beta} e_a^\alpha e_b^\beta$, the induced metric on $\partial\mathcal{M}$, is held fixed during the variation.

2.11.3 Proof of main result

We have computed the variation of the Einstein-Hilbert term, which had an additional term that converts into a boundary term that does not vanish when one has boundary. We will add in the variation of the GHY-boundary term, and show that their sum results in:

$$\delta S_{TOTAL} = \delta S_{EH} + \delta S_{GHY} = \frac{1}{16\pi}\int_{\mathcal{M}} \Big(G_{\alpha\beta} + \Lambda g_{\alpha\beta}\Big)\, \delta g^{\alpha\beta}\sqrt{-g}d^4x \tag{2.268}$$

where $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is the Einstein tensor, which together with the cosmological term, produces the correct left-hand side to the Einstein field equations.

Variation of Einstein-Hilbert term

Recall that varying the Einstein-Hilbert action resulted in an additional term given by

$$\int_{\mathcal{V}} g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}d^4x \tag{2.269}$$

(see (2.244)). By use of (2.237) and the Palatini identity we had derived,

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\mu \delta V^\mu, \qquad \delta V^\mu = g^{\alpha\beta}\delta\Gamma^\mu_{\alpha\beta} - g^{\alpha\mu}\Gamma^\beta_{\alpha\beta} \tag{2.270}$$

(see (2.245)). We will need Stokes theorem in the form:

$$\begin{aligned}
\int_{\mathcal{V}} \nabla_\mu A^\mu \sqrt{-g}d^4x &= \int_{\mathcal{V}} \partial_\mu \left(\sqrt{-g}A^\mu\right) d^4x \\
&= \oint_{\partial\mathcal{V}} A^\mu d\Sigma_\mu \\
&= \oint_{\partial\mathcal{V}} \epsilon A^\mu n_\mu \sqrt{|h|}d^3y
\end{aligned} \tag{2.271}$$

where n_μ is the unit normal to $\partial\mathcal{V}$, and $\epsilon \equiv n_\mu n^\mu = \pm 1$ and y^a are cordinates intrinsic to the boundary. And $d\Sigma_\mu = \epsilon n_\mu d\Sigma$ where $d\Sigma = |h|^{\frac{1}{2}}d^3y$ where $h = \det[h_{ab}]$, is an invariant three-dimensional volume element on the hypersurface. In our particulr case we take $A^\mu = \delta V^\mu$.

We now evaluate $\delta V^\mu n_\mu$ on the boundary $\partial\mathcal{M}$, keeping in mind that on $\partial\mathcal{M}$, $\delta g_{\alpha\beta} = 0 = \delta g^{\alpha\beta}$. Taking this into account we have

$$\delta\Gamma^{\mu}_{\alpha\beta}\big|_{\partial\mathcal{M}} = \frac{1}{2}g^{\mu\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}). \tag{2.272}$$

It is useful to note that

$$\begin{aligned} g^{\alpha\mu}\delta\Gamma^{\beta}_{\alpha\beta}\big|_{\partial\mathcal{M}} &= \frac{1}{2}g^{\alpha\mu}g^{\beta\nu}(\delta g_{\nu\alpha,\beta} + \delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu}) \\ &= \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}(\delta g_{\nu\alpha,\beta} + \delta g_{\alpha\beta,\nu} - \delta g_{\nu\beta,\alpha}) \end{aligned} \tag{2.273}$$

where in the second line we have swapped around α and ν and used that the metric is symmetric. It is then not difficult to work out $\delta V^{\mu} = g^{\mu\nu}g^{\alpha\beta}(\delta g_{\nu\beta,\alpha} - \delta g_{\alpha\beta,\nu})$.

So now

$$\begin{aligned} \delta V^{\mu}n_{\mu}\big|_{\partial\mathcal{M}} &= n^{\mu}g^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\ &= n^{\mu}(\epsilon n^{\alpha}n^{\beta} + h^{\alpha\beta})(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \\ &= n^{\mu}h^{\alpha\beta}(\delta g_{\mu\beta,\alpha} - \delta g_{\alpha\beta,\mu}) \end{aligned} \tag{2.274}$$

where in the second line we used the identity $g^{\alpha\beta} = \epsilon n^{\alpha}n^{\beta} + h^{\alpha\beta}$, and in the third line we have used the anti-symmetry in α and μ. As $\delta g_{\alpha\beta}$ vanishes everywhere on the boundary, $\partial\mathcal{M}$, its tangential derivatives must also vanish: $\delta g_{\alpha\beta,\gamma}e^{\gamma}_{c} = 0$. It follows that $h^{\alpha\beta}\delta g_{\mu\beta,\alpha} = h^{ab}e^{\alpha}_{a}e^{\beta}_{b}\delta g_{\mu\beta,\alpha} = 0$. So finally we have

$$n^{\mu}\delta V_{\mu}\big|_{\partial\mathcal{M}} = -h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^{\mu}. \tag{2.275}$$

Gathering the results we obtain

$$(16\pi)\delta S_{EH} = \int_{\mathcal{V}} G_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x - \oint_{\partial\mathcal{V}} \epsilon h^{\alpha\beta}\delta g_{\alpha\beta,\mu}n^{\mu}\sqrt{|h|}d^3y. \tag{2.276}$$

We next show that the above boundary term will be cancelled by the variation of S_{GHY}.

Variation of the boundary term

We now turn to the variation of the S_{GHY} term. Because the induced metric is fixed on $\partial\mathcal{V}$, the only quantity to be varied is K is the trace of the extrinsic curvature.

We have

$$
\begin{aligned}
K &= n^{\alpha}{}_{;\alpha} \\
&= g^{\alpha\beta} n_{\alpha;\beta} \\
&= (\epsilon n^{\alpha} n^{\beta} + h^{\alpha\beta}) n_{\alpha;\beta} \\
&= h^{\alpha\beta} n_{\alpha;\beta} \\
&= h^{\alpha\beta} (n_{\alpha,\beta} - \Gamma^{\gamma}_{\alpha\beta} n_{\gamma})
\end{aligned} \tag{2.277}
$$

where we have used that $0 = (n^{\alpha} n_{\alpha})_{;\beta}$ implies $n^{\alpha} n_{\alpha;\beta} = 0$. So the variation of K is

$$
\begin{aligned}
\delta K &= -h^{\alpha\beta} \delta\Gamma^{\gamma}_{\alpha\beta} n_{\gamma} \\
&= -h^{\alpha\beta} n^{\mu} g_{\mu\gamma} \delta\Gamma^{\gamma}_{\alpha\beta} \\
&= -h^{\alpha\beta} n^{\mu} g_{\mu\gamma} \frac{1}{2} g^{\gamma\sigma} \left(\delta g_{\sigma\alpha;\beta} + \delta g_{\sigma\beta;\alpha} - \delta g_{\alpha\beta;\sigma} \right) \\
&= -\frac{1}{2} h^{\alpha\beta} \left(\delta g_{\mu\alpha;\beta} + \delta g_{\mu\beta;\alpha} - \delta g_{\alpha\beta;\mu} \right) n^{\mu} \\
&= \frac{1}{2} h^{\alpha\beta} \delta g_{\alpha\beta;\mu} n^{\mu}
\end{aligned} \tag{2.278}
$$

where we have use the fact that the tangential derivatives of $\delta g_{\alpha\beta}$ vanish on $\partial\mathcal{V}$. We have obtained

$$
(16\pi)\delta S_{GHY} = \oint_{\partial\mathcal{V}} \epsilon h^{\alpha\beta} \delta g_{\alpha\beta,\mu} n^{\mu} \sqrt{|h|} d^3y \tag{2.279}
$$

which cancels the second integral on the right-hand side of (2.276). The total variation of the gravitational action is:

$$
\delta S_{TOTAL} = \frac{1}{16\pi} \int_{\mathcal{V}} \left(G_{\alpha\beta} + \Lambda g_{\alpha\beta} \right) \delta g^{\alpha\beta} \sqrt{-g} d^4x. \tag{2.280}
$$

This produces the correct left-hand side of the Einstein equations.

The non-dynamical term

We elaborate on the role of

$$
S_0 = \frac{1}{8\pi} \oint_{\partial\mathcal{M}} \epsilon K_0 |h|^{\frac{1}{2}} d^3y
$$

in the gravitational action. As already mentioned above, because this term only depends on h_{ab}, its variation with respect to $g_{\alpha\beta}$ gives zero and so does not effect the field equations, its purpose is to change the numerical value of the action. As such we will refer to it as the non-dynamical term.

Let us assume that $g_{\alpha\beta}$ is a solution of the vacuum field equations, in which case the Ricci scalar R vanishes. The numerical value of the gravitational action is then

$$S = \frac{1}{8\pi} \oint_{\partial\mathcal{M}} \epsilon K |h|^{\frac{1}{2}} d^3y, \tag{2.281}$$

where we are ignoring the non-dynamical term for the moment. Let us evaluate this for flat spacetime. Choose the boundary $\partial\mathcal{M}$ to consist of two hyper-surfaces of constant time value $t = t_1, t_2$ and a large three-cylinder at $r = r_0$ (that is, the product of a finite interval and a three-sphere of radius r_0). We have $K = 0$ on the hyper-surfaces of constant time. On the three cylinder, in coordinates intrinsic to the hypersurface, the line element is

$$\begin{aligned} ds^2 &= -dt^2 + r_0^2 d\Omega^2 \\ &= -dt^2 + r_0^2(d\theta^2 + \sin^2\theta d\varphi^2), \end{aligned} \tag{2.282}$$

so that the induced metric h_{ab} is

$$h_{ab} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & r_0^2 & 0 \\ 0 & 0 & r_0^2 \sin^2\theta \end{pmatrix} \tag{2.283}$$

so that $|h|^{\frac{1}{2}} = r_0^2 \sin\theta$. The unit normal is $n_\alpha = \partial_\alpha r$, so $K = n^\alpha{}_{;\alpha} = 2/r_0$. Then

$$\oint_{\partial\mathcal{M}} \epsilon K |h|^{\frac{1}{2}} d^3y = \int_{t_1}^{t_2} dt \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \left(\frac{2}{r_0}\right) (r_0^2 \sin\theta) = 8\pi r_0 (t_2 - t_1) \tag{2.284}$$

and diverges as $r_0 \to \infty$, that is, when the spatial boundary is pushed to infinity, even when the $\mathcal{M}$ is bounded by two hyper-surfaces of constant time. One would expect the same problem for curved spacetimes that are asymptotically flat (there is no problem if the spacetime is compact). This problem is remedied by the non-dynamical term. The difference $S_{GHY} - S_0$ will be well defined in the limit $r_0 \to \infty$.

2.11.4 Variation of the Matter action

Matter actions only depend on the field and its first derivatives,

$$S_M = \int_{\mathcal{V}} \mathcal{L}[\phi, \phi_{,a}; g_{\alpha\beta}]\sqrt{-g}d^4x \tag{2.285}$$

Variation of $S_M[\phi; g_{\alpha\beta}]$ yields

$$\begin{aligned}
\delta S_M &= \int_{\mathcal{V}} \delta\left(\mathcal{L}\sqrt{-g}\right) d^4x \\
&= \int_{\mathcal{V}} \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}\sqrt{-g} + \mathcal{L}\delta\sqrt{-g}\right) d^4x \\
&= \int_{\mathcal{V}} \left(\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \frac{1}{2}\mathcal{L}g_{\alpha\beta}\right) \delta g^{\alpha\beta}\sqrt{-g}d^4x.
\end{aligned} \tag{2.286}$$

If we define the stress-energy tensor by

$$T_{\alpha\beta} := -2\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} + \mathcal{L}g_{\alpha\beta} \tag{2.287}$$

then

$$\delta S_M = -\frac{1}{2}\int_{\mathcal{V}} T_{\alpha\beta}\delta g^{\alpha\beta}\sqrt{-g}d^4x, \tag{2.288}$$

and this produces the correct right-hand side of the Einstein field equations, so that $\delta(S_G + S_M) = 0$ implies

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{2.289}$$

because the variation $\delta g^{\alpha\beta}$ is arbitrary within $\mathcal{V}$.

2.11.5 Invariance of the Einstein-Hilbert Action

Electromagnetism

a gauge transformation in Electromagnetism

$$A_a \to A_a + \nabla_a \Phi$$

$$\delta S_E \;=\; \frac{2}{8\pi}\int F^{ab}\delta F_{ab} d^4x \tag{2.290}$$

we get the identity

$$\partial_{[a}F_{bc]} = 0$$

which is equivalent to

$$\partial_a F_{bc} + \partial_c F_{ab} + \partial_b F_{ca} = 0$$

General relativity

Do a gauge transformation

$$g_{\mu\nu} \to g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \tag{2.291}$$

by direct substitution

$$16\pi G \delta S_G = \int_V G_{\mu\nu}(\delta g^{\mu\nu})\sqrt{-g}d^4x = -2\int G^{\mu\nu}(\nabla_\mu \xi_\nu)\sqrt{-g}d^4x \tag{2.292}$$

Integrating by parts using Gauss's law gives

$$8\pi G \delta S_G = -\int_S G^{\mu\nu}\xi_\nu d^3\Sigma_\mu + \int_V \xi_\nu \nabla_\mu G^{\mu\nu}\sqrt{-g}d^4x. \tag{2.293}$$

As the diffeomorphism reduces to identity at the boundary and by the contracted Bianchi identity

$$\nabla_\mu G^{\mu\nu} = 0$$

the action to be invariant.

2.12 Palatini Method in the Metric Formulation

The Palatini approach is based on the idea of treating both the metric and the connection separately as dynamical variables in the Einstein Lagrangian. The specific choose we make is to write $\mathcal{L}_G$ as a functional of $\tilde{g}^{\mu\nu}$ ($\tilde{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$) and a symmetric connection $\Gamma^{\mu}_{\nu\rho}$ and it's derivatives, i.e.

$$\mathcal{L}_G = \mathcal{L}_G(\tilde{g}^{\mu\nu}, \Gamma^{\mu}_{\nu\rho}, \Gamma^{\mu}_{\nu\rho,\sigma}) \tag{2.294}$$

where

$$\begin{aligned} \mathcal{L}_G &= \tilde{g}^{\mu\nu} R_{\mu\nu} \\ &= \tilde{g}^{\mu\nu}(\Gamma^{\rho}_{\mu\nu,\rho} - \Gamma^{\sigma}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\nu}\Gamma^{\sigma}_{\rho\sigma} - \Gamma^{\sigma}_{\mu\rho}\Gamma^{\rho}_{\nu\sigma}) \end{aligned} \tag{2.295}$$

so that the Ricci tensor depends on $\Gamma^{\mu}_{\nu\rho}$ and its derivatives only. Then, if we carry out a variation of the action

$$S = \int_{\Sigma} \tilde{g}^{\mu\nu} R_{\mu\nu} d\Omega \tag{2.296}$$

with respect to $\tilde{g}^{\mu\nu}$ only,

$$\delta S = \int_{\Sigma} \delta\tilde{g}^{\mu\nu} R_{\mu\nu} d\Omega \tag{2.297}$$

and the principle of stationary action gives immediately the vacuum field equations $R_{\mu\nu} = 0$.

