Chapter 8

Introduction to Quantum General Relativity

I choose the title of this chapter carefully. Quantum gravity, for many people, invokes 'string theory'. Here I wish to talk of the attempts to do a canonical quantization of the Hamiltonian formulation of the equations of general relativity directly (as well as attempts to formulate a path integral version). This approach only uses principles of general relativity and quantum mechanics and no experimentally unverified assumptions such as the existsence of strings (or higher dimensional objects), extra dimensions or supersymmetry. This represents the original and arguably the most natural approach to quantum gravity. Progress in this approach had been slow for a period of time because of technical and conceptual reasons difficulties - but then this is the reason why the theory is so difficult. I hope to convey to the reader the true profoundness and interest of the problem of quantum general relativity and the excitment of recent huge progress made (last 25 years), and optimism that a viable theory of quantum gravity may be obtainable in our lifetime, as apposed to popular opinion.

8.1 The Problem of Quantising General Relativity

The basic idea of canonical quantisation is

(i) to take the states of the system to be described by wavefunctions $\Psi(q)$ of the configuration variables,

(ii) to replace each momentum variable p by differentiation with respect to the conjugate configuration variable, and

(iii) to determine the time evolution of Ψ via the Schrödinger equation,

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{\mathcal{H}} \Psi,$$

where $\hat{\mathcal{H}}$ is an operator corresponding to the classical Hamiltonian $\mathcal{H}(p,q)$.

General relativity can be cast in Hamiltonian form, however the Hamiltonian is a linear combination of constraints on phase space. One could consider solving these constraints to find the true physical degrees of freedom and quantise them. However, solving the constaint equations of general relativity is an impossible task - it is equivalent to finding a general solution to the classical field equations.[1] Dirac developed a method for the canonical quantisation of systems with constraints without having to first solve the constraints at the classical level. It involves the imposition of the constraints as an additional condition on the Hilbert space. Seemingly unsummountable difficulties arised from trying to carry out this programme for GR. Not so well known is that in the mid 80's Ashtekar introduced the so-called "new-variables" which led to a mathematically rigorous canonical quantisation of general relativity which goes by the name of loop quantum gravity.

Particle physists developed a different approach to the quantisation of gravity (however it maximally comprimises background independece!) which led ultimately to string theory.

8.1.1 The Problem of Time in Canonical Quantum Gravity

8.2 Introduction to Loop Qauntum Gravity (LQG)

8.3 ADM Metric 3+1 Formulation

First you need a canonical formualtion of your theroy, in this case general relativity.

One starts by considering the Einstein-Hilbert action,

$$S_{EH} = \int dt\mathcal{L} = \int d^4xR(g)\sqrt{-det(g)} \tag{8.1}$$

and considers foliationsof space-time into space and time,

$$t^a = Nn^a + N^a \tag{8.2}$$

where N is referred to as the lapse function and N^a the shift function.

The line element is written

$$ds^2 = -N^2dt^2 + q_{ab}(dx^a + N^a)(dx^b + N^b) \tag{8.3}$$

[1]Recently an approximation scheme has been developed that can be employed to carry out such a quantisation, an approach which was previously thought to be hopeless.

The exrinsic curvature of the $t = Const$ surfaces is given by

$$K_{ab} = \frac{1}{2N}(\dot{q}_{ab} - 2\mathcal{D}_{(a}N_{b)}) \tag{8.4}$$

where the dot is indicates time derivative (specifically $\dot{q}_{ab} = \mathcal{L}_{\vec{t}}q_{ab}$) and $\mathcal{D}_a$ is the covariant derivative of the three-metric. The action in terms of these variables is

$$S[N, \vec{N}, q] = \int dt \int d^3x \sqrt{q} N \left(K_{ab}K^{ab} - K^2 + R[q] \right) \tag{8.5}$$

where $K = k_a{}^a$ and $\sqrt{q} = \sqrt{\det q}$. The Lagrangian density is

$$\mathcal{L}[N, \vec{N}, q] = \frac{\sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})(\dot{q}_{ab} - 2\mathcal{D}_{(a}N_{b)})(\dot{q}_{cd} - 2\mathcal{D}_{(c}N_{d)})}{4N} + \sqrt{q}NR[q] \tag{8.6}$$

In this form the Hamiltonian analysis is easy. The canonical momentum of the Lapseand shift function vanish because $\dot{N}$ and $\dot{N}_a$ do not appear in the action. The canonical momentum of the three-metric

$$\pi^{ab} = \frac{\partial \mathcal{L}}{\partial \dot{q}_{ab}} = \sqrt{q}(q^{ac}q^{bd} - q^{ab}q^{cd})K_{cd} = \sqrt{\det q}(K^{ab} - Kq^{ab}) \tag{8.7}$$

(exercise). The Hamiltonian form reads

$$S[N, \vec{N}, q] = \int dt \int d^3x \sqrt{q} N \left(\pi^{ab}\dot{q}_{ab} - NC(\pi, q) - N^a C_a(\pi, q) \right) \tag{8.8}$$

where

$$C = G_{abcd}\pi^{ab}\pi^{cd} - \sqrt{q}R[q] \tag{8.9}$$

is called the scalar constraint, or Hamiltonian constraint, where

$$G_{abcd} = \frac{1}{2\sqrt{q}}(q_{ac}q_{bd} + q_{ad}q_{bc} - q_{ab}q_{cd}) \tag{8.10}$$

which is called the DeWitt super metric, and

$$C^a = \mathcal{D}_b\pi^{ab} \tag{8.11}$$

is called the vector, or diffeomorphism constraints, (exercise).

8.4 The New Variables

8.4.1 Triad and connection formulation of GR

Let us introduce a set of three vector fields E_i^a, $i = 1, 2, 3$ that are orthogonal, that is,

$$\delta_{ij} = q_{ab} E_i^a E_j^b. \tag{8.12}$$

There are now two types of indices, "space" indices a, b, c and that behave like regular indices in curved space, and "internal" indices i, j, k which behave like indices of flat-space (the corresponding "metric" which raises and lowers internal indices is simply δ_{ij}). Define the dual-triad E_a^i as

$$E_a^i := q_{ab} E_i^b. \tag{8.13}$$

We then have the orogonality relationships

$$\delta^{ij} = q^{ab} E_a^i E_b^j \tag{8.14}$$

where q^{ab} is the inverse matrix of the metric q_{ab}. This comes from

$$\begin{aligned}
q^{ab} E_a^i E_b^j &= q^{ab}(q_{ac} E_i^c)(q_{bd} E_j^d) \\
&= \delta_c^b q_{bd} E_i^c E_j^d \\
&= q_{bd} E_i^d E_j^d = \delta_{ij}.
\end{aligned}$$

The second orthogonality is

$$E_i^a E_b^i = \delta_a^b. \tag{8.15}$$

This comes from by first contracting (8.12) with E_c^i,

$$\begin{aligned}
0 = (\delta_{ij} - q_{ab} E_i^a E_j^b) E_c^i &= E_c^j - q_{ab} E_j^b (E_i^a E_c^i) \\
&= (\delta_c^a - E_i^a E_c^i) E_a^j
\end{aligned}$$

Now as the E_a^i are linearly indepndent it implies what is inside the bracket vanishes. It is then easy to verify from (8.14), employing (8.15),

$$
\begin{aligned}
\delta_{ij} E_i^a E_j^b &= (q^{cd} E_c^i E_d^j) E_i^a E_j^b \\
&= q^{cd} \delta_c^a \delta_d^b = q^{ab}.
\end{aligned}
$$

That is, we can write the inverse metric in terms of the triads,

$$
q^{ab} = \sum_{i,j=1}^{3} \delta_{ij} E_i^a E_j^b = \sum_{i}^{3} E_i^a E_i^b. \tag{8.16}
$$

Actually what is really considered is

$$
det(q) q^{ab} = \sum_{i}^{3} \tilde{E}_i^a \tilde{E}_i^b, \tag{8.17}
$$

which involves the densitized triad instead (densitized as $\tilde{E}_i^a = \sqrt{det(q)} E_i^a$). One recovers from $\tilde{E}_i^a$ the metric times a factor given by its determinant. It is clear that $\tilde{E}_i^a$ and E_i^a contain the same information, just rearranged. Notice the choice for $\tilde{E}_i^a$ is not unique, and in fact one can perform a local in space rotation with respect to the internal indices. This is the origin of the $SU(2)$ gauge symmetry. Now if one is goingt to operate on objects that also have internal indices one needs to introduce an appropriate derivative (generalised covariant derivative), for example the covariant derivative for the object V_i^b will be

$$
D_a V_i^b = \partial_a V_i^b - \Gamma_{a\ i}^{\ j} V_j^b + \Gamma_{ac}^b V_i^c \tag{8.18}
$$

where Γ_{ac}^b is the usual Levi-Civita connection and $\Gamma_{a\ i}^{\ j}$

The canonically conjugate variable is related to the exstrinsic curvature, $K_a^i = K_{ab} E^{bi}$.

$$
\epsilon_{ijk} K_a^j \tilde{E}^{ak} = 0 \tag{8.19}
$$

Consider the following functions on the extended phase space

$$
\begin{aligned}
q_{ab} &:= \tilde{E}_a^j \tilde{E}_b^j |det(\tilde{E}_l^C)| \\
P^{ab} &:= 2 |det(\tilde{E}_l^C)|^{-1} \tilde{E}_k^a \tilde{E}_k^d K_{[d}^j \delta_{c]}^b \tilde{E}_j^c
\end{aligned} \tag{8.20}
$$

The constraints become,

$$
\begin{aligned}
C_a &:= \mathcal{D}_a [\tilde{E}_k^a K_b^k - \delta_b^a \tilde{E}_k^c K_c^k] \\
H &:= -\zeta \sqrt{det(q)} R + \frac{2}{\sqrt{det(q)}} \tilde{E}_k^{[c} \tilde{E}_l^{d]} K_c^k K_d^l
\end{aligned} \tag{8.21}
$$

where $\zeta = -1$ for Lorentzian and $\zeta = 1$ for Euclidean space-time.

Let us equip the extended phase space coordinatised by $(K_a^i, \tilde{E}_i^a)$ with the syplectic structure (formally, that is without smearing) defined by

$$\begin{aligned} \{\tilde{E}_j^a(x), \tilde{E}_k^b(y)\} &:= \{K_a^j(x), K_b^k(y)\} = 0 \\ \{\tilde{E}_j^a(x), K_b^j(y)\} &:= \frac{\kappa}{2}\delta_b^a \delta_i^j \delta^3(x,y). \end{aligned} \tag{8.22}$$

We now prove that symplectic reduction with respect to the constraints (8.19), (8.21) results precisely in the ADM phase space of section 8.3 together with the original spatial diffeomorphism and Hamiltonian constraint.

We start by defining the smeared 'rotation constraints'

$$G(\Lambda) = \int_\Sigma d^3x \Lambda^{jk} K_{aj} E_k^a \tag{8.23}$$

where is an arbitrary test field $\Lambda^{jk} = -\Lambda^{kj}$ with internal indices. We will calculate the Poisson bracket $\{G(\Lambda), G(\Lambda')\}$ using the identity $\{AB, C\} = A\{B, C\} + B\{A, C\}$ and the (8.22),

$$\begin{aligned} \{G(\Lambda), G(\Lambda')\} &= \int_\Sigma \int_\Sigma d^3x d^3y \Lambda^{jk}(x) \Lambda'^{j'k'}(y) \{K_{aj}(x) E_k^a(x), K_{bj'}(y) E_{k'}^b(y)\} \\ &= \int_\Sigma \int_\Sigma d^3x d^3y \Lambda^{jk}(x) \Lambda'^{j'k'}(y) \Big(K_{aj}(x) K_{bj'}(y) \{E_k^a(x), E_{k'}^b(y)\} \\ &\quad + E_k^a(x) K_{bj'}(y) \{K_{aj}(x), E_{k'}^b(y)\} + K_{aj}(x) E_{k'}^b(y) \{E_k^a(x), K_{bj'}(y)\} \\ &\quad + E_k^a(x) E_{k'}^b(y) \{K_{aj}(x), K_{bj'}(y)\} \Big) \\ &= \int_\Sigma \int_\Sigma d^3x d^3y \Lambda^{jk}(x) \Lambda'^{j'k'}(y) \frac{\kappa}{2} \delta_b^a \delta^3(x,y) \Big(- E_k^a(x) K_{bj'}(y) \delta_{jk'} + K_{aj}(x) E_{k'}^b(y) \delta_{kj'} \Big) \\ &= \frac{\kappa}{2} \int_\Sigma d^3x (\Lambda^{jj'} \Lambda'^{j'k} - \Lambda'^{jj'} \Lambda^{j'k})(x) K_{aj}(x) E_k^a(x) \end{aligned} \tag{8.24}$$

So we see that

$$\{G(\Lambda), G(\Lambda')\} = \frac{\kappa}{2} G([\Lambda, \Lambda']) \tag{8.25}$$

This means that it generats infintesimal $SO(3)$ rotationsas expected. Since the functions (8.20) are manifestly $SO(3)$-invariant by inspection, they Poisson commute with $G(\Lambda)$, that is, they constitute a completet set of rotationally-ivariant Dirac observableswith respect to $G(\lambda)$ for any λ. As the constraints defined in (8.21) are in turn fucntions of these, $G(\lambda)$ also Poisson commutes with the constraints (8.21), the total system of constraints consisting of (8.23) and (8.21) are first class.

