

Volume I
Advanced and Modern
General Relativity



Draft version

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Contents:

- Chapter 1: Einstein's hole argument Complete, Partial and Dirac Observables
- Chapter 2: Introduction to General Relativity and its physical Observables
- Chapter 3: Black holes - Event, Isolated and Dynamical Horizons
- Chapter 4: Cosmology
- Chapter 5: Hawking Penrose Singularity Theorems
- Chapter 6: Consistent Discrete Classical GR
- Chapter 7: Quantum Field Theory on Curved Spacetimes
- Chapter 8: Introduction to Quantum General Relativity
- Glossary
- Many Detailed Appendices with Worked Exercises

Chapter 1: Classical GR, Einstein's hole argument and Physical geometry

- Einstein's hole argument.
- Measurements of geometry.
- Conceptual issues.
- Relational mechanics.
- Partial, complete and Dirac observables.

Chapter 2: Introduction to General Relativity and its Physical Observables

- Special Relativity
- Principles of General Relativity
- Spacetime Measurements
- GPS Observables
- Action Principle
- Perturbations Around Exact Solutions

Chapter 3: Black holes - Event, Isolated and Dynamical Horizons

- Event Horizons and Penrose Diagrams
- Non-Expanding Horizons
- Weakly Isolated Horizons and Generalisations of the Laws of Black Hole Mechanics
- Isolated Horizons and Rotating Isolated Horizons
- Dynamical Horizons

Chapter 4: Cosmology

- Classical Cosmology
- Homogeneous and Isotropic Cosmology
- Outline of the Singularity Theorems
- Gauge Invariant Perturbations
- .

Chapter 5: Hawking Penrose Singularity Theorems.

- Basic Definitions
- Strong Causality
- The Space of Causal Curves
- Conjugate Points
- Singularity Theorems
- Collapse of a Star
- The Big Bang
- Initial Value Problem

Chapter 6: Consistent Discrete Classical GR.

- Introduction.
- .
-
- .

Chapter 7: Quantum Field Theory on Curved Spacetimes

- Introduction
- Quantum Field Theory on Flat Spacetime
- Quantum Field Theory on Curved Spacetime
- Quantum Fields in an Expanding spacetime
- Quantum Fields During Inflation
- Hawking Radiation
- Algebraic Approach
- Back Reaction
- The Need for Quantum Gravity
- Stochastic Gravity

Chapter 8: Introduction to Quantum Gravity.

- Introduction to Quantum General Relativity
- Ashtekar-Barbero Variables
- Quantum Constraints - the Equations of Canonical Quantum Gravity
- The Loop Representation
- Geometric Operators
- Spin Networks
- The Hamiltonian Constraint and the Modern formalism
- Spin Foams
- Semi-Classical Limit

- The Master Constraint Programme
- Physical Applications: Black Hole Entropy, Loop Quantum Cosmology, Quantum Gravity Phenomenology, and Background-independent Scattering Amplitudes
- The problem of Time
- Other Approaches

Appendices

- A Physics Glossary
- B Mathematics Glossary
- C Mathematics
- D Physical Geometry and Derivation of Einstein's Field Equations
- E Constrained Hamiltonian Systems, Dirac Observables and the Constraint Algebra
- E ADM and First Order Formulation of Einstein's Equations
- F Basic Functional Analysis
- H Details of Hawking's Calculation

Contents

1	Classical GR, Einstein’s hole argument and Physical Geometry	52
1.1	Einstein’s Hole Argument	54
1.2	Background Independence - A Farewell to Spacetime	67
1.2.1	Comparison of GR with the Rubber Sheet Analogy	67
1.2.2	The View of the World that Emerges	69
1.2.3	Common Misunderstandings	72
1.2.4	The Blessing of background independence - Non-Perturbative Quantum Gravity Finite and Requires No Renormalization!	73
1.3	Physical Geometry	76
1.3.1	Physical GPS Coordinates	77
1.3.2	Physical Area	78
1.3.3	Description of a Measurement of Area	79
1.3.4	Einstein’s Field Equations	81
1.3.5	The velocity composition law	82
1.3.6	Dust as Matter Reference System	82
1.4	Some conceptual issues	82
1.5	Relational Mechanics	83
1.5.1	Covariant Hamiltonian Formulation	83
1.5.2	Deparametrizable Mechanics: Identification of a “time” variable	86
1.5.3	Fully Constrained Hamiltonian Systems	87

1.6	Partial, Complete and Dirac Observables	88
1.6.1	Infinitely Many Constraints	91
1.6.2	Observables for Canonical General Relativity	92
1.6.3	Approximate Observables for Canonical General Relativity	93
1.7	The Problem of Time	94
1.8	The problem of Quantum Cosmology	96
2	Introduction to General Relativity and its Physical Observables	97
2.1	Introduction	97
2.2	Special Relativity	97
2.2.1	Simultaneity	98
2.2.2	Time Dialation	98
2.2.3	Length Contraction	98
2.2.4	Lorentz Transformations	99
2.2.5	Velocities	100
2.2.6	Acceleration	102
2.2.7	Doppler Effect	102
2.2.8	Relativistic Mass and Energy	103
2.2.9	The Twin Paradox	103
2.3	The Principles of General Relativity	103
2.3.1	The Principle of Equivalence	103
2.3.2	The Gravitation Red-shift: Warping Time	104
2.3.3	The Curvature of Spacetime	105
2.3.4	Curvature in a Weak Uniform Gravitation field	106
2.3.5	The Principle of General Relativity	108
2.3.6	Background Independent Theories	108
2.3.7	Einstein's Hole Argument	108

2.4	Observables	110
2.5	Tensor Calculus	112
2.5.1	Tensors	112
2.5.2	Covariant Derivative	113
2.5.3	The Metric Connection	114
2.5.4	Curvature Tensor	114
2.6	Space-Time Measurements	115
2.6.1	Measurements of Time Intervals and Space Distances	115
2.6.2	Measurement Tools	117
2.6.3	Geodesic Deviation	118
2.6.4	World Function	120
2.6.5	Measurement of Relative Velocities	123
2.6.6	Derivation of Lorentz Transformation Formula in GR	123
2.7	Derivation of Vacuum Field Equations	123
2.7.1	The Newtonian Equation of Deviation	123
2.7.2	Equation of geodesic deviation	124
2.7.3	The Newtonian Correspondence	129
2.7.4	The Vacuum Field Equations	131
2.8	GPS Observables	134
2.9	Measurement of an Area	134
2.10	Matter	135
2.10.1	Dust	135
2.10.2	Perfect Fluid	139
2.10.3	Maxwell's Equations	141
2.10.4	Scalar Field	142
2.10.5	Yang-Mills	143
2.10.6	Fermionic Matter	143

