Chapter 4

Classical Cosmology

4.1 Classical Cosmology

4.1.1 Fluid Flow Equations

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx_i}{dt}\frac{\partial}{\partial x_i}, \tag{4.1}$$

The accelaration of the element of fluid is given by the **Euler equation**

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla P - \nabla\Phi_{grav}. \tag{4.2}$$

Here Φ_{grav} is the gravitaional potential, which satisfies the **Poisson equation**

$$\nabla^2\Phi_{grav} = 4\pi G\rho. \tag{4.3}$$

$$\frac{1}{\mathcal{V}}\frac{d\mathcal{V}}{dt} = \nabla\cdot\mathbf{u} \tag{4.4}$$

$$H(x,t) = \frac{1}{3}\nabla\cdot\mathbf{u} \tag{4.5}$$

If we integrate an element of gas at position x and time t, we can integrate this using the divergence theorem to find

$$3H\mathcal{V} = \int_{\mathcal{V}}\mathbf{u}\cdot\mathbf{dS}. \tag{4.6}$$

$$\frac{d^2x^a}{d\tau^2} + \Gamma^a_{bc}\frac{dx^b}{d\tau}\frac{dx^c}{d\tau} = 0. \tag{4.7}$$

$$\begin{aligned} d\tau^2 &= -dt^2 + a^2(t)[dx^2 + dy^2 + dz^2] \\ &= -dt^2 + a^2(t)\eta_{ij}dx^i dx^j \end{aligned} \tag{4.8}$$

$$= \frac{1}{2}\int \left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} \right] d\tau \tag{4.9}$$

$$\frac{d^2t}{d\tau^2} + 2\dot{a}a\eta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0. \tag{4.10}$$

$$\Gamma^0_{00} = 0, \quad \Gamma^0_{i0} = \Gamma^0_{0i} = 0, \quad \Gamma^0_{ij} = a\dot{a}\eta_{ij} \tag{4.11}$$

$$\frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\frac{dt}{d\tau}\frac{dx^i}{d\tau} = 0. \tag{4.12}$$

$$\Gamma^i_{00} = 0, \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j, \quad \Gamma^i_{jk} = 0. \tag{4.13}$$

4.1.2 Newtonian Cosmology

cosmological force acting on the ith galaxy

$$F_i = \frac{1}{3}\Lambda m_i r_i \tag{4.14}$$

where Λ is the cosomological constant. the cosmological potential energy

$$V_c = -\frac{1}{6}\Lambda\sum_{i=1}^{n} m_i r_i \tag{4.15}$$

total energy

$$E = \frac{1}{2}\sum_{i=1}^{n} m_i\dot{r}_i^2 - G\sum_{i,j=1(i<j)}^{n}\frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|} - \frac{1}{6}\Lambda\sum_{i=1}^{n} m_i r_i^2. \tag{4.16}$$

Motions compatible with homogeneity and isotropy are uniform expansion or contraction, a scaling up or down by a time-dependent scale factor.

$S(t)$ scale factor

$$r_i(t) = S(t) r_i(t_0) \tag{4.17}$$

radial velocity of the ith galaxy is then

$$\dot{r}_i(t) = \dot{S}(t) r_i(t_0) = \frac{\dot{S}(t)}{S(t)} r_i(t) \tag{4.18}$$

the Hubble parameter $H(t)$

$$H(t) = \dot{S}(t)/S(t), \tag{4.19}$$

then

$$\dot{r}_i(t) = H(t) r_i(t) \tag{4.20}$$

Hubble's law. This states that, in an expanding universe, at any one epoch, the radial velocity of recession of a galaxy from a given point is proportional to the distance of the galaxy from the point. The value of the Hubble parameter at our epoch is know as the Hubble constant.

$$E = A[\dot{S}(t)]^2 - \frac{B}{s(t)} - D[S(t)]^2 \tag{4.21}$$

where the coefficients are positive defined by

$$\begin{aligned} A &= \frac{1}{2}\sum_{i=1}^{n} m_i [r_i(t_0)]^2, && (4.22) \\ B &= G \sum_{i,j=1(i<j)}^{n} \frac{m_i m_j}{|\mathbf{r}_i(t_0) - \mathbf{r}_j(t_0)|}, && (4.23) \\ D &= \frac{1}{6}\Lambda \sum_{i=1}^{n} m_i [r_i(t_0)]^2 = \frac{1}{3}\Lambda A. && (4.24) \end{aligned}$$

4.1.3 Relativistic Cosmology

$$ds^2 = dt^2 - h_{ij} dx^i dx^j$$

4.1.4 Spaces of Constant Curvature

$$R_{abcd} = K(g_{ac}g_{bd} - g_{ad}g_{bc}) \tag{4.25}$$

$$R_{\alpha\beta\gamma\delta} = K(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}) \tag{4.26}$$

Contracting with $g^{\alpha\gamma}$, we get

$$\begin{aligned} g^{ac}R_{abcd} &= R_{bd} \\ &= Kg^{ac}(g_{ac}g_{bd} - g_{ad}g_{bc}) \\ &= K(3g_{bd} - g_{bd}) \\ &= 2Kg_{bd}. \end{aligned} \tag{4.27}$$

since 3-space is isotropic about every point, it must be spherically symmetric about every point the line element

$$d\sigma^2 = g_{ij}dx^i dx^j = e^{\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.28}$$

We have

$$d\sigma^2 = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{4.29}$$

4.2 Homogeneous and Isotropic Cosmology

4.2.1 The Luminosity Distance

In a Euclidean universe, if a source of absolute luminosity L is at a distance d then the flux that we receive is

$$F = \frac{L}{4\pi d^2}.$$

Now suppose that we are actually in an expanding FRW spacetime and we know that the source has luminosity L and we observe a flux F. We define the *luminosity distance* as

$$d_L := \left(\frac{L}{4\pi d^2}\right)^{1/2}. \tag{4.30}$$

Consider an emmiting source E with a fixed comoving coordinate χ relative to an observer O. We assume that the luminosity E as a function of cosmic time is $L(t)$ and that the light pulse it emmits are detected by O at at cosmic time t_0. Te light pulses had been emmited at the early time t_e.Asssuming the light pulses were emmitted isotropically, the radiation will be spread evenly over a sphere centered at E and passing through O (see diagram). The proper area of this sphere is

$$A = 4\pi R^2(t_0)S^2(\chi). \tag{4.31}$$

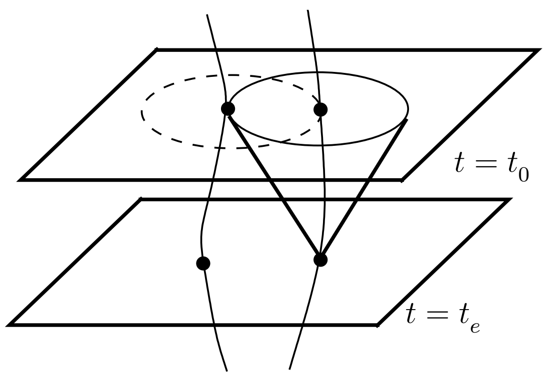

Figure 4.1: LumDist. Luminosity distance

However, each light pulse received by O is red shifted in frequency, so that

$$\nu_0 = \frac{\nu_e}{1+z}, \tag{4.32}$$

also the arrive rate of the light pulse is also reduced by the same factor. Thus, the observed flux at O is

$$F(t_0) = \frac{L(t_e)}{4\pi[R_0 S(\chi)]^2}\frac{1}{(1+z)^2}. \tag{4.33}$$

The luminosity distance defined above is

$$d_L = R_0 S(\chi)(1+z). \tag{4.34}$$

4.3 The Singularity Theorems

there was a big bang and there are black holes. They are strong motivations for attempting to quantize general relativity.