8.4.2 Ashtekar's new variables

Ashtekar's new insight was to introduce a new variable canonoically conjuagte to the triad as,

$$A_a^i = \Gamma_a^i + iK_a^i \tag{8.26}$$

where $\Gamma_a^i = \Gamma_{ajk}\epsilon^{jki}$. A_a^i is known as the chiral spin connection.

The constraints then became,

$$\begin{aligned} G_i &:= \mathcal{D}_a \tilde{E}_i^a = 0 \\ V_a &:= F_{ab}^i \tilde{E}_i^b = 0 \\ \tilde{H} &:= \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} = 0. \end{aligned} \tag{8.27}$$

we see that with the choice of spin connection, the constraints become polynomial in the fundamental variables.

8.4.3 Derivation of Ashtekar's Formalism from the Self-dual Action

The Palatini action

$$S(e, A) = \int d^4x e e_I^a e_J^b \Omega_{ab}{}^{IJ}[\omega] \tag{8.28}$$

where the Ricci tensor, $\Omega_{ab}{}^{IJ}$, is thought of as constructed purely from the connection ω_a^{IJ}, not using the frame field. Variation with espect to the tetrad gives Einstein's equations written in terms of the tetrads, but for a Ricci tentor constructed from the connection that has no a priori relationship with the tetrad. Variation with respect to the connection tells us the connection satifies the usual compatitibility condition

$$D_b e_a^I = 0.$$

This determines the connection in terms of the teterad and we recover the usual Ricci tensor.

The self-dual action for general relativity is given by (see appendix E)

$$S(e, A) = \int d^4x e e_I^a e_J^b F_{ab}{}^{IJ}[A] \tag{8.29}$$

where F is the curvature of the A, the self-dual part of ω,

$$A_a^{IJ} = \frac{1}{2}(\omega_a^{IJ} - \frac{i}{2}\epsilon^{IJ}{}_{MN}\omega_a^{MN}). \tag{8.30}$$

It can be shown that $F[A]$ is the self-dual part of $\Omega[\omega]$ (see appendix E).

Define vector fields orthogonal to n^α,

$$E_I^b = q_a^b e_b^a = (\delta_b^a + n_a n^b) e_I^a$$

Writing $E_I^a = (\delta_b^a + n_b n^a) e_I^b$ then

$$\begin{aligned}
& \int d^4x (e E_I^a E_J^b F_{ab}{}^{IJ} - 2e E_I^a e_J^d n_d n^b F_{ab}{}^{IJ}) \\
= & \int d^4x (e(\delta_c^a + n_c n^a) e_I^c (\delta_d^b + n_d n^b) e_J^d F_{ab}{}^{IJ} - 2e(\delta_c^a + n_c n^a) e_I^c e_J^d n_d n^b F_{ab}{}^{IJ}) \\
= & \int d^4x (e e_I^a e_J^b F_{ab}{}^{IJ} + e n_c n^a e_I^c e_J^b F_{ab}{}^{IJ} + e e_I^a n_d n^b e_J^d F_{ab}{}^{IJ} + e n_c n^a n_d n^b E_I^c E_J^d F_{ab}{}^{IJ} \\
& - 2e e_I^a e_J^d n_d n^b F_{ab}{}^{IJ} - 2 n_c n^a e_I^c e_J^d n_d n^b F_{ab}{}^{IJ}) \\
= & \int d^4x e e_I^a e_J^b F_{ab}{}^{IJ} = S(E, A)
\end{aligned} \tag{8.31}$$

where we used $F_{ab}{}^{IJ} = F_{ba}{}^{JI}$ and $n^a n^b F_{ab}^i = 0$. So the action can be written

$$S(E, A) = \int d^4x (e E_I^a E_J^b F_{ab}{}^{IJ} - 2e E_I^a e_J^d n_d n^b F_{ab}{}^{IJ}) \tag{8.32}$$

We have $e = N\sqrt{q}$. We now define

$$\tilde{E}_I^a = \sqrt{q} E_I^a$$

An internan tensor S^{IJ} is self-dual if and only if

$$*S^{IJ} := \frac{1}{2}\epsilon^{IJ}{}_{MN} S^{MN} = i S^{IJ}$$

and given the curvature $F_{ab}{}^{IJ}$ is self-dual we have

$$F_{ab}{}^{IJ} = -i\frac{1}{2}\epsilon^{IJ}{}_{MN} F_{ab}{}^{MN}$$

Substituting this into the action (8.32) we have,

$$S(E,A) = \int d^4x(-i\frac{1}{2}(\frac{N}{\sqrt{q}})\tilde{E}^a_I\tilde{E}^b_J\epsilon^{IJ}{}_{MN}F_{ab}{}^{MN} - 2Nn^b\tilde{E}^a_I n_J F_{ab}{}^{IJ}) \tag{8.33}$$

where we denoted $n_J = e^d_J n_d$. We pick the gauge $\tilde{E}^a_0 = 0$ and $n^I = \delta^I_0$ (this means $n_I = \eta_{IJ}n^J = \eta_{00}\delta^I_0 = -\delta^I_0$). Writing $\epsilon_{IJKL}n^L = \epsilon_{IJK}$, which in this gauge $\epsilon_{IJK0} = \epsilon_{IJK}$. Therefore,

$$\begin{aligned} S(E,A) &= \int d^4x(-i\frac{1}{2}(\frac{N}{\sqrt{q}})\tilde{E}^a_I\tilde{E}^b_J(\epsilon^{IJ}{}_{M0}F_{ab}{}^{M0} + \epsilon^{IJ}{}_{0M}F_{ab}{}^{0M}) - 2Nn^b\tilde{E}^a_I n_J F_{ab}{}^{IJ}) \\ &= \int d^4x(-i(\frac{N}{\sqrt{q}})\tilde{E}^a_I\tilde{E}^b_J\epsilon^{IJ}{}_M F_{ab}{}^{M0} + 2Nn^b\tilde{E}^a_I F_{ab}{}^{I0}) \end{aligned} \tag{8.34}$$

The indices I, J, M range over $1, 2, 3$ and we denote them with lower case letters in a moment. By the self-duality of A^{IJ}_a,

$$\begin{aligned} A^{i0}_a &= -i\frac{1}{2}\epsilon^{i0}{}_{jk}A^{jk}_a \\ &= i\frac{1}{2}\epsilon^i{}_{jk}A^{jk}_a \\ &= iA^i_a. \end{aligned} \tag{8.35}$$

where we used $\epsilon^{i0}{}_{jk} = -\epsilon^i{}_{0jk} = -\epsilon^i{}_{jk0} = -\epsilon^i{}_{jk}$. This implies

$$\begin{aligned} F_{ab}{}^{i0} &= \partial_a A^{i0}_b - \partial_a A^{i0}_b - A^{ik}_a A_{bk}{}^0 - A^{ik}_b A_{ak}{}^0 \\ &= i(\partial_a A^i_b - \partial_a A^i_b - A^{ik}_a A_{bk} - A^{ik}_b A_{ak}) \\ &= i(\partial_a A^i_b - \partial_a A^i_b - \frac{1}{2}\epsilon^{ik}{}_j A^j_a A_{bk} + \frac{1}{2}\epsilon^{ik}{}_j A^j_b A_{ak} \\ &= i(\partial_a A^i_b - \partial_a A^i_b + \epsilon_{ijk}A^j_a A^k_b) \\ &= iF^i_{ab}. \end{aligned} \tag{8.36}$$

We replace in the second term Nn^b by $t^b - n^b$. We need

$$\mathcal{L}_t A^i_b = t^a\partial_a A^i_b + A^i_a\partial_b t^a$$

and

$$\mathcal{D}_b(t^a A^i_a) = \partial_b(t^a A^i_a) + \epsilon_{ijk}A^j_b(t^a A^k_a)$$

to obtain

$$
\begin{aligned}
\mathcal{L}_t A_b^i - \mathcal{D}_b(t^a A_a^i) &= t^a(\partial_a A_b^i - \partial_b A_a^i + \epsilon_{ijk} A_a^j A_b^k) \\
&= t^a F_{ab}^i.
\end{aligned}
$$

The action bcomes

$$
\begin{aligned}
S &= \int d^4x(-i(\frac{N}{\sqrt{q}})\tilde{E}_I^a \tilde{E}_J^b \epsilon^{IJ}{}_M F_{ab}{}^{M0} - 2(t^a - N^a)\tilde{E}_I^b F_{ab}{}^{I0}) \\
&= \int d^4x(-2i\tilde{E}_i^b \mathcal{L}_t A_b^i + 2i\tilde{E}_i^b \mathcal{D}_b(t^a A_a^i) + 2iN^a \tilde{E}_i^b F_{ab}^i - (\frac{N}{\sqrt{q}})\epsilon_{ijk}\tilde{E}_i^a \tilde{E}_j^b F_{ab}^k) \qquad (8.37)
\end{aligned}
$$

where we swapped the dummy variables a and b in the second term of the first line. Integrating by parts on the second term,

$$
\begin{aligned}
\int d^4x \tilde{E}_i^b \mathcal{D}_b(t^a A_a^i) &= \int dt d^3x \tilde{E}_i^b(\partial_b(t^a A_a^i) + \epsilon_{ijk} A_b^j (t^a A_a^k)) \\
&= -\int dt d^3x t^a A_a^i(\partial_b \tilde{E}_i^b + \epsilon_{ijk} A_b^j \tilde{E}_k^b) \\
&= -\int d^4x t^a A_a^i \mathcal{D}_b \tilde{E}_i^b \qquad (8.38)
\end{aligned}
$$

where we have thrown away the boundary term and where we used the formula for the covariant derivative on a vector density $\tilde{V}_i^b$:

$$
\mathcal{D}_b \tilde{V}_i^b = \partial_b \tilde{V}_i^b + \epsilon_{ijk} A_b^j \tilde{V}_k^b.
$$

The final form of the action we require is

$$
S = \int d^4x(-2i\tilde{E}_i^b \mathcal{L}_t A_b^i - 2i(t^a A_a^i)\mathcal{D}_b \tilde{E}_i^b + 2iN^a \tilde{E}_i^b F_{ab}^i + (\frac{N}{\sqrt{q}})\epsilon_{ijk}\tilde{E}_i^a \tilde{E}_j^b F_{ab}^k) \qquad (8.39)
$$

There is a term of the form "$p\dot{q}$" thus the quantity $\tilde{E}_i^a$ is the conjugate momentum to A_a^i. Hence, we can immediately write

$$
\{A_a^i(x), \tilde{E}_j^b(y)\} = \frac{i}{2}\delta_a^b \delta_j^i \delta^3(x,y). \qquad (8.40)
$$

Variation of action with respect to the non-dynamical quantities $(t^a A_a^i)$, that is the time component of the four-connection, the shift fucntion N^b, and lapse function N give the constraints

$$\begin{aligned}
\mathcal{D}_a \tilde{E}^a_i &= 0, \\
F^i_{ab} \tilde{E}^b_i &= 0, \\
\epsilon_{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} &= 0.
\end{aligned} \tag{8.41}$$

Varying with respect to N actually gives the last constraint in (8.41) divided by $\sqrt{q}$, it has been rescaled to make the constraint polynomial in the fundamental variables. The connection A^i_a can be written

$$\begin{aligned}
A^i_a &= \frac{1}{2} \epsilon^i{}_{jk} A^{jk}_a \\
&= \frac{1}{2} \epsilon^i{}_{jk} \big(\omega^{jk}_a - i \frac{1}{2} (\epsilon^{jk}{}_{m0} \omega^{m0}_a + \epsilon^{jk}{}_{0m} \omega^{0m}_a) \big) \\
&= \Gamma^i_a - i \omega^{0i}_a
\end{aligned} \tag{8.42}$$

$$\begin{aligned}
E_{ci} \omega^{0i}_a &= -q^b_a E_{ci} \omega^{i0}_b \\
&= -q^b_a E_{ci} e^{di} \nabla_b e^0_d \\
&= q^b_a q^d_c \nabla_b n_d = K_{ac}
\end{aligned} \tag{8.43}$$

where we used $e^0_d = \eta^{0I} g_{dc} e^c_I = -g_{dc} e^c_0 = -n_d$, therefore $\omega^{0i}_a = K^i_a$. So the connection reads

$$A^i_a = \Gamma^i_a - i K^i_a. \tag{8.44}$$

8.5 Ashtekar-Barbero Variables

8.5.1 The Holst Action

The Holst action is a generalisation of the Palatini actio (8.28),

$$S(e, \omega) = \frac{1}{16 \pi G} \int d^4 x |e| e^a_I e^b_J P^{IJ}{}_{KL} \Omega_{ab}{}^{KL}(\omega) \tag{8.45}$$

where

$$P^{IJ}{}_{KL} = \delta^{[I}_K \delta^{J]}_L - \frac{1}{2\beta} \epsilon^{IJ}{}_{KL}. \tag{8.46}$$

Assume $\beta \neq \pm i$, then $P^{IJ}{}_{KL}$ has an inverse given by

$$(P^{-1})_{IJ}{}^{KL} = \frac{\beta^2}{\beta^2+1}\left(\delta_I^{[K}\delta_J^{L]} + \frac{1}{2\beta}\epsilon_{IJ}{}^{KL}\right). \tag{8.47}$$

As the inverse exists we get equivalent conditions from variation with respect to the connection.