2.10.7	Energy-Momentum Tensor	146
2.11	Action Principle	146
2.11.1	Invariance of the Einstein-Hilbert Action	146
2.12	Palatini Method in the Metric Formulation	148
2.13	Cosmological Definition of Distance	150
2.14	Relativistic Material Reference Systems	151
2.15	Linearized Equations of General Relativity	152
2.15.1	Gauge Transformations	154
2.15.2	Linearized Einstein Equations in the Temporal Gauge	156
2.15.3	Gravitational Wave Solutions	156
2.15.4	Waves Emitted by Oscillating Masses	158
2.16	Classical Cosmology	158
2.16.1	Fluid Flow Equations	158
2.16.2	Newtonian Cosmology	159
2.16.3	Relativistic Cosmology	161
2.16.4	Spaces of Constant Curvature	161
2.17	Homogeneous and Isotropic Cosmology	162
2.17.1	Friedmann's Equation - Universe with Dust	162
2.17.2	The Luminosity Distance	165
2.18	Models with a Cosmological Constant	166
2.18.1	Flat Universe	166
2.18.2	The de Sitter Model	167
2.19	Perturbations of Exact Solutions	167
2.19.1	Gauge Dependency in Perturbation Theory	168
2.20	Cosmological Perturbation Theory	176
2.20.1	Scalar-Vector-Tensor Decomposition	176
2.20.2	Choice of Gauge	178

2.21	Perturbations of Black Holes	178
2.22	Approximation to Observables of the Full Theory	178
2.23	Gravity from Gravitons?	179
2.24	Some Things of String Theory	180
2.25	Biblioliographical notes	181
2.26	Worked Exercises and Details	181
2.27	Backreaction Issues in Relativistic Cosmology and the Dark Energy Debate	184
2.27.1	Cosmological Perturbation Theory	184
2.28	Gauge Invariant Perturbations Around Symmetry Reduced Sectors of Gen- eral Relativity: Applications to Cosmology	185
2.28.1	Introduction	185
2.29	Approximate Complete Observables	187
2.30	Application to Cosmology	191
3	Isolated and Dynamical Horizons - Generalizations of Stationary Black Holes	198
3.1	Review of Stationary Black Holes	199
3.1.1	Mass and Angular Momentum of Bodies in Newtonian Gravity and Special Realtivity	199
3.2	The Schwarzschild Metric	201
3.3	Schwarzschild Black Hole	211
3.3.1	Eddington-Finkelstein Coordinates	211
3.4	Internal Schwarzschild Solution	214
3.5	Penrose-Carter diagrams	217
3.5.1	Penrose-Carter Diagram for Minkowski Spacetime.	218
3.5.2	Maximal Extensions	220
3.5.3	The Kruskal Solution	220
3.6	Charged Balck Holes	220

3.6.1	Event Horizons	226
3.6.2	Analogue of Eddington-Finkelstein Coordinates	227
3.6.3	Penrose Diagram	229
3.6.4	Double null coordinates	229
3.6.5	Maximal extension	231
3.7	Rotating Black Holes	232
3.7.1	Field Equations	232
3.7.2	The Kerr Solution	261
3.7.3	Independence of Metric on Angular Variable φ	266
3.7.4	Boyer-Lindquist Coordinates	269
3.7.5	Interpretation as Rotating Body	272
3.7.6	Basic Properties of the Kerr Solution	272
3.7.7	Singularities and Event Horizons	273
3.8	Raychaudhuri equations	274
3.8.1	Null geodesic congruences	274
3.9	Properties of Null Surfaces	274
3.9.1	geodesics: Expansion, Rotation, and Shear	279
3.9.2	Frobenius' Theorem	280
3.9.3	Null Case	281
3.9.4	Killing Horizons	282
3.10	Laws of (Stationary) Black Hole Mechanics	283
3.10.1	Zeroth law	283
3.10.2	First law	283
3.10.3	Second law	284
3.10.4	Third law	289
3.10.5	Quasi-Local Generalizations	289
3.10.6	Black Hole Thermodynamics	289

3.11	Definitions	290
3.12	Non-Expanding Horizons (NEH)	291
3.13	Weakly Isolated Horizons (WIH) and Generalization of the Laws of Black Hole Mechanics	292
3.13.1	Zeroth law	292
3.13.2	First law	293
3.13.3	Second law	295
3.14	Isolated Horizons	295
3.14.1	Boundary Conditions for Isolated Horizons	295
3.15	Rotating Isolated Horizons	296
3.15.1	Basic Review of Multipoles	297
3.15.2	Spherical Harmonics	302
3.15.3	Invariant Coordinates on the Horizon	303
3.15.4	Definition of Geometric Multipoles	306
3.16	Dynamical Horizons	307
3.16.1	Gravitational Energy Flux	309
3.16.2	Rotating Dynamical Horizons	310
3.17	Null Tetrads and Spinor Analysis	310
3.17.1	Spinor Analysis in GR	312
3.17.2	Curvature Spinors	322
3.17.3	Curvature in spinors	332
3.17.4	Spinor Form of the Ricci Identities	335
3.17.5	Einstein's equations	338
3.17.6	Spinor form of the Bianchi identity	340
3.17.7	Newman-Penrose Formalism in Spinor Form	341
3.17.8	Petrov Classification	348
3.17.9	Focussing and Shearing of Null Curves	352

3.17.10	Goldberg Sachs Theorem	354
3.17.11	Tetrad Formulism and the Cartan Structure Equations	364
3.17.12	Specialisation to Null Tetrads	369
3.18	Summary	374
3.19	Biblioliographical notes	374
3.20	Worked Exercises and Details	375
3.20.1	Non-Expanding Horizons	378
3.20.2	Weakley Isolated Horizons	380
3.20.3	Isolated Horizons	380
3.20.4	Rotating Isolated Horizons	386
3.20.5	Dynamical Horizons	389
4	Classical Cosmology	420
4.1	Classical Cosmology	420
4.1.1	Fluid Flow Equations	420
4.1.2	Newtonian Cosmology	421
4.1.3	Relativistic Cosmology	422
4.1.4	Spaces of Constant Curvature	423
4.2	Homogeneous and Isotropic Cosmology	423
4.2.1	The Luminosity Distance	423
4.3	The Singularity Theorems	424
4.3.1	Application of the Singularity Theorem: Cosmological Singularity .	426
4.3.2	Energy Conditions	427
4.3.3	Causality and Chronology	427
4.3.4	Existence of maximum geodesic	429
4.3.5	The Significance of Conjugate Points: The Singularity Theorems . .	432
4.4	Backreaction Issues in Relativistic Cosmology and the Dark Energy Debate	432

