Chapter 7

Quantum Field Theory on Curved Spacetimes

7.1 Motivation

Quantum Field Theory on Curved Spacetimes is an approximate theory in which quantum matter fields are assumed to propagate in a fixed classical background gravitational field.

Matter fields are quantum mechanical, we would wish to understand quantum effects in gravity.

Gravity is classical whereas the world is fundamentally quantum mechanical.

Must recover Quantum Field Theory on a curved spacetime from a theory of quantum gravity to be a viable theory.

Results such as Hawking radiation give hints and consistency tests for a theory of quantum gravity.

Studing Quantum Field Theory on curved spacetimes is a first step toward quantum gravity.

7.2 Introduction

Except for issues related to back-reaction the dynamics of the metric is not considered.

Classical mechanics admits two different kinds of formulation: Hamiltonian and Lagrangian. So does quantum mechanics. The one corresponding to the Hamiltonian formulation of classical mechanics is called canonical quantisation and involves a Hilbert space and operators.

The basic idea of canonical quantisation is

(i) to take the states of the system to be described by wavefunctions $\Psi(q)$ of the configuration variables,

(ii) to replace each momentum variable p by differentiation with respect to the conjugate configuration variable:

$$p \to -i\hbar \frac{\partial}{\partial q},$$

and

(iii) to determine the time evolution of Ψ via the Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{\mathcal{H}}\Psi,$$

where $\hat{\mathcal{H}}$ is an operator corresponding to the classical Hamiltonian $\mathcal{H}(p,q)$.

7.3 Quantum Field Theory in Flat Spacetime

7.3.1 The Simple Harmonic Oscillator

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{m}\hat{q}^2 \tag{7.1}$$

We define two non-Hertitian operators, which are called the annihilation and creation operators, respectively

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{q} + \frac{i}{m\omega}\hat{p}\right) \tag{7.2}$$

$$\hat{a}^\dagger = \sqrt{\frac{m\omega}{2}}\left(\hat{q} - \frac{i}{m\omega}\hat{p}\right) \tag{7.3}$$

It is straightforward to show that $[\hat{q}, \hat{p}] = i$, that

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{7.4}$$

define the number operator

$$\hat{N} = \hat{a}^\dagger \hat{a} \tag{7.5}$$

$$\hat{N}|n >= n|n > \tag{7.6}$$

$$\hat{N}\hat{a}^{\dagger}|n>=$$

7.3.2 Quantisation of the Klein-Gordon field in Flat Spacetime

Consider a free, real Klein-Gordon field in flat spacetime,

$$\partial^a \partial_a \phi - m^2 \phi = 0. \tag{7.7}$$

We decompose the field ϕ via a Fourier transform into a series of modes of spacial wave vector $\vec{k}$, so that the amplitude of each mode satisfies the same equation as a classical harmonic oscillator and then treat each mode by the rules of ordinary quantum mechanics. It is convenient to imagine the field is in a cubic box of side L with periodic boundary conditions. We can then decompose ϕ as a Fourier series in terms of the modes

$$\phi_k := L^{-3/2} \int e^{-ik\cdot x} \phi(t,x) d^3x \tag{7.8}$$

where

$$k = \frac{2\pi}{L}(n_1, n_2, n_3).$$

The Hamiltonian of the system is then given by

$$\mathcal{H} = \sum_k \frac{1}{2}(|\dot{\phi}_k| + \omega_k^2 |\phi_k|) \tag{7.9}$$

where

$$\omega_k^2 = |k|^2 + m^2. \tag{7.10}$$

Thus, a free Klein-Gordon field, ϕ, in flat spacetime is explicitly seen to be simply an infinite collection of decoupled harmonic oscillators.

In flat spacetime we are able to pick out a natural set of modes by demanding that they be positive frequency with respect to the time coordinate. The time coordinate is not unique, since we are free to perform Lorentz transformations; but the vacuum state and total number operators are invariant under such transformations. Thus, every inertial observer will agree on what is the vacuum state, and how many particles there are.

