Chapter 6

Consistent Discrete Classical GR

Given data on an initial time slice which fulfill the constaint equations, will the constraint equations hold on later time slices? That is, do the evolution equations preserve the constraint equations?

The usual starting point of numerical general relativity are: the six evolution equations for h_{pq} and K_{pq}

$$\dot{h}_{pq} = 2NK_{pq} + \mathcal{L}_{N^p} h_{pq} \tag{6.1}$$

$$\dot{K}_{pq} = N_{|pq} - N(R_{pq} + KK_{pq} - 2K^r{}_p K_{qr}) + \mathcal{L}_{N^p} K_{pq}. \tag{6.2}$$

and the four constraint equations which put conditions on the initial data:

$$\begin{aligned} C &= R + K^2 - K^{ab} K_{ab} = 0 \\ C_m &= \nabla^a (K_{am} - K q_{am}) = 0. \end{aligned} \tag{6.3}$$

The discrete formulations of constrained systems are often *inconsistent*. The discrete equations you get cannot be solved simultaneously. If you solve the constraint equations at the beginning they will fail to be solved when you evolve according to the discrete evolution equations, so the discretized evolution equations produce solutions in the future that do not satisfy the discretized constraints. This is a well known problem in numerical relativity. The discrete theory is also inconsistent with regards to the Poisson bracket algebra, the discrete versions of the constraints fail to close as an algebra.

Pullin et al developed a genral technique allowing to define consistent discrete theroies. One can define a consistent discrete theory for general relativity. What you do is you discretize the **action** of the theory and then you work out a canonical theory for the discrete action. Instead of just taking the EQMs and disretizing them discretize the action. Since derrived from an action, they are going to be consistent!

The lapse and shift are not free but are determined by imposing the preservation of constraints.

The resulting theory is different from GR, yet it will generically include solutions that approximate continuum general relativity very well.

For cosmological models the generic behaviour far from the big bang approximates very well the continuum.

$$L(n,n+1) \equiv L(q_n,q_{n+1}) \equiv \epsilon \hat{L}(q,\dot{q}) \tag{6.4}$$

$$q + q_0 \quad \text{and} \quad \dot{q} \equiv \frac{q_{n+1}-q_n}{\epsilon}. \tag{6.5}$$

$$S = \sum_{n=0}^{N} L(q_n,q_{n+1}) \tag{6.6}$$

$$\frac{\partial S}{\partial q_n} = \frac{\partial L(q_{n-1},q_n)}{\partial q_n} + \frac{\partial L(q_n,q_{n+!})}{\partial q_n} = 0. \tag{6.7}$$

$$p_{n+1} = \frac{\partial L(q_n,q_{n+!})}{\partial q_{n+1}}, \quad p_n = -\frac{\partial L(q_n,q_{n+!})}{\partial q_n}. \tag{6.8}$$

Relations that define a type I canonical transformation

$$L(Q_n,q_{n+1}) = m\frac{(q_{n+1}-q_n)^2}{2\epsilon} - V(q_n)\epsilon \tag{6.9}$$

$$\begin{aligned}
p_{n+1} &= m\frac{(q_{n+1}-q_n)}{\epsilon} \\
p_n &= m\frac{(q_{n+1}-q_n)}{\epsilon} + V'(q_n)\epsilon \\
q_{n+1} &= q_n + \frac{p_n}{m}\epsilon - V'(q_n)\frac{\epsilon}{m} \\
p_{n+1} &= p_n - V'(q_n)\epsilon.
\end{aligned} \tag{6.10}$$

$$U = \exp\left(i\frac{V(q_{i-1})\epsilon}{\hbar}\right)\exp\left(i\frac{p_{i-1}^2\epsilon}{2m\hbar}\right) = \exp H. \tag{6.11}$$

$$\mathcal{H} = \epsilon\left(\frac{p^2}{2m} + \frac{1}{2m\omega^2 q^2} - \frac{1}{2\epsilon p q\omega^2}\right) f(\epsilon) \tag{6.12}$$