A singularity happens when a cloud of particles collapses to form a singularity from an initially spherical state. However, no singularity forms from the general, non-symmetric inital state.

A singularity in the finite past. The theory does not allow for it.

They derived general results about singularities by using braod physically reasonable assumptions, say about the energy-momentum tensor or causal violations, without considering the field equations in detail.

This secton is only meant to be a brief introduction to this work. We shall attempt to provide a flavor of the argument while at the same time evading most of the technicalities.

The singularity theorems show that singularities are inevitable in many physical situations in general relativity. They are the reason why we are so confident in the big bang singularity at the beginning of the universe.

The large scale structure of space-time Hawking-Ellis

differential methods in gravity David Learner- who took notes when Penrose gave a talk - pretty much the same as Penroe's book. This is the article I leared the most from however the future horizmos I understood when reading Hawking Penrose paper future horizmos.

Definition of a singularity:

A spacetime singularity is singular if it is timelike or null geodesically incomplete but cannot be embedded in a larger spacetime

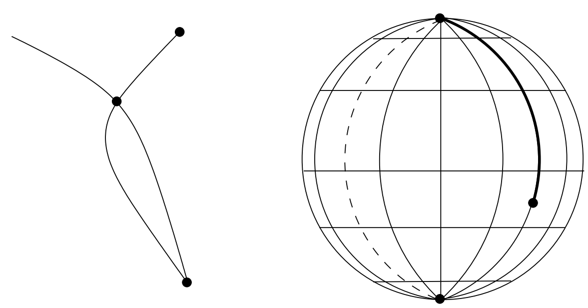

Figure 4.2: nonmingeodesic.

Singularity Theorems

(i) Energy condition

(ii) Gravity strong enough to trap a region

(iii) Condition on the global structure

If the following conditions hold, a space time is singular:

(i) $R_{ab}l^a l^b \leq 0$ *for each future pointing null vector* l^a*,*
(ii) *There exists a closed trapped surface.*

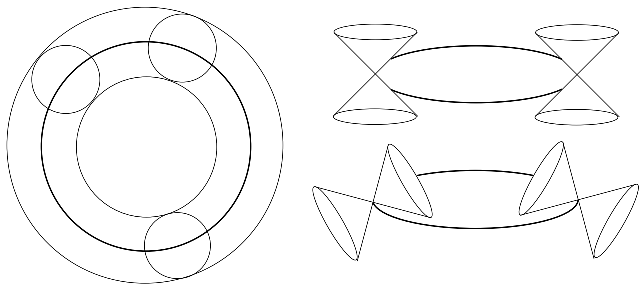

Figure 4.3: trapped.two-surface such that the areas of pulses of light emmited from each little element of surface decrease in both directions.

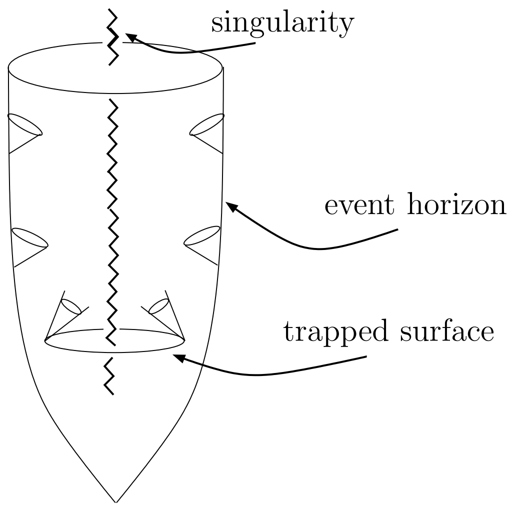

Figure 4.4: illustrating a trapped surface.

(iii) *There exists a noncompact, global Cauchy hypersurface, that is, the space time is predictable.*

but subsequent singularity theorems have been able to weaken this condition by replacing it with a nonglobal predictablity, which is more physically reasonable.

While this theorem was designed for predicting the occurence of a singularity for gravitational collapse, the next theorem has a more cosmological flavour.

4.3.1 Application of the Singularity Theorem: Cosmological Singularity

$$N = -(-g(\nabla T, \nabla T))^{-1/2} \nabla T \tag{4.35}$$

We define the *mean curvature* H_S of S in $\mathcal{M}$ by

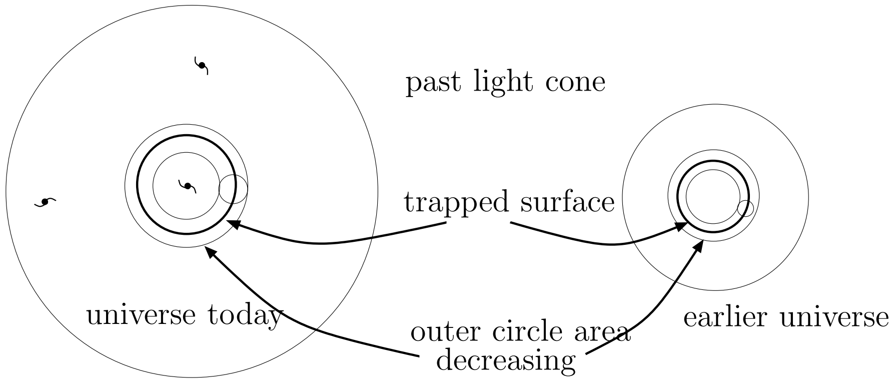

Figure 4.5: illustrating a past-directed trapped surface, correspondimg to a cosmological singularity.

$$H_S := \mathrm{div}N|_S \equiv \theta|_S \tag{4.36}$$

Hawking's Theorem: Let $\mathcal{M}$ be a (time orientable, stably causal, globally hyperbolic) spacetime which satsifies

(1) $R_{ab}V^aV^b \geq 0$ for all time-like tangent vector V, and
(2) There xists a Cauchy surface S in $\mathcal{M}$ on which the mean curvature is bounded below by some positive constant k, i.e., $H_S(p) \geq k$ for each p in S.