7.3.3 Mode Expansion

$$\hat{\phi}_k(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{ik\cdot x} \hat{\phi}_k(t) \tag{7.11}$$

$$\hat{\phi}_k(t) = \frac{1}{\sqrt{2\omega_k}} (\hat{a}_k^- e^{-i\omega_k t} + \hat{a}_{-k}^+ e^{i\omega_k t}) \tag{7.12}$$

substituting this into ()

$$\hat{\phi}_k(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_k}} (e^{-i\omega_k t + ik\cdot x} \hat{a}_k^- + e^{i\omega_k t - ik\cdot x} \hat{a}_{-k}^+) \tag{7.13}$$

where we replaced $-k$ by k in the second term.

In fact the sum in equation () does not converge pointwise and must interprted in a distributioal sense, i.e., $\hat{\phi}$ can be defined only as an operator-valued distribution on spacetime.

We take $\phi(f)$

$$\phi(f) = \int_{\mathbb{R}^4} \phi(x) f(x) dx$$

Bohr and Rosenfeld pointed out that it is impossible to measure the electric field strengh at a point because of the uncertainty principle. Thus from a physical point of view it is reasonable to consider the smeared field $\phi(f)$. We will see in the next chapter that for the entropy of a black hole to be acounted for by the statistical mechanical origin the necessary degrees of freedom must be quantum mechanical in origin, which can only be so if quantum fields have arbitrary discontinuities.

7.3.4 Behaviour of Fock Basis Under Lorentz Transformations

Our ability to find positive- and negative-frequency solutions can be traced to the existence of a timelike Killing vector in Minkowski spacetime.

7.3.5 Fock Particles

Energy eigenstates are collections particleswith definite momenta.

The Poincare group plays an important role in the particle interpretation of states of the field. Wigner showed that the above states are the irreducible representations of the Poicare group in the QFT state space. The defining properties of the particles, mass and spin (helicity), are indeed the invariants of the Poincare group.

On one hand, a particle is a local object detected by a local apparatus, such as a photoelectric detector or a high-energy-experiment bubble chamber. On the other hand, the n-particles states of quantum fiels theory, namely the eigenstates of the particle-number operator in Fock space, have a non-local character.

7.4 Quantum Field Theory on Curved Spacetimes

Note that () has the form

$$\hat{\phi}(t,x) = \sum_i (v_i(t,x)a_i + v_i(t,x)a_i^\dagger) \tag{7.14}$$

where $\{u_i\}$ is an orthonormal basis of $\mathcal{H}$.

In flat space at this point we would now introduce a set of positive and negative frequency modes forming a complete basis for solutions of the equations of motion, expand the field operator $\hat{\phi}$ in terms of these modes, and interpret the operator coefficients as creation and annihilation operators.

7.4.1 Bogolyubov Transformations

The functions

$$u_k(t) = \alpha_k v_k(t) + \beta_k v_k^*(t), \tag{7.15}$$

where α_k and β_k are time-independent complex coefficients also satisfy the equations of motion.

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{7.16}$$

In terms of the mode functions $u_k(t)$ the field operator expansion takes the following form,

$$\hat{\phi}(x,t) = \frac{1}{\sqrt{2}} \int \frac{d^3k}{(2\pi)^{2/3}} \left[e^{ik\cdot x} u_k^*(t) \hat{b}_k^- + e^{-ik\cdot x} u_k(t) \hat{b}_k^+ \right] \tag{7.17}$$

The two expressions for the field operator $\hat{\phi}(x,t)$ agree only if

$$u_k^*(t)\hat{b}_k^- + u_k(t)\hat{b}_{-k}^+ = v_k^*(t)\hat{a}_k^- + v_k(t)\hat{a}_{-k}^+$$

$$
\begin{aligned}
u_k^*(t)\hat{b}_k^- + u_k(t)\hat{b}_{-k}^+ &= (\alpha_k^* v_k^*(t) + \beta_k^* v_k(t))\hat{b}_k^- + (\alpha_k v_k(t) + \beta_k v_k^*(t))\hat{b}_{-k}^+ \\
&= v_k^*(t)(\alpha_k^*\hat{b}_k^- + \beta_k\hat{b}_{-k}^+) + v_k(t)(\alpha_k\hat{b}_{-k}^+ + \beta_k^*\hat{b}_k^-)
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}_k^- &= \alpha_k^*\hat{b}_k^- + \beta_k\hat{b}_{-k}^+ \\
\hat{a}_k^+ &= \alpha_k\hat{b}_k^+ + \beta_k^*\hat{b}_{-k}^-
\end{aligned}
\tag{7.18}
$$