Next, we have to show that variation with respect to the connection results in the usual dependency of the connection on the metric. To do this we need the tensor identity,

$$\delta R_{\nu\sigma} = \nabla_{\mu}(\delta\Gamma^{\mu}_{\nu\sigma}) - \nabla_{\sigma}(\delta\Gamma^{\mu}_{\nu\mu}). \tag{2.298}$$

relating a variation in the Ricci tensor $\delta R_{\nu\sigma}$ to a variation in the connection $\delta\Gamma^{\mu}_{\nu\rho}$. This is the equation which shall be employed in the derivation of Einstein equations from the Palatini version of the Einstein-Hilbert action.

The equation (2.298) is obtained from Paltini equation

$$\delta R^{\mu}_{\nu\rho\sigma} = \nabla_{\rho}(\delta\Gamma^{\mu}_{\nu\sigma}) - \nabla_{\sigma}(\delta\Gamma^{\mu}_{\nu\rho}) \tag{2.299}$$

by contraction on μ and ρ.

To derive the Paltini equation we use the technique of geodesic coordinates, where we choose an arbitrary point P at which $\Gamma^{\mu}_{\nu\rho} = 0$. Then, in particular, covariant derivatives reduce to ordinary derivatives at the point P. The Riemann tensor (C.590) reduces to

$$R^{\mu}_{\nu\rho\sigma} \stackrel{*}{=} \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho}. \tag{2.300}$$

Now we know that any connection transforms as a tensor but with a homogeneous part (see ()). Now, this homogeneous part does not involve the particular connection in question so the difference, $\delta\Gamma^{\mu}_{\nu\rho}$, of any two connections from the space of connections,

$$\delta\Gamma^{\mu}_{\nu\rho} = \bar{\Gamma}^{\mu}_{\nu\rho} - \Gamma^{\mu}_{\nu\rho} \tag{2.301}$$

transforms as a tensor. These two connections correspond to two different Riemann tensors. The difference is between them is:

$$\begin{aligned} \delta R^{\mu}_{\nu\rho\sigma} &= \overline{R}^{\mu}_{\nu\rho\sigma} - R^{\mu}_{\nu\rho\sigma} \\ &\stackrel{*}{=} \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} \\ &\stackrel{*}{=} \nabla_{\rho}(\delta\Gamma^{\mu}_{\nu\sigma}) - \nabla_{\sigma}(\delta\Gamma^{\mu}_{\nu\rho}) \end{aligned} \tag{2.302}$$

since partial derivatives commute with variations and is equivalent to covariant derivatives in geodesic coordinates. Now both $\delta R^{\mu}_{\nu\rho\sigma}$, being the difference of two tensors, and the quantities the right-hand side of the last equation are tensors, and so by the fundamental result (if a tensor equation holds in one coordinate system it most hold in all coordinate systems) we can deduce the Palatini equation.

We now carry out the variation with respect to $\Gamma^{\mu}_{\nu\sigma}$ with the use of (2.298),

$$\begin{aligned} \delta S &= \int_{\Sigma} \tilde{g}^{\mu\nu}\delta R_{\mu\nu} d\Omega \\ &= \int_{\Sigma} \tilde{g}^{\mu\nu}\left[\nabla_{\rho}(\delta\Gamma^{\rho}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\rho}_{\mu\rho})\right] d\Omega \end{aligned} \tag{2.303}$$

Integrating by parts and discarding the divergence term by we get

$$\begin{aligned} \delta S &= \int_{\Sigma} \left[\nabla_{\nu}\tilde{g}^{\mu\nu}\delta\Gamma^{\rho}_{\mu\rho} - \nabla_{\rho}\tilde{g}^{\mu\nu}\delta\Gamma^{\rho}_{\mu\nu}\right] d\Omega \\ &= \int_{\Sigma} \left[\delta^{\nu}_{\rho}\nabla_{\sigma}\tilde{g}^{\mu\sigma} - \nabla_{\rho}\tilde{g}^{\mu\nu}\right] \delta\Gamma^{\rho}_{\mu\nu} d\Omega \end{aligned} \tag{2.304}$$

Since δS vanishes for arbitrary volumes Ω, the integrand must vanish,

$$\left[\delta^{\nu}_{\rho}\nabla_{\sigma}\tilde{g}^{\mu\sigma} - \nabla_{\rho}\tilde{g}^{\mu\nu}\right]\delta\Gamma^{\rho}_{\mu\nu} = 0.$$

The variations of $\delta\Gamma^{\rho}_{\mu\nu}$ are arbitrary, but symmetric in μ and ν and so only the symmetric part of the expression in brackets vanishes,

$$\frac{1}{2}\delta^{\nu}_{\rho}\nabla_{\sigma}\tilde{g}^{\mu\sigma} + \frac{1}{2}\delta^{\mu}_{\rho}\nabla_{\sigma}\tilde{g}^{\nu\sigma} - \nabla_{\rho}\tilde{g}^{\mu\nu} = 0. \tag{2.305}$$

We now show in turn that this implies that the covariant derivatives of $\tilde{g}^{\mu\nu}$, $(-g)^{\frac{1}{2}}$, $g^{\mu\nu}$, and $g_{\mu\nu}$ vanish.

First by contracting the indices ρ and ν in (2.305) gives

$$\frac{1}{2}4\nabla_{\sigma}\tilde{g}^{\mu\sigma} + \frac{1}{2}\delta^{\mu}_{\rho}\nabla_{\sigma}\tilde{g}^{\rho\sigma} - \nabla_{\rho}\tilde{g}^{\mu\rho} = \frac{3}{2}\nabla_{\rho}\tilde{g}^{\mu\rho} = 0.$$

Now substituting $\nabla_{\rho}\tilde{g}^{\mu\rho} = 0$ into (2.305) gives

$$\nabla_{\rho}\tilde{g}^{\mu\nu} = 0.$$

Now as the determiant of $\tilde{g}^{\mu\nu}$ involves the sum of products of $\tilde{g}^{\mu\nu}$ and by Leibniz rule its covariant derivative is also zero, therefore

$$0 = \nabla_{\rho}\det\tilde{g} = \nabla_{\rho}(-g)^{\frac{1}{2}}\det g^{\mu\nu} = -\nabla_{\rho}\frac{(-g)^{\frac{1}{2}}}{(-g)} = -\nabla_{\rho}\frac{1}{(-g)^{\frac{1}{2}}}$$

where we have used that $g^{\mu\nu}$ is the inverse metric $g_{\mu\nu}$. Now

$$\begin{aligned}
\nabla_{\rho}(-g)^{\frac{1}{2}} &= -\nabla_{\rho}\frac{(-g)}{(-g)^{\frac{1}{2}}} \\
&= -\frac{1}{(-g)^{\frac{1}{2}}}\nabla_{\rho}[(-g)^{\frac{1}{2}}]^{2} \\
&= -2\nabla_{\rho}(-g)^{\frac{1}{2}},
\end{aligned}$$

implying that $\nabla_{\rho}(-g)^{\frac{1}{2}} = 0$. Then

$$
\begin{aligned}
0 &= \nabla_\rho \tilde{g}^{\mu\nu} \\
&= \nabla_\rho[(-g)^{\frac{1}{2}} g^{\mu\nu}] \\
&= (\nabla_\rho (-g)^{\frac{1}{2}}) g^{\mu\nu} + (-g)^{\frac{1}{2}} \nabla_\rho g^{\mu\nu} \\
&= (-g)^{\frac{1}{2}} \nabla_\rho g^{\mu\nu}
\end{aligned}
$$

implies $\nabla_\rho g^{\mu\nu} = 0$. Now

$$
\begin{aligned}
0 &= g_{\sigma\nu} \nabla_\rho (\delta^\sigma_\mu) \\
&= g_{\sigma\nu} \nabla_\rho (g_{\mu\delta} g^{\delta\sigma}) \\
&= g_{\sigma\nu} [g^{\delta\sigma} \nabla_\rho g_{\mu\delta} + g_{\mu\delta} \nabla_\rho g^{\delta\sigma}] \\
&= g^{\delta\sigma} g_{\sigma\nu} \nabla_\rho g_{\mu\delta} \\
&= \delta^\delta_\nu \nabla_\rho g_{\mu\delta} \\
&= \nabla_\rho g_{\mu\nu}
\end{aligned} \tag{2.306}
$$

We know that it folows from this condition, $\nabla_\rho g_{\mu\nu} = 0$, that the connection is necessarily the metric connection

$$
\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma} [\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}].
$$

We have found that variation with respect to the densitised metric gives the vacuum field equations, and variation with respect to the connection reveals that it is necessarily the metric connection.

2.13 Cosmological Definition of Distance

A frequency shift can be thought of as induced by both gravity and by proper motion of the source relative to the observer.

$$
d_A = R(t) \int_0^{r_1} \frac{dr}{(1 - kr^2)^{1/2}}. \tag{2.307}
$$

[115]

they are not gauge invariant and therefore should not be observable in obvious contradiction to reality. Moreover, the time evolution described by the FRW equations is obtained from the Hamiltonian equations of motion generated by the Hamiltonian constraint and not by an actual Hamiltonian. This is due to the fact that the "Hamiltonian" used to derive the FRW equations is actually constrained to vanish by one of the Einstein equations. The "evolution" equations generated by a constraint must therefore be interpreted as gauge transformations and those, by the very definition of gauge transformations, are also not observable, again in sharp contradiction to observation.

All textbooks on classical GR incorrectly describe the Friedmann equations as physical evolution equations rather than what they really are, namely gauge transformation equations. The true evolution equations acquire possibly observable modifications to the gauge transformation equations whose magnitude depends on the physical clock that one uses to deparamterize the gauge transformation equations.

One could think that what cosmologists usually do in order to describe measurable quantities mathematically is actually precisely correct, that is "relational". For instance the redshift factor

$$z(t_1, t_2) = \frac{\omega_1}{\omega_2} \approx \frac{a(t_2)}{a(t_2)} \tag{2.308}$$

is the ratio between the emission frequency ω_1 of a spectral line (known from a table top experiment on Earth) and the absorption frequency ω_2 observed on Earth is certainly measurable. Formula (2.308) relates this observable quantity to the ratio of the scale factors at unphysical emission time t1 and absorption time t2 respectively. We will now show that (2.308) is in fact incorrect:

The reason is that the quantities $a(t)$ are not observable. In order to see what is going on, we have to go through the derivation of the redshift formula. Consider a star at comoving distance r from Earth. For light the geodesic is null and due to

$$ds^2 = -dt^2 + a(t)dx^a dx^b \delta_{ab}$$

we get as an expression of motion $a(t)\dot{r}(t) = 1$. Formula (2.308) then results from the fact that the beginning and the end of the wave travel the same comoving distance

$$r = \int_{t_1}^{t_2} \frac{dt}{a(t)} = \int_{t_1+T_1}^{t_2+T_2} \frac{dt}{a(t)}$$

with $\omega_i 2\pi/T_j$. This is certainly mathematically correct, however, the quantities $a(t_j)$ are not observable. In order to express z in terms of observable quantities $O_a(\tau)$ we express the line element in terms of τ (see (6.30))

$$ds^2 = -d\tau^2(1 + \frac{1}{x}) + O_a(\tau)^2 dx^a dx^b \delta_{ab} \tag{2.309}$$

Notice that τ is no gauge parameter but a physical observable associated with the physical Hamiltonian, hence the factor $1 + 1/x$ cannot be transformed away by an active diffeomorphism $\tau \mapsto \varphi(\tau)$ without changing the Hamiltonian. We now obtain the null geodesic equation of motion $O_a(\tau) dO_r(\tau)/d\tau = \sqrt{1 + 1/x}$. The same argument now leads to the modified redshift factor relation

$$z(\tau_1, \tau_2) = \frac{\omega_1}{\omega_2} = \frac{O_a(\tau_2)}{O_a(\tau_1)} \sqrt{\frac{1 + \frac{1}{x(\tau_1)}}{1 + \frac{1}{x(\tau_2)}}}, \qquad x(\tau) = \frac{E^2}{\alpha^2 O_a(\tau)^6} \tag{2.310}$$

and now all displayed quantities are observable. Hence we see that as long as x is large, (2.310) and (2.308) agree in the following sense: What one incorrectly does in cosmology is to identify the unobservable gauge pair $(t, a(t))$ with the observable physical pair $(\tau, O_a(\tau))$. With this interpretation, the wrong relation (2.308) is a good approximation to the correct relation (2.310) as long as x is large. However, there are large deviations especially in the late universe and of course the modification (2.310) may have an observable effect on the interpretation of supernovae type Ia observations (standard candles) which provide evidence for recent accelarated expansion of the universe.

2.14 Relativistic Material Reference Systems

[41]

The use of material reference systems in general relativity has a long and noble history. Beginning with the systems of rods and clocks conceived by Einstein [40]and Hilbert [**?**, **?**] material systems have been used as a physical means of specifying events in spacetime and for addressing conceptual questions in classical gravity. That such systems also provide important tools for quantum gravity was pointed out by DeWitt [3], who used them to analyze the implications of the uncertainty principle for measurements of the gravitational field.

2.15 Linearized Equations of General Relativity

We wish to find a solution to Einstein's equations of the form

$$g_{ab} = \eta_{ab} + \epsilon h_{ab} \tag{2.311}$$

where ϵ is small. Of course, as with any solution to Einstein's equations, physical quantities of solution are those which are invariant under *all* active diffeomorphisms. As before a quantity which isn't an observable in the pure gravity case can become an observable in a gravity + matter system, for example the test particles of a gravitational interferometer

Throughout the following we neglect terms of second order and higher in ϵ. We want to find the equations of motion for the perturbations h_{ab} from Einstein's equations keeping first order in ϵ. We begin with the Christoffel symbols.

$$\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2}g^{ad}(g_{dc,b} + g_{db,c} - g_{bc,d}) \\
&= \frac{1}{2}\epsilon\eta^{ad}(h_{dc,b} + h_{db,c} - h_{bc,d}) \\
&= \frac{1}{2}\epsilon(h^a{}_{c,b} + h^a{}_{b,c} - h_{bc,}{}^a) \qquad (2.312)
\end{aligned}$$

Since the connection coefficients are first order in ϵ, the only contribution to the Riemann tensor will come from the derivatives of Γ's, but not the Γ^2 terms. We obtain

$$\begin{aligned}
R_{abcd} &= \eta_{ae}\partial_c\Gamma^e_{bd} - \eta_{ae}\partial_d\Gamma^e_{bc} \\
&= \frac{1}{2}\epsilon(h_{ad,bc} + h_{bcad} - h_{ac,bd} - h_{bd,ac}). \qquad (2.313)
\end{aligned}$$

the Bianchi identities

$$R_{ab[cd;e]} = 0 \qquad (2.314)$$

become

$$R_{ab[cd,e]} = 0 \qquad (2.315)$$

The Ricci tensor is obtained by contracting over a and c of the Riemann tensor, giving

$$R_{ab} = \eta^{cd}R_{cadb} = \frac{1}{2}(h^c{}_{a,bc} + h^c{}_{b,ac} - \Box h_{ab} - h_{,ab}), \qquad (2.316)$$

where we defined the trace of the perturbation $h := \eta^{cd}h_{cd} = h^c{}_c$ and $\Box$ is the d'Alembertian operator of Minkowski spacetime

$$
\begin{aligned}
\Box &= \eta^{ab}\partial_a\partial_b \\
&= \partial^a\partial_a \\
&= \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 \\
&= \frac{\partial^2}{\partial t^2} - \nabla^2
\end{aligned}
\tag{2.317}
$$

Finally we obtain the Ricci scalar

$$
R = \epsilon(h^{cd}{}_{,cd} - \Box h) \tag{2.318}
$$

and Einstein's tensor

$$
G_{ab} = \frac{1}{2}\epsilon(h^c{}_{a,bc} + h^c{}_{b,ac} - \Box h_{ab} - h_{,ab} - \eta_{ab}h^{cd}{}_{,cd} + \eta_{ab}\Box h). \tag{2.319}
$$

$$
\Box h_{ab} - \partial_\mu\partial^c h_{cb} - \partial_b\partial^c h_{ca} + \partial_a\partial_b h^c{}_c = 0 \tag{2.320}
$$

h_{ab} is a small correction that we hope to interpret as a gravitational wave propagating through Minkowski space (or some other 'background' space-time). Since h_{ab} is symmetric, it has ten independent components.