Then the spacetime is singular in the past, to be more specific:

$\mathcal{M}$ is time-like gedesically incomplete. More precisely, if $\mu : (-u_0, 0] \to \mathcal{M}$ is any future-directed time-like geodesic such that $g(\mu', \mu') = -1$, $\mu(0) \in S$ and $\mu'(0)$ is normal to S, then $-u_0 \geq -3/k$.

a direction in time and therefore we will only consider spacetimes that are time orientable, i.e. there exists a nowhere vanishing smooth timelike vector field t^a on $\mathcal{M}$.

4.3.2 Energy Conditions

(i) WEC weak energy condition energy density is positive.

(ii) The strong energy condition, SEC. Energy density is positive and the local speed of sound doe not exceed speed of light.

4.3.3 Causality and Chronology

Avoid paradoxes traveling into the past. Heierarky of conditions:

1) Chronology: no closed timelike curves;

2) Causality: no closed causal curves;

3) Strong causality: no "almost closed" causal curves;

4) Stably causality: close metrics are causal.

we write $p \ll q$ if there is a timelike curve from p to q; we write $p < q$ if either $p = q$ or there is a causal curve from p to q. If $p < q$ but not $p \ll q$, then there is a null geodesic from p to q, or else $p = q$. If $p \ll q$ and $q < r$, or $p < q$ and $q \ll r$, then $p \ll r$ - we proved the latter statement in section 3.10.3, the former can be proved similarly. We do not have $p \ll p$ unless $\mathcal{M}$ contains *closed timelike curves.*

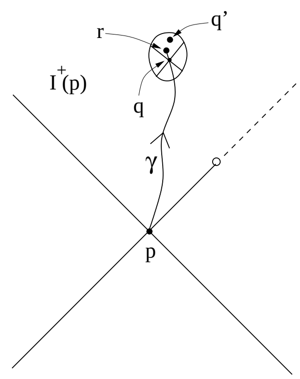

Figure 4.6: $I^+(p)$ is open.

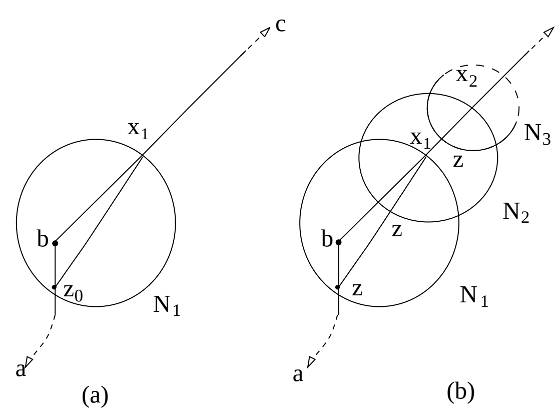

Figure 4.7: trivialsing.

The set $I^+(p)$ is open because for $r \in I^+(p)$, one could choose q on γ sufficiently near to r in a neighbourhood of r, then the set of points $q' \in I^+(q)$ forms an open neighbourhood of r in $I^+(p)$.

Proposition 4.3.1 *$a \ll b$, $b < c$ $\Rightarrow a \ll c$ and $a < b$, $b \ll c$ $\Rightarrow a \ll c$.*

Proof: To prove this we show that if a and b are joined by a time-like geodesic , b and c by a null geodesic, then there is a trip from a to c.

Consider a simple set N_1 at b. Let x_1 be the future intersection of $[bc]$ with ∂N_1, see fig (4.3.3(a)). We now make use of fact that our intuition from Minkowskian spacetime can be applied to simple sets: b lies on the past light cone of x_1 and hence there is a point z_0 on the segment of $[a, b]$ lying inside N_1, $[ab] \cap N_1$, such that $z_0 \in I^-(x_1)$.

We now proceed iterativley: Cover $[bc]$ by finitely many simple stes $N_1, \ldots, N_k$. Let x_i be the future intersection of $[bc]$ with ∂N_i, see fig (4.3.3(a)). Thus $[az_0 z_1 \cdots z_{k-1} c]$ is the required trip.

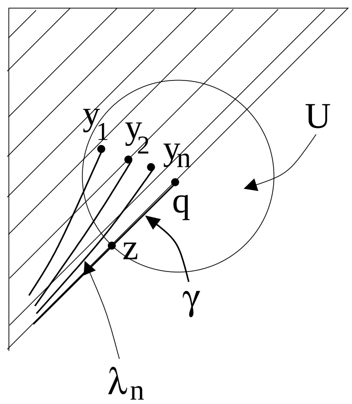

Figure 4.8: The points y_n converge to the point q in the boundary of $L^+(S)$. From each y_n there is a past directed timelike curve λ_n to S. These curves converge to the past directed null geodesic segment γ through q.

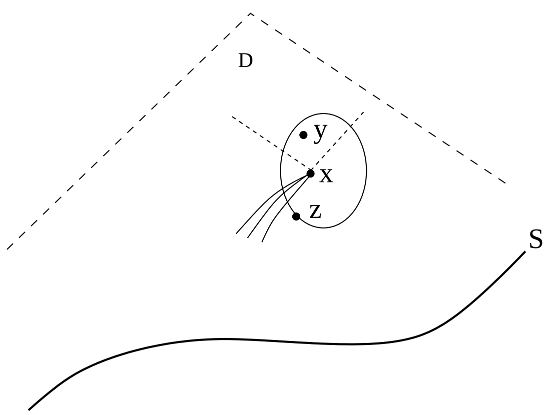

Figure 4.9: $K = J^+(S) \cap J^-(p)$ K is compact.

4.3.4 Existence of maximum geodesic

asserts that any continuous, real-valued function on a closed, bounded interval $[a, b]$ in R. In real analysis the same theorem can be proved for any subspace of any Euclideanspace. In fact, one can show that a continuous, real valued function on any campact metric space achieves maximum and minimum values.

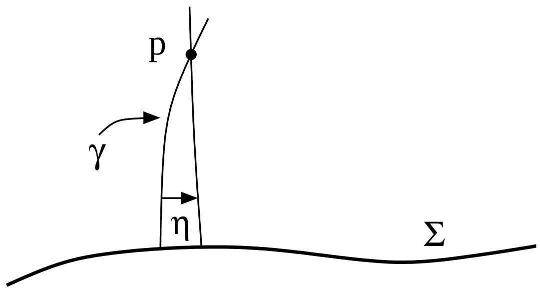

Figure 4.10: trivialsing.

: Consider the collection $\mathcal{C}$ of all causal curves which join a point in S with fixed point p in say, int$D_{-}(S)$. Define a metric on the set $\mathcal{C}$ in such a way that

(1) the metric space is compact, and
(2) the real-valued function on $\mathcal{C}$ which asigns to every element of $\mathcal{C}$ its Lorentzian length is continuous.