7.4.2 a-Particles in the b-Vacuum

The a-particle number operator is

$$
\hat{n}_k^{(a)} = \hat{a}_k^+\hat{a}_k^- .
$$

$$
\begin{aligned}
< 0_{(b)}|\hat{n}_k^{(a)}|0_{(b)} > &= < 0_{(b)}|\hat{a}_k^+\hat{a}_k^-|0_{(b)} > \\
&= < 0_{(b)}|(\alpha_k\hat{b}_k^+ + \beta_k^*\hat{b}_{-k}^-)(\alpha_k^*\hat{b}_k^- + \beta_k\hat{b}_{-k}^+)|0_{(b)} > \\
&= < 0_{(b)}|(\beta_k^*\hat{b}_{-k}^-)(\beta_k\hat{b}_{-k}^+)|0_{(b)} > \\
&= |\beta_k|^2 < 0_{(b)}|\hat{b}_{-k}^+\hat{b}_{-k}^- + \delta^3(0)|0_{(b)} > \\
&= |\beta_k|^2\delta^3(0)
\end{aligned}
\tag{7.19}
$$

The divergent factor $\delta^3(0)$ is due to the volume. The total mean density of particles is then

$$
n = \int d^3k|\beta_k|^2 \tag{7.20}
$$

If $\beta_k \neq 0$, we have mixing of positive and negative frequences leading to particle production.

7.5 Quantum Fields in an Expanding Universe

7.5.1 Particle Interpretation

The Hilbert space first makes its appearance in the attempt to define a Hilbert space for a single relativistic particle, and Fock space initially appears in an attempt to allow multiparticle states.

The notions of energy and vacuum play an important role in Minkowskian physics. However, a preferred notion of a vacuum is not needed for a quantum theory to be well defined.

7.6 Quantum Fields During Inflation

The de Sitter universe plays an important role in cosmology, it can be used as a good approximation for the stage of accelerated expansion - inflation.

7.7 The Unruh Effect

An accelerating in the Minkowski vacuum sees particles which have a thermal spectrum, with temperature being proportional to the acceleration. This is called the Unruh effect.

7.8 Hawking Radiation

In January 1974 Hawking announced that black holes emit radiation with a thermal spectrum because of quantum effects.

With all the knowledge of quantum field theory in curved spacetime we are now in a position to derive Hawking's result.

There are negative energy states inside the horizon, and therefore one of the virtual particles (inside the horizon) can have negative energy while the other one (outside the horizon) has positive energy. The second particle outside the horizon can move away from the black hole to infinity thus becoming a real particle. As a result, the black hole can emit radiation and its mass decrease.

If the spacetime were time independent, a solution of the wave equation that contained only positive frequences on $\mathcal{I}^-$ would also be of positive frequency on $\mathcal{I}^+$.

However, if the metric is time dependent during the collapse, a solution that is positive frequency on $\mathcal{I}^-$ to be partly negative frequency when it gets to $\mathcal{I}^+$.

7.8.1 Hand Wavy Calculation

In 1974, Steven Hawking showed that black holes do radiate quantum mechanically, thereby shinking in area. Only matter was treated quantum mechanically; there were no quanta of geometry.

$$\left(i\frac{\partial}{\partial x^\mu} - eA_\mu\right)^2 \psi(x) = m^2\psi(x). \tag{7.21}$$

$$i\frac{\partial\varphi}{\partial\lambda} = -\frac{1}{2}\left(i\frac{\partial}{\partial x^\mu} - eA_\mu\right)^2 \varphi \tag{7.22}$$

Because the functions A_μ are independent of λ, (7.22) has solutions of the form $\varphi(x, \lambda) = \exp(im^2/2)\psi(x)$ with satisfying (7.21).