Variation with respect to the tetrad yields Einstein's equation plus an additional term. However, this extra termvanishes by symmetries of the Riemann tensor.

8.5.2 3+1 Decomposition of the Holst Action

We first rewrite the action as in (8.32)

$$S(E,A) = \frac{1}{16\pi G}\int d^4x e P^{IJ}{}_{KL}F_{ab}{}^{KL}(E_I^a E_J^b - 2eE_J^b e_I^d n_d n^a) \tag{8.48}$$

We substitute $e = N\sqrt{q}$ and $Nn^a = t^a - N^a$ and denote $n_I = e_I^d n_d$,

$$S(E,A) = \frac{1}{16\pi G}\int d^4x\sqrt{q}P^{IJ}{}_{KL}F_{ab}{}^{KL}(E_I^a - 2n_I t^a + 2N^a n_I)E_J^b \tag{8.49}$$

The first term will give the Hamiltonian constraint, the last term the spatial diffeomorphism constraint, and the middle term will provide the symplectic structure. Here we just examine the symplectic structure. Calcualtion is moreinvolved and will be reproduced in appendix E. Define

$$\tilde{E}_i^a = \frac{\sqrt{q}E_i^a}{8\pi\beta G} \tag{8.50}$$

and using the same gauge as before (namely $E_0^a = 0$ and $n_I = (-1,0,0,0)$) then the middle term can be written

$$\begin{aligned}
-\beta\int d^4x n_I t^a \tilde{E}_J^b P^{IJ}{}_{KL}F_{ab}{}^{KL} &= \beta\int d^4x t^a \tilde{E}_j^b P^{0j}{}_{KL}F_{ab}{}^{KL} \\
&= \beta\int d^4x t^a \tilde{E}_j^b\left(F_{ab}{}^{0j} - \frac{1}{2\beta}\epsilon^{0j}{}_{KL}F_{ab}{}^{KL}\right) \\
&= \beta\int d^4x t^a \tilde{E}_j^b\Big(\partial_a\omega_b^{0j} - \partial_b\omega_a^{0j} + 2\omega_{[a}^{0k}\omega_{b]k}{}^j \\
&\quad + \frac{1}{\beta}\epsilon^j{}_{kl}(\partial_{[a}\omega_{b]}^{kl} + \omega_{[a}^{Kk}\omega_{b]K}{}^l)\Big)
\end{aligned} \tag{8.51}$$

Consider the first two terms and integrate by parts the second term,

$$
\begin{aligned}
\beta \int d^4 x t^a \tilde{E}^b_j (\partial_a \omega^{0j}_b - \partial_b \omega^{0j}_a) &= \beta \int d^4 x (t^a \tilde{E}^b_j \partial_a \omega^{0j}_b + \omega^{0j}_a \partial_b (t^a \tilde{E}^b_j)) \\
&= \beta \int d^4 x \tilde{E}^b_j (t^a \partial_a \omega^{0j}_b + \omega^{0j}_a \partial_b t^a) + \beta \int d^4 x \omega^{0j}_a t^a \partial_b \tilde{E}^b_j \\
&= \beta \int d^4 x \tilde{E}^b_j \mathcal{L}_t \omega^{0j}_b + \beta \int d^4 x \omega^{0j}_a t^a \partial_b \tilde{E}^b_j . \qquad (8.52)
\end{aligned}
$$

Similatly from the last line of (8.51) we obtain the 'time' Lie derivative of

$$
\frac{1}{2\beta} \epsilon^j{}_{kl} \omega^{kl}_b .
$$

Since these are the only time derivatives appearing in the action, and since both are contracted with $\beta \tilde{E}^b_j$, the canonical variables are $\tilde{E}^b_j$ and

$$
A^i_a = \frac{1}{2} \epsilon^j{}_{kl} \omega^{kl}_b + \beta \omega^{0i}_a , \qquad (8.53)
$$

as before we have $\Gamma^i_a = \frac{1}{2} \epsilon^j{}_{kl} \omega^{kl}_a$ and $\omega^{0i}_a = K^i_a$, so the canonical variable is

$$
A^i_a = \Gamma^i_a + \beta K^i_a , \qquad (8.54)
$$

called the Ashtekar-Barbero connection.

8.5.3 Introduction to Canonical Transformations

We can arrive at the Ashtekar-Barbero formalism from the 3+1 Palatini analysis. In this subsection we introduce necessary ideas.

We can perform a change of variables on phase space

$$
\begin{aligned}
Q_i &= Q_i(q_i, p^i, t) \\
P^i &= P^i(q_i, p^i, t)
\end{aligned} \qquad (8.55)
$$

such that

$$
\dot{Q}_i = \frac{\partial \mathcal{K}}{\partial P^i}, \quad \dot{P}^i = -\frac{\partial \mathcal{K}}{\partial Q_i} \qquad (8.56)
$$

and $\mathcal{K}$ is a new Hamiltonian, that is it obeys Hamiltonian's principle,

$$\delta \int_{t_1}^{t_2} \left(\sum_i P^i \dot{Q}_i - \mathcal{K}(Q,P,t) \right) dt = 0. \tag{8.57}$$

It is easy to see that these two Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ by a total time derivative yield exactly the same equations of motion,

$$\begin{aligned} S'(q,\dot{q}) &= \int_{t_0,q_0}^{t_1,q_1} \mathcal{L}' dt = \int_{t_0,q_0}^{t_1,q_1} \left(\mathcal{L} + \frac{dF}{dt} \right) dt \\ &= S(q,\dot{q}) + [F]_{t_0,q_0}^{t_1,q_1} \end{aligned} \tag{8.58}$$

If we take $F = F_1(q,Q)$, then

$$\begin{aligned} \sum_i p^i \dot{q}_i - \mathcal{H} &= \sum_i P_i \dot{Q}_i - \mathcal{K} + \frac{dF_1}{dt}(q,Q) \\ &= \sum_i P_i \dot{Q}_i - \mathcal{K} + \sum_i \frac{\partial F_1}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F_1}{\partial Q_i} \dot{Q}_i + \frac{\partial F_1}{\partial t} \end{aligned} \tag{8.59}$$

Comparing coefficients we get

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \tag{8.60}$$

and

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_1}{\partial t}. \tag{8.61}$$

We can do a Legendre transformation to change variables

$$F_2(q,P,t) = F_1(q,Q,t) + \sum_i P_i Q_i \tag{8.62}$$

Remember $P_i = -\frac{\partial F_1}{\partial Q_i}$.

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} \\ Q_i &= \frac{\partial F_2}{\partial P_i} \end{aligned} \tag{8.63}$$

and

$$\mathcal{K} = \mathcal{H} + \frac{\partial F_2}{\partial t}. \tag{8.64}$$

Take $F_2(q, P, t)$. With $F_2 = \beta q P$, then

$$p = \frac{\partial F_2}{\partial q} = \beta P; \quad Q = \frac{\partial F_2}{\partial P} = \beta q.$$

So this canonical transform $F_2 = \beta q P$, produces the constant rescaling,

$$Q = \beta q, \quad P = p/\beta; \quad \mathcal{K} = \mathcal{H}. \tag{8.65}$$

$F_4 = F_4(p, P)$

8.5.4 Canonical Transformation on Extended Phase Spase: Obtaining the Gauss Law

It is easy to see that for any non-vanishing complex number β that the rescaling $(K_a^j, \tilde{E}_j^a) \mapsto ({}^\beta K_a^j := \beta K_a^i,\ {}^\beta \tilde{E}_j^a := \tilde{E}_j^a/\beta)$ is a canonical transformation, as Poisson brackets are obviously invariant under this map. The notation K_a^j and $\tilde{E}_j^a$ still refers to the variables with $\beta = 1$. The rotational constraint, which we write in the equivalent form, $G^i = \epsilon_{ijk} K_a^k \tilde{E}_l^a$ is invariant under this rescaling, that is,

$$G^i = \epsilon_{ijk} K_a^k \tilde{E}_l^a = \epsilon_{ijk}\ {}^\beta K_a^k\ {}^\beta \tilde{E}_l^a \tag{8.66}$$

We come to the other two constraints (8.21) in a moment.

We demand that the covariant derivative D_a satisfy the compatibility condition

$$D_a E_b^j = 0$$

This implies

$$\Gamma_{ajk} = -E_k^b \big[\partial_a E_b^j - \Gamma_{ab}^c E_c^j\big]. \tag{8.67}$$

As $D_a \delta_{ij} = 0$, we have

$$D_a \delta_{ij} = \partial_a \delta_{ij} + \Gamma_{aik} \delta_{kj} + \Gamma_{akj} \delta_{ik} = 0$$

or $\Gamma_{a(ik)} = 0$.

We now consider the affine transformation.

We will need to recal the equation for the covariant derivative of a vector V^a of weight W,

$$\nabla_a V^b = \partial_a V^b + \Gamma^b_{ac} V^c - W \Gamma^c_{ca} V^b.$$

In the special case when $W = 1$ (as we have with $\tilde{E}^a = \sqrt{det(q)} E^a$) and when $a = b$ this reduces to

$$\nabla_a V^a = \partial_a V^a + \Gamma^a_{ac} V^c - W \Gamma^c_{ca} V^b = \partial_a V^a.$$

We notice that $\mathcal{D}_a \tilde{E}^b_j = 0$. In particular we have

$$\begin{aligned} D_a \tilde{E}^b_j &= \partial_a \tilde{E}^a_j + \Gamma_{aj}{}^k \tilde{E}^a_k \\ &= \partial_a \tilde{E}^a_j + \epsilon_{jkl} \Gamma^k_a = 0 \end{aligned} \tag{8.68}$$

Next we explicitly solve the spin connection in terms of $\tilde{E}^a_j$ using (8.67),

$$\begin{aligned} \Gamma^i_a &= \frac{1}{2} \epsilon^{ijk} \Gamma_{ajk} \\ &= \frac{1}{2} \epsilon^{ijk} E^b_k [\Gamma^c_{ab} E^j_c - \partial_a E^j_b] \\ &= \frac{1}{2} \epsilon^{ijk} (E^b_k E^j_c \frac{1}{2} q^{cd} [q_{ad,b} + q_{bd,a} - q_{ab,d}] - E^b_k E^j_{b,a}) \\ &= \frac{1}{2} \epsilon^{ijk} (\frac{1}{2} E^b_k E^c_j [(E^l_a E^l_c)_{,b} + (E^l_b E^l_c)_{,a} - (E^l_a E^l_b)_{,c}] - E^b_k E^j_{b,a}) \\ &= \frac{1}{2} \epsilon^{ijk} (\frac{1}{2} E^b_k E^c_j [E^l_{a,b} E^l_c + E^l_a E^l_{c,b} + E^l_{b,a} E^l_c + E^l_b E^l_{c,a} - E^l_{a,c} E^l_b - E^l_a E^l_{b,c}] - E^b_k E^j_{b,a}) \\ &= \frac{1}{2} \epsilon^{ijk} (\frac{1}{2} E^{[b}_k E^{c]}_j [(E^l_{a,b} E^l_c - E^l_{a,c} E^l_b + E^l_{b,a} E^l_c + E^l_{c,a} E^l_b) + (E^l_a E^l_{c,b} - E^l_a E^l_{b,c})] - E^b_k E^j_{b,a}) \\ &= \frac{1}{2} \epsilon^{ijk} (\frac{1}{2} E^b_k E^c_j [2 E^l_{a,b} E^l_c + 2 E^l_a E^l_{c,b}] - E^b_k E^j_{b,a}) \\ &= \frac{1}{2} \epsilon^{ijk} E^b_k (E^j_{a,b} - E^j_{b,a} + E^c_j E^l_a E^l_{c,b}) \end{aligned} \tag{8.69}$$