4.4.1	Cosmological Perturbation Theory	432
4.5	Gauge Invariant Perturbations Around Symmetry Reduced Sectors of General Relativity: Applications to Cosmology	433
4.5.1	Introduction	433
4.6	Approximate Complete Observables	435
4.7	Application to Cosmology	439
5	Proof of the Hawking-Penrose Singularity Theorems	445
5.1	Proof of the Hawking-Penrose Singularity Theorem	445
5.1.1	Introduction	445
5.1.2	Some Basic Terminology	446
5.1.3	The Singularity Theorem of Hawking and Penrose	450
5.1.4	Basic Definitions	452
5.1.5	Achronal Sets	455
5.1.6	Strong Causality	459
5.1.7	The Space of Causal Curves	469
5.1.8	Conjugate Points	479
5.1.9	The Singularity Theorems	497
5.1.10	Implication of the “Displayed” Singularity Theorem from the Established Version	500
5.2	Black Holes	526
5.2.1	Collapse of a Star	527
5.2.2	The Cauchy Problem - Existence and Uniqueness	531
5.2.3	Non-Linear Hyperbolic Differential Equations	532
5.2.4	Existence and Uniqueness	536
5.2.5	Cauchy-Kowalewski Theorem	537
5.2.6	Reduced Einstein Equations	538
5.2.7	Sobolev Inequalities	541

5.2.8	Developments for the Empty Space Einstein Equations	542
5.2.9	Stability of Closed Trapped Surfaces	543
5.2.10	Apparent Horizons	543
5.3	The Big Bang	543
5.4	Biblioliographical notes	544
5.5	Sobolev Spaces	544
5.5.1	Review of L^P Spaces	544
5.5.2	Hardy-Littlewood-Sobolev Inequality	546
5.5.3	Sobolev Inequalities	549
5.5.4	Classical Sobolev Spaces	555
5.5.5	Fractional H^s - Sobolev Spaces	558
5.6	Worked Exercies and Details	558
6	Consistent Discrete Classical GR	563
6.0.1	“Dirac’s” canonical approach to general discrete systems	565
7	Quantum Field Theory on Curved Spacetimes	567
7.1	Motivation	567
7.2	Introduction	567
7.3	Quantum Field Theory in Flat Spacetime	568
7.3.1	The Simple Harmonic Oscillator	568
7.3.2	Quantisation of the Klein-Gordon field in Flat Spacetime	569
7.3.3	Mode Expansion	570
7.3.4	Behaviour of Fock Basis Under Lorentz Transformations	570
7.3.5	Fock Particles	570
7.4	Quantum Field Theory on Curved Spacetimes	571
7.4.1	Bogolyubov Transformations	571
7.4.2	a -Particles in the b -Vacuum	572

7.5	Quantum Fields in an Expanding Universe	572
7.5.1	Particle Interpretation	572
7.6	Quantum Fields During Inflation	573
7.7	The Unruh Effect	573
7.8	Hawking Radiation	573
7.8.1	Hand Wavy Calculation	573
7.8.2	Hawking's Calculation	574
7.8.3	Rotating Black Holes and Higher Spin Fields	577
7.8.4	Black Hole Thermodynamics	578
7.8.5	Information Loss Paradox	578
7.9	Backreaction	578
7.10	The Algebraic Approach	578
7.10.1	Algebraic Quantum Theory	578
7.10.2	Wightman Axioms in Minkowski Spacetime	578
7.10.3	Revision	579
7.10.4	Operator Product Expansion	579
7.10.5	Algebraic Quantum Field Theory	579
7.10.6	Viewpoint on QFT	579
7.10.7	Global and Local Particles	579
7.11	The Need For Quantum Gravity	580
7.11.1	Backreaction	580
7.11.2	Information Loss Paradox	580
7.12	Stochastic Gravity	580
8	Introduction to Quantum General Relativity	581
8.1	The Problem of Quantising General Relativity	581
8.1.1	The Problem of Time in Canonical Quantum Gravity	582

8.2	Introduction to Loop Quantum Gravity (LQG)	582
8.3	ADM Metric 3+1 Formulation	582
8.4	The New Variables	583
8.4.1	Triad and connection formulation of GR	583
8.4.2	Ashtekar's new variables	586
8.4.3	Derivation of Ashtekar's Formalism from the Self-dual Action	586
8.5	Ashtekar-Barbero Variables	590
8.5.1	The Holst Action	590
8.5.2	3+1 Decomposition of the Holst Action	591
8.5.3	Introduction to Canonical Transformations	592
8.5.4	Canonical Transformation on Extended Phase Space: Obtaining the Gauss Law	594
8.5.5	Poisson Brackets for New Variables	597
8.5.6	The constraints in the New Variables	598
8.5.7	The Constraints	598
8.5.8	The Poisson bracket algebra	600
8.5.9	Observables	601
8.6	Quantisation of the Constraints - the Equations of Quantum General Relativity	601
8.6.1	Quantum Constraints as the Equations of Quantum General relativity	602
8.7	The Loop Representation	603
8.7.1	The loop transform	604
8.7.2	Solutions to all the Constraints	605
8.8	Geometric operators	605
8.8.1	The Area Operator	605
8.8.2	The Volume Operator	607
8.8.3	Physical Meaning of these Results	607

8.9	Spin Networks	607
8.10	Hamiltonian Constraint and the Modern Formulism	607
	8.10.1 Deriving Thiemann’s Identity and Other equations	609
	8.10.2 Quantising the Hamiltonian Constraint	609
8.11	Spin Foams	611
	8.11.1 Spin Foam from the Hamiltonian Constraint	611
	8.11.2 Spin Foam from BF Theory	612
8.12	Semiclassical Limit	613
	8.12.1 Why might LQG not have General Relativity as its Semiclassical limit?	613
	8.12.2 Difficulties Checking the Semiclassical Limit of LQG	614
	8.12.3 Progress in demonstrating LQG has the Correct Semiclassical Limit	614
8.13	Master Constrain Programme	615
	8.13.1 The Master Constraint	615
	8.13.2 Quantising the Master Constraint	616
	8.13.3 Anomalies	617
	8.13.4 Solving the Master Constraint	617
	8.13.5 Testing the Master Constraint	617
	8.13.6 Applications of the Master Constraint	618
8.14	Black Hole Entropy	618
	8.14.1 Introduction	618
	8.14.2 The LQG Calculation	618
8.15	Loop Quantum Cosmology	619
	8.15.1 Traditional Wheeler-De Witt Quantization	619
	8.15.2 Introduction to Loop Quantum cosmology	619
8.16	Loop Quantum Cosmology Phenomenology	620
8.17	Background Independent Scattering Amplitudes	620