$$\mathcal{L} = \frac{1}{2}\left(\frac{dx^\mu}{d\lambda}\right)^2 + e\frac{dx^\mu}{d\lambda}A_\mu \tag{7.23}$$

$$G(x_A, x_B) = \int_0^\infty e^{-im^2\lambda/2} G(x_A, x_B; \lambda)\, d\lambda \tag{7.24}$$

with $G(x_A, x_B; \lambda)$ given by the path integral in the Schwarzschild metric

$$G(x_A, x_B; \lambda) = \int_{x(0)=x_A}^{x(\lambda)=x_B} \mathcal{D}[x] \exp\left(i \int_0^\lambda \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\, d\lambda\right). \tag{7.25}$$

$$\mathcal{H}_s \otimes \mathcal{H}_B \tag{7.26}$$

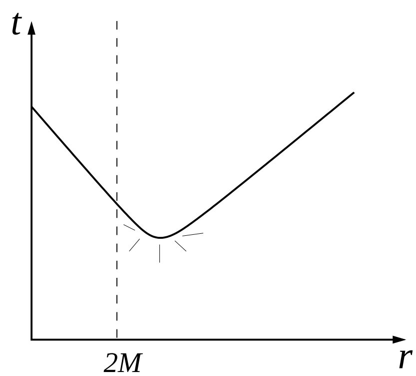

Figure 7.1: Reflects forward in time by the strong gravitaional field outside the event horizon.

Feynmann told us that we can interpret particles that going backwards in time as anti-particles (see details on page 430). So anti-particles get swallowed by the black hole leaving a particle which has a chance of escaping off to infinity. That is, the black hole evapourates particles reducing its mass.

7.8.2 Hawking's Calculation

We assume that in the remote past the spacetime is approximately flat. In which case the standard Minkowski space vacuum state describes the situation. Positive frequency modes are partially converted into negative frequency modes near the horizon (pair creation). Whereas positive frequency modes scatter off the horizon, negative frequency modes are absorbed.

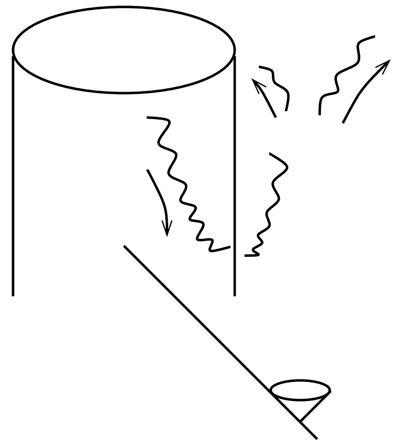

Figure 7.2: timereflection2. The antiparticle mode falling into the black hole can be interpreted as a particle travelling backwards in time, form the singularity down to the horizon.

The operators a_i and $a_i^\dagger$ are respectively regarded as the annihilation and creation operators on $\mathcal{I}^-$. The vacuum at $\mathcal{I}^-$ is thus defined as

$$a_i|0_- >= 0. \tag{7.27}$$

Similarly, in the future null infinity $\mathcal{I}^+$, the Schwarzschild metric is asymptotically flat.

The operators b_i and $b_i^\dagger$ respectively stand for the annihilation and creation operators on $\mathcal{I}^+$.

Calculation of the coefficients β_{ij}

Modes solutions of the wave equations $\Box\phi = 0$ in the Schwarzschild spacetime have the for

$$r^{-1}R'(r_*)Y_{lm}(\theta,\varphi)e^{-i\omega t} \tag{7.28}$$

where $Y_{lm}(\theta,\varphi)$ is a spherical harmonic function, and where $R'(r_*)$ satisfies the equation

$$\frac{\partial^2}{\partial r_*^2}R'(r_*) + \left[\omega^2 - \frac{1}{r^2}\left\{\frac{2M}{r} + l(l+1)\right\}\left(1 - \frac{2M}{r}\right)\right]R'(r_*) = 0. \tag{7.29}$$

A solution that is positive frequency on $\mathcal{I}^-$ to be partly negative frequency when it gets to $\mathcal{I}^+$. One can calculate this mixing by taking a wave with time dependence

$$e^{-i\omega u}$$

on $\mathcal{I}^+$ and propagating it back to $\mathcal{I}^-$.