Then this length function would have not achieve a maximum value on some causal curve in $\mathcal{C}$ which we might then try to show is necessarily a geodesic

there is no reasonable way to define a compact metric on the set of all smooth curves from p to S since limits of smooth curves (functions) are generally not smooth.

Completedness is an important property to have around if you are counting on an equation having a solution.

A metric space is **complete** if every Cauchy sequence in M converges to a point in M.

but is semicontinuous so that, although it will not achieve a minimum in general, it will achieve a maximum as we require.

Now consider a Cauchy surface S in $\mathcal{M}$ and a point p in, say, int$D^{-}(S)$. We set

$$K = J^{+}(p) \cap J^{-}(S) \tag{4.37}$$

recall that K is compact and denote by $C_K(p, S)$ the set of all future causal curves in K from p to a point in S. Let d denote the natural metric on $\mathcal{M}$

Contimuity and boundedness on compact domain

Continuity is often loosely described as a function you can draw without taking your pen from the paper

Let f be a continuous function on the compact interval $[a, b]$. Then there are points x_0 and x_1 such that $f(x_0) < f(x) < f(x_1)$ for all $x \in [a, b]$.

The result is false if compactness is ommited; for example, consider the function $f(x) = 1/x$ on the open interval $(0, 1)$.

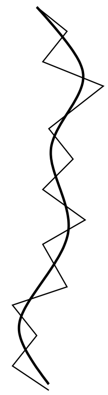

Figure 4.11: A timelike curve can be approximated by null geodesic segments. This approximating null curve fails to have a well defined tangent vector anywhere.

compactness of a closed, bounded interval of the real line

Take an arbitrary open cover $\mathcal{U}$ of $[a, b]$. We denote by G the points $x \geq a$ that are covered by a finite subcollection of $\mathcal{U}$. We will call such points good (for $\mathcal{U}$). If x is good and if $a \leq y \leq x$ then y good too as it is covered by the same finite subcollection that covers $[a, x]$ since $[a, y] \subset [a, x]$. Next we show that $G \neq \emptyset$. The point a is in some U in $\mathcal{U}$, since U is open $[a, a + \delta] \subset U$ for some $\delta > 0$. Hence if $[a, x] \subset U$ for all $x \in [a, a + \delta]$

$g = \sup G$ Since $g \in [a, b]$, g must belong to some $U_0 \in \mathcal{U}$. $B_\epsilon(g) \subset U_0$ for some $\epsilon > 0$ and since $g > a$ wemay suppose that $\epsilon < g - a$. As g is good there is a point c $c > g - \epsilon$. This means $[a, c]$ is covered by a finite subcollection in $\mathcal{U}$, say $\{U_1, U_2, \ldots, U_k\}$.

Lemma

Let E, E' be metric spaces, $f : E \to E'$ a continuous function. Then if E is compact, so is its image $f(E)$.

Proof:

Corollary 2. A continuous real-valued function on a nonempty compact metric space attains a maximum at some point, and also attains a minimum at some point.

Corollary 2 is false if the compactness condiion condition is ommited, even for bounded functions; for example, consider the function $f(x) = x$ on the open interval $(0, 1)$.

Uniform continuity $\to$ continuity in space of functions.

4.3.5 The Significance of Conjugate Points: The Singularity Theorems

The existence of a Cauchy surface is too strong a condition. We replace this by a condition that implies *global hyperbolicity.* In a region of spacetime which is globally hyperbolic, given a closed set without edge and a point p within this region, there exist timelike geodesics such as γ (fig. (4.3.4)) orthogonal to S, which maximise the lengths of all non-spacelike curves from p to S.

The global hyperbolicity of $\mathcal{M}$ is closely related to the future or past development of initial data from a given spacelike hypersurface.

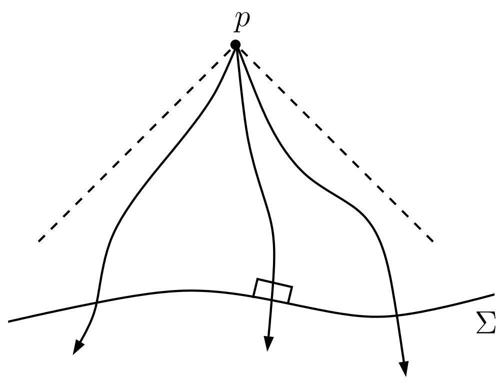

Figure 4.12: The global hyperbolicity of $\mathcal{M}$ is closely related to the future or past development of initial data from a given spacelike hypersurface.

4.4 Backreaction Issues in Relativistic Cosmology and the Dark Energy Debate

We exclude inhomogeneities from the outset. Assume the metric and matter fields are spherically symetric, substitute them into Einsteins equations producing ordinary differential equations with independent variable time t.

Instead we calculate spherically averaged quantities. The equation governing them have the usual ones but also with additioinal terms that can be interpreted as the effects of the course-grained inhomogeneities on the large scale dynamics.

4.4.1 Cosmological Perturbation Theory

cosmological perturbation theory one expands the Einstein equations to linear order about a background metric.

We begin by expanding the metric about the FRW background metric $g_{ab}^{(0)}$ given by (4.3.4)

$$g_{ab} = g_{ab}^{(0)} + \delta g_{ab}. \tag{4.38}$$

$$\delta g_{ab} = a^2 \begin{pmatrix} 2\phi & -B_{,i} \\ -B_{,i} & 2(\psi\delta_{ij} - E_{,ij}), \end{pmatrix} \tag{4.39}$$

4.5 Gauge Invariant Perturbations Around Symmetry Reduced Sectors of General Relativity: Applications to Cosmology

4.5.1 Introduction

Challenging Features of GR

General relativity has two very challenging features: firstly the dynamics of the theory is highly non-linear, secondly general relativity is a diffeomorphism invariant and background independent theory. These two features make it very difficult to construct gauge invariant observables, that is to extract physical predictions. Diffeomorphism invariance of the theory includes invariance under time reparametrizations, therefore observables have to be constants of motions.

Hence finding gauge invariant observables is intimately related to solving the dynamics of the theory. But because of the highly non-linear structure of the theory it is quite hopeless to solve general relativity exactly. Indeed so far there are almost no gauge invariant observables known.

Gauge Independent Perturbation Theory

One might wonder why we attempt to develop a perturbation theory in the canonical formalism, where one would expect the problem to be even worse due to the foliation for the physical and background universe one has to choose in the canonical framework.

The resolution is that we use observables as central objects, i.e. we attempt to approximate directly a gauge invariant observable of the full theory and do not consider (the difference of) fields on two different manifolds representing the perturbed and unperturbed spacetime. Observables in the canonical formalism correspond to phase space functions, gauge invariant observables are invariant under the action of the constraints (the gauge generators).