2.15.1 Gauge Transformations

To analyze the linearized equations of GR we will consider a restricted set of active diffeomorphisms; those generated by vector fields first order in ϵ. Solutions related by such transformations represent the same physical situation while maintaining the requirement that the metric is Minkowski with small perturbation. These will be the gauge symmetries of the linearized Einstein equations, in analogy to the gauge symmetries of electromagnetism. In this analysis we will see closely resemble the analysis of the covariant form of Maxwell's equations. As with the field A_μ in electromagnetism, it is possible to use the gauge invariance to require some particular condition of the field h_{ab}. There are analogies with the temporal and Lorentz gauges.

There are four infinitesimal gauge degrees of freedom so the number of physical degrees is six out of the ten independent components of h_{ab}.

Let us see the effect of a gauge transformation on the field h_{ab}. Under the point transformation

$$x^a \to x'^a = x^a + \epsilon\xi^a. \tag{2.321}$$

the consequent transformation of h_{ab} is

$$h_{ab} \to h'_{ab} = h_{ab} - 2\xi_{(a,b)}. \tag{2.322}$$

This represents the change in the metric perturbation h_{ab} under an infinitesimal active diffeomorphism along the vector field $\epsilon\xi$ and is a gauge transformation in the linearized theory.

The invariance of our theory under such transformations is analogous to gauge invariance in electromagnetism under the transformation in $A_\mu \to A_\mu + \partial_\mu\Lambda$. This is because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which encodes all the physical information about the electro-dynamical field, is left unchanged by such a transformation.

As (2.322) is the effect of an active diffeomorphism it does not alter the form of equation of motion (2.320), (this can also be seen by direct substitution into (2.320)), so that $h_{ab}(x)$ and $h'_{ab}(x)$ satisfy the same equations of motion, as does $A_\mu(x)$ and $A_\mu(x) + \partial_\mu\Lambda(x)$ in electromagnetism.

Just as two vector potentials related by a gauge transformation with an arbitrary function $\Lambda(x)$ represent the same physical situation, because they give the same electric and magnetic fields, so two metrics related by the gauge transformation (2.322) describe the same physical situation, because they give the same geometry up to active diffeomorphisms generated by vector fields of first order in ϵ. By the analogy between the field strength tensor $F_{\mu\nu}$ and the curvature tensor $R^a{}_{bcd}$, it can be shown that the curvature tensors are equal. However, unlike electromagnetism the curvature tensor is not have direct physical significance because it is not invariant under all active diffeomorphisms, only infinitesimal ones.

Each time we fix the constraint by restricting the the gauge field h_{ab}, we need to check that there exists a choice of $\epsilon\xi^a(x)$ such that the gauge condition is possible.

introduce new variables ψ_{ab}

$$\psi_{ab} := h_{ab} - \frac{1}{2}\eta_{ab}h, \tag{2.323}$$

then

$$R_{ab} = \frac{1}{2}\epsilon(\psi^c{}_{a,bc} + \psi^c{}_{b,ac} - \Box h_{ab}) \tag{2.324}$$

and

$$R = \frac{1}{2}\epsilon(2\psi^{cd}{}_{,cd} - \Box h) \tag{2.325}$$

and

$$G_{ab} = \frac{1}{2}\epsilon(\psi^c{}_{a,bc} + \psi^c{}_{b,ac} - \Box\psi_{ab} - \eta_{ab}\psi^{cd}{}_{,cd}) \tag{2.326}$$

these field equations will reduce to wave equations if we impose the gauge condition

$$\psi^a{}_{b,a} = 0, \tag{2.327}$$

that is,

$$h^a{}_{b,a} - \frac{1}{2}h_{,b} = 0, \tag{2.328}$$

other stuff

Einstein's full equations become

$$\frac{1}{2}\epsilon\Box\psi_{ab} = -\kappa T_{ab}. \tag{2.329}$$

other stuff

$$\Box h_{ab} = 0 \tag{2.330}$$

that is

$$\partial_a h^a{}_b - \frac{1}{2}\partial_b h = 0. \tag{2.331}$$

2.15.2 Linearized Einstein Equations in the Temporal Gauge

The GR analogue of the temporal gauge is know as "gaussian normal coordinate", or the "lapse=1, Shift=0", or "proper-time" gauge temporal gauge $h_{0\mu} = 0$

$$
\begin{aligned}
h_0 &= \frac{1}{2}h_{kl}D_{kl}, \qquad h_L = h_{kl}\frac{p_k p_l}{p^2} \\
h_i^T &= h_{kl}\frac{p_k}{p}D_{il}, \qquad h_{ij}^{TT} = h_{kl}\Big(D_{ik}D_{jl} - \frac{1}{2}D_{ij}D_{kl}\Big).
\end{aligned} \tag{2.332}
$$

$$
\partial_\alpha \partial^\alpha h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \partial_\mu \partial_\nu h^\alpha{}_\alpha = \frac{\rho}{2}\delta_{\mu\nu}. \tag{2.333}
$$

written in the temporal gauge as

$$
\begin{aligned}
\ddot{h}^i{}_i &= \frac{\rho}{2}, \\
-\partial^j \dot{h}_{ij} + \partial_i \dot{h}^k{}_k &= 0, \\
h_{ij} - \partial_i \partial^k h_{kj} - \partial_j \partial^k h_{ji} + \partial_i \partial_j h^k{}_k &= \frac{\rho}{2}\delta_{ij},
\end{aligned} \tag{2.334}
$$

2.15.3 Gravitational Wave Solutions

In analogy to electromagnetism, plane waves are solutions of this equation

$$
\Box h_{ab} = 0 \tag{2.335}
$$

with the gauge fixing condition

$$
\partial^b h_{ab} = 0 \tag{2.336}
$$

analogous to the Lorentz gauge (5.24). We write

$$
h_{ab} = A_{ab}\exp(ik\cdot x), \tag{2.337}
$$

and find

$$
k_a k^a = 0 \quad \text{and} \quad k^b A_{ab} = 0. \tag{2.338}
$$

Next we use residual gauge freedom

$$
A_{ab} \to A'_{ab} = A_{ab} - ik_a B_b - ik_b B_a + i\eta_{ab}k^c B_c \tag{2.339}
$$

choose B^a so that

$$A'_{ab}\eta^{ab} = A_{ab}\eta^{ab} + 2ik^c B_c = 0 \qquad \text{Traceless} \tag{2.340}$$

We can make more constraints on B^a

$$h_{ab}^{TT} = A_{ab}\exp(ik\cdot x), \tag{2.341}$$

where A_{ab} is a constant symmetric (0,2) tensor, which is purely spatial and traceless

$$A_{0a} = 0, \quad \eta^{ab}A_{ab} = 0. \tag{2.342}$$

convenient to express the metric tensor in this transverse traceless gauge

$$h_{ab}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -2s_{ij} & \\ 0 & & & \end{pmatrix} \tag{2.343}$$

it is purely spatial, traceless and transverse

$$\begin{aligned} h_{00}^{TT} &= 0 && (2.344)\\ \eta^{ab}h_{ab}^{TT} &= 0 && (2.345)\\ \partial^a h_b^{TT} &= 0 && (2.346) \end{aligned}$$

2.15.4 Waves Emitted by Oscillating Masses

$$\Box_x G(x^c - y^c) = \delta^4(x^c - y^c) \tag{2.347}$$

where $\Box_x$ denotes the D'Alembertian with respect to the $x-$coordinates.

$$\overline{h}_{ab}(x^c) = -16\pi G \int G(x^c - y^c)T_{ab}(y^c)d^4y, \tag{2.348}$$

$$G(x^c - y^c) = -\frac{1}{|x-y|}\delta[|x-y| - (x^0 - y^0)]\theta(x^0 - y^0). \tag{2.349}$$

$$\overline{h}_{ab}(t,x) = 4G\int \frac{1}{|x-y|}T_{ab}(t-|x-y|,y)d^3y, \tag{2.350}$$

where . The "retarded time" is

$$t_t = t - |x-y|. \tag{2.351}$$

2.16 Classical Cosmology

2.16.1 Fluid Flow Equations

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx_i}{dt}\frac{\partial}{\partial x_i}, \tag{2.352}$$

The accelaration of the element of fluid is given by the **Euler equation**

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla P - \nabla\Phi_{grav}. \tag{2.353}$$

Here Φ_{grav} is the gravitaional potential, which satisfies the **Poisson equation**

$$\nabla^2\Phi_{grav} = 4\pi G\rho. \tag{2.354}$$

$$\frac{1}{\mathcal{V}}\frac{d\mathcal{V}}{dt} = \nabla\cdot\mathbf{u} \tag{2.355}$$

$$H(x,t) = \frac{1}{3}\nabla\cdot\mathbf{u} \tag{2.356}$$

If we integrate an element of gas at position x and time t, we can integrate this using the divergence theorem to find

$$3H\mathcal{V} = \int_{\mathcal{V}} \mathbf{u}\cdot\,\mathbf{dS}. \tag{2.357}$$

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0. \tag{2.358}$$

$$\begin{aligned} d\tau^2 &= -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \\ &= -dt^2 + a^2(t)\eta_{ij}dx^i dx^j \end{aligned} \tag{2.359}$$

$$= \frac{1}{2}\int \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] d\tau \tag{2.360}$$

$$\frac{d^2t}{d\tau^2} + 2\dot{a}a\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0. \tag{2.361}$$

$$\Gamma^0_{00} = 0, \quad \Gamma^0_{i0} = \Gamma^0_{0i} = 0, \quad \Gamma^0_{ij} = a\dot{a}\eta_{ij} \tag{2.362}$$

$$\frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0. \tag{2.363}$$

$$\Gamma^i_{00} = 0, \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j, \quad \Gamma^i_{jk} = 0. \tag{2.364}$$

2.16.2 Newtonian Cosmology

cosmological force acting on the ith galaxy

$$F_i = \frac{1}{3}\Lambda m_i r_i \tag{2.365}$$

where Λ is the cosomological constant. the cosmological potential energy

$$V_c = -\frac{1}{6}\Lambda\sum_{i=1}^{n} m_i r_i \tag{2.366}$$

total energy

$$E = \frac{1}{2}\sum_{i=1}^{n} m_i \dot{r}_i^2 - G\sum_{i,j=1(i<j)}^{n}\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{1}{6}\Lambda\sum_{i=1}^{n} m_i r_i^2. \tag{2.367}$$

Motions compatible with homogeneity and isotropy are uniform expansion or contraction, a scaling up or down by a time-dependent scale factor.

$S(t)$ scale factor

$$r_i(t) = S(t) r_i(t_0) \tag{2.368}$$

radial velocity of the ith galaxy is then

$$\dot{r}_i(t) = \dot{S}(t) r_i(t_0) = \frac{\dot{S}(t)}{S(t)} r_i(t) \tag{2.369}$$

the Hubble parameter $H(t)$

$$H(t) = \dot{S}(t)/S(t), \tag{2.370}$$

then

$$\dot{r}_i(t) = H(t) r_i(t) \tag{2.371}$$

Hubble's law. This states that, in an expanding universe, at any one epoch, the radial velocity of recession of a galaxy from a given point is proportional to the distance of the galaxy from the point. The value of the Hubble parameter at our epoch is know as the Hubble constant.

$$E = A[\dot{S}(t)]^2 - \frac{B}{s(t)} - D[S(t)]^2 \tag{2.372}$$

where the coefficients are positive defined by

$$A = \frac{1}{2}\sum_{i=1}^{n} m_i [r_i(t_0)]^2, \tag{2.373}$$

$$B = G \sum_{i,j=1 (i<j)}^{n} \frac{m_i m_j}{|\mathbf{r}_i(t_0) - \mathbf{r}_j(t_0)|}, \tag{2.374}$$

$$D = \frac{1}{6}\Lambda \sum_{i=1}^{n} m_i [r_i(t_0)]^2 = \frac{1}{3}\Lambda A. \tag{2.375}$$

2.16.3 Relativistic Cosmology

$$ds^2 = dt^2 - h_{ij} dx^i dx^j$$

2.16.4 Spaces of Constant Curvature

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{2.376}$$

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{2.377}$$

Contracting with $g^{\alpha\gamma}$, we get

$$\begin{aligned} g^{ac}R_{abcd} &= R_{bd} \\ &= Kg^{ac}(g_{ac}g_{bd} - g_{ad}g_{bc}) \\ &= K(3g_{bd} - g_{bd}) \\ &= 2Kg_{bd}. \end{aligned} \tag{2.378}$$

since 3-space is isotropic about every point, it must be spherically symmetric about every point the line element

$$d\sigma^2 = g_{ij}dx^i dx^j = e^\lambda dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.379}$$

We have

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.380}$$

2.17 Homogeneous and Isotropic Cosmology

2.17.1 Friedmann's Equation - Universe with Dust

$$ds^2 = dt^2 - [R(t)]^2 \frac{[d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^\theta d\phi^2)]}{[1 + \frac{1}{4}k\tilde{r}^2]^2} \tag{2.381}$$

$$T_{ab} = (\rho + p)u_a u_b - pg_{ab} \tag{2.382}$$

Eistein's equations

$$G_{ab} - \Lambda g_{ab} = 8\pi T_{ab} \tag{2.383}$$

$$
\begin{aligned}
&g_{00} = 1, \quad g_{11} = -R^2(t)/(1-kr^2), \quad g_{22} = -r^2R^2(t)\\
&g_{33} = \sin^2\theta R^2(t),\\
&g^{00} = 1, \quad g^{11} = -(1-kr^2)/R^2(t), \quad g^{22} = -(rR(t))^{-2}\\
&g^{33} = -(r\sin\theta R(t))^{-2}
\end{aligned}
\tag{2.384}
$$

$$
\begin{aligned}
&\Gamma^0_{11} = R(t)\dot{R}(t)/(1-kr^2), \quad \Gamma^0_{22} = r^2R(t)\dot{R}(t)\\
&\Gamma^0_{33} = r^2\sin^2\theta R(t)\dot{R}(t)\\
&\Gamma^1_{01} = R(t)/\dot{R}(t), \quad \Gamma^1_{01} = kr/(1-kr^2), \quad \Gamma^1_{22} = -r(1-kr^2),\\
&\Gamma^1_{33} = -r(1-kr^2)\sin^2\theta,\\
&\Gamma^2_{02} = R(t)/\dot{R}(t), \quad \Gamma^2_{12} = 1/r, \quad \Gamma^2_{33} = -\sin\theta\cos\theta,\\
&\Gamma^3_{03} = R(t)/\dot{R}(t), \quad \Gamma^3_{13} = 1/r, \quad \Gamma^3_{23} = \cot\theta.
\end{aligned}
\tag{2.385}
$$

$$
\begin{aligned}
&R_{00} = -3\ddot{R}(t)/R(t),\\
&R_{11} = (R(t)\ddot{R}(t) + 2\dot{R}^2(t) + 2k)/(1-kr^2)\\
&R_{22} = r^2(R(t)\ddot{R}(t) + 2\dot{R}^2(t) + 2k),\\
&R_{22} = r^2\sin\theta(R(t)\ddot{R}(t) + 2\dot{R}^2(t) + 2k)
\end{aligned}
\tag{2.386}
$$

$$
R = g^{ab}R_{ab} = -6(R(t)\ddot{R}(t) + \dot{R}^2(t) + k)/R^2(t) \tag{2.387}
$$