Geometric optics is a good approxiamtion when the wavelength small compared with the size of structures with which the wave interacts.

Modes which had an extremely high frequency during their passage through the body, we may describe their propagation by use of geometric optics. It means that the scattering of the wavefunction by the gravitational field can be ignored.

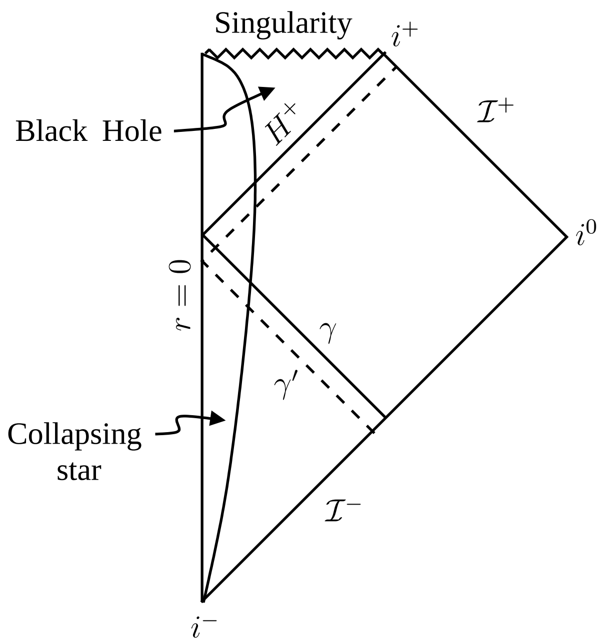

Figure 7.3: Penrose diagram of a star that collapses to form a black hole.

$$\phi(v) = \begin{cases} 0 & (v > 0) \\ \phi_0 \exp\left[\frac{i\omega}{\kappa}\ln(-\alpha v)\right] & (v < 0) \end{cases} \tag{7.30}$$

Note that although we started with the purely positive frequency mode $e^{-i\omega u}$ at $\mathcal{I}^+$, the solution at $\mathcal{I}^-$ is not purely positive frequency.

Greybody factor

$$\frac{\partial^2}{\partial r_*^2}R'(r_*) + \omega^2 R'(r_*) - V_l(r)R'(r_*) = 0. \tag{7.31}$$

where

$$V_l(r) = \frac{1}{r^2}\left\{\frac{2M}{r} + l(l+1)\right\}\left(1 - \frac{2M}{r}\right) = 0. \tag{7.32}$$

A wave escaping the black hole needs to propagate through the potential $V_l(r)$, and this decreases the intensity of the wave and changes the resulting spectrum by a greybody factor. The greybody factor is just the absorption cross section of the black hole for that mode. Thus () is precisely the formula for a perfect blackbody emmiter.

Thermal spectrum

As with all blackbody emmiters, the more energetic modes have a lower probability of emmision. From Heisenberg's principle $\Delta E \Delta t \sim h$ it follows that high energy particles have smaller lifetime, hence the particle-antiparticle pair exists for less time Δt, and therefore there is less chance of the negative energy particle to be absorbed and positive energy particle to escape from the black hole.

A complete analysis of the density matrix describing the outgoing state at infinity shows that it is identical in all aspects to a thermal density matrix at temperature ().

7.8.3 Rotating Black Holes and Higher Spin Fields

Rotating black holes

The frequency ω is replaced by the frequency with respect to the horizon Killing field

$$\partial_t + \Omega_H \partial_\phi,$$

where ∂_t and ∂_ϕ are the asymptotic time translation and rotation Killing vectors, and Ω_H is the angular velocity of the horizon.

Higher spin fields

The Hawking effect also occurs to higher spin fields, the difference being

1. the greybody factors are different;

2. for half-integer spin fields the Fermi distribution rather than the Bose distribution arises.

$$< N >= \Gamma \frac{1}{\exp[2\pi\kappa^{-1}(\omega - m\Omega - Q\Phi)] \mp 1} \tag{7.33}$$

-1 for bosons and $+1$ for fermions.