We write this in terms of the densitised triads noting that by $|det(\tilde{E})| = |det(E)|^2$, we have $E^a_i = |det(\tilde{E})|^{-1/2} \tilde{E}^a_i$ and $E^i_a = |det(\tilde{E})|^{1/2} \tilde{E}^i_a$,

$$
\begin{aligned}
\frac{1}{2}\epsilon^{ijk}E^b_k(E^j_{a,b}-E^j_{b,a}+E^c_jE^l_aE^l_{c,b}) &= \frac{1}{2}\epsilon^{ijk}(|det(\tilde{E})|^{-1/2}\tilde{E}^b_k)\big[(|det(\tilde{E})|^{1/2}\tilde{E}^j_a)_{,b}-(|det(\tilde{E})|^{1/2}\tilde{E}^j_b)_{,a} \\
&\quad +\tilde{E}^c_j\tilde{E}^l_a(|det(\tilde{E})|^{1/2}E^l_c)_{,b}\big] \\
&= \frac{1}{2}\epsilon^{ijk}\tilde{E}^b_k(\tilde{E}^l_{a,b}-\tilde{E}^j_{b,a}+\tilde{E}^c_j\tilde{E}^l_a\tilde{E}^l_{c,b}) \\
&\quad +\frac{1}{4}\epsilon^{ijk}\tilde{E}^b_k\Big[\tilde{E}^j_a\frac{|det(\tilde{E})|_{,b}}{|det(\tilde{E})|}-\tilde{E}^j_b\frac{|det(\tilde{E})|_{,a}}{|det(\tilde{E})|}+\tilde{E}^c_j\tilde{E}^l_a\tilde{E}^l_c\frac{|det(\tilde{E})|_{,b}}{|det(\tilde{E})|}\Big] \\
&= \frac{1}{2}\epsilon^{ijk}\tilde{E}^b_k(\tilde{E}^l_{a,b}-\tilde{E}^j_{b,a}+\tilde{E}^c_j\tilde{E}^l_a\tilde{E}^l_{c,b}) \\
&\quad +\frac{1}{4}\epsilon^{ijk}\tilde{E}^b_k\Big[2\tilde{E}^j_a\frac{|det(\tilde{E})|_{,b}}{|det(\tilde{E})|}-\tilde{E}^j_b\frac{|det(\tilde{E})|_{,a}}{|det(\tilde{E})|}\Big].
\end{aligned}
\tag{8.70}
$$

We summerise the two equations,

$$
\begin{aligned}
\Gamma^i_a &= \frac{1}{2}\epsilon^{ijk}E^b_k(E^l_{a,b}-E^j_{b,a}+E^c_jE^l_aE^l_{c,b}) \\
&= \frac{1}{2}\epsilon^{ijk}\tilde{E}^b_k(\tilde{E}^l_{a,b}-\tilde{E}^j_{b,a}+\tilde{E}^c_j\tilde{E}^l_a\tilde{E}^l_{c,b}) \\
&\quad +\frac{1}{4}\epsilon^{ijk}\tilde{E}^b_k\Big[2\tilde{E}^j_a\frac{|det(\tilde{E})|_{,b}}{|det(\tilde{E})|}-\tilde{E}^j_b\frac{|det(\tilde{E})|_{,a}}{|det(\tilde{E})|}\Big].
\end{aligned}
\tag{8.71}
$$

Note that as ${}^\beta\tilde{E}^a_i=\tilde{E}^a_i/\beta$ and ${}^\beta\tilde{E}^i_a=\beta\tilde{E}^a_i$, and as by the above formula Γ^j_a is a homogeneous function of degree zero in $\tilde{E}^a_j$ and its derivatives, we arrive at the important conclusion that

$$
{}^\beta\Gamma^j_a := \Gamma^j_a({}^\beta\tilde{E}) = \Gamma^j_a = \Gamma^j_a(E)
\tag{8.72}
$$

is itself invariant under the rescaling transformation. This is also true for the Christofell symbol since this is a homogeneous function of degree zero in q_{ab} and its derivatives and $q_{ab}=|det(\tilde{E})|\tilde{E}^j_a\tilde{E}^j_b$

Thus the derivative D_a is indepndent of β and therefore we have

$$
D_a\,{}^\beta\tilde{E}^a_j=0.
$$

We can then rewrite the rotation constraint $G_i=\epsilon_{jkl}K^k_a\tilde{E}^a_l=0$ as

$$
\begin{aligned}
0 &= \epsilon_{jkl}K^k_a\tilde{E}^a_l \\
&= \epsilon_{jkl}\,{}^\beta\,{}^\beta K^k_a\tilde{E}^a_l \\
&= D_a({}^\beta E^a_j)+\epsilon_{jkl}\,{}^\beta K^k_a\,{}^\beta\tilde{E}^a_l \\
&= \partial_a\,{}^\beta\tilde{E}^a_j+\epsilon_{jkl}[\Gamma^k_a+\,{}^\beta K^k_a]\,{}^\beta\tilde{E}^a_l \\
&=: {}^\beta\mathcal{D}_a\,{}^\beta\tilde{E}^a_j.
\end{aligned}
\tag{8.73}
$$

We see that the rotation constraint takes the form of the Gauss consraint of Yang-Mills theory,

$$^{\beta}\mathcal{D}_a \, ^{\beta}\tilde{E}^a_j = 0,$$

if we take our connection to be

$$^{\beta}A^i_a := \Gamma^k_a + \, ^{\beta}K^k_a, \tag{8.74}$$

that is,

$$^{\beta}A^i_a := \Gamma^k_a + \beta K^k_a \tag{8.75}$$

8.5.5 Poisson Brackets for New Variables

Here we will prove that the new varaibles have the same structure as the original symplectic space, namely, we have

$$\begin{aligned} \{ \, ^{\beta}A^i_a(x), \, ^{\beta}A^j_b(y)\} &= \{ \, ^{\beta}\tilde{E}^a_i(x), \, ^{\beta}\tilde{E}^b_j(y)\} = 0 \\ \{ \, ^{\beta}\tilde{E}^a_i(x), \, ^{\beta}A^j_b(y)\} &= \frac{\kappa}{2}\delta^a_b\delta^j_i\delta^3(x,y). \end{aligned} \tag{8.76}$$

This is quite surprising given the complicated structure of (8.71)

We wish to prove (8.76) by means of (8.22). First it is easy to see that

$$\{ \, ^{\beta}\tilde{E}^a_i(x), \, ^{\beta}\tilde{E}^b_j(y)\} = \{\tilde{E}^a_i(x)/\beta, \tilde{E}^b_j(y)/\beta\} = 0.$$

Let us consider

$$\begin{aligned} \{ \, ^{\beta}\tilde{E}^a_i(x), \, ^{\beta}A^j_b(y)\} &= \{\tilde{E}^a_i(x)/\beta, \Gamma^j_b + \beta K^j_b\} \\ &= \{\tilde{E}^a_i(x)/\beta, \beta K^j_b\} \\ &= \{\tilde{E}^a_i(x), K^j_b\}. \end{aligned} \tag{8.77}$$

where we used that Γ is a function of E

$$\begin{aligned} \{ \, ^{\beta}A^i_a(x), \, ^{\beta}A^j_b(y)\} &= \{\Gamma^j_b + \beta K^j_b, \Gamma^j_b + \beta K^j_b\} \\ &= \end{aligned} \tag{8.78}$$

8.5.6 The constraints in the New Variables

$$\tilde{H} = -\zeta\epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{abk} + \frac{2(\beta^2\zeta - 1)}{\beta^2}\tilde{E}_{[i}^a\tilde{E}_{j]}^b(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j). \tag{8.79}$$

We start with phase space $\Gamma(K_a^i, \tilde{E}_i^a)$

$$\{f, g\} = \frac{\delta f}{\delta K_a^i}\frac{g}{\delta \tilde{E}_i^a} - \frac{f}{\delta \tilde{E}_i^a}\frac{g}{\delta K_a^i} \tag{8.80}$$

Consider the transformation

$$\begin{aligned} {}_\beta\tilde{E}_i^a &= -\frac{1}{\beta}\tilde{E}_i^a \\ {}_\beta A_a^i &= \Gamma_a^i + \beta K_a^i. \end{aligned} \tag{8.81}$$

The Poisson brackets of the new variables are

$$\begin{aligned} \{\,{}_\beta A_a^i(x),\ {}_\beta A_b^j(y)\} &= \{\,{}_\beta\tilde{E}_a^i(x),\ {}_\beta\tilde{E}_b^j(y)\} = 0 \\ \{\,{}_\beta A_a^i(x),\ {}_\beta\tilde{E}_i^a(y)\} &= \delta_j^i\delta_a^b\delta^3(x, y). \end{aligned} \tag{8.82}$$

We end up with phase space $\Gamma(A_a^i, \tilde{E}_i^a)$

8.5.7 The Constraints

The constraints of classical canonical general relativity In the Hamiltonian formulation of ordinary classical mechanics the Poisson bracket is an important concept. A "canonical coordinate system" consists of canonical position and momentum variables that satisfy canonical Poisson-bracket relations,

$$\{q_i, p_j\} = \delta_{ij}$$

Hamilton's equations can be rewritten as,

$$\begin{aligned} \dot{q}_i &= \{q_i, H\}, \\ \dot{p}_i &= \{p_i, H\}. \end{aligned} \tag{8.83}$$

These equations describe a "flow" or orbit in phase space generated by the Hamiltonian H. Let us consider constrained systems, of which General relativity is an example. In a similar way the Poisson bracket between a constraint and a phase space function generates a flow along an orbit in (the unconstrained) phase space. There are three types of constraints in Ashtekar's reformulation of classical general relativity:

1. $SU(2)$ Gauss gauge constraints

$$G_j(x) = 0. \tag{8.84}$$

These come about from re-expressing General relativity as an $SU(2)$ Yang-Mills type gauge theory (Yang-Mills is a generalization of Maxwell's theory where the gauge field transforms as a vector under Gauss transformations, that is, the Gauge field is of the form $A_a^i(x)$ where i is an internal index. See Ashtekar variables). These infinite number of Gauss gauge constraints can be smeared with test fields with internal indices, $\lambda^j(x)$,

$$G(\lambda) = \int d^3x G_j(x)\lambda^j(x). \tag{8.85}$$

which we demand vanish for any such function. These smeared constraints defined with respect to a suitable space of smearing functions give an equivalent description to the original constraints.

In fact Ashtekar's formulation may be thought of as ordinary $su(2)$ Yang-Mills theory together with the following special constraints, resulting from diffeomorphism invariance, and a Hamiltonian that vanishes. The dynamics of such a theory are thus very different from that of ordinary Yang-Mills theory.

2. Spatial diffeomorphisms constraints

$$C_a(x) = 0 \tag{8.86}$$

of which there and infinite number, can be smeared by the so-called shift functions $\vec{N}(x)$ to give an equivalent set of smeared spatial diffeomorphism constraints,

$$C(\vec{N}) = \int d^3x C_a(x)N^a(x). \tag{8.87}$$

These generate spatial diffeomorphisms along orbits defined by the shift function $N^a(x)$.

3. Hamiltonian constraints

$H(x) = 0$ of which there are an infinite number, can be smeared by the so-called lapse functions N (x) to give an equivalent set of smeared Hamiltonian constraints,

$$H(N) = \int d^3x H(x)N(x). \tag{8.88}$$

These generate time diffeomorphisms along orbits defined by the lapse function $N(x)$. In Ashtekar formulation the gauge field $A_a^i(x)$ is the configuration variable (the configuration variable being analogous to q in ordinary mechanics) and its conjugate momentum is the (densitized) triad (electrical field) $\tilde{E}_i^a(x)$. The constraints are certain functions of these phase space variables.