With $u_a = (1,0,0,0)$ the non-zero components of T_{ab} are:

$$
\begin{aligned}
&T_{00} = \rho, \quad T_{11} = pR^2(t)/(1-kr^2), \quad T_{22} = pr^2R^2(t),\\
&T_{22} = pr^2(\sin^2\theta)R^2(t)
\end{aligned}
\tag{2.388}
$$

$R_{00} - \frac{1}{2}g_{00}R = 8\pi T_{00}$:

$$
-3\ddot{R}(t)/R(t) - \frac{1}{2} \times -6(R(t)\ddot{R}(t) + \dot{R}^2(t) + k)/R^2(t) = 8\pi\rho
$$

or

$$
3(\dot{R}^2(t) + k) = 8\pi\rho R^2(t) \tag{2.389}
$$

$R_{11} - \frac{1}{2}g_{11}R = 8\pi T_{11}$:

$$(R(t)\ddot{R}(t) + 2\dot{R}^2(t) + 2k)/(1 - kr^2) - \frac{1}{2} \times -R^2(t)/(1 - kr^2) \times$$
$$\times - 6(R(t)\ddot{R}(t) + \dot{R}^2(t) + k)/R^2(t) = 8\pi p R^2(t)/(1 - kr^2)$$

or

$$R(t)\ddot{R}(t) + \dot{R}^2(t) + k = -8\pi p R^2(t) \tag{2.390}$$

The 22 and 33−components yield equations equivalent to (2.390).

$$p = 0 \tag{2.391}$$

so dust

$$\int (2R\ddot{R} + \dot{R}^2 + k)dR - \Lambda \int R^2 dR = C$$

which becomes

$$\int_0^t (2R\ddot{R} + \dot{R}^2)\dot{R}dt + kR - \frac{1}{3}\Lambda R^3 = C$$

or

$$\int_0^t \frac{d}{dt}[R\dot{R}^2]dt + kR - \frac{1}{3}\Lambda R^3 = C$$

so

$$R(\dot{R}^2 + k) - \frac{1}{3}\Lambda R^3 = C \tag{2.392}$$

This is Friedmann's equation in the absence of pressure.

Let us solve this for vanishing cosmological constant.

We have to solve

$$\dot{R}^2 = C/R - k. \tag{2.393}$$

$k = +1$**:**

$$u^2 = R/C \tag{2.394}$$

Then $2u\dot{u} = \dot{R}/C$

$$\dot{u}^2 = \frac{\dot{R}^2}{4C^2u^2} = \frac{1}{4C^2u^2}\left(\frac{C}{R} - 1\right) = \frac{1}{4C^2u^2}\left(\frac{1}{u^2} - 1\right)$$

It follows

$$2\int_0^u \frac{u^2}{(1-u^2)^{1/2}}du = \frac{1}{C}\int_0^t dt = \frac{t}{C}$$

$$\begin{aligned}
2\int_0^u \frac{u^2}{(1-u^2)^{1/2}}du &= 2\int_0^\theta \frac{\sin^2\theta\cos\theta}{(1-\sin^2\theta)^{1/2}}d\theta \\
&= 2\int_0^\theta \sin^2\theta d\theta \\
&= \int_0^\theta (1-\cos 2\theta)d\theta \\
&= \theta - \frac{1}{2}\sin 2\theta \\
&= \theta - \sin\theta\cos\theta \\
&= \sin^{-1} u - u(1-u^2)^{1/2}
\end{aligned} \tag{2.395}$$

$$C[sin^{-1}(R/C)^{1/2} - (R/C)^{1/2}(1-R/C)^{1/2}] = t \tag{2.396}$$

2.17.2 The Luminosity Distance

In a Euclidean universe, if a source of absolute luminosity L is at a distance d then the flux that we receive is

$$F = \frac{L}{4\pi d^2}.$$

Now suppose that we are actually in an expanding FRW spacetime and we know that the source has luminosity L and we observe a flux F. We define the *luminosity distance* as

$$d_L := \left(\frac{L}{4\pi d^2}\right)^{1/2}. \tag{2.397}$$

Consider an emmiting source E with a fixed comoving coordinate χ relative to an observer O. We assume that the luminosity E as a function of cosmic time is $L(t)$ and that the light pulse it emmits are detected by O at at cosmic time t_0. Te light pulses had been emmited at the early time t_e.Asssuming the light pulses were emmitted isotropically, the radiation will be spread evenly over a sphere centered at E and passing through O (see diagram). The proper area of this sphere is

$$A = 4\pi R^2(t_0)S^2(\chi). \tag{2.398}$$

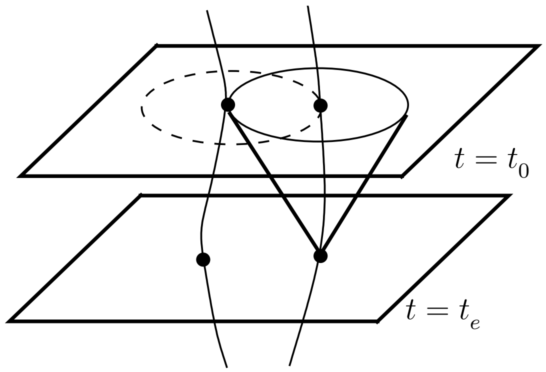

Figure 2.29: LumDist. Luminosity distance

However, each light pulse received by O is red shifted in frequency, so that

$$\nu_0 = \frac{\nu_e}{1+z}, \tag{2.399}$$

also the arrive rate of the light pulse is also reduced by the same factor. Thus, the observed flux at O is

$$F(t_0) = \frac{L(t_e)}{4\pi[R_0 S(\chi)]^2}\frac{1}{(1+z)^2}. \tag{2.400}$$

The luminosity distance defined above is

$$d_L = R_0 S(\chi)(1+z). \tag{2.401}$$

2.18 Models with a Cosmological Constant

$$\begin{aligned} 3(\dot{R}^2 + k) &= 8\pi G\rho R^2 + \Lambda R^2, \\ 2R\ddot{R} + \dot{R}^2 + k &= -8\pi G\rho R^2 + \Lambda R^2 \end{aligned} \tag{2.402}$$

the Einstein static model is the one in which the gravitational attraction is exactly counterbalanced by the cosmic repulsion. It didin't actually work to keep the universe from contracting. If you perturbed the universe the balance between the contraction caused by matter against the expansion caused by the cosmological constant is destabalised and the universe starts to grow or shrink. That is, either the attractive force of the matter or the repulsive force due to the cosmological constant takes over.

2.18.1 Flat Universe

$$\dot{R}^2 = C/R + \frac{1}{3}\lambda R^2. \tag{2.403}$$

$\Lambda > 0$

$$u = \frac{2\Lambda}{3C}R^3.$$

$$\dot{u} = \frac{2\Lambda}{C}R^2\dot{R},$$

$$\begin{aligned} \dot{u}^2 &= \frac{4\Lambda^3}{C^2}R^4\left(\frac{C}{R} + \frac{1}{3}\Lambda R^2\right) \\ &= \frac{4\Lambda^2}{C}R^3 + \frac{4\Lambda^3}{3C^2}R^6 \\ &= 6\Lambda u + 3\Lambda u^2 \\ &= 3\Lambda(2u + u^2) \end{aligned} \tag{2.404}$$

$$\dot{u} = (3\Lambda)^{1/2}(2u + u^2)^{1/2}$$

$$\int_0^u \frac{du'}{(2u + u'^2)^{1/2}} = \int_0^t (3\Lambda)^{1/2}dt' = (3\Lambda)^{1/2}t.$$

$$
\begin{aligned}
\int_0^u \frac{du'}{[(u'+1)^2-1]^{1/2}} &= \int_1^v \frac{dv'}{(v'^2-1)^{1/2}} \\
&= \int_0^w \frac{\sinh w' dw'}{(\cosh^2 w'-1)^{1/2}} \\
&= \int_0^w dw' = w. \qquad (2.405)
\end{aligned}
$$

$$
R^3 = \frac{3C}{2\Lambda}[\cosh(3\Lambda)^{1/2}t - 1]. \qquad (2.406)
$$

2.18.2 The de Sitter Model

the line element becomes

$$
ds^2 = dt^2 - [\exp 2(\frac{1}{3}\Lambda)^{1/2}][dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)]
$$

2.19 Perturbations of Exact Solutions

The Einstein equation in general relativity is a set of nonlinear equations, but many exact solutions to this equation are known. However, these exact solutions are most often too idealized to properly represent the realm of natural phenomena. In such situations, the perturbative approach is one of the powerful techniques to investigate physical systems and is one of the popular techniques in any theory of physics.

In relativistic perturbation theory one tries to find approximate solutions of Einstein's equations, regarding them as "small" deviations from know exact solution - the socalled background.

In general relativistic perturbations, gauge freedom, which is unphysical degree of freedom, arises due to general covariance. To obtain physically meaningful results, we have to fix these gauge freedom or to extract gauge invariant part of perturbations.

In linear perturbation theory, this gauge freedom is regarded as the freedom of the infinitesimal active diffeomorphism (which can be thought of as corresponding to an infinitesimal coordinate transformation). This understanding of gauge freedom is correct when we concentrate only on the linear order. However, it is known that this understanding of gauge freedom

Gauge theory in GR is connected with mapping a spacetime onto itself

$$x'^a = f^a(x^b). \tag{2.407}$$

These transformations can be genrated by a smooth vector field ξ:

$$x'^a = x^a + \xi^a(x) + \frac{1}{2!}\xi^b \xi^a_{'b} + \frac{1}{3!}\xi^c(\xi^b \xi^a_{,b})_{,c} + \dots \tag{2.408}$$

After the transformation, the metric, for example, is transfromed as $g_{ab}(x) \to g'_{ab}(x')$. Then return to the original coordinates. After this one has to compare geometric objects of the initial spacetime and of the mapped spacetime

$$g'_{ab}(x) = g_{ab}(x) + \delta_f g_{ab} = g_{ab}(x) + \sum_{k=1}^{\infty} \frac{1}{k!}\mathcal{L}^k_\xi g_{ab}(x). \tag{2.409}$$

2.19.1 Gauge Dependency in Perturbation Theory

In the general relativistic theory of perturbations one is always dealing with two spacetimes, the "physical" (perturbed) universe one, and an idealised (unperturbed) "background" universe (for example, we will be interested in the case where the idealised "background" universe is a symmetric universe).

We formally denote the spacetime metric an other tensor fields on the perturbed spacetime by Q and its background value on the background spacetime by Q_0.

write equation for the perturbation of the variable Q in the from

$$Q(\text{“}p\text{”}) = Q_0(p) + \delta Q(p) \tag{2.410}$$

Recall that no spacetime structure, i.e. a manifold $\mathcal{M}$ with a metric g, is assumed a priori in genaral relativity, unlike the situation in Newtonian theory, where $\mathcal{M}$ and $\mathcal{M}_0$ are identified, thus making possible a straightforward formulation of Newtonian cosmology.

$(\mathcal{M}, g)$

The main diffculty here is to control the gauge dependence of the results. This gauge dependence can be understood from the fact, that one has to identify spacetime points in the "physical" (non-symmetric) universe with spacetime points in the "background" universe, around which the perturbation is taken. This identification can be related to a choice of coordinates for the "physical" universe.

If one manifold is a replica of another we need a point-identification map which relates points in the two manifolds which are to be regarded as the "same".

To obtain physically meaningful results, we have to fix these gauge freedom or to extract gauge invariant part of perturbations.

How to describe the equivalence classes of perturbations.

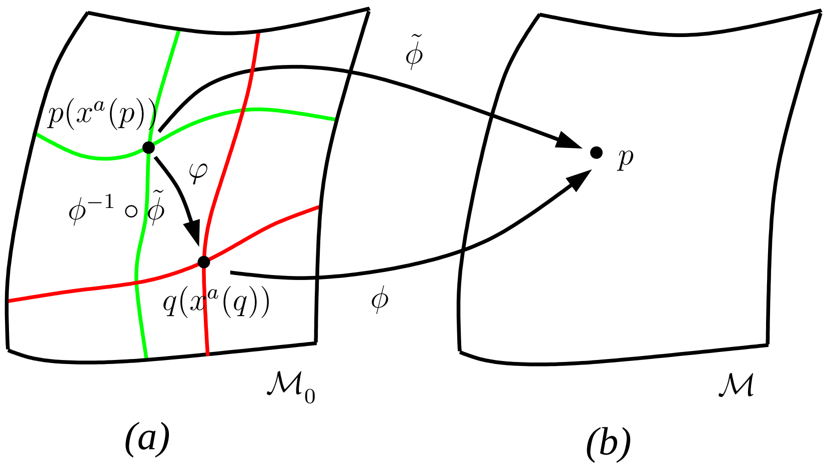

Figure 2.30: pertCosGauge. A diffeomorphism on the perturbed manifold $\mathcal{M}$ induces a change in coordinates of the background manifold $\mathcal{M}_0$. The issue of perturbative gauge invariance is closely related, though not equivalent to, the coordinate independence of General Relativity.

Viewed alternatively, there is no prefered point-to-point identification mapping between the background manifold and perturbed manifold, so that the comparison of two tensor fields on two different manifolds is not an invariantly defined concept.

If there is a preferred coordinate system on both these spacetimes, we can accomplish this identification using this preferred coordinate system. However, there is no such coordinate system due to general covariance and the gauge choice is not unique when we consider theories in which general covariance is imposed. This arbitrariness is just "gauge freedom" of perturbations. This freedom is arisen by the relation between the physical spacetime and the background. Hence, this gauge freedom should have nothing to do with physical quantities which appear in observations or in experiments. Actually, some linear order variables which is independent of this gauge freedom relate variables in observations. These are called gauge invariant variables.

Consider some relavent quantity $\mathcal{Q}$ on $\mathcal{M}$ represented by a tensor field, and the corresponding iquantity on $\mathcal{Q}^{(0)}$ on $\mathcal{M}_0$. The perturbation ΔQ of Q, is defined as the difference between the value Q has in the physical spacetime, and the background value Q_0. There is an arbitrariness of the perturbation of Q at any given spacetime point, unless Q is gauge-invariant.