Perturbation Theory for Symmetry Reduced Models

Using an approximation scheme around a whole (symmetry reduced) sector of the theory allows one to explore properties of gauge independent observables better than in a perturbative scheme around a fixed phase spce point. This is because one can now incorporate results from symmetry reduced (exactly solvable) models. The degrees of freedom describing these sectors are treated non-perturbatively.

Key feature

We keep the zeroth order variables as full dynamical phase space variables and not just parameters describing the background universe as one does in perturbation around a fixed phase space point.

Indeed we have to keep the zeroth variables as canonical variables to allow for a consistent gauge invariant framework to higher in linear order. Moreover this provides a very natural description for backreaction effects.

Backreaction Effects

These come from higher order corrections to observables arising through avergaing of (time evolved) phase space variables. Since this approach is gauge invariant it could shed light on the discussion whether these backreactions are measurable effects or caused by a specific choice of gauge, see for instance [].

Which Variables are Small?

In order to order to approximate phase space functions we have to declare which varables are to be considered small. This choice is done in such a way that the appoximate observables coincide with the exact observables if elvaluated on teh symmetry reduced sector of the phase space.

Indeed the zeroth order variables can be defined by an averaging procedure. First order phase space functions vanish on symmetric sapcetimes, higher order phase space functions are products of first order phase space functions. Note that the splitting of phase space variables into zeroth and first order is done on the gauge variant level. Generically a gaige invariant phase space function is a sum of terms of different order.

The Approximation

We have to choose clocks, which define also the hypersurfaces (by physical criteria, e.g. by demanding that a scalar field is constant on these hypersurfaces) over which the averaging is performed. Therefore the observables describing the backreaction effect depend on the choice of clocks.

However, as we will see, one can find relations between the gauge invariant observables coresponding to one choice of clocks and the gauge invariant observables corresponding to another choice of clocks.

4.6 Approximate Complete Observables

$$\begin{aligned}
\chi^a &= \mathcal{P}\cdot\chi^a + (Id-\mathcal{P})\cdot\chi^a \\
\pi^a &= \mathcal{P}\cdot\pi^a + (Id-\mathcal{P})\cdot\pi^a.
\end{aligned} \tag{4.40}$$

Gauge Invariant Observables of Order k

recall the notation :

${}^{(k)}f$ denotes all terms which are of order k in f,

${}^{[k]}f$ denotes all terms which are of order less than or equal to k.

We define gauge invariant observables of order k as phase space function which commute with the constraints modulo terms of order k (and modulo constraints). Gauge invariant functions of order k can be obtained from phase space functions F which are exactly gauge invariant by ommiting all terms of order higher than k, i.e. by truncating to ${}^{[k]}F$

$$F = {}^{[k]}F + {}^{(k+1)}F + {}^{(k+2)}F + \cdots$$

$$\{{}^{[k]}F, C_j\} = \underbrace{\{F, C_j\}}_{\simeq 0} + \{\mathcal{O}(k+1),\ {}^{(0)}C_j + {}^{(1)}C_j + \cdots\}$$

$$\{\mathcal{O}(k+1),\ {}^{(0)}C_j\} \simeq \mathcal{O}(k+1)$$

$$\{\mathcal{O}(k+1),\ {}^{(1)}C_j\} \simeq \mathcal{O}(k)$$

all other terms are of higher order. Hence

$$\{{}^{[k]}F, C_j\} \simeq \mathcal{O}(k). \tag{4.41}$$

In particular we can find approximate complete observables of order k by considering their trunction to order k. In the following we will assume that the constraints $\tilde{C}_K$ and the clocks T^K can be divided into two subsets

$$\{\{\tilde{C}_H\}_{H\in\mathcal{H}}, \{\tilde{C}_I\}_{I\in\mathcal{I}}\} \ \text{ and } \ \{\{T_H\}_{H\in\mathcal{H}}, \{T_I\}_{I\in\mathcal{I}}\},$$

such that

$$T^H \quad \text{are of zeroth order}$$
$$T^I \quad \text{are of first order}$$

Note that

$$\{T^H, \tilde{C}_I\} = 0 \quad \text{and} \quad \{T^I, \tilde{C}_H\} = 0. \tag{4.42}$$

For the constraints $\tilde{C}_H$ we assume that for a first order function ${}^{(1)}f$,

$$\{{}^{(1)}f, \tilde{C}_H\} = \mathcal{O}(1), \tag{4.43}$$

which is satisfied if the constaints $\tilde{C}_H$ do not have a first order term ${}^{(1)}\tilde{C}_H$, however they may have a zeroth order term. For the constraints $\tilde{C}_I$ we will assume that the zeroth order terms vanish and that the first order terms do not vanish

$$ {}^{(0)}\tilde{C}_I = 0, \qquad {}^{(1)}\tilde{C}_I \neq 0 \tag{4.44}$$

Rewritting the Series Solution

$$\sum_{K_1}\{f, \tilde{C}_{K_1}\}(\tau^{K_1} - T^{K_1}) = \sum_{H_1\in\mathcal{H}}\{f, \tilde{C}_{H_1}\}(\tau^{H_1} - T^{H_1}) + \sum_{I_1\in\mathcal{I}}\{f, \tilde{C}_{I_1}\}(\tau^{I_1} - T^{I_1}) \tag{4.45}$$

The next term in the series expansion (??)

$$\begin{aligned}
& \sum_{K_1}\sum_{K_2}\{\{f, \tilde{C}_{K_1}\}, \tilde{C}_{K_2}\}(\tau^{K_1} - T^{K_1})(\tau^{K_2} - T^{K_2}) \\
= \ & \sum_{K_1}(\tau^{K_1} - T^{K_1})\Big(\sum_{H_2}\{\{f, \tilde{C}_{K_1}\}, \tilde{C}_{H_2}\}(\tau^{H_2} - T^{H_2}) + \\
& \sum_{I_2}\{\{f, \tilde{C}_{K_1}\}, \tilde{C}_{I_2}\}(\tau^{I_2} - T^{I_2})\Big) \\
\simeq \ & \sum_{H_1}\sum_{H_2}\{\{f, \tilde{C}_{H_1}\}, \tilde{C}_{H_2}\}(\tau^{H_1} - T^{H_1})(\tau^{H_2} - T^{H_2}) \\
+ \ & 2\sum_{H_1}\sum_{I_1}\{\{f, \tilde{C}_{H_1}\}, \tilde{C}_{I_1}\}(\tau^{H_1} - T^{H_1})(\tau^{I_1} - T^{I_1}) \\
+ \ & \sum_{I_1}\sum_{I_2}\{\{f, \tilde{C}_{I_1}\}, \tilde{C}_{I_2}\}(\tau^{I_1} - T^{I_1})(\tau^{I_2} - T^{I_2})
\end{aligned} \tag{4.46}$$

where we used that

$$\{\{f,\tilde{C}_{I_1}\},\tilde{C}_{H_1}\} \simeq \{\{f,\tilde{C}_{H_1}\},\tilde{C}_{I_1}\}$$