8.5.8 The Poisson bracket algebra

Of particular importance is the Poisson bracket algebra formed between the (smeared) constraints themselves. An important notion here is the Lie derivative, $\mathcal{L}_V$, which is basically a derivative operation that infinitesimally "shifts" functions along some orbit with tangent vector V. In terms of the above the smeared constraints the constraint algebra amongst the Gauss' law reads,

$$\{G(\lambda), G(\mu)\} = G([\lambda, \mu]) \tag{8.89}$$

And so we see that the Poisson bracket of two Gauss' law is equivalent to a single Gauss' law evaluated on the commutator of the smearings. The Poisson bracket amongst spatial diffeomorphisms constraints reads

$$\{C(\vec{N}), C(\vec{M})\} = C(\mathcal{L}_{\vec{N}}\vec{M}) \tag{8.90}$$

and we see that its effect is to "shift the smearing". The reason for this is that the smearing functions are not functions of the canonical variables and so the spatial diffeomorphism does not generate diffeomorphims on them. They do however generate diffeomorphims on everything else. This is equivalent to leaving everything else fixed while shifting the smearing. The action of the spatial diffeomorphism on the Gauss law is

$$\{C(\vec{N}), G(\lambda)\} = G(\mathcal{L}_{\vec{N}}\lambda), \tag{8.91}$$

again, it shifts the test field λ. The Gauss law has vanishing Poisson bracket with the Hamiltonian constraint. The spatial diffeomorphism constraint with a Hamiltonian gives a Hamiltonian with its smearing shifted,

$$\{C(\vec{N}), H(M)\} = H(\mathcal{L}_{\vec{N}}M), \tag{8.92}$$

Finally, the poisson bracket of two Hamiltonians is a spatial diffeomorphism,

$$\{H(N), H(M)\} = C(K) \tag{8.93}$$

where K is some phase space function.

A (Poisson bracket) Lie algebra, with constraints C_I, is of the form

$$\{C_I, C_J\} = f_{IJ}^K C_K \tag{8.94}$$

where f_{IJ}^K

are constants (the so-called structure constants). The above Poisson bracket algebra for General relativity does not form a true Lie algebra as we have structure functions rather than structure constants for the Poisson bracket between two Hamiltonians. This leads to difficulties.

8.5.9 Observables

The constraints define a constraint surface in the original phase space. The gauge motions of the constraints apply to all phase space but have the feature that they leave the constraint surface where it is, and thus the orbit of a point in the hypersurface under gauge transformations will be an orbit entirely within it. Dirac observables are defined as phase space functions, O, that Poisson commute with all the constraints when the constraint equations are imposed,

$$\{G_j, O\}_{G_j=C_a=H=0} = \{C_a, O\}_{G_j=C_a=H=0} = \{H, O\}_{G_j=C_a=H=0} = 0, \tag{8.95}$$

that is, they are quantities defined on the constraint surface that are invariant under the gauge transformations of the theory.

Then, solving only the constraint $G_j = 0$ and determining the Dirac observables with respect to it leads us back to the ADM phase space with constraints H, C_a. The dynamics of general relativity is generated by the constraints, it can be shown that six Einstein equations describing time evolution (really a gauge transformation) can be obtained by calculating the Poisson brackets of the three-metric and its conjugate momentum with a linear combination of the spatial diffeomorphism and Hamiltonian constraint. The vanishing of the constraints, giving the physical phase space, are the four other Einstein equations.

8.6 Quantisation of the Constraints - the Equations of Quantum General Relativity

Many of the technical problems in canonical quantum gravity revolve around the constraints. Canonical general relativity was originally formulated in terms of metric variables, but there seemed to be insurmountable mathematical difficulties in promoting the constraints to quantum operators because of highly non-linear dependence on the canonical variables. The equations were muchsimplified with the introduction of Ashtekar variables. The constraints are polynomial in the fundamental variables, unlike as with the constraints in the metric formulation. This dramatic simplification seemedto open up the way to quantising the constraints.

With Ashtekar variables, given the configuration variable A_a^i, it is natural to consider wavefunctions $\Psi[A_a^i$. This is the connection representation. It is analogous to ordinary quantum mecahnaics with configuration variables q and wavefunctions $\psi(q)$. The configuration varaible gets promoted to a quantum operator via:

$$\hat{A}_a^i \Psi[A] = A_a^i \Psi[A], \tag{8.96}$$

(analogous to $\hat{q}\psi(q) = q\psi(q)$ and triads are (fucntional) derivatives,

$$\hat{\tilde{E}}_i^a \Psi[A] = -i \frac{\delta \Psi[A]}{\delta A_a^i}, \tag{8.97}$$

(analogous to $\hat{p}\psi(q) = -i\hbar d\psi(q)/dq$). In passing over to the quantum theory the constraints become operators on a kinematic Hilbert space (the unconstrained $su(2)$ Yang-Mills Hilbert space). Note that different ordering of the A's and $\tilde{E}$'s when replacing the $\tilde{E}$'s with derivatives give rise to different operators - the choice made is called the factor orderingand should be choosen via physical reasoning. Formally they read:

$$\begin{aligned}\hat{G}_j|\psi > &= 0,\\ \hat{C}_a|\psi > &= 0,\\ \hat{\tilde{H}}|\psi > &= 0.\end{aligned} \tag{8.98}$$

There are still problems in properly defing all these equations and solving them. Ashtekar's formalism presents difficulties. For example the Hamiltonian constraint Ashtekar worked with was the densitised version instead of the original Hamiltonian, that is, he worked with $\tilde{H} = \sqrt{\det(q)}H$. There were serious difficulties in promotingthis quantity to a quantum operator. Moreover, although Ashtekar variables had the virtueof simplifying the Hamiltonian, it has the problem that the variables become complex. When one quantisesthe theoryit is a difficult task to ensure that one recovers real general relativity as apposed to complex general relativity.

8.6.1 Quantum Constraints as the Equations of Quantum General relativity

We demonstrate an important aspect of the quantum constraints. We consider Gauss' law only. First we state the classical result that the Poisson bracket of the smeared Gauss' law $G(\lambda) = \int d^3x \lambda^j (D_a E^a)^j$ with the connection is

$$\{G(\lambda), A_a^i\} = \partial_a \lambda^i + g\epsilon^{ijk} A_a^j \lambda^k = (D_a \lambda)^i.$$

The quantum Gauss constraint reads

$$\hat{G}_j \Psi = -iD_a \frac{\delta \Psi[A]}{\delta A_a^j} = 0. \tag{8.99}$$

If one smears the quaqntum Gauu's law and study its action on the quantum state one finds that the action of the constraint on the quantum state is equivalent to shifting the argument of Ψ by an infinitesimal (in the sense of the parameter λ small) gauge transformation,

$$\Big[1 + \int d^3x \lambda^j(x) \hat{G}_j\Big] \Psi[A] = \Psi[A + D\lambda] = \Psi[A], \tag{8.100}$$

and te last identity come from the fact that the constraint annihilates the state. It is telling us that the functions $\Psi[A]$ have to be gauage invariant functions of the connection. The same idea is true for the other constraints.

Therefore the two step process in the classical theory of solving the constraints $C_I = 0$ (equivalent to solving the admissibility conditions for the initial data) and looking for the gauge orbits (solving the 'evolution' equations) is replaced by a one step process in the quantum theory, namely looking for solutions Ψ to the quantum constraints $\hat{C}_I \Psi = 0$. This is because it obviously solves the constraints at the quantum level and it simultaneously looks for states that are gauge invariant because $\hat{C}_I$ is the quantum generator of gauge transformations. Recall that, at the classical level, solving the admissibility conditions and evolution equations was equivalent to solving all of Einstein's field equations, this underlines the central role of the quantum constraint equations in canonical quantum gravity.

8.7 The Loop Representation

The holonomy is a measure of how much the initialand final values of a spinor or vector have been parallel propagated around a closed loop γ and is denoted,

$$h_\gamma[A]$$

Knowledge of the holonomies is equivalent to knowledge of the connection,up to gauge equivalence. An holonomy can also be defined for an edge, denoted $h_e[A]$, then under a Gauss law these tranform as

$$(h'_e)_{\alpha\beta} = U^{-1}(x)(h_e)_{\gamma\sigma} U_{\sigma\beta}(y)$$

For a closed loop $x = y$ andif we take the trace we obtain,

$$Trh'_e = Trh_e.$$

The trace of the holonomy around a closed loop is a called a Wilson loop. Thus Wilson loops are Guass gauge invariant.

8.7.1 The loop transform

As Wilson loops form a basis we can formally expand any Gauus gauge invariant function as,

$$\Psi[A] = \sum_\gamma \Psi[\gamma] W_\gamma[A] \tag{8.101}$$

We can see the analogy with the momentum representation in ordinary quantum mechanics. There one has a basis of states $\exp(ikx)$ labelledby a number k andone expands

$$\psi(x) = \int dk \psi(k) \exp(ikx)$$

The loop transform is given by

$$\Psi[\gamma] = \int [dA] \Psi[A] W_\gamma[A] \tag{8.102}$$

This defines the loop representation. Given any operator $\hat{O}$ in the connection representation,

$$\Phi[A] = \hat{O}\Psi[A]. \tag{8.103}$$

We define $\Phi[\gamma]$ by the transform

$$\Phi[\gamma] = \int [dA] \Phi[A] W_\gamma[A] \tag{8.104}$$

We should define the corresponding operator $\hat{O}'$ in the loop representation as

$$\Phi[\gamma] = \hat{O}'\Psi[\gamma] \tag{8.105}$$

Equating (8.104) and (8.105) and substituting (8.103) we finally get

$$\hat{O}'\Psi[\gamma] = \int [dA] W_\gamma[A] \hat{O} \Psi[A]. \tag{8.106}$$

This can be rewritten as,

$$\hat{O}'\Psi[\gamma] = \int [dA] (\hat{O}^\dagger W_\gamma[A]) \Psi[A]. \tag{8.107}$$

where by $\hat{O}^\dagger$ we mean the operator $\hat{O}$ but with the reverse factor ordering. We evaluate the action of this operator on the Wilson loop as a calculation in the connection representation and rearranging the result as a manipulation purely in terms of loops (one should remember that when considering the action on the Wilson loop one shouldchoose the operator ne wishes to transform with the opposite factor ordering to the one chosen for its action on the waefunctions $\Psi[A]$). This gives the physical meaning of the operator $\hat{O}'$. For example if the action of $\hat{O}^\dagger$ was a spatial diffeomorphism, then this can be thought of as keeping the connection field A of $W_\gamma[A]$ where it is while performing a spatial diffeomorphsim on γ instead.

8.7.2 Solutions to all the Constraints

In the loop representation it is immediate to solve the spatial diffeomorphim constraint by considering functions of loop $\Psi[\gamma]$ that are invariant under spatial diffeomorphisms of the loop γ. That is, we construct what mathematicians call knot invariants. This opened up an unexpected link between knot theory and quantum gravity.

What about Ashtekar's Hamiltonian constraint? Let us briefly go back to the connection representation. For suitible factor ordering any Wilson loops without intersections or discontinuities satisfy the Hamiltonian constraint. Consider

$$\hat{\tilde{H}}^\dagger W_\gamma[A] = -\epsilon_{ijk} \hat{F}^k_{ab} \frac{\delta}{A^i_a} \frac{\delta}{A^j_b} W_\gamma[A] \tag{8.108}$$

When a derivative is taken it bring down, amongst other things, the tangent vector $\dot{\gamma}^a$, of the loop γ. So we have something like

$$\hat{F}^k_{ab} \dot{\gamma}^a \dot{\gamma}^b \times ...$$

However, as $\hat{F}^k_{ab}$ is anti-symmetric in the indices a and b this vanishes (this assumes that γ is not discontinuous anywhere and so the tangent is unique).And when we say it vanishes this isn't perfectly correct as the other stuff involves $\delta-$functions,and so the solution is only formal.

We consider wavefunctions $\Psi[\gamma]$ that vanishes if the loop has discontinuities and that are knot invariants. Such functions solve the Gauss law, the spatial diffeomorphism constraint and (formally) the Hamiltonian constraint. We have thus identified an infinite number of exact (if only formal) solutions to all the equations of quantum gravity! This result generated a lot of interest in the approach and matured into LQG.