Then in the first coordinate system given by the mapping ϕ, the perturbation $\delta\mathcal{Q}$ of $\mathcal{Q}$ at the point $p \in \mathcal{M}$ is defined by

$$\delta\mathcal{Q}(p) = \mathcal{Q}(p) - \mathcal{Q}^{(0)}(\phi^{-1}(p)). \tag{2.411}$$

In the second coordinate system given by $\tilde{\phi}$.

$$\delta\tilde{\mathcal{Q}}(p) = \mathcal{Q}(p) - \mathcal{Q}^{(0)}(\tilde{\phi}^{-1}(p)). \tag{2.412}$$

The difference

$$\Delta\mathcal{Q}(p) = \delta\tilde{\mathcal{Q}}(p) - \delta\mathcal{Q}(p) \tag{2.413}$$

is a gauge quantity and carries no physical significance.

Let $\mathcal{N}$ be a 5-dimensional smooth manifold containing a 1-parameter family of smooth non-intersecting 4-manifolds $\mathcal{M}_\epsilon$. The manifolds are perturbations of the manifold $\mathcal{M}_0$. The point-indentification is supplied by a vector field V on $\mathcal{N}$ which is every transverse, i.e., non-parallel to the $\mathcal{M}_\epsilon$. Points in the various $\mathcal{M}_\epsilon$ which lie on the same integral curves of V are regarded as the same. The choice of V is a choice of gauge.

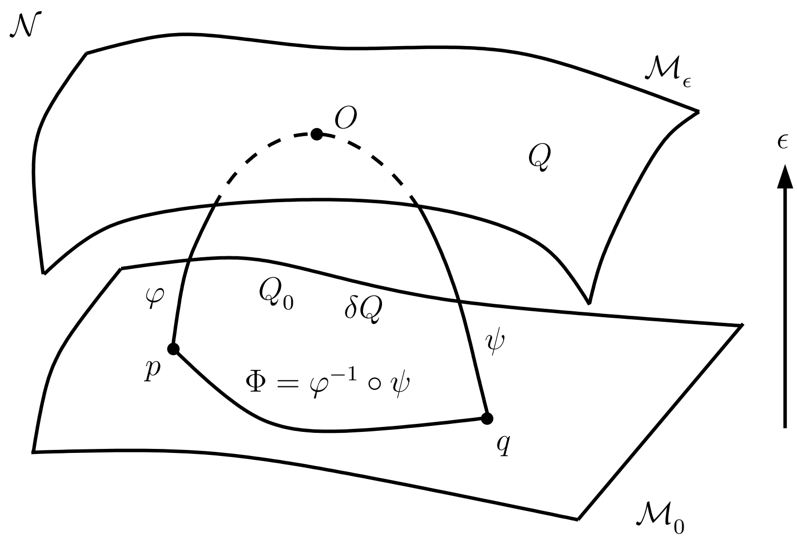

Figure 2.31: pertManifolds. 5-dimensional manifold $\mathcal{N}$ containing a 1-parameter family of smooth non-intersecting 4-manifolds $\mathcal{M}_\epsilon$. $\mathcal{N} = \mathcal{M} \times \mathbb{R}$

Now let Q_0 be some geometric quantity on $\mathcal{M}$ and Q_ϵ the corresponding quantity on $\mathcal{M}_\epsilon$. This defines a field on Q on $\mathcal{N}$. Along each integral curve of V we have a Taylor series

$$Q_\epsilon = \psi_{\epsilon *}[Q_0 + \epsilon(\mathcal{L}_V Q_\epsilon)_{\epsilon=0} + \mathcal{O}(\epsilon^2)] \tag{2.414}$$

where ϵ is small. The first order term is the linear perturbation of Q. In general Q will satisfy some complicated non-linear equation, but if all quantities are expanded as a Taylor sereis in ϵ and non-linear terms are discarded, the resulting linear equations for the perturbation may be both simple to solve and relevant. There is however a problem, because there is no preferred choice of V.

Higher Order Perurtbations

reliable measure the accuracy of linearized theory is the calculation to the next order in ϵ.

We write the metric as

$$g_{ab} = g_{ab}^{(0)} + \epsilon g_{ab}^{(1)} + \epsilon^2 g_{ab}^{(2)} + \mathcal{O}(\epsilon^3)$$

We write the sourceless Einstein equations as

$$\mathcal{G}(g_{ab}^{(0)} + \epsilon g_{ab}^{(1)} + \epsilon^2 g_{ab}^{(2)}) = 0, \tag{2.415}$$

where $\mathcal{G}$ represents the actions of taking partial derivatives and algebraic combinations to ofrm the components of the Einstein tensor. If we expand (2.415) in ϵ, the term of order ϵ^0 automatically vanishes if $g_{ab}^{(0)}$ is a background solution. The terms of first order in ϵ can be written in the form

$$\epsilon \mathcal{L}(g_{ab}^{(1)}) = 0, \tag{2.416}$$

where $\mathcal{L}$ is a set of differentiations and combinations with details that depends on $g_{ab}^{(0)}$. These opertors are all linear in ϵ, and they are linearized perturbation theory.

The part of (2.415) that is proportional to ϵ^2 has two kinds of terms. There are terms that are linear in $g_{ab}^{(2)}$, and terms that are quadratic in $g_{ab}^{(1)}$. The former terms occur in precisely the same form as do the $g_{ab}^{(1)}$ terms in (2.416). The set of ϵ^2 terms can then be written as

$$\epsilon^2 \mathcal{L}(g_{ab}^{(2)}) = \epsilon^2 \mathcal{J}(g_{ab}^{(1)}, g_{ab}^{(1)}), \tag{2.417}$$

where $\mathcal{J}$ is quadratic in the first order perturbation. In solving for the second order perturbations, one treats the first order perturbations as already known, so $\mathcal{J}$ plays the role of a source term in (2.417).

The operator $\mathcal{J}$ is precisely the same operator in (5.24), so for ach linerized theory equation for $g_{ab}^{(1)}$

Gauge Dependence

We work a coordinate system $(x^1, \ldots, x^m)$, such that x^1 is the parameter along the integral curves and the other coordinates are choosen any way. In this coordinate system the Lie derivative becomes

$$\mathcal{L}_\xi Q^{a_1 \cdots a_k}_{b_1 \cdots c_l} \stackrel{*}{=} \frac{\partial}{\partial x^1} Q^{a_1 \cdots a_k}_{b_1 \cdots c_l} \tag{2.418}$$

and

$$\mathcal{L}^k_\xi Q^{a_1 \cdots a_k}_{b_1 \cdots c_l} \stackrel{*}{=} \partial^k_1 Q^{a_1 \cdots a_k}_{b_1 \cdots c_l} \tag{2.419}$$

$$\begin{aligned} Q^{a_1 \cdots a_k}_{b_1 \cdots c_l}(x^1 + \lambda, x^2, \ldots, x^m) \quad \stackrel{*}{=} \quad & Q^{a_1 \cdots a_k}_{b_1 \cdots c_l}|_{\lambda=0} + \lambda \partial_1 Q^{a_1 \cdots a_k}_{b_1 \cdots c_l}|_{\lambda=0} + \frac{\lambda^2}{2!} \partial^2_1 Q^{a_1 \cdots a_k}_{b_1 \cdots c_l}|_{\lambda=0} + \\ & + \mathcal{O}(\lambda^3) \end{aligned} \tag{2.420}$$

This in any coordinates is

$$Q(r) = Q(\varphi_\lambda(p)) = Q(p) + \lambda \mathcal{L}_\xi Q|_p + \frac{1}{2} \lambda^2 \mathcal{L}^2_\xi Q|_p + \mathcal{O}(\lambda^3), \tag{2.421}$$

where $r = \varphi_\lambda(p) \in \mathcal{M}_\lambda$.

This can be written sybolically as

$$Q(r) = \exp(\lambda \mathcal{L}_\xi) Q(p) \tag{2.422}$$

A diffeomorphism φ between the two manifolds $\mathcal{M}_0$ and $\mathcal{M}_\lambda$ naturally defines a linear map from the point $p \in \mathcal{M}_0$ to the point $\varphi(p) \in \mathcal{M}_\lambda$.

$$\varphi_*|_p : T_p \mathcal{M}_0 \to T_{\varphi(p)} \mathcal{M}_\lambda$$

between the tangent spaces, called the push-forward (see fig (2.19.1)), and a linear map

$$\varphi^*|_p : T^*_p \mathcal{M}_0 \to T^*_{\varphi(p)} \mathcal{M}_\lambda$$

between the cotangent spaces, called the pull-back.

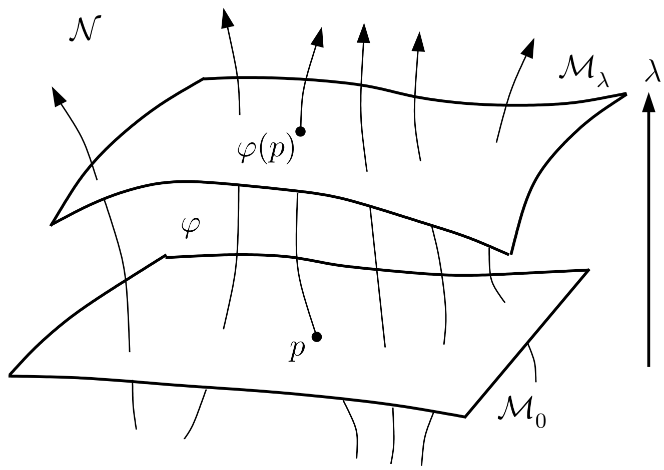

Figure 2.32: pertManFlow. The diffeomorphism φ.

The pull-back $\varphi_\lambda^* Q$ maps a tensor field Q on the perturbed manifold $\mathcal{M}_\lambda$ to a tensor field $\varphi_\lambda^* Q$ on the background spacetime. In terms of this enerator ξ_X^a the pull-back $\varphi_\lambda^* Q$ is represented by the Tayler expansion

$$Q(r) = Q(\varphi_\lambda(p)) = \varphi_\lambda^* Q(p) = Q(p) + \lambda \mathcal{L}_\xi Q\big|_p + \frac{1}{2}\lambda^2 \mathcal{L}_\xi^2 Q\big|_p + \mathcal{O}(\lambda^3), \tag{2.423}$$

where $r = \varphi_\lambda(p) \in \mathcal{M}_\lambda$.

Flows

the integral curves of these vector fields intersect each $\mathcal{M}_\lambda$. Therefore, points lying on the same integral curve is to be regarded as the same point within the particular gauge.

Recall that the Lie derivative is the rate of change of a vector or tensor field along the flow of another vector field.

The first derivative is, by definition, just the Lie derivative of T with respect to ξ:

$$\left.\frac{d}{d\lambda}\right|_0 \phi_\lambda^* T = \lim_{\lambda \to 0} \frac{1}{\lambda}(\phi_\lambda^* T - T) := \mathcal{L}_\xi T. \tag{2.424}$$

We have

$$\phi_\lambda^* T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \left.\frac{d^k}{d\lambda^k}\right|_0 \phi_\lambda^* T \tag{2.425}$$

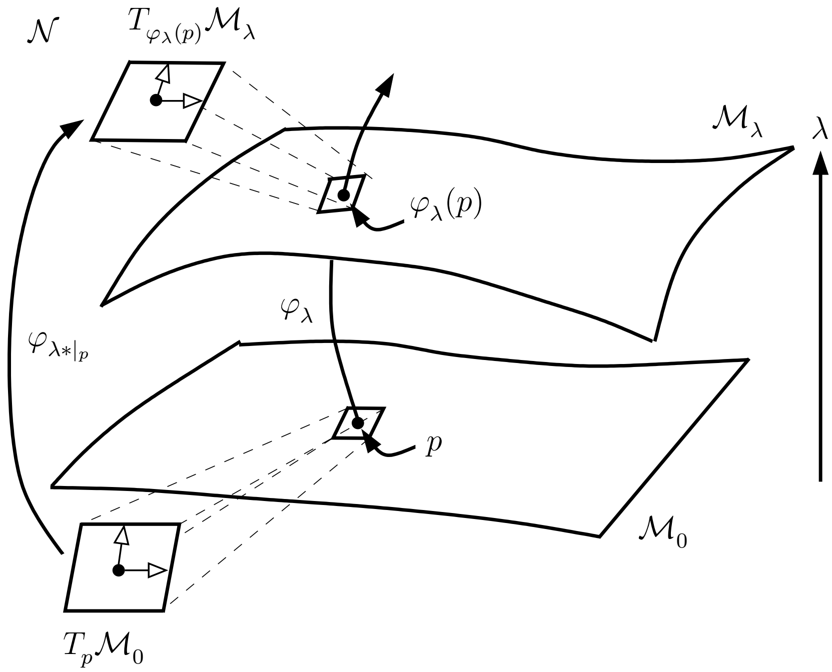

Figure 2.33: pertManPush. The push-forward $\varphi_{\lambda*|_p}$ is the natural linear map between the tangent spaces $T_p\mathcal{M}_0$ and $T_{\varphi(p)}\mathcal{M}_\lambda$ induced by the diffeomorphism φ. The push-forward $\varphi^*_{\lambda|_p}$ is the linear map between the co-tangent spaces $T^*_p\mathcal{M}_0$ and $T^*_{\varphi(p)}\mathcal{M}_\lambda$. Push-forwrads and pull-backs are related by .

This can be written as

$$\phi^*_\lambda T = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k}\Big|_0 \mathcal{L}^k_\xi T. \tag{2.426}$$

In order to prove (2.426), we need to show that, for all k,

$$\frac{d^k}{d\lambda^k}\Big|_0 \phi^*_\lambda T = \mathcal{L}^k_\xi T. \tag{2.427}$$

This can be done by induction over k. Suppose that (2.427) is true for some k. Then

$$\begin{aligned} \frac{d^{k+1}}{d\lambda^{k+1}}\Big|_0 \phi^*_\lambda T &= \lim_{\epsilon\to 0} \left(\frac{d^k}{d\lambda^k}\Big|_\epsilon \phi^*_\lambda T - \frac{d^k}{d\lambda^k}\Big|_0 \phi^*_\lambda T \right) \\ &= \lim_{\epsilon\to 0} \left(\frac{d^k}{d\tau^k}\Big|_0 \phi^*_{\tau+\epsilon} T - \frac{d^k}{d\lambda^k}\Big|_0 \phi^*_\lambda T \right) \\ &= \lim_{\epsilon\to 0} \left(\phi^*_\lambda \mathcal{L}^k_\xi T - \mathcal{L}^k_\xi T \right) = \mathcal{L}^{k+1}_\xi T \end{aligned} \tag{2.428}$$

where $\tau := \lambda - \epsilon$, and we have used $\phi_{\tau+\epsilon} = \phi_\tau \circ \phi_\epsilon$.

A Gauge Transformation

$$\delta^k T := \left[\frac{d^k \varphi_\lambda^* T}{d\lambda^k}\right]_{\lambda=0} \tag{2.429}$$

$$T_\lambda^X := \varphi_\lambda^* T|_0, \qquad T_\lambda^Y := \psi_\lambda^* T|_0, \tag{2.430}$$

Gauge Invariant Quantities

If $T_\lambda^X = T_\lambda^Y$, for any pair of gauges X and Y, we say that T is totally gauge-invariant. This is a very strong condition, because then (??) and (??) imply that $\delta^k T^X = \delta^k T^Y$, for all gauges X and Y and for any k. In any practical case one is however interested in perturbations to a fixed order n; it is thus convenient to weaken the definition above, saying that T is *gauge-invariant to order* n if and only $\delta^k T^X = \delta^k T^Y$ for any two gauges X and Y, and for all $k \leq n$.