We can write (4.46) as

$$\begin{aligned}&\sum_{s=0}^{r=2}\frac{2!}{(2-s)!s!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_{(2-s)}}\}\,,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_s}\}\times\\ &\quad(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_{(2-s)}}-T^{H_{(2-s)}})\times(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_s}-T^{I_s})\end{aligned} \tag{4.47}$$

as should be checked. We have in general

$$\begin{aligned}&\{\cdots\{f,\tilde{C}_{K_1},\cdots,\},\tilde{C}_{K_q}\}(\tau^{K_1}-T^{K_1})\cdots(\tau^{K_q}-T^{K_q})\\ &\quad\simeq\sum_{s=0}^{r}\frac{r!}{(r-s)!s!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_{(r-s)}}\}\,,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_s}\}\times\\ &\quad(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_{(r-s)}}-T^{H_{(r-s)}})\times(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_s}-T^{I_s})\end{aligned} \tag{4.48}$$

where we have used that we can rearrange the constraints in any order.

$$\begin{aligned}F_{[f;T^K]}(\tau) \;\simeq\; &\sum_{r=0}^{\infty}\sum_{s=0}^{r}\frac{1}{(r-s)!s!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_{(r-s)}}\}\,,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_s}\}\times\\ &(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_{(r-s)}}-T^{H_{(r-s)}})\times(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_s}-T^{I_s})\end{aligned} \tag{4.49}$$

$$\begin{aligned}F_{[f;T^K]}(\tau) \;\simeq\; &\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{p!q!}\{\cdots\{f,\tilde{C}_{H_1},\cdots\},\tilde{C}_{H_q}\}(\tau^{H_1}-T^{H_1})\cdots(\tau^{H_q}-T^{H_q})\,,\tilde{C}_{I_1}\},\cdots\},\tilde{C}_{I_p}\}\\ &(\tau^{I_1}-T^{I_1})\cdots(\tau^{I_p}-T^{I_p})\end{aligned} \tag{4.50}$$

(recall that $\{T^{H_r},\tilde{C}_{I_s}\}=0$).

0th Order Complete Obs. Associated to a 0th order Function: Symmetry Reduced Sector

Now set the parameters τ^I to zero. Then for zeroth order complete observable associated to with a zeroth order function ${}^{(0)}f$ we have

$$
\begin{aligned}
{}^{(0)}F_{[f;T^K]}(\tau^H, \tau^I = 0) &\simeq \sum_{q=0}^{\infty} \frac{1}{q!}\, {}^{(0)}(\{\cdots\{{}^{(0)}f, \tilde{C}_{H_1}\}, \cdots\}, \tilde{C}_{H_q}\})(\tau^{H_1} - T^{H_1}) \cdots (\tau^{H_q} - T^{H_q}) \\
&\simeq \sum_{q=0}^{\infty} \frac{1}{q!} \{\cdots\{{}^{(0)}f, {}^{(0)}\tilde{C}_{H_1}\}, \cdots\}, {}^{(0)}\tilde{C}_{H_q}\}(\tau^{H_1} - T^{H_1}) \cdots (\tau^{H_q} - T^{H_q})
\end{aligned}
\tag{4.51}
$$

where we only kept the $p = 0$ term and the second equation holds because of our assumption that the constraints $\tilde{C}_H$ to have vanishing first order parts. There only appear zeroth order variables in the second line (T^H are zeroth order), hence we can say that the zeroth order complete complete observables associated with a zeroth order function are complete observables of the symmetry reduced sector.

2nd Order Complete Obs. Associated to a 0th order Function: Backreaction

The next higher order correction to this complete observable is a second order term and can be considered as the correction (backreaction) term to the dynamics of the reduced symmetry sector due to deviations from symmtry (in the inital values).

1st Order Complete Obs. Associated to a 1st order Function: Propagation of Linear Perturbations

One can also consider for instance the first order complete observable associated to a first order function. As we will see these observables decribe the proagation of linear perturbations (which are linearly gauge invariant) on the symmetry reduced sector.

Gauge Invaraint Observables to any Order k

Note that this approach allows to find gauge invariant observables to any order k by omitting in the serie for the complete observables all terms higher order than k. For this assumptions we made on the clocks $\{\{T_H\}_{H\in\mathcal{H}}, \{T_I\}_{I\in\mathcal{I}}\}$ and the constraints $\{\{\tilde{C}_H\}_{H\in\mathcal{H}}, \{\tilde{C}_I\}_{I\in\mathcal{I}}\}$ are not strictly necessary.

Standard Perturbative Calculations

However we will see that with these conditions the computation of complete is similar to the usual perturbative calculations involving the "free" propagation of perturbations and their interaction as well as the interaction of the zeroth order variables with the perturbations.

4.7 Application to Cosmology

To set up linear metric perturbations [??], one perturbs the background metric

$$ds^2 = a^2(\eta)(-d\eta^2 + \delta_{ab}dx^a dx^b), \tag{4.52}$$

here chosen as a flat isotropic metric written in conformal time η and with spatial coordinates x^a. There are initially ten perturbation functions for the ten metric components, but some of them can be absorbed simply by redefining coordinates. The remaining functions, in gauge-invariant combinations, comprise scalar, vector and tensor modes. We are here primarily interested in scalar modes which in longitudinal gauge lead to a perturbed metric

Perturbed canonical variables: Ashtekar variables [??] due to their transformation properties. First, one introduces a co-triad e_a^i instead of the spatial metric q_{ab}, related to it by $e_a^i e_b^i = q_{ab}$. (Unlike the position of spatial indices $a, b, \ldots$, the upper or lower positions of indices i are not relevant, and summing over i is understood even though it appears twice in the same position.) An oriented co-triad contains the same information as a metric but has more components as it is not a symmetric tensor. This corresponds to freedom one has in rotating the triple of triad co-vectors which does not change the metric. Not being of geometrical relevance, this freedom is removed in a canonical formalism by implementing the Gauss constraint introduced below. By inverting the matrix (e_a^i), one obtains the triad e_i^a, a set of vector fields related to the inverse metric by $e_i^a e_i^b = q^{ab}$. Just as the metric determines a compatible Christoffel connection Γ_{ab}^c, a triad determines a compatible spin connection

$\Gamma_a^i = -\epsilon^{ijk} e_j^b (\partial_{[a} e_{b]}^k + \frac{1}{2} e_k^c e_a^l \partial_{[c} e_{b]}^l)$.