8.8 Geometric operators

8.8.1 The Area Operator

The easiest geometric quantity is the area. Let us choose coordinates so that the surface Σ is characterized by $x^3 = 0$. The area of small parallelogram of the surface Σ is the product of length of each side times $\sin\theta$ where θ is the angle between the sides. Say one edge is given by the vector $\vec{u}$ and the other by $\vec{v}$ then,

$$A = \|\vec{u}\|\|\vec{v}\|\sin\theta = \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2(1-\cos^2\theta)} \quad = \sqrt{\|\vec{u}\|^2\|\vec{v}\|^2 - (\vec{u}\cdot\vec{v})^2} \tag{8.109}$$

From this we get the area of the surface Σ to be given by

$$A_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{det\ q^{(2)}} \tag{8.110}$$

where $detq^{(2)} = q_{11}q_{22} - q_{12}^2$ and is the determinant of the metric induced on Σ. This can be rewritten as

$$det\ q^{(2)} = \frac{\epsilon^{3ab}\epsilon^{3cd}q_{ac}q_{bc}}{2}. \tag{8.111}$$

The standard formula for an inverse matrix is

$$q^{ab} = \frac{\epsilon^{acd}\epsilon^{bef}q_{ce}q_{df}}{3!det(q)} \tag{8.112}$$

Note the similarity between this and the expression for $det\ q^{(2)}$. But in Ashtekar variables we have $\tilde{E}^a_i\tilde{E}^{bi} = det(q)q^{ab}$. Therefore

$$A_\Sigma = \int_\Sigma dx^1 dx^2 \sqrt{\tilde{E}^3_i\tilde{E}^{3i}}. \tag{8.113}$$

According to the rules of canonical quantization we should promote the triads $\tilde{E}^3_i$ to quantum operators, $\hat{\tilde{E}}^3_i \sim \frac{\delta}{\delta A^i_3}$. It turns out that the area A_Σ can be promoted to a well defined quantum operator despite the fact that we are dealing with product of two functional derivatives and worse we have a square-root to contend with as well.

Putting $N = 2J$, we talk of being in the $J-$th representation. We note that $\sum_i T^iT^i = J(J+1)1$. This quantity is important in the final formula for the area spectrum. We simply state the result below,

$$\hat{A}_\Sigma W_\gamma[A] = 8\pi\ell^2_{Planck}\beta\sum_I\sqrt{j_I(j_I+1)}W_\gamma[A] \tag{8.114}$$

where the sum is over all edges I of the Wilson loop that pierce the surface Σ.

8.8.2 The Volume Operator

The formula for the volume of a region R is given by

$$V = \int_R d^3x\sqrt{det(q)} = \frac{1}{6}\int_R dx^3\sqrt{\epsilon_{abc}\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}. \tag{8.115}$$

The quantization of the volume proceeds the same way as with the area. As we take the derivative, and each time we do so we bring down the tangent vector $\dot{\gamma}^a$, when the volume operator acts on non-intersecting Wilson loops the result vanishes. Quantum states with non-zero volume must therefore involve intersections. Given that the anti-symmetric summation is taken over in the formula for the volume we would need at least intersections with three non-coplanar lines. Actually it turns out that one needs at least four-valent vertices for the volume operator to be non-vanishing.

8.8.3 Physical Meaning of these Results

8.9 Spin Networks

We now consider Wilson loops with intersections. Wilson loops are an overcomplete basis as there are identities relating different Wislon loops. These come about from the fact that Wilson loops are based on matrices (the holonomy) and these matrices satisfy identities, the so-called Madelstam identities. Given two $SU(2)$ matrices A and B it is easy to check that

$$Tr(A)Tr(B) = Tr(AB) + Tr(AB^{-1}). \tag{8.116}$$

This implies that given two loops γ and η that intersect, we will have

$$W_\alpha[A]W_\beta[A] = W_{\alpha\circ\beta}[A] + W_{\alpha\circ\beta^{-1}}[A] \tag{8.117}$$

where β^{-1} we mean the loop β tranversed in the opposite direction and $\alpha\circ\beta$ means the loop obtained by going around the loop α and then along β. See fig. Spin networks are certain linear combinations of intersecting Wilson loops designed to address the overcompleteness introduced by the Madelstam identities.

8.10 Hamiltonian Constraint and the Modern Formulism

With real connection

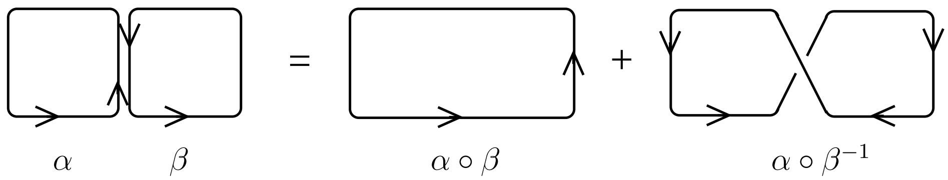

Figure 8.1: Graphical representation of the Mandelstam identity (8.117) relating different Wilson loops.

$$A_a^i = \Gamma_a^i + \beta K_a^i$$

In real Ashtekar variables the full Hamiltonian is

$$H = \frac{\epsilon_{ijk} F_{ab}^k \tilde{E}_i^a \tilde{E}_j^b}{\sqrt{det(q)}} + 2\frac{\zeta\beta^2 + 1}{\beta^2} \frac{(\tilde{E}_i^a \tilde{E}_j^b - \tilde{E}_j^a \tilde{E}_i^b)}{\sqrt{det(q)}} (A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) = H_E + H'. \tag{8.118}$$

where the constant β is the Barbero-Immirzi parameter. The constant $\zeta is +1$ for Lorentzian signature and -1 for Euclidean signature. The Γ_a^i have a complicated relationship with the desitized triads and causes serious problems upon quantization. Ashtekar variables can be seen as choosing $\beta = i$ to make the second more complicated term was made to vanish (the first term is denoted H_E because for the Euclidean theory this term remains for the real choice of $\beta = \pm 1$). Also we still have the problem of the $1/\sqrt{det(q)}$ factor.

Thiemann was able to make it work for real β. First he could simplify the troublesome $1/\sqrt{det(q)}$ by using the identity

$$\{A_c^k, V\} = \frac{\epsilon_{abc} \epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b}{\sqrt{det(q)}} \tag{8.119}$$

where V is the volume. The first term of the Hamiltonian constraint becomes

$$H_E = \{A_c^k, V\} F_{ab}^k \tilde{\epsilon}^{abc} \tag{8.120}$$

upon using Thiemann's identity. This Poisson bracket is replaced by a commutator upon quantization. It turns out that a similar trick can be used to teat the second term. Why are the Γ_a^i given by the densitized triads $\tilde{E}_i^a$? It actually comes about from the compatitibility condition

$$D_b E_i^a = 0. \tag{8.121}$$

We can solve this in much the same way as the Levi-Civita connection can be calculated from the equation $\nabla_c g_{ab} = 0$; by rotating the various indices and then adding and subtracting them.

We the rewrite this in terms of the densitised triad using that $\det(\tilde{E}) = |\det(E)|^2$. The result is complicated and non-linear,

$$\Gamma_a^i = \frac{1}{2}\epsilon^{ijk}\tilde{E}_k^b\left[\tilde{E}_{a,b}^j - \tilde{E}_{b,a}^j + \tilde{E}_j^c\tilde{E}_a^l\tilde{E}_{c,b}^l\right] + \frac{1}{4}\epsilon^{ijk}\tilde{E}_k^b\left[2\tilde{E}_a^j\frac{(\det(\tilde{E}))_{,b}}{\det(\tilde{E})} - \tilde{E}_b^j\frac{(\det(\tilde{E}))_{,a}}{\det(\tilde{E})}\right] \quad (8.122)$$

To circumvent the problems introduced by this complicated relationship Thiemann first defines the Gauss gauge invariant quantity

$$K = \int d^3x K_a^i \tilde{E}_i^a \quad (8.123)$$

where $K_a^i = K_{ab}\tilde{E}^{ai}/\sqrt{det(q)}$, and notes that

$$K_a^i = \{A_a^i, K\}.$$

(this is because $\{\Gamma_a^i, K\} = 0$ which comes about from the fact that βK is the generator of the canonical transformation of constant rescaling, $\tilde{E}_i^a \mapsto \tilde{E}_i^a/\beta$,and Γ_a^i is a homogeneous function of order zero). We are then able to write

$$A_a^i - \Gamma_a^i = \beta K_a^i = \beta\{A_a^i, K\}$$

and as such find an expression in terms of the configuration variable A_a^i and K. We obtain for the second term of the Hamiltonian

$$H' = \epsilon^{abc}\epsilon_{ijk}\{A_a^i, K\}\{A_b^j, K\}\{A_c^k, V\}. \quad (8.124)$$

Why is it easier to quantize K? This is because

$$K = -\{V, \int d^3x H_E\}$$

where we have used that the integrated densitized trace of the extrinsic curvature is the "time derivative of the volume".

8.10.1 Deriving Thiemann's Identity and Other equations

8.10.2 Quantising the Hamiltonian Constraint

The constraints in their primitive form are rather singular, and so should be 'smeared' by appropriate test functions. The Hamiltonian is the written as

$$H(N) = \int d^3x N\{A_c^k, V\}F_{ab}^k\epsilon^{abc}.$$

For simplicity we are only considering the "Euclidean" part of the Hamiltonian constraint, extension to the full constraint can be found in the literature. There are actually many different choices for functions, and so what one then ends up with an (smeared) Hamiltonians constraints. Demanding them all to vanish is equivalent to the original description.

We promote the Hamiltonian constraint to a quantum operator in the loop representation. One introduces a lattice regularization procedure. we assume that space has been divided into tetrahedra Δ. One builds an expression such that the limit in which the tetrahedra shrink in size approximates the expression for the Hamiltonian constraint. For each tetrahedron pick a vertex and call $v(\Delta)$. Let $s_i(\Delta)$ with $i = 1, 2, 3$ be three edges ending at $v(\Delta)$. We now construct a loop

$$\alpha_{ij} = s_i(\Delta) \cdot s_{ij}(\Delta) \cdot s_j(\Delta)^{-1} \tag{8.125}$$

by moving along $s_i(\Delta)$ then along the line joining the points s_i and s_j that are not $v(\Delta)$ (which we have denoted s_{ij}) and then returning to $v(\Delta)$ along s_j. The holonomy

$$h_\gamma[A] = \mathcal{P}\exp\Big\{ -\int_{s_0}^{s_1} ds\dot{\gamma}^a A_a^i(\gamma(s))T_i\Big\} \approx I - (s_k^a)A_a^i T_i \tag{8.126}$$

along a line in the limit the tetraherdon shrinks approximates the connection via

$$\lim_{\Delta \to v(\Delta)} h_{s_k} = I - A_c s_k^c \tag{8.127}$$

where s_k^c is a vector in the direction of edge s_k. It can be shown that

$$\lim_{\Delta v \to (\Delta)} h_{\alpha_{ij}} = I + \frac{1}{2}F_{ab}s_i^a s_j^b. \tag{8.128}$$

We are led to trying

$$H_\Delta(N) = \sum_\Delta N(v(\Delta))\epsilon^{ijk}Tr\big(h_{\alpha_{ij}} h_{s_k}\{h_{s_k}^{-1}, V\}\big) \tag{8.129}$$

where the sum is over all tetrahedra Δ. Substituting for the holonomies,

$$H_\Delta(N) = \sum_\Delta N(v(\Delta))\epsilon^{ijk}Tr\big((I + \frac{1}{2}F_{ab}s_i^a s_j^b)(I - A_c s_k^c)\{(I + A_d s_k^d), V\}\big). \tag{8.130}$$

The identity will have vanishing Poisson bracket with the volume, so the only contribution will come from the connection. As the Poisson bracket is already proportional to s_k^c only the identity part of the holonomy h_{s_k} outside the bracket contributes. Finally we have that the holonomy around α_{ij}; the identity term doesn't contribute as the Poisson bracket is proportional to a Pauli matrix (since $A_c = A_c^i T_i$ and the constant matrix T_i can be taken outside the Poisson bracket) and one is taking the trace. The remaining term of $h_{\alpha_{ij}}$ yields the F_{ab}. The three lengths s's that appear combine with the summation in the limit to produce an integral.

This expression immediately can be promoted to an operator in the loop representation, both holonomies and volume promote to well defined operators there.