Proposition 2.19.1 *A tensor field T is gauge-invariant to order $n \geq 1$ if and only if $\mathcal{L}_\xi \delta^k T = 0$, for any vector field ξ on $\mathcal{M}$ and*

Proof: Let us first show that the statement is true for $n = 1$. In fact, if $\delta T^X = \delta T^Y$,

$$\begin{aligned} \delta T^X - \delta T^Y &= \frac{d}{d\lambda}\varphi_\lambda^* T|_0 - \frac{d}{d\lambda}\psi_\lambda^* T|_0 \\ &= \frac{d}{d\lambda}(\varphi - \psi)_\lambda^* T|_0 = 0 \\ &= \mathcal{L}_{X-Y} T|_0 = 0, \end{aligned} \tag{2.431}$$

we have $\mathcal{L}_{X-Y} T|_0 = 0$. But since X and Y define arbitrary gauges, it follows that $X - Y$ is an arbitrary vector field ξ with $\xi^m = 0$, i.e., tangent to $\mathcal{M}$.

Let us now suppose that the statement is true for some n. Then, if we also have $\delta^{n+1} T^X|_0 = \delta^{n+1} T^Y|_0$

$$\frac{d}{d\lambda}\frac{d^n}{d\lambda^n}(\varphi - \psi)_\lambda^* T|_0 = 0,$$

it follows that

$$\frac{d}{d\lambda}(\varphi - \psi)^*_\lambda(\delta^n T)|_0 = 0$$

$\mathcal{L}_{X-Y}\delta^n T^X = 0$

$\square$

As a consequence, T is gauge-invariant to order n if and only if T_0 and all its perturbations of order lower than n are, in any gauge, either vanishing, or constant scalars, or a combination of Kronecker deltas with constant coefficients. Thus, this generalizes to an arbitrary order n the results of references [??]. Further, it then follows that T is totally gauge-invariant if and only if it is a combination of Kronecker deltas with coefficients depending only on λ.

2.20 Cosmological Perturbation Theory

The backbone of most current cosmology in the theory of perturbation equations for metric modes around an isotropic spacetime [??]. It is used particular for cosmologicasl structure formation and for testing alternative theories beyond general relativity such as quantum gravity candidates

2.20.1 Scalar-Vector-Tensor Decomposition

In linear perturbation theory, the metric perturbations h_{ab} are regarded as a tensor field residing on the backgound Robertson-Walker spacetime. As a symmetric 4×4 matrix, h_{ab} has ten degrees of freedom. However, as we are able to perform active diffeomorphisms, four of degrees of freedom are gauge dependent, leaving us with six physical degrees of freedom. We require a clear separation between physical degrees of freedom.

$$h_{00} \equiv -2\psi, \quad h_{0i} \equiv w_i, \quad h_{ij} \equiv (\phi\gamma_{ij} + S_{ij}) \text{ with } \gamma^{ij}S_{ij} = 0. \tag{2.432}$$

where γ^{ij} is the inverse matrix of γ_{ij}.

The scalar-vector-tensor spilt is based on the decomposition of a vector into longituidinal and transverse parts. For any three-vector field $w_i(\vec{x})$, we may write

$$w_i = w_i^{\parallel} + w_i^{\perp} \quad \text{where } \vec{\nabla}\times\vec{w}^{\parallel} = \vec{\nabla}\cdot\vec{w}^{\perp} = 0 \tag{2.433}$$

where the curl and divergence are defined with the spacial covariant deriviative, e.g. $\vec{\nabla}\cdot\vec{w} := \gamma^{ij}\nabla_i w_j$.

One may always add a constant to $w_i^{\parallel}$

Note that $w_i^{\parallel} = \nabla_i \phi_w$ for some scalar field ϕ_w. Thus, the longitudinal/transverse decomposition allows us to write a vector field in terms of a scalar (the longitudinal or irrotational part) and a part that cannot be obtained from a scalar (the transverse or rotational part).

A similar decomposition holds for a two-index tensor, but now each index can be either longitudinal or transverse. For a symmetric tensor, there are three possibilities: both indices are longitudinal, one is transverse, or two are transverse. These are written as follows:

$$S_{ij} = S_{ij}^{\parallel} + S_{ij}^{\perp} + S_{ij}^{T}, \tag{2.434}$$

where

$$\gamma^{jk}\nabla_k S_{ij} = \gamma^{jk}\nabla_k S_{ij}^{\parallel} + \gamma^{jk}\nabla_k S_{ij}^{\perp}, \tag{2.435}$$

The first term in equation (2.435) is a longitudinal vector while the second term is a transverse vector. The divergence of the doubly-transverse part, S_{ij}^{T}, is zero.

i) The tensor mode S_{ij}^{T} represents the part of h_{ij} that cannot be obtained from the grdient of a scalar or vector. It is a gauge-invariant. Physically, it rpresents gravitational radiation.

ii) The vector mode corresponds to the transverse vector parts of the metric, which are found in $w_i^{\perp}$ and $S_{ij}^{\perp}$. It behaves like a spin-1 field under spatial rotation. Each part has two degrees of freedom, but by imposing a gauge condition, it is possible to eliminate two of them.

iii) The scalar mode is spin-0 under spatial rotation and correponds physically to Newtotian graviation with relativistic modifications. Any two of the scalar parts of the metric $\phi, \psi, w_i^{\parallel}$ and $S_{ij}^{\parallel}$ can be set to zero by a gauge tranformation.

2.20.2 Choice of Gauge

i) The *synchronous gauge* for which $w_i^{\perp} = w_i^{\parallel} = w_i = 0$. ... The line element is given by

$$ds^2 = a^2(\tau)[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j]. \tag{2.436}$$

ii) The *conformal Newtonian* gauge for which $w_i^{\parallel} = S_{ij}^{\parallel} = 0$, corresopnds to the scalar mode in the transverse gauge, defined by the gauge conditions [??]

$$\gamma^{ij}\nabla_i w_j = 0, \qquad \gamma^{ik}\nabla_k S_{ij} = 0. \tag{2.437}$$

In this gauge, perturbations are charaterized by two scalar potentials ψ and ϕ which are respectively perturbation to the Newtonian potential and to the spacial curvature. The line element is

$$ds^2 = a^2(\tau)[-(1+2\psi)d\tau^2 + (1+2\phi)dx^i dx^j]. \tag{2.438}$$

2.21 Perturbations of Black Holes

Black hole perturabion theory astrophysical situations without symmetries. In the study of gravitational radiation from processes like the infall of matter into a balck hole, [??], and in the collision of two black holes [].

2.22 Approximation to Observables of the Full Theory

We will apply this approximation scheme to general relativity. In doing so, we have to make certain choices the most important being the choice of the clock variables. Here our guide line is that we want to have a good approximation to the observables of field theory on a fixed background which in this work will be the flat Minkowski background. As we will see this results in observables which in the zero gravity limit (i.e. for $\kappa = 0$ and vanishing gravitational fluctuations) coincide with the usual observables of field theory on a flat background.

The gravity corrections can be calculated explicitly order by order and are connected to the standard perturbation theory. The first order observables are given by the observables of linearized general relativity, hence this method gives us a precise understanding of the observables of the linearized theory, for instance the graviton, as approximations to observables of the full theory.

Moreover the approximation scheme in this work gives a precise proposal how to compute higher order corrections to the observables of the linearized theory. These higher order corrections are in a consistent way gauge invariant to a certain order - to make these corrections completely gauge invariant one would have to add terms which are of higher order than the corrections themselves.

2.23 Gravity from Gravitons?

[111]

Somehow the whole idea of the gravataional interaction as a result of graviton exchange on a backgroud metric contradicts Einstein's original and fundamental idea that gravity is geometry and not a force in the usual sense. Therefore such a perturbative description of the theory is very unnatural from the outset and can have at most a semi-classical meaning when the metric fluctuations are very tiny.

Einstein' s theory can be obtained by coupling h_{ab} to itself self consistently.

central role in the quantization of the gravitational field.

massless particle of spin-2 is the mediator of the gravitational interaction, as in the photon in quantum electrodynamics.

• Einstein' s theory can be obtained by coupling h_{ab} to itself self consistently.

• Requirement of conformal symmetry leads to a consistency equation that is Einstein's vacuum field equations $R_{ab} = 0$.

• Equivalence principle comes out from gravitons.

Always doubted by general relativists. Ashtekar: "how do you get a stationary space-time from gravitational radiation?"

[151], *From Gravitons to Gravity: Myths and Reality.*

[http://www.lns.cornell.edu/spr/1999-04/msg0016118.html]:

"If I remember right, these papers show that the flat metric you start with is unobservable and only an apparently curved effective metric is observable."

[http://www.lns.cornell.edu/spr/1999-04/msg0016062.html]:

"If you start with string theory on flat Minkowski spacetime you get a massless spin-2 particle in the low-energy limit. This is not the same as getting Einstein's equation. Most importantly, space-time is still flat! The lightcones and thus the notion of causality are just those of flat Minkowski spacetime."

derive that the background metric must satisfy the correct Einstein equations if the world-sheet theory is conformal i.e. consistent. They also explain that the same effective action is seen by the scattering of the perturbations - namely by the gravitons. This is such a basic feature of string theory - and a key motivation to study it.

[http://www.lns.cornell.edu/spr/1999-04/msg0016062.html]:

"But there are other hints that Einstein's equations *are* actually lurking in string theory. For example, suppose you try to formulate string theory on a *curved* space-time with a fixed metric. In the low-energy limit, the theory you get turns out to be inconsistent unless the Ricci tensor vanishes. This is precisely the same as saying that the vacuum Einstein equations hold."

Conformal symmetry put in by hand to be factorized out later. Doubious???

Energy-momentum

$$T^{ab}(x) = \left[\frac{2}{\sqrt{-\gamma}} \left\{ \frac{\partial \mathcal{L}\sqrt{-\gamma}}{\partial \gamma_{ab}} - \partial_c \left(\frac{\partial \mathcal{L}\sqrt{-\gamma}}{\partial(\partial_c \gamma_{ab})} \right) \right\} \right]_{\gamma=\eta} \tag{2.439}$$

We do it the other way round and identify the correct form of the tensor to which h_{ab} couples to in Einstein s theory. Start with an action functional $A_g[g_{ab}]$ which leads to Einstein s field equations for the metric tensor g_{ab}.

$\lambda^2 = 4\pi G$

$$A_{EH} = \frac{1}{4\lambda^2} \int R\sqrt{-g} d^4x = \frac{1}{4\lambda^2} \tag{2.440}$$

$$S^{ab} = \frac{1}{2} \left[\frac{\partial \sqrt{-\gamma} M^{cdeijk}(\gamma^{mn})}{\partial \gamma_{ab}} \right]_{\gamma=\eta} \partial_d h_{ef} \partial_i h_{jk}. \tag{2.441}$$

$$M^{abcijk} := [\eta^{ai}\eta^{bc}\eta^{jk} - \eta^{ai}\eta^{bj}\eta^{ck} + 2\eta^{ak}\eta^{bj}\eta^{ci} - 2\eta^{ak}\eta^{bc}\eta^{ij}]_{\text{symm}} \tag{2.442}$$

2.24 Some Things of String Theory

Peter Woit - Not even Wrong [?]:

> This conjecture that superstring theory gives finite numbers each term in the expansion is what leads people to say that it is a consistent theory of gravity, but it ignores the fact that this is not a convergent expansion. While all terms in the expansion may be finite, trying to add them all together is almost certain to give an infinite result.

Smolin:

Superstring theory is only well defined in stationary spacetimes which are measure zero in the space of solutions to Einstein's equations. Superspace symmetry requires a time-like killing vector field, if there is no superspace symmetry then the spectrum contains unphysical Tachyons.

[**?**]:

"As we have seen more than once, supersymmetry plays a fundamental role in string theory. String theories built without supersymmetries have instabilitites; left alone, they will take off, emmiting more and more tachyons in a procces that has no end, untilthe theory breaks down. This is very unlike our world. Super string theory eliminates this behaviour and stabilizes the theories. But in some respects, it does that too well. This is because supersymmetry implies there is a symmetry in time, the upshot being that a supersymmetric theory cannot be built on a spacetime *that is evolvong in time*. Thus, the aspect of the thoery required to stabilize it also makes it difficult to study questions we would most like a quantum thoery of gravity to answer, like what happened in the universe just after the big bang, or what happens deep inside the horizon of a black hole. Both circumstances where the geometry is evolving rapidly in time."

2.25 Biblioliographical notes

In this chapter I have relied on the following references: Ray D'Invero, [2].

2.26 Worked Exercises and Details

World Function

The geodesic equation

$$\frac{d^2x^c}{ds^2} + \Gamma^c_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} = 0 \tag{2.443}$$

yields the power series

$$\begin{aligned} x^c &\approx x'^a + U^c ds - \frac{1}{2}\Gamma'^{c'}_{de}U^dU^e ds^2 + \frac{1}{6}(\Gamma'^d_{ef}\Gamma'^e_{gh})ds^3 \\ &- \frac{1}{6}\Gamma'^d_{ef,g}U^eUUds^3 + \ldots \end{aligned} \tag{2.444}$$

where

$$U^c = \frac{dx^c}{ds} \tag{2.445}$$

and

$$U^c ds = dx^c \approx \xi^c = (\tau - \tau')\delta^c_0. \tag{2.446}$$

Proof.

$$x^c = x'^c + U^c s + \frac{1}{2!}M^c_{de}U^dU^e s^2 + \frac{1}{3!}M^c_{def}U^dU^eU^f s^3 + \frac{1}{4!}M^c_{defg}U^dU^eU^fU^g s^4 + \dots \tag{2.447}$$

where $M^c_{de\dots g} = M^c_{de\dots g}(x')$ is symmetric in its lower indices.

$$\frac{dx^b}{ds} = U^b + M^a_{de}U^dU^e s + \frac{1}{2}[M^a_{def} + M^a_{de,f}]U^dU^eU^f s^2 + \frac{1}{6}[M_{defg} + M_{def,g}]U^dU^eU^fU^g s^3 + \dots \tag{2.448}$$

where we have written $dM^c_{de\dots h}/ds = U^g(x')\partial_g M^c_{de\dots h}(x')$.

$$\begin{aligned} \frac{d^2x^c}{ds^2} &= M_{de}U^dU^e + \big([M_{def} + M_{de,f}]U^dU^eU^f\big)s + \big([M_{defg} + M_{def,g} + \frac{1}{2}M_{de,fg}]U^dU^eU^fU^g\big)s^2 + \\ &\quad + \big([M_{defgh} + \frac{1}{3}M_{defg,h} + \frac{1}{6}M_{def,gh}]U^dU^eU^fU^gU^h\big)s^3 + \dots \end{aligned} \tag{2.449}$$

$$\begin{aligned} \Gamma^c_{ab}\frac{dx^a}{ds}\frac{dx^b}{ds} &= \Gamma^c_{ab}\Big(U^a + M^a_{de}U^dU^e s + \frac{1}{2}[M^a_{def} + M^a_{de,f}]U^dU^eU^f s^2 + \dots\Big) \\ &\quad \Big(U^b + M^b_{de}U^dU^e s + \frac{1}{2}[M^b_{def} + M^b_{de,f}]U^dU^eU^f s^2 + \dots\Big) + \dots \\ &= \Gamma^c_{ab}U^aU^b + \big(2\Gamma^c_{ab}M^b_{de}U^dU^eU^a\big)s \\ &+ \big(\Gamma^c_{ab}[M^a_{de}M^b_{fg} + (M^a_{def} + M^a_{de,f})U^b]U^dU^eU^fU^g\big)s^2 + \\ &+ \big(\Gamma^c_{ab}[]U^dU^eU^fU^gU^h\big)s^3 + \dots. \end{aligned} \tag{2.450}$$