7.8.4 Black Hole Thermodynamics

The Kerr-Newmann family of solutions describes completely all the stationary black holes which can possibly occur in general relativity. All black hole solutions are completely characterised by only three externally observable classical parameters; mass M, total angular momentum J, and electric charge Q.

7.8.5 Information Loss Paradox

An initially pure quantum state before the collapsing to a blackhole and then evaporating completetly evolves into a mixed state, i.e. information gets lost.

7.9 Backreaction

The back-reaction of a classical gravitational field interacting with quantum matter fields is described by the semiclassical Einstein equation, which has the expectation value of the quantum matter fields stress tensor as a source.

$$G_{\mu\nu} = 8\pi < T_{\mu\nu} > \tag{7.34}$$

7.10 The Algebraic Approach

We can formulate quantum field theory where symmetries and "preferred states" play no role whatsoever.

By adopting the algebraic approach, one may simultaneously consider all states arising in all Hilbert space constructions of the theory without having to make a particular choice of representation from the outset.

Such an algebraic approach is also advantageous in canonical quantum gravity where there is no immediate physical reason to pick out a particular representation.

7.10.1 Algebraic Quantum Theory

7.10.2 Wightman Axioms in Minkowski Spacetime

1. The states of the theory are unit rays in a Hilbert space, $\mathcal{H}$, that carries a unitary representation of the Poincare group.

2. The 4-momentum (defined by the action of the Poincare group on the Hilbert space) is positive, i.e., its spectrum is contained within the closed future light cone ("spectrum condition").

3. There exists a unique, Poincare invariant state ("the vacuum").

4. The quantum fields are operator-valued distributions defined on a dense domain $\mathcal{D} \subset \mathcal{H}$ that is both Poincare invariant and invariant under the action of Poincare transformations.

5. At spacetime separations, quantum fields either commute or anticommute.

7.10.3 Revision

Let $\sigma(x_1, x_2)$ is the squared distance along the shortest geodesic connecting two points.

7.10.4 Operator Product Expansion

An operator product expansion (OPE) is a short-distance asymptotic formula for products of fields near point x in terms of fields defined at x. For example, for a free Klein-Gordon field in curved spacetime, we have

$$\phi(x)\phi(y) = H(x,y)\mathbf{1} + \phi^2(x) + \dots \tag{7.35}$$

where H is a locally and covariantly constructed Hadamard distribution and "..." has higher scaling degree than the other terms (i.e., it goes to zero more rapidly in the coincidence limit).

7.10.5 Algebraic Quantum Field Theory

In the algebraic approach, instead of starting with a Hilbert space of states and then defing the field observables on this Hilbert space, one starts with an algebra, $\mathcal{A}$, of observables.

7.10.6 Viewpoint on QFT

For $\mathcal{M}$, we have an algebra $\mathcal{A}(\mathcal{M})$ of local field observables.

7.10.7 Global and Local Particles

Thus in the absense of well-defined global particle states, Poincare invariance, or a preferred vacuum state, has no bearing on the possibility of interpreting QFT in terms of particles.

7.11 The Need For Quantum Gravity

It is problematic to combine classical geometry with quantum matter.

It is expected that in extreme astrophysical and cosmological situations (blackholes, big bang) the notion of a classical, smooth spacetime breaks down altogether.

7.11.1 Backreaction

The resulting iteration does not converge in general. Thus such a procedure is inconsistent, whence we must quantise the gravitational field as well.

7.11.2 Information Loss Paradox

Quantum field theory in curved spacetime is not a realistic theory - need a theory of quantum gravity to address the situation.

7.12 Stochastic Gravity

The back-reaction of a classical gravitational field interacting with quantum matter fields is described by the semiclassical Einstein equation, which has the expectation value of the quantum matter fields stress tensor as a source. Stocastic gravity goes beyond the semmicalssical limit and allows for a systematic and self-consistent description of the metric fluctuations induced by these quantum fluctuations.