8.11 Spin Foams

8.11.1 Spin Foam from the Hamiltonian Constraint

A way of trying to directly link the canonical theory of LQG to a path intgeral is by expressing the (generalised) projector

$$\prod_{x \in \Sigma} \delta\big(\hat{H}(x)\big), \tag{8.131}$$

with anaology to

$$\delta(x) = \int e^{ikx} dk,$$

formally as

$$\int [dN] e^{i \int d^3x N(x) \hat{H}(x)} \tag{8.132}$$

Using this the physical inner product is formally given by

$$< \prod_{x \in \Sigma} \delta\big(\hat{H}(x)\big) s, s' >_{Diff} \tag{8.133}$$

The exponetial can be expanded

$$< \int [dN] \big(1 + i \int d^3x N(x) \hat{H}(x) + \frac{i^2}{2!} [N(x)\hat{H}(x)][N(x')\hat{H}(x')] + \dots \big) s, s' >_{Diff} \tag{8.134}$$

and each time a Hamiltonian constraint acts it does so to add a new line to a vertex. The summation over different sequences of actions of $\hat{H}$ visualized as a summation over different histories of ‘inetraction vertices’min the time evolution sending thwe initial spin network to the final spin network. We elvolve forward the initial spin network sweeping out a surface, the action of the Hamiltonian is to produce a new planar surface at the vertex. See fig (4.3.4)

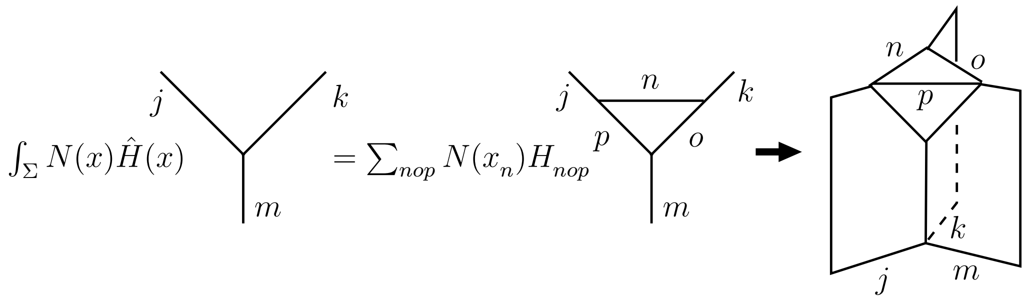

Figure 8.2: The action of the Hamiltonian constraint transalted to the ‘path-integral’ or spin foam description. Where $N(x_n)$ is the value of N at the vertex and H_{nop} are the matrix elements of the operator $\hat{H}$.

So this gives rise to a two-complex (a combinatorial set of faces that join along edges, which in turn join on vertices). The action of the Hamiltonian constraint on a vertex of a spin network associates an amplitude to each “interaction”.

There are serious difficulties with the above scheme:

(i) The $\hat{H}(N)$ do not preserve $\mathcal{H}_{Diff}$ and so $< s, Ps' >_{Diff}$ is ill defined.

(ii) There are problems chosing the appropriate measure so that there are not anomilies.

(iii) The Hamiltonian constraint operator is not self-adjoint so the exponential cannot be well defined in general.

(iv) The most serious problem is that the $\hat{H}(x)$ are not mutually commuting - it can then be shown that $\prod_{x\in\Sigma} \delta\big(\hat{H}(x)\big)$ cannot even define a (generalised) projector.

(v) Somewhat subtle matter that the $\hat{H}(x)$, while defined on the Hibert space $\mathcal{H}_{Kin}$ are not explicitly known (they are known up to spatial diffeomorphisms; they exist by the axiom of choice).

8.11.2 Spin Foam from BF Theory

It turns out that there are alternative routes to formulating the path integral

It turns out there are alternative routes to formulating the path integral, however their connection to the Hamiltonian formalism is less clear. One way is to start with the so-called BF theory. This is a simpler theory to general relativity. It has no local degrees of freedom and as such depends only on topological aspects of the fields. Surprisingly, it turns out that general relativity can be obtained from BF theory by imposing a constraint, BF theory involves a field B_{ab}^{IJ} and if one chooses the field B to be the (anti-symmetric) product of two tetrads

$$B_{ab}^{IJ} = E_{[a}^{I} E_{b]}^{J} \tag{8.135}$$

one recovers general relativity. The condition that the B field be given by the product of two tetrads is called the simplicity constraint. BF theory is what is known as a topological field theory. It might seem strange that imposing a constraint on a theory with no local degrees of freedom should then give general relativity which does have local (relational) degrees of freedom. What is actually happening when one imposes the simplicity constraints is that one is imposing constraints on the Lagrangean multipliers of the original theory, and so freeing up new degrees of freedom.

A rigorous path integral formulation can be given for this topological field theory and is well understood. One then tries to implement the simplicity conditions to obtain a path integral for general relativity. The first attempt at this was the famous Barrett-Crane model.

8.12 Semiclassical Limit

There are a couple of basic requirements in establishing the semiclassical limit

i) repreoduction fo the Poisson bracket structure;

ii) the specification of a complete set of observables whose corresponding operators when acted on by appropriate semi-classcial states reproduce the same classical variables with small quantum corrections.

8.12.1 Why might LQG not have General Relativity as its Semi-classical limit?

Any candidate theory of quantum gravity must be able to reproduce Einstein's theory of general relativity as a classical limit of a quantum theory. This is not guaranteed because of a feature of quantum field theories which is that they have different sectors, these are analogous to the different phases that come about in the thermodynamical limit of statistical systems. Just as different phases are physically different, so are different sectors of a quantum field theory. It may turn out that LQG belongs to an unphysical sector - one in which you do not recover general relativity in the semi classical limit (in fact there might not be any physical sector at all).

Theorems establishing the uniqueness of the loop representation as defined by Ashtekar et al (i.e. a certain concrete realization of a Hilbert space and associated operators reproducing the correct loop algebra - the realization that everybody was using) have been given by two groups (Lewandowski, Okolow, Sahlmann and Thiemann)[27] and (Christian Fleischhack).[28] Before this result was established it was not known whether there could be other examples of Hilbert spaces with operators invoking the same loop algebra, other realizations, not equivalent to the one that had been used so far. These uniqueness theorems imply no others exist and so if LQG does not have the correct semiclassical limit then this would mean the end of the loop representation of quantum gravity altogether.

8.12.2 Difficulties Checking the Semiclassical Limit of LQG

There are difficulties in trying to establish LQG gives Einstein's theory of general relativity in the semi classical limit. There are a number of particular difficulties in establishing the semi-classical limit

(i) There is no operator corresponding to infinitesimal spacial diffeomorphisms (it is not surprising that the theory has no generator of infinitesimal spatial 'translations' as it predicts spatial geometry has a discrete nature, compare to the situation in condensed matter). Instead it must be approximated by finite spatial diffeomorphisms and so the Poisson bracket structure of the classical theory is not exactly reproduced. This problem can be circumvented with the introduction of the so-called Master constraint (see below)

(ii) There is the problem of reconciling the discrete combinatorial nature of the quantum states with the continuous nature of the fields of the classical theory.

(iii) There are serious difficulties arising from the structure of the Poisson brackets involving the spatial diffeomorphism and Hamiltonian constraints. In particular, the algebra of (smeared) Hamiltonian constraints does not close, it is proportional to a spatial diffeomorphism constraint (which, as we have just noted, does not exist in the infinitesimal form). Moreover, the coefficient of proportionality is not a constant, it is a non-trivial function on phase space - as such it does not form a Lie algebra. However, the situation is much improved by the introduction of the Master constraint.

(iv) The semi-classical machinery developed so far is only appropriate to non-graph-changing operators, however, Thiemann's Hamiltonian constraint is a graph-changing operator - the new graph it generates has degrees of freedom upon which the coherent state does not depend and so their quantum fluctuations are not suppressed.

(v) Formulating observables for classical general relativity is a formidable problem by itself because of its non-linear nature and diffeomorphism invariance. In fact a systematic approximation scheme to calculate observables has only been recently developed.

Difficulties in trying to examine the semi classical limit of the theory should not be confused with it having the wrong semi classical limit.

8.12.3 Progress in demonstrating LQG has the Correct Semiclassical Limit

Much details here to be written up...

Concerning issue number 2 above one can consider so-called weave states. Ordinary measurements of geometric quantities are macroscopic, and planckian discreteness is smoothed out. The fabric of a T-shirt is analogous. At a distance it is a smooth curved two-dimensional surface. But a closer inspection we see that it is actually composed of thousands of one-dimensional linked threads. The image of space given in LQG is similar, consider a very large spin network formed by a very large number of nodes and links, each of Plank scale. But probed at a macroscopic scale, it appears as a three-dimensional continuous metric geometry.

As far as the author knows problem 4 of having semiclassical machinery for non-graph changing operators is as the moment still out of reach.

8.13 Master Constrain Programme

8.13.1 The Master Constraint

The Master constraint programme for LQG was proposed as a classically equivalent way of imposing the infinite number of Hamiltonian constraint operators

$$H(x) = 0$$

(x being a continuous index) in terms of a single Master constraint

$$M = \int d^3x \frac{[H(x)]^2}{\sqrt{det(q)}} \tag{8.136}$$

which involves the square of the constraints in question. Note $\hat{H}(x)$ were infinitely many whereas the Master constraint is only one. It is clear that if M vanishes then so do te infnitely many $\hat{H}(x)$'s. Conversely if all the $\hat{H}(x)$'s vanish then so does M, therefore classically they are equivalent. The Master constraint involves an appropriate averaging over all space of and so is invariant under spatial diffeomorphisms. Hence its Poisson bracket with a (smeared) spatial diffeomorphim constraint, $C(\vec{N})$, is simple

$$\{M, C(\vec{N})\} = 0 \tag{8.137}$$

(it is $su(2)$ invariant as well). Also as any quantity Poison commutes with itself, and the Master constraint being a single constraint, it satisfies

$$\{M, M\} = 0. \tag{8.138}$$

We also have the usual algebra between spatial diffeomorphisms. This represents a dramatic simplificationof the Poisson bracket structure, and raises new hope in understanding the dynamics and establishing the semiclassical limit.

An initial objection of the useof the Master constraintwas that on first sight it did not seem to encode information about observables; because the Master constraint is quadratic in the constraint, when you commute its Poisson bracket with any quantity, the result is proportional to the constraint, therefore it always vanishes when the constraints are imposed. However it was realsied that the condition

$$\{\{M, O\}, O\}_{M=0} = 0. \tag{8.139}$$

is equivalent to O being a Dirac observable. So the Master constraint does capture information about the onservables. Because of its significance this is known as the Master equation.

8.13.2 Quantising the Master Constraint

Let us write the classical expression in the form

$$M = \int d^3x \frac{H(x)^2}{\sqrt{det(q)}(x)} = \int d^3x (\frac{H}{[det(q)]^{1/4}})(x) \int d^3y \delta(x,y) (\frac{H}{[det(q)]^{1/4}})(y). \tag{8.140}$$

This expression is regulated by a one parameter function $\chi_\epsilon(x,y)$ such that $\lim_{\epsilon \to 0} \chi_\epsilon(x,y)/\epsilon^3 = \delta(x,y)$ and $\chi_\epsilon(x,x) = 1$. Define

$$V_{\epsilon,x} = \int d^3y \chi_\epsilon(x,y) \sqrt{det(q)}(y).$$

Both terms will be similar to the expression for the Hamiltonian constraint except now it will involve $\{A, \sqrt{V_\epsilon}\}$ rather than $\{A, V\}$ which comes from the additional factor $[det(q)]^{1/4}$. That is,

$$M = \int d^3x \epsilon^{abc} \{A_c^k, \sqrt{V}_\epsilon\} F_{ab}^k(x) \int d^3y \chi_\epsilon(x,y) \epsilon^{a'b'c'} \{A_{c'}^{k'}, \sqrt{V}_\epsilon\} F_{a'b'}^{k'}(y). \tag{8.141}$$

Thus we proceed exactly as for the Hamiltonian constraint and introduce a partition into tetrahedra, splitting both integrals into sums,

$$M = \lim_{\epsilon \to 0} \sum_{\Delta, \Delta'} \chi(v(\Delta), v(\Delta')) \overline{C_\epsilon(\Delta)} C_\epsilon(\Delta'). \tag{8.142}$$

where the meaning of $C_\epsilon(\Delta)$ is similar to that of H_Δ. This is a huge simplification as $C_\epsilon(\Delta)$ can be quantized precisely as the H_Δ with a simple change in the power of the volume operator. However, it can be shown that graph-changing, spatially diffeomorphism invariant operators such as the Master constraint cannot be defined on the kinematic Hilbert space $\mathcal{H}_{Kin}$. The way out is to define $\hat{M}$ not on $\mathcal{H}_{Kin}$ but on $\mathcal{H}_{Diff}$.

What is done first is, we are able to compute the matrix elements of the would-be operator $\hat{M}$, that is, we compute the quadratic form Q_M. We would like there to be a unique, positive, self-adjoint operator $\hat{M}$ whose matrix elements reproduce Q_M. It has been shown that such an operator exists and is given by the Friedrichs extension.