Comparing (2.449) and(2.450) we read off

$$M^c_{ab} = -\Gamma^c_{ab} \tag{2.451}$$

$$M^c_{def} = \Gamma^c_{dg}\Gamma^g_{ef} + \Gamma^c_{gf}\Gamma^g_{de} - 2\Gamma^c_{de,f} \tag{2.452}$$

(5.24) can be inverted and used in the definition

$$2\sigma(x, x') = ds^2 g'_{ab} U^a U^b \tag{2.453}$$

giving

$$\begin{aligned} 2\sigma(x, x') &= \xi^a \xi^b g'_{ab} + \frac{1}{2} \xi^a \xi^c \xi^d g'_{ab} \Gamma^b_{cd} g'_{ab} - \frac{2}{3} \\ &+ \frac{1}{2} \xi^d \xi^e \xi^c \Gamma^a_{de} g'_{ab} + \frac{1}{4} g'_{ab} \end{aligned} \tag{2.454}$$

World Function

$$h^c{}_{c,a} h^d{}_{d,a} = \partial_a h^i{}_i \partial^a h^j{}_j \tag{2.455}$$

$$h^{cd,c} h_{cd,a} = \partial^a h^{ij} \partial_a h_{ij} \tag{2.456}$$

$$h^{ab,c} h_{ab,c} = \partial^c h^{ij} \partial_i h_{cj} = \partial^k h^{ij} \partial_i h_{kj} = \frac{1}{2}[\partial^k h^{ij} \partial_i h_{jk} + \partial^k h^{ij} \partial_j h_{ik}] \tag{2.457}$$

$$h^{ab}{}_{,b} h^c{}_{c,a} = \partial_j h^{ij} \partial_i h^k{}_k = \frac{1}{2}[\partial^j h_{ij} \partial^i h + \partial_i h^{ij} \partial_j h^k{}_k] \tag{2.458}$$

Flows

Problem. By considering the difference between two choices for V show that the linearized perturbation of Q is gauge invariant if and only if

$$\mathcal{L}_\xi Q_0 = 0, \tag{2.459}$$

for all 4-vactors ξ on $\mathcal{M}_0$. Verify that this is the case if one of the following holds:

i) Q_0 vanishing identically,

ii) Q_0 is a constant scalar field,

iii) Q_0 is a linear combination of products of Kronecker deltas with constant coefficients.

2.27 Backreaction Issues in Relativistic Cosmology and the Dark Energy Debate

We exclude inhomogeneities from the outset. Assume the metric and matter fields are spherically symetric, substitute them into Einsteins equations producing ordinary differential equations with independent variable time t.

Instead we calculate spherically averaged quantities. The equation governing them have the usual ones but also with additioinal terms that can be interpreted as the effects of the course-grained inhomogeneities on the large scale dynamics.

2.27.1 Cosmological Perturbation Theory

cosmological perturbation theory one expands the Einstein equations to linear order about a background metric.

We begin by expanding the metric about the FRW background metric $g_{ab}^{(0)}$ given by (5.24)

$$g_{ab} = g_{ab}^{(0)} + \delta g_{ab}. \tag{2.460}$$

$$\delta g_{ab} = a^2 \begin{pmatrix} 2\phi & -B,_i \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}), \end{pmatrix} \tag{2.461}$$

2.28 Gauge Invariant Perturbations Around Symmetry Reduced Sectors of General Relativity: Applications to Cosmology

2.28.1 Introduction

Challenging Features of GR

General relativity has two very challenging features: firstly the dynamics of the theory is highly non-linear, secondly general relativity is a diffeomorphism invariant and background independent theory. These two features make it very difficult to construct gauge invariant observables, that is to extract physical predictions. Diffeomorphism invariance of the theory includes invariance under time reparametrizations, therefore observables have to be constants of motions.

Hence finding gauge invariant observables is intimately related to solving the dynamics of the theory. But because of the highly non-linear structure of the theory it is quite hopeless to solve general relativity exactly. Indeed so far there are almost no gauge invariant observables known.

Gauge Independent Perturbation Theory

One might wonder why we attempt to develop a perturbation theory in the canonical formalism, where one would expect the problem to be even worse due to the foliation for the physical and background universe one has to choose in the canonical framework.

The resolution is that we use observables as central objects, i.e. we attempt to approximate directly a gauge invariant observable of the full theory and do not consider (the difference of) fields on two different manifolds representing the perturbed and unperturbed spacetime. Observables in the canonical formalism correspond to phase space functions, gauge invariant observables are invariant under the action of the constraints (the gauge generators).

Perturbation Theory for Symmetry Reduced Models

Using an approximation scheme around a whole (symmetry reduced) sector of the theory allows one to explore properties of gauge independent observables better than in a perturbative scheme around a fixed phase spce point. This is because one can now incorporate results from symmetry reduced (exactly solvable) models. The degrees of freedom describing these sectors are treated non-perturbatively.

Key feature

We keep the zeroth order variables as full dynamical phase space variables and not just parameters describing the background universe as one does in perturbation around a fixed phase space point.

Indeed we have to keep the zeroth variables as canonical variables to allow for a consistent gauge invariant framework to higher in linear order. Moreover this provides a very natural description for backreaction effects.

Backreaction Effects

These come from higher order corrections to observables arising through avergaing of (time evolved) phase space variables. Since this approach is gauge invariant it could shed light on the discussion whether these backreactions are measurable effects or caused by a specific choice of gauge, see for instance [].

Which Variables are Small?

In order to order to approximate phase space functions we have to declare which varables are to be considered small. This choice is done in such a way that the appoximate observables coincide with the exact observables if elvaluated on teh symmetry reduced sector of the phase space.

Indeed the zeroth order variables can be defined by an averaging procedure. First order phase space functions vanish on symmetric sapcetimes, higher order phase space functions are products of first order phase space functions. Note that the splitting of phase space variables into zeroth and first order is done on the gauge variant level. Generically a gaige invariant phase space function is a sum of terms of different order.

The Approximation

We have to choose clocks, which define also the hypersurfaces (by physical criteria, e.g. by demanding that a scalar field is constant on these hypersurfaces) over which the averaging is performed. Therefore the observables describing the backreaction effect depend on the choice of clocks.

However, as we will see, one can find relations between the gauge invariant observables coresponding to one choice of clocks and the gauge invariant observables corresponding to another choice of clocks.

2.29 Approximate Complete Observables

$$\begin{aligned} \chi^a &= \mathcal{P} \cdot \chi^a + (Id - \mathcal{P}) \cdot \chi^a \\ \pi^a &= \mathcal{P} \cdot \pi^a + (Id - \mathcal{P}) \cdot \pi^a. \end{aligned} \tag{2.462}$$

Gauge Invariant Observables of Order k

recall the notation :

${}^{(k)}f$ denotes all terms which are of order k in f,

${}^{[k]}f$ denotes all terms which are of order less than or equal to k.

We define gauge invariant observables of order k as phase space function which commute with the constraints modulo terms of order k (and modulo constraints). Gauge invariant functions of order k can be obtained from phase space functions F which are exactly gauge invariant by ommiting all terms of order higher than k, i.e. by truncating to ${}^{[k]}F$

$$F = {}^{[k]}F + {}^{(k+1)}F + {}^{(k+2)}F + \cdots$$

$$\{{}^{[k]}F, C_j\} = \underbrace{\{F, C_j\}}_{\simeq 0} + \{\mathcal{O}(k+1),\ {}^{(0)}C_j + {}^{(1)}C_j + \cdots\}$$

$$\{\mathcal{O}(k+1),\ {}^{(0)}C_j\} \simeq \mathcal{O}(k+1)$$

$$\{\mathcal{O}(k+1),\ {}^{(1)}C_j\} \simeq \mathcal{O}(k)$$

all other terms are of higher order. Hence

$$\{{}^{[k]}F, C_j\} \simeq \mathcal{O}(k). \tag{2.463}$$

In particular we can find approximate complete observables of order k by considering their trunction to order k. In the following we will assume that the constraints $\tilde{C}_K$ and the clocks T^K can be divided into two subsets

$$\{\{\tilde{C}_H\}_{H\in\mathcal{H}}, \{\tilde{C}_I\}_{I\in\mathcal{I}}\} \ \text{ and } \ \{\{T_H\}_{H\in\mathcal{H}}, \{T_I\}_{I\in\mathcal{I}}\},$$

such that

$$\begin{aligned} T^H &\ \text{ are of zeroth order} \\ T^I &\ \text{ are of first order} \end{aligned}$$

Note that

$$\{T^H, \tilde{C}_I\} = 0 \quad \text{and} \quad \{T^I, \tilde{C}_H\} = 0. \tag{2.464}$$

For the constraints $\tilde{C}_H$ we assume that for a first order function ${}^{(1)}f$,

$$\{{}^{(1)}f, \tilde{C}_H\} = \mathcal{O}(1), \tag{2.465}$$

which is satisfied if the constaints $\tilde{C}_H$ do not have a first order term ${}^{(1)}\tilde{C}_H$, however they may have a zeroth order term. For the constraints $\tilde{C}_I$ we will assume that the zeroth order terms vanish and that the first order terms do not vanish

$${}^{(0)}\tilde{C}_I = 0, \qquad {}^{(1)}\tilde{C}_I \neq 0 \tag{2.466}$$

Rewritting the Series Solution

$$\sum_{K_1}\{f, \tilde{C}_{K_1}\}(\tau^{K_1} - T^{K_1}) = \sum_{H_1 \in \mathcal{H}}\{f, \tilde{C}_{H_1}\}(\tau^{H_1} - T^{H_1}) + \sum_{I_1 \in \mathcal{I}}\{f, \tilde{C}_{I_1}\}(\tau^{I_1} - T^{I_1}) \tag{2.467}$$

The next term in the series expansion (??)

$$\begin{aligned}
& \sum_{K_1}\sum_{K_2}\{\{f, \tilde{C}_{K_1}\}, \tilde{C}_{K_2}\}(\tau^{K_1} - T^{K_1})(\tau^{K_2} - T^{K_2}) \\
= \; & \sum_{K_1}(\tau^{K_1} - T^{K_1})\Big(\sum_{H_2}\{\{f, \tilde{C}_{K_1}\}, \tilde{C}_{H_2}\}(\tau^{H_2} - T^{H_2}) + \\
& \sum_{I_2}\{\{f, \tilde{C}_{K_1}\}, \tilde{C}_{I_2}\}(\tau^{I_2} - T^{I_2})\Big) \\
\simeq \; & \sum_{H_1}\sum_{H_2}\{\{f, \tilde{C}_{H_1}\}, \tilde{C}_{H_2}\}(\tau^{H_1} - T^{H_1})(\tau^{H_2} - T^{H_2}) \\
+ \; & 2\sum_{H_1}\sum_{I_1}\{\{f, \tilde{C}_{H_1}\}, \tilde{C}_{I_1}\}(\tau^{H_1} - T^{H_1})(\tau^{I_1} - T^{I_1}) \\
+ \; & \sum_{I_1}\sum_{I_2}\{\{f, \tilde{C}_{I_1}\}, \tilde{C}_{I_2}\}(\tau^{I_1} - T^{I_1})(\tau^{I_2} - T^{I_2})
\end{aligned} \tag{2.468}$$

where we used that

$$\{\{f, \tilde{C}_{I_1}\}, \tilde{C}_{H_1}\} \simeq \{\{f, \tilde{C}_{H_1}\}, \tilde{C}_{I_1}\}$$

We can write (4.46) as

$$\sum_{s=0}^{r=2} \frac{2!}{(2-s)!s!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_{(2-s)}}\}\;,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_s}\}\times$$
$$(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_{(2-s)}}-T^{H_{(2-s)}})\times(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_s}-T^{I_s}) \quad (2.469)$$

as should be checked. We have in general

$$\begin{aligned}
&\{\cdots\{f,\tilde{C}_{K_1},\cdots,\},\tilde{C}_{K_q}\}(\tau^{K_1}-T^{K_1})\cdots(\tau^{K_q}-T^{K_q}) \\
&\quad\simeq \sum_{s=0}^{r} \frac{r!}{(r-s)!s!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_{(r-s)}}\}\;,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_s}\}\times \\
&\quad(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_{(r-s)}}-T^{H_{(r-s)}})\times(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_s}-T^{I_s})
\end{aligned} \quad (2.470)$$

where we have used that we can rearrange the constraints in any order.

$$\begin{aligned}
F_{[f;T^K]}(\tau) \;\simeq\; &\sum_{r=0}^{\infty}\sum_{s=0}^{r} \frac{1}{(r-s)!s!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_{(r-s)}}\}\;,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_s}\}\times \\
&(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_{(r-s)}}-T^{H_{(r-s)}})\times(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_s}-T^{I_s})
\end{aligned} \quad (2.471)$$

$$\begin{aligned}
F_{[f;T^K]}(\tau) \;\simeq\; &\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \frac{1}{p!q!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_q}\}(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_q}-T^{H_q})\;,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_p}\} \\
&(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_p}-T^{I_p})
\end{aligned} \quad (2.472)$$

(recall that $\{T^{H_r},\tilde{C}_{I_s}\}=0$).

0th Order Complete Obs. Associated to a 0th order Function: Symmetry Reduced Sector

Now set the parameters τ^I to zero. Then for zeroth order complete observable associated to with a zeroth order function ${}^{(0)}f$ we have

$$
\begin{aligned}
{}^{(0)}F_{[f;T^K]}(\tau^H,\tau^I=0) &\simeq \sum_{q=0}^{\infty}\frac{1}{q!}\,{}^{(0)}(\{\cdots\{{}^{(0)}f,\tilde{C}_{H_1}\},\cdots\},\tilde{C}_{H_q}\})(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_q}-T^{H_q}) \\
&\simeq \sum_{q=0}^{\infty}\frac{1}{q!}\{\cdots\{{}^{(0)}f,{}^{(0)}\tilde{C}_{H_1}\},\cdots\},{}^{(0)}\tilde{C}_{H_q}\}(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_q}-T^{H_q})
\end{aligned}
\tag{2.473}
$$

where we only kept the $p = 0$ term and the second equation holds because of our assumption that the constraints $\tilde{C}_H$ to have vanishing first order parts. There only appear zeroth order variables in the second line (T^H are zeroth order), hence we can say that the zeroth order complete complete observables associated with a zeroth order function are complete observables of the symmetry reduced sector.

2nd Order Complete Obs. Associated to a 0th order Function: Backreaction

The next higher order correction to this complete observable is a second order term and can be considered as the correction (backreaction) term to the dynamics of the reduced symmetry sector due to deviations from symmtry (in the inital values).

1st Order Complete Obs. Associated to a 1st order Function: Propagation of Linear Perturbations

One can also consider for instance the first order complete observable associated to a first order function. As we will see these observables decribe the proagation of linear perturbations (which are linearly gauge invariant) on the symmetry reduced sector.

