

# Volume I

## Advanced and Modern General Relativity



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- Chapter 2: Introduction to General Relativity and its physical Observables
- Chapter 3: Black holes - Event, Isolated and Dynamical Horizons
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# Terminology and Notation

Here is a list of symbols.

$[ , ]$	commutator
$\{ , \}$	Poisson bracket
$\dagger$	Hermitian conjugation
$:=$	definition
$\equiv$	identity
$\stackrel{*}{=}$	only true in a special coordinate system
iff	If and only if
$\eta_{ab}$	Minkowski metric
$\eta(x)$	test function of a variation of action
$\mathcal{A}$	space of gauge fields or area
$A_\mu(x)$	Yang-Mills connection
$D_\mu$	covariant derivative
$\mathcal{M}$	spacetime manifold
$\mathbf{M}$	The Master constraint
$\hat{\mathbf{M}}$	The Master constraint operator
$\omega_{\mu\beta}^\alpha$	spin connection
$\mathcal{C}$	constraint surface in phase space
$S$	labells spin-network
$s$	equivalent class of spin-networks under the action of Diff denoted $s$ - knots
$s(S)$	denotes equivalent class $S$ to which belongs
$g_{ab}$	spacetime metric
$K_{ab}$	extrinsic curvature of $\Sigma$
$G_{ab}$	Einstein tensor
$T_{ab}$	The energy-momentum tensor
$e_I^a, E_i^a$	tetrad and triad
$\mathcal{L}_t$	Lie derivative with respect to $t$
$n_a$	unit normal to $\Sigma_t$
$N, (\tilde{N})$	lapse function (density)
$N^a$	shift vector on $\Sigma$
$\Omega_{\alpha\beta}$	symplectic form
$\mathcal{A}/\mathcal{G}$	space of gauge fields moduli gauge transformations
$[A]$	gauge equivalence classe of the connection $A$
$\mathcal{HA}$	the holonomy algebra
$\overline{\mathcal{HA}}$	the completion of the holonomy algebra in the norm $\ f\  := \sup_{[A] \in \mathcal{A}/\mathcal{G}}  f([A]) $
$\overline{\mathcal{A}/\mathcal{G}}$	spectrum of $\overline{\mathcal{HA}}$

# Preface



## Acknowledgments

Discussions with Tong,Pun Wai on parts of the proof of the singularity theorems.

# Introduction

the beginning of the revolutionary contributions to physics by Einstein,

# Paths through the report

Introductory book on general relativity [?]