8.18	The Problem of Time in Quantum Gravity	621
8.19	Outlook	621
8.20	Other Approaches to Quantum Gravity	621
8.20.1	String Theory	621
8.20.2	Causal Dynamical Triangulations	621
8.20.3	Consistent Discrete Quantum Gravity	622
8.20.4	Noncommutative Geometry	622
8.20.5	Twistor Theory	622
A	Physics Glossary	623
B	Mathematics Glossary	688
C	Mathematics	743
C.1	Introduction	743
C.2	Action Principle	743
C.2.1	Variational Derivative and the Euler-Lagrange Equation	743
C.2.2	Action Principle for a Particle	744
C.2.3	Action Principle for Fields	745
C.2.4	Noether's Theorem	745
C.2.5	Action Principle with Variation of Boundary	747
C.2.6	Lagrange multipliers	749
C.3	Some Inequalities	749
C.4	Curvilinear Coordinates	751
C.5	Totally Antisymmetric symbol and Determinates	752
C.5.1	Variation of a Determinate	756
C.5.2	Tensor Densities	757
C.5.3	Divergence Theorem	757

C.5.4	Summary of Tensor Calculus	758
C.5.5	Linear operators and Matrices	759
C.6	Group Theory	760
C.6.1	Examples of Groups	760
C.6.2	Unitary Representations of Groups	765
C.6.3	Schur's First Lemma	766
C.6.4	Schur's Second Lemma	768
C.6.5	Orthogonality relations	770
C.6.6	The Characters of a Representation	771
C.6.7	Direct Products	773
C.7	Continuous Groups, Lie Groups and Lie algebras	774
C.7.1	Infinitesimal Generating Technique	775
C.7.2	General Structure of Lie Groups	778
C.7.3	Rotations $SO(3)$ and $SU(2)$	779
C.7.4	Spin Direct Products	781
C.7.5	Direct Products and Clebsch-Gordan Coefficients	788
C.7.6	Recoupling Theory	791
C.7.7	$SO(3,1)$ and $SL(2,C)$	791
C.7.8	$SO(4)$	793
C.7.9	Conformal Group	794
C.7.10	Group Integration: The Haar Measure	795
C.7.11	Peter-Weyl theorem	800
C.7.12	Analogies	802
C.7.13	Clebsch-Gordan	804
C.7.14	Semi-direct Products	804
C.8	Infinite-Dimensional Group Representations	808
C.8.1	Group Actions	808

C.8.2	Countable and Locally Compact Topological Groups	808
C.8.3	Haar Measure	809
C.8.4	Summary of Group theory	809
C.9	Manifolds and Elementary Topology	810
C.9.1	Sets and Mappings Between Sets	810
C.9.2	Continuity	811
C.10	Elementary Tensor Analysis	813
C.10.1	Affine Connection	815
C.10.2	Affine Geodesic	817
C.10.3	The Metric Connection	818
C.10.4	Curvature	820
C.10.5	Gaussian Normal Coordinates	822
C.10.6	Bianchi Identities	823
C.10.7	Conformal Tensor, Ricci tensor and Ricci Scalar	824
C.10.8	The Weyl Tensor	824
C.10.9	Index Free Formulism	824
C.11	Differential Geometry	827
C.11.1	Tangent Vectors	827
C.11.2	Covectors	829
C.11.3	Induced Metric and Other Objects on Sub-manifolds	830
C.12	Active Diffeomorphisms and the Lie Derivative	832
C.12.1	Mapping a Manifold to Itself Along Integral Curves	834
C.12.2	The Lie Derivative	835
C.12.3	Pull-back and Lie Derivative of a co-vector	842
C.12.4	More on Lie Derivative	843
C.12.5	Isometries and Killing Vector Fields	844
C.12.6	Conserved Quantities	846

C.12.7	Adapted Coordinates	847
C.12.8	Properties of Killing Fields	848
C.12.9	Diffeomorphism Gauge Group - Symmetry of GR Under Active Diffeomorphisms	848
C.13	Frame Fields	849
C.14	The Spin Connection	850
C.14.1	Curvature Associated with the Spin Connection.	852
C.15	Differential Forms	854
C.15.1	Exterior Calculus	859
C.15.2	Exterior Derivatives	859
C.15.3	Maxwell's equations in differential forms	865
C.15.4	Integration on a Manifold	870
C.15.5	Cartan Structure Equations	874
C.15.6	A Differential Geometry Translator	875
C.16	More on Lie groups	876
C.16.1	Discrete Groups	879
C.16.2	Universal Covering Group	879
C.17	Group Actions on Sets	880
C.17.1	Transitive Actions	882
C.17.2	Faithful Actions	882
C.17.3	Free Actions	882
C.17.4	Introduction to Gauge Invariance of the Yang-Mills Equations	883
C.18	Principle Bundles and Connections	886
C.19	Fibre Bundles	887
C.19.1	The Structure Group of a Bundle	893
C.19.2	Frame Bundle	893
C.19.3	The Idea of a Principal Bundle	894

C.19.4	Principal Bundle	895
C.19.5	Action of the structure Group on a Principal Bundle	896
C.19.6	Connections on Principal Bundles	896
C.19.7	Gauge Fields	900
C.19.8	Parallel Transport in a Principal Bundle	906
C.19.9	Curvature on a Principal Bundle	906
C.19.10	Extension and Reduction of Principal Bundles	907
C.19.11	The Complex Line Bundle	909
C.20	Summary of Differential Geometry	909
C.21	Summary	909
C.22	Biblioliographical notes	910
C.23	Worked Exercises and Details	910
C.23.1	Dynamical and Non-Dynamical Symmetries	910

D Constrained Hamiltonian Systems, Dirac observables and the Constraint Algebra 914

D.0.2	Introduction	914
D.1	Hamiltonian Mechanics	915
D.1.1	Poisson Brackets	917
D.1.2	Symplectic Geometry and Phase Space	919
D.1.3	Canonical Transformations	922
D.1.4	Infinitesimal Contact Transformations	926
D.1.5	Noether's Theorem	928
D.2	Geometry of Configuration Space and Phase Space	929
D.2.1	Vector Fields on Configuration Space and Phase Space	929
D.2.2	The Lie Derivative	929
D.2.3	Definition of Hamiltonian System	930