Gauge Invaraint Observables to any Order k

Note that this approach allows to find gauge invariant observables to any order k by omitting in the serie for the complete observables all terms higher order than k. For this assumptions we made on the clocks $\{\{T_H\}_{H\in\mathcal{H}},\{T_I\}_{I\in\mathcal{I}}\}$ and the constraints $\{\{\tilde{C}_H\}_{H\in\mathcal{H}},\{\tilde{C}_I\}_{I\in\mathcal{I}}\}$ are not strictly necessary.

Standard Perturbative Calculations

However we will see that with these conditions the computation of complete is similar to the usual perturbative calculations involving the "free" propagation of perturbations and their interaction as well as the interaction of the zeroth order variables with the perturbations.

2.30 Application to Cosmology

To set up linear metric perturbations [??], one perturbs the background metric

$$ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ab}dx^a dx^b), \tag{2.474}$$

here chosen as a flat isotropic metric written in conformal time η and with spatial coordinates x^a. There are initially ten perturbation functions for the ten metric components, but some of them can be absorbed simply by redefining coordinates. The remaining functions, in gauge-invariant combinations, comprise scalar, vector and tensor modes. We are here primarily interested in scalar modes which in longitudinal gauge lead to a perturbed metric

Perturbed canonical variables: Ashtekar variables [??] due to their transformation properties. First, one introduces a co-triad e_a^i instead of the spatial metric q_{ab}, related to it by $e_a^i e_b^i = q_{ab}$. (Unlike the position of spatial indices $a, b, \ldots$, the upper or lower positions of indices i are not relevant, and summing over i is understood even though it appears twice in the same position.) An oriented co-triad contains the same information as a metric but has more components as it is not a symmetric tensor. This corresponds to freedom one has in rotating the triple of triad co-vectors which does not change the metric. Not being of geometrical relevance, this freedom is removed in a canonical formalism by implementing the Gauss constraint introduced below. By inverting the matrix (e_a^i), one obtains the triad e_i^a, a set of vector fields related to the inverse metric by $e_i^a e_i^b = q^{ab}$. Just as the metric determines a compatible Christoffel connection Γ_{ab}^c, a triad determines a compatible spin connection

$\Gamma_a^i = -\epsilon^{ijk} e_j^b (\partial_{[a} e_{b]}^k + \frac{1}{2} e_k^c e_a^l \partial_{[c} e_{b]}^l)$.

The configuration varibles are given by a (complex) connection $\{A_a^j\}_{j,a=1}^3$:

$$A_a^i = \Gamma_a^j + \beta K_a^j. \tag{2.475}$$

Recall the Poisson bracket between the phase space variables

$$\{A_a{}^j(\sigma), E^b{}_k(\sigma')\} = \kappa \delta_k^j \; \delta_a^b \delta(\sigma, \sigma') \tag{2.476}$$

where $\kappa = 8\pi G_N/c^3$ is the gravitational coupling constant. Furthermore we have a scalar field φ and its conjugated momentum π which satisfy the commutation relation

$$\{\varphi(\sigma), \pi(\sigma')\} = \gamma \; \delta(\sigma, \sigma') \tag{2.477}$$

Expanding variables

We will expand the canonical variables around homogeneous and isotropic field configurations in the following way:

$$
\begin{aligned}
A_a{}^j(\sigma) = A\beta\delta_a^j + a_a{}^b\beta\delta_b^j \quad &, \qquad E^a{}_j(\sigma) = E\beta^{-1}\delta_a^j + e^a{}_b(\sigma)\beta^{-1}\delta_b^j \\
\varphi(\sigma) = \Phi + \phi(\sigma) \quad &, \qquad \pi(\sigma) = \Pi + \rho(\sigma)
\end{aligned} \tag{2.478}
$$

The Poisson brackets between the homogeneous variables and between the fluctuation variables can be found by using the (5.24)

$$
\begin{aligned}
A\beta = \mathcal{P}\cdot A_a{}^j := \frac{1}{3}\int_\Sigma \delta_j^a A_a{}^j\, \mathbf{d}\sigma \quad &, \qquad E\beta^{-1} = \mathcal{P}\cdot E^a{}_j := \frac{1}{3}\int_\Sigma \delta_a^j E^a{}_j\, d\sigma \\
\Phi = \mathcal{P}\cdot\varphi := \int_\Sigma \varphi\, \mathbf{d}\sigma \quad &, \qquad \Pi = \mathcal{P}\cdot\pi := \int_\Sigma \pi\, \mathbf{d}\sigma
\end{aligned} \tag{2.479}
$$

Working's outs.

It ensures that the kinematics of the symmetry reduced system and of the symmetry reduced sector embedded into the full phase space coincide. [**?**]

Working's outs.

Show that A and E are real if evaluated on a homogeneous cosmology with flat slicing.

Proof:

$$
A_a^j = \Gamma_a^j + \beta K_a^j
$$

Flat slicing means that the extrinsic curvature K_{ab} vainshes which implies $K_a^j = 0$.

Working's outs.

Prove $\{A, E\}$

$$
\begin{aligned}
\{A,E\} &= \{\mathcal{P}\cdot {A_a}^j, \mathcal{P}\cdot {E^a}_k\} \\
&= \frac{1}{9}\,\delta^a_j\;\delta^k_b\;\{\int_\Sigma {A_a}^j(\sigma)\;\mathbf{d}\sigma, \int_\Sigma {E^b}_k(\sigma')\;d\sigma'\} \\
&= \frac{1}{9}\,\delta^a_j\;\delta^k_b\int_\Sigma\int_\Sigma \{{A_a}^j(\sigma), {E^b}_k(\sigma')\}\;\mathbf{d}\sigma\;\mathbf{d}\sigma' \\
&= \frac{1}{9}\kappa\;(\delta^a_j\;\delta^k_b)\;(\delta^j_k\;\delta^b_a)\int_\Sigma\int_\Sigma \delta(\sigma,\sigma')\;\mathbf{d}\sigma\;\mathbf{d}\sigma' \\
&= \frac{1}{9}\delta^a_k\delta^k_a\;\int_\Sigma \mathbf{d}\sigma \\
&= \frac{1}{3}\kappa
\end{aligned} \tag{2.480}
$$

where we used $\int_\Sigma\;\mathbf{d}\sigma = 1$.

Now $\{\Phi,\Pi\} = \gamma$

$$
\begin{aligned}
\{\Phi,\Pi\} &= \{\mathcal{P}\cdot\varphi, \mathcal{P}\cdot\pi\} \\
&= \int_\Sigma\int_\Sigma \{\varphi(\sigma), \pi(\sigma')\}\;\mathbf{d}\sigma\;\mathbf{d}\sigma' \\
&= \int_\Sigma\int_\Sigma \gamma\delta(\sigma,\sigma')\;\mathbf{d}\sigma\;\mathbf{d}\sigma' \\
&= \gamma
\end{aligned} \tag{2.481}
$$

Now $\{\phi(\sigma),\rho(\sigma')\} = \gamma\delta(\sigma,\sigma') - \gamma$

$$
\begin{aligned}
\{\phi(\sigma),\rho(\sigma')\} &= \{\varphi(\sigma)-\Phi, \pi(\sigma')-\Pi\} \\
&= \{\varphi(\sigma),\pi(\sigma')\} + \{\Phi,\Pi\} - \{\varphi(\sigma),\Pi\} - \{\Phi,\pi(\sigma')\} \\
&= \gamma\delta(\sigma,\sigma') + \gamma - \{\varphi(\sigma), \int_\Sigma \pi(\sigma')\;\mathbf{d}\sigma'\} - \{\int_\Sigma \varphi(\sigma)\;\mathbf{d}\sigma, \pi(\sigma')\} \\
&= \gamma\delta(\sigma,\sigma') - \gamma
\end{aligned} \tag{2.482}
$$

Now $\{{a_a}^b(\sigma), {e^c}_d(\sigma')\}$

From equation ()

$$
{a_a}^c(\sigma)\delta^j_c = \beta^{-1}{A_a}^j(\sigma) - A\delta^j_a
$$

affecting δ^b_j to both sides of this we find

$$
{a_a}^b(\sigma) = \beta^{-1}{A_a}^j\delta^b_j - A\delta^b_a.
$$

From a similar calculation we find for ${e^a}_b(\sigma)$

$$
{e^a}_b(\sigma) = \beta {E^a}_j(\sigma)\delta^j_b - E\delta^a_b.
$$

$$
\begin{aligned}
\{{a_a}^b(\sigma), {e^c}_d(\sigma')\} &= \{\beta^{-1}{A_a}^j(\sigma)\delta^b_j - A\delta^b_a \ , \ \beta {E^c}_j(\sigma')\delta^j_d - E\delta^a_b\} \\
&= \text{and so on...}
\end{aligned}
\tag{2.483}
$$

Fourier transforms

(i) Show the homogeneous variables are given by the ($\frac{1}{3}\times$ trace of the) k modes of the fields.

(ii) Show the Poisson brackets for the Fourier modes of the fluctuation variables are

$$
\begin{aligned}
\{a_{ab}(k), e^{cd}(k')\} &= \kappa\delta^c_a\delta^d_b\delta_{k,-k'} - \frac{\kappa}{3}\delta_{ab}\delta^{cd}\delta_{k,0}\delta_{k',0} \\
\{\phi(k), \rho(k')\} &= \gamma\delta_{k,-k'} - \gamma\delta_{k,0}\delta_{k',0}.
\end{aligned}
\tag{2.484}
$$

Proof:

$$
f(k) = \int_\Sigma \exp(ik\cdot\sigma) f(\sigma)\mathbf{d}\sigma
$$

where $k\cdot\sigma := k_a\sigma^a$. The inverse transform is

$$
f(\sigma) = \sum_{k\in\{2\pi\mathbb{Z}^3\}} \exp(ik\cdot\sigma) f(k).
$$

$$
\begin{aligned}
{A_a}^j(k) &= \int_\Sigma \exp(ik\cdot\sigma){A_a}^j(\sigma)\ \mathbf{d}\sigma \\
&= \beta\delta^j_a A\int_\Sigma \exp(ik\cdot\sigma)\ \mathbf{d}\sigma + \beta\int_\Sigma \exp(ik\cdot\sigma){a_a}^j(\sigma)\ \mathbf{d}\sigma \\
&= \beta\delta^j_a A\delta_{k,0} + \beta {a_a}^j(k)
\end{aligned}
\tag{2.485}
$$

$$
\begin{aligned}
\frac{1}{3}\delta^a_j {A_a}^j(k) &= \frac{1}{3}\beta\delta^a_j\delta^j_a A + \frac{1}{3}\beta\delta^a_j\delta^j_b {a_a}^b(k) \\
&= \beta A + \beta {a_a}^a(k) \\
&= \beta A
\end{aligned}
\tag{2.486}
$$

$$
\begin{aligned}
\frac{1}{3}\delta^j_a E^a{}_j(k) &= \frac{1}{3}\beta^{-1}\delta^j_a\delta^a_j E + \frac{1}{3}\beta^{-1}\delta^j_a\delta^b_j e^a{}_b(k) \\
&= \beta^{-1}E + \beta^{-1}e^a{}_a(k) \\
&= \beta^{-1}E
\end{aligned} \tag{2.487}
$$

(i)

$$
\begin{aligned}
\{\phi(k),\rho(k')\} &= \{\int_\Sigma \exp(ik\cdot\sigma)\phi(\sigma)\mathbf{d}\sigma, \int_\Sigma \exp(ik'\cdot\sigma')\rho(\sigma)\mathbf{d}\sigma'\} \\
&= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\{\phi(\sigma),\rho(\sigma')\}\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\left(\gamma\delta(\sigma,\sigma') - \gamma\right)\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \gamma\int_\Sigma \exp(i(k-k')\cdot\sigma)\mathbf{d}\sigma - \gamma\int_\Sigma \exp(ik\cdot\sigma)\,\mathbf{d}\sigma\int_\Sigma \exp(ik'\cdot\sigma')\,\mathbf{d}\sigma' \\
&= \gamma\delta_{k,-k'} - \gamma\delta_{k,0}\delta_{k',0}
\end{aligned} \tag{2.488}
$$

as $\int_\Sigma \exp(ik\cdot\sigma) = 0$ when $k \neq 0$ and is equal to 1 when $k = 0$.

(ii)

$$
\begin{aligned}
\{a_{ab}(k), e^{cd}(k')\} &= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\{a_{ab}(\sigma), e^{cd}(\sigma')\}\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\left(\kappa\delta^c_a\delta^d_b(\sigma,\sigma') - \frac{\kappa}{3}\delta_{ab}\delta^{cd}\right)\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \kappa\delta^c_a\delta^d_b\delta_{k,-k'} - \frac{\kappa}{3}\delta_{ab}\delta^{cd}\delta_{k,0}\delta_{k',0}
\end{aligned} \tag{2.489}
$$

Note that the additional terms on the right hand side implement that $a_a{}^a(0) = e^a{}_a(0) = \phi(0) = \rho(0) = 0$.

Working's outs.

Fourier transformed variables can be used to define the symplectic coordinates used in section ?? in which the projection operator $\mathcal{P}$ maps part of the symplectic coordinates to zero and leaves the other coordinates invariant.

show that homogeneous part of the coordinates are given by

$$
(\sqrt{3}A, \sqrt{3}E; \Phi, \Pi).
$$

$$
\begin{aligned}
\mathcal{P}\cdot\sqrt{3}A &= \frac{\sqrt{3}}{3\beta}\int_\Sigma \delta^a_j\;\mathcal{P}\cdot {A_a}^j\;d\sigma \\
&= \frac{1}{\sqrt{3}\beta}\int_\Sigma \delta^a_j\left(\beta\delta^j_a A\right)\,d\sigma \\
&= \sqrt{3}A
\end{aligned}
\tag{2.490}
$$

$$
\begin{aligned}
\mathcal{P}\cdot\sqrt{3}E &= \frac{\sqrt{3}\beta}{3}\int_\Sigma \delta^j_a\;\mathcal{P}\cdot {E^a}_j\;d\sigma \\
&= \frac{\beta}{\sqrt{3}}\int_\Sigma \delta^a_j\left(\beta^{-1}\delta^a_j E\right)\,d\sigma \\
&= \sqrt{3}A
\end{aligned}
\tag{2.491}
$$

$$
\begin{aligned}
\mathcal{P}\cdot\Phi &= \int_\Sigma \mathcal{P}\cdot\varphi\;d\sigma \\
&= \int_\Sigma \Phi\;d\sigma \\
&= \Phi
\end{aligned}
\tag{2.492}
$$

$$
\begin{aligned}
\mathcal{P}\cdot\Pi &= \int_\Sigma \mathcal{P}\cdot\pi\;d\sigma \\
&= \int_\Sigma \Pi\;d\sigma \\
&= \Phi
\end{aligned}
\tag{2.493}
$$

The symplectic pairs that are mapped to zero are given by $(a_{ab}(k), e^{ab}(-k))$ and $(\phi(k), \pi(-k))$ for $k \neq 0$ and

$$
\begin{aligned}
\mathcal{P}\cdot a_{ab}(k) &= \beta^{-1}\mathcal{P}\cdot {A_a}^j\delta^b_j - \mathcal{P}\cdot A\delta^b_a \\
&= \beta^{-1}(\beta A)\delta^b_j -
\end{aligned}
\tag{2.494}
$$

$$
{e^a}_b(\sigma) = \beta {E^a}_j(\sigma)\delta^j_b - E\delta^a_b.
$$