The configuration varibles are given by a (complex) connection $\{A_a^j\}_{j,a=1}^3$:

$$A_a^i = \Gamma_a^j + \beta K_a^j. \tag{4.53}$$

Recall the Poisson bracket between the phase space variables

$$\{{A_a}^j(\sigma), {E^b}_k(\sigma')\} = \kappa \delta_k^j \, \delta_a^b \delta(\sigma, \sigma') \tag{4.54}$$

where $\kappa = 8\pi G_N/c^3$ is the gravitational coupling constant. Furthermore we have a scalar field φ and its conjugated momentum π which satisfy the commutation relation

$$\{\varphi(\sigma), \pi(\sigma')\} = \gamma\ \delta(\sigma, \sigma') \tag{4.55}$$

Expanding variables

We will expand the canonical variables around homogeneous and isotropic field configurations in the following way:

$$\begin{aligned} A_a{}^j(\sigma) &= A\beta\delta_a^j + a_a{}^b\beta\delta_b^j \quad , \qquad & E^a{}_j(\sigma) &= E\beta^{-1}\delta_a^j + e^a{}_b(\sigma)\beta^{-1}\delta_b^j \\ \varphi(\sigma) &= \Phi + \phi(\sigma) \quad , \qquad & \pi(\sigma) &= \Pi + \rho(\sigma) \end{aligned} \tag{4.56}$$

The Poisson brackets between the homogeneous variables and between the fluctuation variables can be found by using the (4.3.4)

$$\begin{aligned} A\beta = \mathcal{P}\cdot A_a{}^j &:= \frac{1}{3}\int_\Sigma \delta_j^a A_a{}^j\ \mathbf{d}\sigma \quad , \qquad & E\beta^{-1} = \mathcal{P}\cdot E^a{}_j &:= \frac{1}{3}\int_\Sigma \delta_a^j E^a{}_j\ d\sigma \\ \Phi = \mathcal{P}\cdot\varphi &:= \int_\Sigma \varphi\ \mathbf{d}\sigma \quad , \qquad & \Pi = \mathcal{P}\cdot\pi &:= \int_\Sigma \pi\ \mathbf{d}\sigma \end{aligned} \tag{4.57}$$

Working's outs.

It ensures that the kinematics of the symmetry reduced system and of the symmetry reduced sector embedded into the full phase space coincide. [**?**]

Working's outs.

Show that A and E are real if evaluated on a homogeneous cosmology with flat slicing.

Proof:

$$A_a^j = \Gamma_a^j + \beta K_a^j$$

Flat slicing means that the extrinsic curvature K_{ab} vainshes which implies $K_a^j = 0$.

Working's outs.

Prove $\{A, E\}$

$$
\begin{aligned}
\{A,E\} &= \{\mathcal{P}\cdot {A_a}^j, \mathcal{P}\cdot {E^a}_k\} \\
&= \frac{1}{9}\,\delta^a_j\,\delta^k_b\,\{\int_\Sigma {A_a}^j(\sigma)\,\mathbf{d}\sigma, \int_\Sigma {E^b}_k(\sigma')\,d\sigma'\} \\
&= \frac{1}{9}\,\delta^a_j\,\delta^k_b\int_\Sigma\int_\Sigma \{{A_a}^j(\sigma), {E^b}_k(\sigma')\}\,\mathbf{d}\sigma\,\mathbf{d}\sigma' \\
&= \frac{1}{9}\kappa\,(\delta^a_j\,\delta^k_b)\,(\delta^j_k\,\delta^b_a)\int_\Sigma\int_\Sigma \delta(\sigma,\sigma')\,\mathbf{d}\sigma\,\mathbf{d}\sigma' \\
&= \frac{1}{9}\delta^a_k\delta^k_a\int_\Sigma \mathbf{d}\sigma \\
&= \frac{1}{3}\kappa
\end{aligned} \tag{4.58}
$$

where we used $\int_\Sigma \mathbf{d}\sigma = 1$.

Now $\{\Phi,\Pi\} = \gamma$

$$
\begin{aligned}
\{\Phi,\Pi\} &= \{\mathcal{P}\cdot\varphi, \mathcal{P}\cdot\pi\} \\
&= \int_\Sigma\int_\Sigma \{\varphi(\sigma),\pi(\sigma')\}\,\mathbf{d}\sigma\,\mathbf{d}\sigma' \\
&= \int_\Sigma\int_\Sigma \gamma\delta(\sigma,\sigma')\,\mathbf{d}\sigma\,\mathbf{d}\sigma' \\
&= \gamma
\end{aligned} \tag{4.59}
$$

Now $\{\phi(\sigma),\rho(\sigma')\} = \gamma\delta(\sigma,\sigma') - \gamma$

$$
\begin{aligned}
\{\phi(\sigma),\rho(\sigma')\} &= \{\varphi(\sigma)-\Phi, \pi(\sigma')-\Pi\} \\
&= \{\varphi(\sigma),\pi(\sigma')\} + \{\Phi,\Pi\} - \{\varphi(\sigma),\Pi\} - \{\Phi,\pi(\sigma')\} \\
&= \gamma\delta(\sigma,\sigma') + \gamma - \{\varphi(\sigma), \int_\Sigma \pi(\sigma')\,\mathbf{d}\sigma'\} - \{\int_\Sigma \varphi(\sigma)\,\mathbf{d}\sigma, \pi(\sigma')\} \\
&= \gamma\delta(\sigma,\sigma') - \gamma
\end{aligned} \tag{4.60}
$$

Now $\{{a_a}^b(\sigma), {e^c}_d(\sigma')\}$

From equation ()

$$
{a_a}^c(\sigma)\delta^j_c = \beta^{-1}{A_a}^j(\sigma) - A\delta^j_a
$$

affecting δ^b_j to both sides of this we find

$$
{a_a}^b(\sigma) = \beta^{-1}{A_a}^j\delta^b_j - A\delta^b_a.
$$

From a similar calculation we find for ${e^a}_b(\sigma)$

$$
{e^a}_b(\sigma) = \beta {E^a}_j(\sigma)\delta^j_b - E\delta^a_b.
$$

$$
\begin{aligned}
\{{a_a}^b(\sigma), {e^c}_d(\sigma')\} &= \{\beta^{-1}{A_a}^j(\sigma)\delta^b_j - A\delta^b_a \ , \ \beta {E^c}_j(\sigma')\delta^j_d - E\delta^a_b\} \\
&= \text{and so on...}
\end{aligned} \tag{4.61}
$$

Fourier transforms

(i) Show the homogeneous variables are given by the ($\frac{1}{3}\times$ trace of the) k modes of the fields.