D.2.4	Symplectic Geometry of Phase Space	931
D.2.5	Canonical Transformations	933
D.2.6	The Hamiltonian Framework: Résumé	934
D.2.7	Connection to quantum mechanics	934
D.3	Covariant Phase Space	935
D.3.1	Space of Solutions	935
D.3.2	Field Theory	936
D.3.3	Hamiltonian-Jacobi Theory	937
D.3.4	Hamilton principal function	940
D.4	Solving for the Dynamics using the HJ Equation	942
D.4.1	1. Free particle (one-dimension)	942
D.4.2	2. The Harmonic oscillator (one-dimension)	946
D.4.3	Hamiltonian Characteristic Function	948
D.4.4	‘Derivation’ of Schrodinger’s Equation	950
D.5	Constrained Hamiltonian Systems	952
D.5.1	Examples	956
D.5.2	Dirac’s Procedure for Constrained Hamiltonian Systems	958
D.5.3	First Class Constraints and Gauge Symmetries	962
D.5.4	Dirac Method and Electrodynamics	963
D.5.5	Quantization of Constrained Hamiltonian Systems	964
D.5.6	Dirac Observables	964
D.5.7	Darboux’s Theorem	964
D.5.8	Symplectic Reduction	966
D.5.9	Poisson Reduction	982
D.5.10	Symplectic Group Actions	982
D.6	Worked Examples	983
D.7	Open Constraint Algebras	985

D.7.1	The Master Constraint	985
D.8	Partial, Complete and Dirac Observables for Hamiltonian Constrained Systems	985
D.8.1	Weak Dirac Observables	987
D.8.2	Backreaction	989
D.8.3	Different Observers	989
D.8.4	Systems with Several Constraints	989
D.8.5	Partial Differential Equations for Complete Observables	990
D.8.6	Partially Invariant Partial Observables	994
D.9	A Perturbative Approach to Dirac Observables and Their Spacetime Algebra	996
D.9.1	Introduction	996
D.9.2	The Approximate Dirac Observable	996
D.9.3	Abelianization	998
D.9.4	Approximations	999
D.9.5	Dynamics	1003
D.9.6	The second Order Approximation	1004
D.9.7	Application to General Relativity	1005
D.9.8	ADM Clock Variables	1005
D.9.9	Gravity Coupled to a Scalar Field	1006
D.9.10	Control of Gauge Dependence	1007
D.9.11	Outlook and Summary	1007
D.10	Reduced Phase Space Quantization of Constrained Theories	1008
D.10.1	Reduced Phase Space Quantization with Dirac Observables	1008
D.11	Bibliographical notes	1009
D.12	Worked Examples	1010
D.12.1	Poisson Brackets	1010
D.12.2	Worked Examples: Dittrich	1014

D.12.3	Worked Examples: Brute force Thiemann	1027
E	ADM and First order Formalism of Einstein's Theory	1043
E.1	Intrinsic and Extrinsic Curvature	1043
E.2	ADM Metric Formulation	1044
E.3	Stuff	1046
E.4	The Hamiltonian Formulation	1046
E.5	The Cauchy Problem	1046
E.6	Gravitational Hamiltonian	1047
E.6.1	Boundary Term	1049
E.6.2	Constraint Algebra	1049
E.7	First Order Formulation of Einstein Equations	1053
E.8	Palatini Method in the Connection Formulation	1053
E.8.1	Method I	1053
E.8.2	Method II	1055
E.9	Inclusion of Matter	1060
E.9.1	Yang-Mills	1060
E.9.2	Klein-Gordan - Scalar Matter Field	1060
E.9.3	Fermionic Matter	1061
E.9.4	In the Language of Differential Geometry	1061
E.10	Self-dual Connection Formulation	1062
E.10.1	Self-dual Curvature	1062
E.10.2	Self-dual Action	1067
E.11	Ashtekar's Canonical Formalism	1069
E.12	Generators of Symmetry Transformations	1069
E.12.1	The Gauss-law Constraint Generates Gauge Transformations	1070
E.12.2	Incorporating Matter in the Quantum Theory	1071

E.13	Toy Model: Free Particle described using Half-Complex Coordinates.	1071
E.13.1	Complex Variables and Reality Conditions	1071
E.13.2	Quantization in Complex Coordinates.	1072
E.14	The Holst Action	1073
E.14.1	3+1 Decomposition of the Holst Action	1073
E.14.2	The Diffeomorphism Constraint	1073
E.14.3	The Hamiltonian Constraint	1075
E.14.4	Addition Constraints	1075
E.14.5	Final Total Hamiltonian	1077
E.15	Bibliographical notes	1077
E.16	Worked Exercises and Details	1078
F	Basic Functional Analysis	1085
F.1	Finite Hilbert Space.	1085
F.1.1	The Hamilton-Cayley Theorem.	1085
F.1.2	Projection Operators	1086
F.1.3	Spectral Theorem for Finite Spaces	1089
G	Quantum Field Theory	1093
G.1	Elementary Quantum Mechanics	1093
G.1.1	Path Integrals and Functional Integrals	1093
G.1.2	Semi-Classical Limit	1094
G.1.3	Second Quantization	1095
G.1.4	N Real variables	1095
G.1.5	Complex variables	1097
G.2	Grassmann Integration	1098
G.3	Quantization on the Space of Classical Solutions	1106
G.3.1	Harmonic Oscillators	1108

G.3.2	‘Fock Space’ Quantization	1110
G.3.3	The Fock Representation of Field Theory	1118
G.3.4	The Fock Representation of a Free Scalar Field	1119
G.3.5	The Fock Representation of the Maxwell Field	1120
G.3.6	Quantum Field Theory on Curved Spacetime - The Basics	1120
H	Details of Hawking’s Calculation	1121
H.1	Decomposition into Complete Basis	1121
H.2	Solution of Klein-Gordon Equation in Schwarzschild Spacetime	1121
H.3	Bogoliubov Coefficients	1124

List of Figures

- 1.1 Rubbersheet simulation of geodesic motion in special relativity. 53
- 1.2 rubbersheet. It doesnot matter that the coordinates are time-dependent -
it still serves as a physical reference system. 53
- 1.3 $E(x) = Q/x^2$ 58
- 1.4 $E(y) = Q/y^2$ 59
- 1.5 $E(y) = Q/y^2$ 59
- 1.6 Passive spatial diffeomorphism $f : M \rightarrow M$ refers to invariance under
change of coordinates. The same object in a different coordinate system.
Any theory of nature is invariant under passive diffeomorphisms. 61
- 1.7 An active diffeomorphism $f : M \rightarrow M$ drags fields on the manifold while
remaining in the same coordinate system. f is viewed as a map that asso-
ciates one point in the manifold to another one. 62
- 1.8 The value of $\tilde{\phi}(P)$ at P is equated to the value of $\phi(P_0)$ at P_0 , i.e. $\tilde{\phi}(P) =$
 $\phi(P_0)$. Under this transformation f we indentify one point of the manifold
 P_0 to another point P $f : P_0 \rightarrow P$ 62
- 1.9 The value of the metric function \tilde{g}_{ab} at P is defined by the value of the
metric function g_{ab} at P_0 , i.e. $\tilde{g}_{ab}(P) = g_{ab}(P_0)$. We go to a new coordinate
system which assigns P the same coordinate values that P_0 has in the x-
coordinates, so that $\tilde{g}_{ab}(y_1 = u_1, y_1 = u_2) = g_{ab}(x_1 = u_1, x_1 = u_2)$, compare
to (1.31). 63
- 1.10 (a) An active diffeomorphism in which we indentify one point of the man-
ifold to another point. (b) We then go to a coordinate system that assigns
the newly identified points the original coordinate values. 64