8.13.3 Anomalies

The Master constraint algebra

$$\begin{aligned}\vec{C}(\vec{N}), \vec{C}(\vec{N}')\} &= \kappa \vec{C}(\mathcal{L}_{\vec{N}} N') \\ \{\vec{C}(\vec{N}), \mathbf{M}\} &= 0 \\ \{\mathbf{M}, \mathbf{M}\} &= 0\end{aligned} \tag{8.143}$$

The Master constraint must preserve $\mathcal{H}_{Diff}$. This is precisely what is needed to remove the regularization ambiguities.

Whether the resulting theory is satisfactory depends solely on the question of whether the final physical Hilbert space contains enough number of semiclassicalstates in order to recover general relativity as its classical limit.

8.13.4 Solving the Master Constraint

As mentioned above one cannot simply solve the spatial diffeomorphism constraint and then the Hamiltonian constraint, inducing a physical inner product from the spatial diffeomorphism inner product, because the Hamiltonian constraint maps spatially diffeomorphism invariant states onto non-spatial diffeomorphism invariant states. However, as the Master constraint M is spatially diffeomorphism invariant it can be defined on $\mathcal{H}_{Diff}$. Therefore, we are finally able to exploit the full power of the results mentioned above in obtaining $\mathcal{H}_{Diff}$ from $\mathcal{H}_{Kin}$.

In the Master constraint programme one only uses standard spectral theory for normal operators (an operator N is normal if $NN^\dagger = N^\dagger N$) on Hilbert space in order to arrive at a direct integral decomposition (DID) of the hilbert space. The physical Hilbert space is then the induced zero eigenvalue "subspace".

8.13.5 Testing the Master Constraint

The constraints in their primitive form are rather singular, this was the reason for integrating over test functions to obtained smeared constraints. However, it would appear that equation (8.136) is even more singular involving the product of two primative constraints (although averaged over space). Squaring the constraint is dangerous from the perspective of worsened ultraviolet behaviour of the corresponding operator and hence the Master constraint programme must be approached with due care.

In doing so the Master constraint programme has been satisfactorily tested in a number of model sysstems with non-trivial constraint algebras, free and interacting field theories.[160] [161] [162] [163] [164] The Master constraint for LQG was established as a genuine positive self-adjoint operator and the Physical Hilbert space of LQG was shown to be non-empty, [165] an obvious consistency test LQG must pass to be a viable theory of quantum general relativity.

8.13.6 Applications of the Master Constraint

8.14 Black Hole Entropy

8.14.1 Introduction

The no hair theorem of general relativity states that a staionary black hole is characterized by mass, its charge, and its angular momentum; it is then argued that it can have no entropy. An oversight in the application of the no theorem is the assumption that the relevant degrees of freedom accounting for the entropy of the black hole must be classical in nature; what if they were purely quantum mechanical. This is actually what happens in the LQG derivation of BH entropy, and can be seen as a direct consequence of its background independence - the classical black hole spacetime comes about from the semiclassical limit of the quantum gravitational field, there are many are many quantum states that have the black hole in its semiclassical limit.

8.14.2 The LQG Calculation

The interior degrees of freedom of the black hole are indistguishable to an exterior observer - classically because there is a causal barrier at horizon stops the interior effecting the exterior.

Isolated horizons are generalizations of the event horizon of stationary black holes to physically more realistic situations. The generaliztion is in two directons. First, while one needs the entire space-time history to locate an event horizon, isolated horizons are defined using properties of space-time at the horizon. Second, although the horizon itself is stationary, the outside space-time can contain non-stationary fields and admit radiaton.

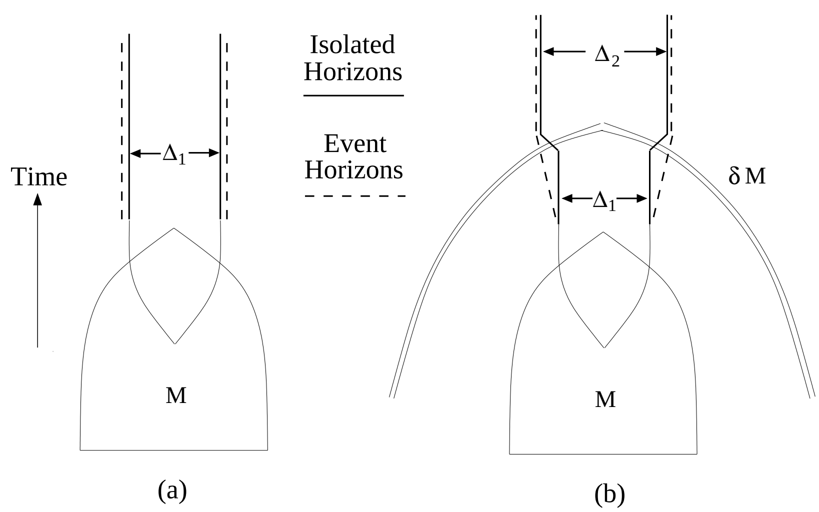

Figure 8.3: a) A spherical star of mass M undergoes collapse. b) Later, a spherical shell of mass δM falls into the resulting black hole. With Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.

the event horizons can only be identified after knowing the complete evolution of the space-time.

Physically, it should be sufficient to impose boundary conditions at the horizon the horizon that ensure only the back hole itself is isolated. That a spacetime possess such a boundary defines a certain subset of the phase space of full general relativity.

One then wishes to consider a microcanonical ensemble, in terms of the area, $(a_0 - \delta a, a_0 + \delta a)$. The quantum boundary conditions dictate that for a given state in the bulk that punctures P times the surface, the Chern-Simons state has its curvature concentrated at the punctures, one therefore has,

$$H^{phys}_{surface} = \oplus_P H^P_{surface} \tag{8.144}$$

Each puncture adds an element of area $8\pi\beta$ and introduces a deficit angle of value $2\pi m_i/k$ Where m is in the interval of $[-j_i, +j_i]$ and k is the "level" of the Chern-simons theory.

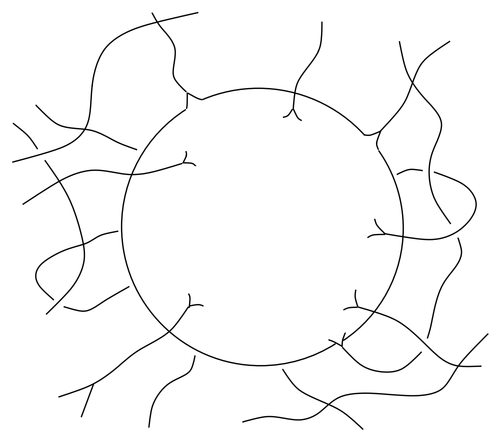

Figure 8.4: Quantum Horizon. Polymer exicitations in the bulk puncture the horizon, endowing it with quantized area. Intrinsically, the horizon is flat except at punctures where it aquires a quantized deficit angle. These angles add up to 4π.

8.15 Loop Quantum Cosmology

8.15.1 Traditional Wheeler-De Witt Quantization

8.15.2 Introduction to Loop Quantum cosmology

Loop quantum cosmology (LQC) was mainly developed by Martin Bojowald. It is a symmetry-reduced model of classical general relativity quantised using methods that mimic those of loop quantum gravity. Achievements of LQC have been a resolution of the Big-Bang singularity, the prediction of a big bounce,and a nutural mechanism for inflation.

LQC models share features of LQG and so is a useful toy model. However, the results obtained are subject to the usual restriction that a truncated classical theory, then quantised, might not display the true behaviour of the full theory due to artificial suppression of degrees of freedom that might have large quantum fluctuations in the full theory. It has been argued that singularity avoidance in LQC are by a mechanism only availablein these restrictive models nad that singularity avoidance in the full theory can still be obtained but by a more subtle feature of LQG.

"On (Cosmological) Singularity Avoidance in Loop Quantum Gravity", Johannes Brunnemann, Thomas Thiemann, Class.Quant.Grav. 23 (2006) 1395-1428.

"Unboundedness of Triad – Like Operators in Loop Quantum Gravity", Johannes Brunnemann, Thomas Thiemann, Class.Quant.Grav. 23 (2006) 1429-1484.

8.16 Loop Quantum Cosmology Phenomenology

Quantum gravity effects are notoriously difficult to measure because the Plank length is so incredibily small. However recently physicists have started to consider the possibility of measuring quantum gravity effects, mostly from astrophysical observations and gravitational wave detectors.

8.17 Background Independent Scattering Amplitudes

Loop quantum gravity is formulated in a background-independent language. No spoacetime is assumed, but rather it is built up by the states of the theory themselves - however scattering amplitudes are derived from $n-$point functions and these, formulated in conventional quantum field theory, are functions of points of a background space-time. The relation between the background-independent formalism and the conventional formalsim of quantum field theory on a given space-time is far from obvious, and it is far from obvioushow to recover low-energy quantities from the full background-independent theory. One would like to derive the $n-$point functions of the theory from the background-independent formalism, in order to compare to them with the standard perturbative expansion of quantumgeneral relativity and therefore check that loop quantum gravity yields the correct low-energy limit.

A strategy for addressing this problem has been suggested; (L Modesto, C Rovelli:"Particle scattering in loop quantum gravity", Phys Rev Lett 95 (2005) 191301) the idea is to study the boundary amplitude, namely a path integral over a finite space-timeregion, seenas a function of the boundary value of the field. (R Oeckl, "A 'general boundary' formulation for quantum mechanics and quantum gravity", Phys Lett B575 (2003) 318-324 ; "Schrodinger's cat and thye clock: Lessons for quantum gravity", Class Quant Grav 20 (2003) 5371-53801) In conventional quantum field theory, this boundaryamplitutde is well-defined (F Conrady, C Rovelli "Generalised Tomonaga-Schwinger equation from the Hadamard formula", Phys Rev D70 (2004) 064037) and codes the physical information of the theory; it does so in quantum gravity as well, but in a full background-independent manner.)(F Conrady, L Dopicher, R Oeckl, C Rovelli, M

Testa, "Minkowski vacuumin background independent quantum gravity", Phys Rev D69 (2004) 064019) A generally covariant definition of $n-$point functions can then be based on the idea that the distance between physical points - arguments of the $n-$point function is determined by the state of the gravitational field on the boundary of the space-time region considered.

Progress has been made in calculating background-independent scattering amplitudes this way-with the use of spin foams. This is a way to extract physical information from the theory. Claims to have reproduced the correctbehaviour for graviton scattering amplitudes and to have recovered classical gravity have been made "We have calculated Newton's law startting from a world with no psace and no time" - Carlo Rovelli

8.18 The Problem of Time in Quantum Gravity

Roughly speaking the problem of time is that there is none in general relativity; since the Hamiltonian in general relativity is a constraint that must vanish, butin any canonical theory the Hamiltonian generates time translations we arrive at the conclusion that 'nothing moes' in general relativity. Since there is no time the usual interpretation of quantum mechanics at a given moment of time breaks down. The problem of time is the broad banner for all interpretational problems of the mathematical formalism.

8.19 Outlook

8.20 Other Approaches to Quantum Gravity

8.20.1 String Theory

"If I were to hazard a guess on future developments, I'd imagine that the background-indpendent techniques developed by the loop quantum gravity community will be adapted to string theory, paving the way for a string formulation that is background indepndent. And that spark, I suspect, that will ignite a third susperstring revolution in which, I'm optimistic, many of the remaining deep mysteries will be solved".

8.20.2 Causal Dynamical Triangulations

Another background independent approach to quantising gravity.

A new approach to the nonperturbative (background independent) quantization of gravity, that of Causal Dynamical Triangulations which recently has produced a number of remarkable results. These include a dynamical derivation of the fact that space-time is four-dimensional (something that can be taken for granted only in classical gravity) and that it has the shape of a de Sitter Universe (like our own universe in the absence of matter), and of the so-called wave function of the universe which plays an important role in understanding the quantum behaviour of the very early universe. Remarkably, one also finds that the dimensionality of spacetime reduces smoothly to two at short distances, indicative of a highly nonclassical behaviour of spacetime geometry near the Planck scale. These results are obtained by superposing elementary quantum excitations of geometry which have a notion of causality ("cause preceding effect") built into them at the very smallest scale.

8.20.3 Consistent Discrete Quantum Gravity

8.20.4 Noncommutative Geometry

8.20.5 Twistor Theory

Twistor Theory was invented by Sir Roger Penrose. : entire light-rays are represented as points, and events by entire Riemann spheres. Twistors defined in terms of a pair of spinors. Twisters are the coordinates of twister space. In twister theory spacetime is a secondary concept.