Canonical formulation for constrained discrete dynamical systems

$$L(n, n+1) = p_n(q_{n+1} - q_n) - \epsilon\mathcal{H}(q_n, p_n) - \lambda_{nB}\phi^B(q_n, p_n) \tag{6.13}$$

$$\begin{aligned}
P^q_{n+1} &= \frac{\partial L(n,n+1)}{\partial q_{n+1}} = p_n, \quad P^q_n = -\frac{\partial L(n,n+1)}{\partial q_n} = p_n + \epsilon\frac{\partial \mathcal{H}(q_n,p_n)}{\partial q_n} + \lambda_{nB}\frac{\partial \phi^B(q_n,p_n)}{\partial q_n} \\
P^p_{n+1} &= \frac{\partial L(n,n+1)}{\partial p_{n+1}} = 0, \quad P^p_n = -\frac{\partial L(n,n+1)}{\partial p_n} = -(q_{n+1} - q_n) + \epsilon\frac{\partial \mathcal{H}(q_n,p_n)}{\partial p_n} + \lambda_{nB}\frac{\partial \phi^B(q_n,p_n)}{\partial p_n} \\
P^{\lambda_B}_{n+1} &= \frac{\partial L(n,n+1)}{\partial \lambda_{(n+1)}B} = 0, \quad P^{\lambda_B}_n = \phi^B(q_n, p_n).
\end{aligned} \tag{6.14}$$

$$\phi^B(q_n, P^q_{n+1}) = 0. \tag{6.15}$$

$$\phi^B(q_{n+1}, P^q_{n+1}, \lambda_{nB}) = 0, \tag{6.16}$$

$$\lambda_{nB} = \lambda_{nB}(q_{n+1}, P^q_{n+1}, v^\alpha) \tag{6.17}$$

The final evolution equations are obtained by substituting the Lagrangian multipliers.

Notice that here the Lagrange multipliers were determined without imposing any gauge fixing. Notice that more precisely what has been determined is "$\lambda \times \epsilon$".

For a completely parametrized theory is no explicit dependence on ϵ, which may be fixed arbitrarily. Once the time interval (or the lattice spacing) is chosen, lapse is determined.

6.0.1 "Dirac's" canonical approach to general discrete systems

They have recently developed a "Dirac's" canonical approach to general discrete systems.

$$\begin{aligned}
L(n, n+1) &\equiv L(q_n, q_{n+1}) \\
{P^q}_{n+1}{}^a &= \frac{\partial L(n,n+1)}{\partial {q^a}_{n+1}} \\
{P^q}_n{}^a &= -\frac{\partial L(n,n+1)}{\partial {q^a}_n}
\end{aligned} \tag{6.18}$$

$$\left|\frac{\partial^2 L(n,n+1)}{\partial q^b{}_{n+1}\partial q^a{}_n}\right| = 0. \tag{6.19}$$

Primary constraints

$$\Phi_A(q^a{}_n, P^q{}_n{}^a) = 0 \tag{6.20}$$

$$q^a_{n+1} = f^a(q^b{}_n, P^q{}_n{}^b, V^A, U^A) \tag{6.21}$$

V and U arbitrary functions. Consistency:

$$\Phi^A(q^a{}_{n+1}, P^q{}_{n+1}{}^a) = \Phi_A(f^a{}_n, \frac{\partial L(q_n, f^a)}{\partial q^a{}_{n+1}}) = 0 \tag{6.22}$$

$$C(q_n, P^q_n) \qquad V^A = V^A(q_n, P^q_n, v^\alpha, u^\rho) \tag{6.23}$$

$$q^a_{n+1} = f^a(q^b{}_n, P^q{}_n{}^b, V^A(q, P^q_n, v, u), U^A(q, P^q_n, v, u)) = \tilde{f}^a(q^b_n, P^{qb}_n, v^\alpha, u^\rho) \tag{6.24}$$

$$P^q{}_{n+1}{}^b = g^a(q^b{}_n, P^q{}_n{}^b, v^\alpha, u^\rho). \tag{6.25}$$

By using a Type II, II, or IV transformation one can show that this evolution is canonical, preserves the Poisson brackets and the canstraint surface. This is equivalent to what happens in the continuum, consistency may be achieved by determing the complete constraint surface and a total Hamiltonian that preserves the Poisson structure.

Finally one can recognise the second class constraints and impose them strongly. While some symmetries of the continuum are broken the discretization others are preserved.