1.11	(a) An active diffeomorphism in which we actively drag the tensor function over the, in doing so indentify one point of the manifold to another. (b) We then go to a coordinate system which assigns the newly identified points their original coordinate values. That is to say - we carry the tensor function over the manifold, keeping the coordinate lines ‘attached’.	65
1.12	Einstein’s hole argument.	65
1.13	Resolution of Einstein’s hole argument.	66
1.14	Illustration of smearing. operator valued distributions.	75
1.15	Regime where gravity is very strong so that the non-perturbative and background independence of GR must be taken into account. That spacetime points have no independent physical reality casts doubt on the hand-wavey argument I gave above.	76
1.16	Labortary walls exemplify Newton’s absolute space and the clock absolute time. We can define positions relative to the wall.	76
1.17	GPS.	78
1.18	GPS3D A spacetime point in Minkoski spacetime can be expressed as a relation amogst 4? measurable variables. This definition of spacetime location retains meaning in the jump to GR.	79
1.19	clock time.	80
1.20		80
1.21	measLocation.	81
1.22	measVelocity.	81
1.23	partialobs. τ is an unphysical parammeter labelling different possible correlations between the time reading t of the clock and the elongation x of the pendulm.	84
1.24	partComptDitt1.	92
1.25	partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^t(x))$ assumes the value τ . (b) The function $F_{[f,T]}(\tau, x)$ gives the value that the function $f(\alpha_C^t(x))$ assumes if the function $T(\alpha_C^t(x))$ assumes the value τ . $F_{[f,T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$.	93
2.1	timedilF.	98
2.2	rocketEarth.	104

2.3	rocketEarth2.	104
2.4	rocket. The clock at the top seems to run faster than the one on the bottom.	105
2.5	rocketaccel. The clock at the top seems to run faster than the one on the bottom.	105
2.6	lightdeflec.	106
2.7	WeakGrav. geodesic.	107
2.8	Hole. Einstein's hole argument.	108
2.9	Hole3. Einstein's hole argument. $\Phi : \mathcal{M} \rightarrow \mathcal{M}' \rightarrow \mathcal{M}$	110
2.10	Hole4. Einstein's hole argument. A gauge transformation which does not change the coordinate label system but moves the points on the manifold, and then evaluate the coordinates of the new point	110
2.11	We display the geometric interpretation of the curvature tensor. Carry a third vector Z , by parallel transport from p to s via q , comparing this with transporting this from p to s' via r we find a discrepancy between the two vectors given in terms of the curvature tensor components $R_{abc}{}^d$ by the formula $\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$	115
2.12	geodesic deviation.	116
2.13	clock time.	116
2.14	measLocation.	117
2.15	tidalforceF.	117
2.16	worldfunc1.	121
2.17	measVelocity.	123
2.18	η is the orthogonal connecting vector.	126
2.19	We find the spatial frame components η^α of the orthogonal connecting vector by projecting onto a spatial frame field. This is the precise analogue of the Newtonian connecting vector.	130
2.20	Different world line passing through P corresponds to different observer with different v^a	133
2.21	GPScoord. s_1 and s_2 are the GPS coordinates of the point p . Σ is a Cauchy surface with p in its future domain of dependence.	134
2.22	crossParea	134

2.23	continuityEM Y and Y	136
2.24	Lfluid. The Lorentz contraction of a fluid element.	139
2.25	LumDist. Luminosity distance	165
2.26	pertCosGauge. A diffeomorphism on the perturbed manifold \mathcal{M} induces a change in coordinates of the background manifold \mathcal{M}_0 . The issue of perturbative gauge invariance is closely related, though not equivalent to, the coordinate independence of General Relativity.	169
2.27	pertManifolds. 5-dimensional manifold \mathcal{N} containing a 1-parameter family of smooth non-intersecting 4-manifolds \mathcal{M}_ϵ . $\mathcal{N} = \mathcal{M} \times \mathbb{R}$	171
2.28	pertManFlow. The diffeomorphism φ	173
2.29	pertManPush. The push-forward $\varphi_{\lambda^* _p}$ is the natural linear map between the tangent spaces $T_p\mathcal{M}_0$ and $T_{\varphi(p)}\mathcal{M}_\lambda$ induced by the diffeomorphism φ . The push-forward $\varphi_{\lambda^* _p}$ is the linear map between the co-tangent spaces $T_p^*\mathcal{M}_0$ and $T_{\varphi(p)}^*\mathcal{M}_\lambda$. Push-forwards and pull-backs are related by	174
3.1	eventhorizon.	198
3.2	Penrose diagram of Minkowski spacetime	220
3.3	Penrose diagram of the Kruskal solution	221
3.4	Penrose diagram of a black hole.	221
3.5	Penrose diagram of a black hole.	233
3.6	Rotating blackhole	274
3.7	Normal and tangent vector to a tangent	275
3.8	(a) Normal to a timelike surface, (b) Normal to a spacelike surface, (c) Normal to a null surface.	275
3.9	Coordinatizing a null surface in Minkowsian spacetime - λ, θ_A	275
3.10	Each two sphere. Coordinates λ, θ, ϕ	276
3.11	A spacial two-sphere S embedded in the spacial slice Σ (which in turn is embedded in spacetime \mathcal{M}), with two sets of orthogonal null vector fields. The vector field n^a is the unit timelike normal to Σ , R^a is the unit spacial normal to S , and n^a and ℓ^a are, respectively, the outgoing and ingoing null vectors orthogonal to S	276
3.12	A spacial two-sphere S	277