(ii) Show the Poisson brackets for the Fourier modes of the fluctuation variables are

$$
\begin{aligned}
\{a_{ab}(k), e^{cd}(k')\} &= \kappa\delta^c_a\delta^d_b\delta_{k,-k'} - \frac{\kappa}{3}\delta_{ab}\delta^{cd}\delta_{k,0}\delta_{k',0} \\
\{\phi(k), \rho(k')\} &= \gamma\delta_{k,-k'} - \gamma\delta_{k,0}\delta_{k',0}.
\end{aligned} \tag{4.62}
$$

Proof:

$$
f(k) = \int_\Sigma \exp(ik\cdot\sigma) f(\sigma)\mathbf{d}\sigma
$$

where $k\cdot\sigma := k_a\sigma^a$. The inverse transform is

$$
f(\sigma) = \sum_{k\in\{2\pi\mathbb{Z}^3\}} \exp(ik\cdot\sigma) f(k).
$$

$$
\begin{aligned}
{A_a}^j(k) &= \int_\Sigma \exp(ik\cdot\sigma){A_a}^j(\sigma)\ \mathbf{d}\sigma \\
&= \beta\delta^j_a A\int_\Sigma \exp(ik\cdot\sigma)\ \mathbf{d}\sigma + \beta\int_\Sigma \exp(ik\cdot\sigma){a_a}^j(\sigma)\ \mathbf{d}\sigma \\
&= \beta\delta^j_a A\delta_{k,0} + \beta {a_a}^j(k)
\end{aligned} \tag{4.63}
$$

$$
\begin{aligned}
\frac{1}{3}\delta^a_j {A_a}^j(k) &= \frac{1}{3}\beta\delta^a_j\delta^j_a A + \frac{1}{3}\beta\delta^a_j\delta^j_b {a_a}^b(k) \\
&= \beta A + \beta {a_a}^a(k) \\
&= \beta A
\end{aligned} \tag{4.64}
$$

$$
\begin{aligned}
\frac{1}{3}\delta_a^j E^a{}_j(k) &= \frac{1}{3}\beta^{-1}\delta_a^j\delta_j^a E + \frac{1}{3}\beta^{-1}\delta_a^j\delta_j^b e^a{}_b(k) \\
&= \beta^{-1}E + \beta^{-1}e^a{}_a(k) \\
&= \beta^{-1}E \qquad (4.65)
\end{aligned}
$$

(i)

$$
\begin{aligned}
\{\phi(k),\rho(k')\} &= \{\int_\Sigma \exp(ik\cdot\sigma)\phi(\sigma)\mathbf{d}\sigma, \int_\Sigma \exp(ik'\cdot\sigma')\rho(\sigma)\mathbf{d}\sigma'\} \\
&= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\{\phi(\sigma),\rho(\sigma')\}\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\,\big(\gamma\delta(\sigma,\sigma') - \gamma\big)\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \gamma\int_\Sigma \exp(i(k-k')\cdot\sigma)\mathbf{d}\sigma - \gamma\int_\Sigma \exp(ik\cdot\sigma)\,\mathbf{d}\sigma\int_\Sigma \exp(ik'\cdot\sigma')\,\mathbf{d}\sigma' \\
&= \gamma\delta_{k,-k'} - \gamma\delta_{k,0}\delta_{k',0} \qquad (4.66)
\end{aligned}
$$

as $\int_\Sigma \exp(ik\cdot\sigma) = 0$ when $k \neq 0$ and is equal to 1 when $k = 0$.

(ii)

$$
\begin{aligned}
\{a_{ab}(k), e^{cd}(k')\} &= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\{a_{ab}(\sigma), e^{cd}(\sigma')\}\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \int_\Sigma\int_\Sigma \exp(ik\cdot\sigma + k'\cdot\sigma')\,\Big(\kappa\delta_a^c\delta_b^d(\sigma,\sigma') - \frac{\kappa}{3}\delta_{ab}\delta^{cd}\Big)\,\mathbf{d}\sigma\mathbf{d}\sigma' \\
&= \kappa\delta_a^c\delta_b^d\delta_{k,-k'} - \frac{\kappa}{3}\delta_{ab}\delta^{cd}\delta_{k,0}\delta_{k',0} \qquad (4.67)
\end{aligned}
$$

Note that the additional terms on the right hand side implement that $a_a{}^a(0) = e^a{}_a(0) = \phi(0) = \rho(0) = 0$.

Working's outs.

Fourier transformed variables can be used to define the symplectic coordinates used in section ?? in which the projection operator $\mathcal{P}$ maps part of the symplectic coordinates to zero and leaves the other coordinates invariant.

show that homogeneous part of the coordinates are given by

$$(\sqrt{3}A, \sqrt{3}E; \Phi, \Pi).$$

$$
\begin{aligned}
\mathcal{P}\cdot\sqrt{3}A &= \frac{\sqrt{3}}{3\beta}\int_\Sigma \delta^a_j\,\mathcal{P}\cdot {A_a}^j\,d\sigma \\
&= \frac{1}{\sqrt{3}\beta}\int_\Sigma \delta^a_j\left(\beta\delta^j_a A\right)d\sigma \\
&= \sqrt{3}A
\end{aligned}
\tag{4.68}
$$

$$
\begin{aligned}
\mathcal{P}\cdot\sqrt{3}E &= \frac{\sqrt{3}\beta}{3}\int_\Sigma \delta^j_a\,\mathcal{P}\cdot {E^a}_j\,d\sigma \\
&= \frac{\beta}{\sqrt{3}}\int_\Sigma \delta^a_j\left(\beta^{-1}\delta^a_j E\right)d\sigma \\
&= \sqrt{3}A
\end{aligned}
\tag{4.69}
$$

$$
\begin{aligned}
\mathcal{P}\cdot\Phi &= \int_\Sigma \mathcal{P}\cdot\varphi\,d\sigma \\
&= \int_\Sigma \Phi\,d\sigma \\
&= \Phi
\end{aligned}
\tag{4.70}
$$

$$
\begin{aligned}
\mathcal{P}\cdot\Pi &= \int_\Sigma \mathcal{P}\cdot\pi\,d\sigma \\
&= \int_\Sigma \Pi\,d\sigma \\
&= \Phi
\end{aligned}
\tag{4.71}
$$

The symplectic pairs that are mapped to zero are given by $(a_{ab}(k), e^{ab}(-k))$ and $(\phi(k), \pi(-k))$ for $k \neq 0$ and

$$
\begin{aligned}
\mathcal{P}\cdot a_{ab}(k) &= \beta^{-1}\mathcal{P}\cdot {A_a}^j\delta^b_j - \mathcal{P}\cdot A\delta^b_a \\
&= \beta^{-1}(\beta A)\delta^b_j -
\end{aligned}
\tag{4.72}
$$

$$
{e^a}_b(\sigma) = \beta {E^a}_j(\sigma)\delta^j_b - E\delta^a_b.
$$