3.13	In Schwarzschild black hole the horizon is generated by the radial light rays, which meet at the center.	284
3.14	(a) An open interval of the real line is the set of points between a and b excluding a and b . (b)	285
3.15	q lies in the chronological future of z	285
3.16	The chronological future $I^+(p)$ of p is an open set; given any point $q \in I^+(p)$, there exists a sufficiently small neighbourhood $V(q)$ contained in $I^+(p)$. Similar statements hold for the p in the chronological past $I^-(q)$ of q	286
3.17	If two points on the event horizon are timelike separated, we can produce a timelike curve starting inside the black hole which joins to a point outside the event horizon.	287
3.18	When neighbouring null geodesics have conjugate points there exists a timelike curve joining the two conjugate points. The dashed line represents a timelike curve joining to the null geodesic. The points q and q' are timelike separated - rounding off the corner. We make this argument more rigouress in the appendix M.	288
3.19	In flat spacetime, when a null geodesic curve joins onto a timlike curve, there exists a timelike curve between p and q	288
3.20	There exists a timlike curve joining the two conjugate points. A timelike curve joining to the null geodesic. (b) Contiuing in this way, we “peel” away a timelike curve that joins r and p	289
3.21	There exists a timlike curve joining the two is way, we a timelike curve that joins r and p	290
3.22	There exists a timlike curve joining the two is way, we a timelike curve that joins r and p	290
3.23	Classical bourndary conditions for weakly isolated horizons.	292
3.24	Conformal spacetime diagram of a WHI.	292
3.25	Construction of a null tetrad for a NEH. Any spatial two surface S determines uniquely, up to rescaling, two null vectors orthogonal to S	293
3.26	multipole is the position of the mass density source and \vec{r} is the requested position for the potential $\Phi(\vec{r})$	297
3.27	dipolemass (a) monopole. (b) dipole (c) quadrapole	298
3.28	magmulty is the position of the current density and \vec{r} is the requested position for the magnetic potential $\Phi_m(\vec{r})$	300

3.29	polarcoo	304
3.30	axicoords. In addaptive coordinates	304
3.31	308
3.32	visualflownull.	353
3.33	Horizons.	419
4.1	LumDist. Luminosity distance	424
4.2	nonmingeodesic.	425
4.3	trapped.two-surface such that the areas of pulses of light emmited from each little element of surface decrease in both directions.	426
4.4	illustrating a trapped surface.	426
4.5	illustrating a past-directed trapped surface, corresponding to a cosmological singularity.	427
4.6	$I^+(p)$ is open.	428
4.7	trivialsing.	428
4.8	The points y_n converge to the point q in the boundary of $L^+(S)$. From each y_n there is a past directed timelike curve λ_n to S . These curves converge to the past directed null geodesic segment γ through q	429
4.9	$K = J^+(S) \cap J^-(p)$ K is compact.	429
4.10	trivialsing.	430
4.11	A timelike curve can be approximated by null geodesic segments. This approximating null curve fails to have a well defined tangent vector anywhere.	431
4.12	The global hyperbolicity of \mathcal{M} is closely related to the future or past development of initial data from a given spacelike hypersurface.	432
5.1	Sir Roger Penrose and Stephen Hawking. Initiated by Penrose, Penrose and Hawking, together with Robert Geroch, contributed much of the work on the existence of spacetime singularities with the use of point-set topological methods.	446
5.2	446
5.3	An example of a non-orientable space.	447

5.4	(a) A closed achronal set with edge. (b) A closed achronal set without edge.	447
5.5	The h -null cone contains more timelike vectors than the g -null cone so there is more likelihood to find closed timelike curves in (\mathcal{M}, h) than in (\mathcal{M}, g) .	448
5.6	Domains of dependence. (a) The future domains of dependence of the achronal set S . (b) The past domain of dependence of the set S . (c) The total domain of dependence of S .	449
5.7	closedtrapArea. A closed trapped surface is when the outgoing light cone also converges.	450
5.8	Focal points. (b) Focal point to a surface.	450
5.9	A path is a connected set in \mathbb{R} to the space-time manifold \mathcal{M} .	452
5.10	Examples in Minkowski spacetime. (a) The curve α is taken to contain its own past end point a and future end point b . (b) The curve α' is now future-inextendable. (c) The curve α'' is now past-inextendable (d) The curve γ is not future-inextendable because it cannot be prolonged any further.	453
5.11	(a) A curve in Minkowski spacetime with point removed. (b) A curve is zig zag that it fails to have a well defined tangent vector at a point. (c) Here we have a time like curve which is null at its end point.	453
5.12	In flat spacetime, when a null geodesic curve joins onto a timlike curve, there exists a timelike curve between p and q .	454
5.13	Say there are two points p and q connected by a null curve and a point r which is connected to q by a timelike curve. A timelike curve joining to the null geodesic. (b) Continuing in this way, (c), we “peel” away a timelike curve that joins r and p .	454
5.14	Artificial example of how the causal future of a point p will not necessarily coincide with the closure of the chronological future of p .	455
5.15	The point q is conjugate to p along null geodesics, so a null geodesic γ that joins p to q will leave the bounadry of the uture of p at q .	455
5.16		456
5.17	The future set cannot be bounded by timelike curves.	457
5.18		457
5.19		458
5.20		458

5.21	The lines ℓ_1 and ℓ_2 have been removed. This is a space-time which is causal but fails to be strongly causal.	460
5.22	(a) . (b) Suitable causal basis.	460
5.23	Minkowski space-time: b is joined to a by a null geodesic we consider $x \in I^+(a)$ and $y \in I^-(b)$. In the first in case (a) $y \ll x$. In case (b) $y < x$ but $y \not\ll x$. In case (c) x and y are not causally related, in particular $y \not\ll x$.	461
5.24	b is joined to a by a null geodesic we consider $x \in I^+(a)$ and $y \in I^-(b)$. In the first case (a) $y \ll x$ there are no closed timelike curves. In case (c) For some x and y	461
5.25	converse. (b) That $y \gg x$ to be true, there must be a timelike curve that enters $U(a)$	462
5.26	proof of	463
5.27	For $i \geq i_0$, $U_i \in I^+(x)$, from which we can conclude $x \ll a_i$. For $i \geq i'_0$, $c_i \in I^-(y)$. If we choose $i > i_0, i'_0$ then we have $x \ll a_i$ and $c_i \ll y$	463
5.28	$b a x \in I^+(a)$ and $y \in I^-(b)$. In the first case (a) $y \ll x$ timelike curves. (b) Points in the open set of b are also in $I^+(x) \cap I^-(y)$ In case (c)	464
5.29	$a \in I^-(y) \cap V(y)$ and $b \in I^+(y) \cap V(y)$. In the first case (a) $y \ll x$ timelike curves. (b) Points in the open set of b are also in $I^+(x) \cap I^-(y)$	464
5.30	$a \in I^-(y) \cap V(y)$ and $b \in I^+(y) \cap V(y)$. In the first case (a) $y \ll x$ timelike curves. (b) Points in the open set of b are also in $I^+(x) \cap I^-(y)$	465
5.31	(a) There is an endless null geodesic along which strong causality is violated. (b) Strong causality is violated everywhere in R	466
5.32	$F := I^+(Q)$ and $P := I^-(Q)$. $Q = F \cap P$, $\partial Q = F\partial + \partial P$	466
5.33	The lines ℓ_1 , ℓ_2 and ℓ_3 have been removed.	467
5.34	The future domain of dependence, $D^+(\Sigma)$, of Σ . p is in $D^+(\Sigma)$, q isn't because there are past-inextendable causal curve through q that don't intersect Σ , e.g. the curve γ	467
5.35	The future domain of dependence, $D^+(\Sigma)$, of a closed Σ in Minkowski spacetime.	468
5.36	Domains of dependence.	468
5.37	The future domain of dependence.	469

5.38	There exists a subsequence γ_m of past inextendible causal curves that do not meet S that converges to a past inextendible C^0 null geodesic γ starting at p	469
5.39	The edge of a closed achronal surface Σ	470
5.40	A simple example of a closed achronal surface without edge can be given by considering the spacetime $\mathbb{R} \times S$ with light cones locally at 45 degrees. For the open neighbourhood there is no $r \in I^-(p)$ and $q \in I^+(p)$ with a timelike curve between them that doesn't intersect S . The closed achronal set Σ has no edge.	470
5.41	We can only get convergence to a cluster point in K if the space wasn't locally finite.	471
5.42	$K = J^+(S) \cap J^-(p)$ K is compact.	471
5.43	$K = J^+(S) \cap J^-(p)$ K is compact.	472
5.44	G is an open set.	472
5.45	approximate a causal curve by a causal trip.	473
5.46	Imposition of the condition $I^+[A_j] \cap A_{j+2} \neq \emptyset$ avoids cases such as the above.	476
5.47	The geodesic λ	477
5.48	The sequence is a Cauchy sequence in $\mathcal{C}(a, b)$, but it is not convergent, since there is a point missing.	478
5.49	Minkowski space time with a point removed is not globally hyperbolic. The point q is not in $D^+(S)$ as there are non-spacelike curves like λ which do not meet S in the past.	479
5.50	(a) An upper semi-continuous function. (b) A lower semi-continuous function.	480
5.51	Two points joined by a timelike curve can be connected by a broken null geodesic.	484
5.52	A sequence of null curves may converge to a timelike curve.	484
5.53	If A and B are parallel in Minkowskia spacetime then γ_1 and γ_2 are maximal. (b) Here there is only one element in $\mathcal{C}_K(A, B)$ which is necessarily maximal. It is not a geoesic. (c) Here the maximal element is a trip.	485
5.54	Displacement vectors for a hypersurface.	493
5.55	497
5.56	Future horozmos	497

5.57	“doubling” future horozmos. $E^+[S]$ is compact	498
5.58	From the fact that H is a Cauchy horizon it follows that through every point of H there passes a maximally extened past-directed null geodesic that remains in H . Since H is compact, such a curve would have to come back arbitrarily close to itself - reentering some Alexandoff neighbourhood and so violating strong causality.	499
5.59	$E^+[S]$ is compact by defintion, however its Cauchy horizon is non-compact. As the two subsets can not homeomorphic there must be at least one trajectory γ which remains in $\text{int}D^+(E^+[S])$	499
5.60	γ is a future endless causal geodesic in $\text{int}D^+(E^+(S))$. $H = H^+(E^+[S])$. As the intersection of the closed set $\dot{J}^-(\gamma)$ with a compact set generated by null geodesic segments from T of some bounded affine length, $E^-[T]$ is compact	500
5.61	The geodesic λ	501
5.62	The limit geodesic γ contains conjugate points.	502
5.63	Since $b_n \rightarrow b$, $tr(b_n) < tr(b)/2$ for all $n > N$	521
5.64	$\eta(S)$ is contained in the bounded segment from $\gamma(s_1)$ to $\gamma(s_5)$	522
5.65	The Jacobi fields which are zero at r must have expansion θ which is positive at p otherwise r would lie in the bounded interval from $\gamma(s_1)$ to $\gamma(s_5)$	524
5.66	With $R_{abcd}V^bV^c \neq 0$. Non-positive expansion at p ($\theta > 0$) implies there is a point q conjugate to r , in the past of p . This is just the <i>time reversed</i> version of the focussing theorem.	525
5.67	The null expansion scalar $\hat{\theta}$	525
5.68	Singularity theorems.	527
5.69	Diagram of collapse of a star.	528
5.70	Penrose diagram of collapse of a star.	529
5.71	starCollaps.	530
5.72	531
5.73	characteristic.	533
5.74	EnergyIneq1.	540
5.75	EnergyCond.	541
5.76	542

7.1	Reflects forward in time by the strong gravitational field outside the event horizon.	574
7.2	timereflection2. The antiparticle mode falling into the black hole can be interpreted as a particle travelling backwards in time, from the singularity down to the horizon.	575
7.3	Penrose diagram of a star that collapses to form a black hole.	576
8.1	Graphical representation of the Mandelstam identity (8.109) relating different Wilson loops.	608
8.2	The action of the Hamiltonian constraint translated to the ‘path-integral’ or spin foam description. Where $N(x_n)$ is the value of N at the vertex and H_{nop} are the matrix elements of the operator \hat{H}	612
8.3	a) A spherical star of mass M undergoes collapse. b) Later, a spherical shell of mass δM falls into the resulting black hole. With Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.	618
8.4	Quantum Horizon. Polymer excitations in the bulk puncture the horizon, endowing it with quantized area. Intrinsically, the horizon is flat except at punctures where it acquires a quantized deficit angle. These angles add up to 4π	619
A.1	Abhay Ashtekar.	624
A.2	Julian Barbour.	626
A.3	LebRien.	627
A.4	Bondi coordinates at future null infinity.	629
A.5	Cauchy Horizon.	630
A.6	We display the geometric interpretation of the curvature tensor. Carry a third vector Z , by parallel transport from p to s via q , and compare this with transporting this from p to s' via r . We find that the two vectors differ according to a rotation given in terms of the curvature tensor components $R_{abc}{}^d$ by the formula $\epsilon^2 X^a Y^b Z^c R_{abc}{}^d$	634
A.7	Bryce DeWitt.	635
A.8	Paul Dirac.	636
A.9	Bianca Dittrich.	637

A.10	The future domain of dependence, $D^+(\Sigma)$, of Σ . p is in $D^+(\Sigma)$, q isn't because there are past-inextendable causal curves through q that don't intersect Σ , e.g. the curve γ .	637
A.11	Albert Einstein (1879-1955).	639
A.12	The charge renormalization.	642
A.13		645
A.14	Rodolfo Gambini.	645
A.15	geons.	648
A.16	Stephen Hawking.	652
A.17	An expression for the conserved mass, evaluated when the spacial boundary is pushed to infinity. (b) $\partial\Sigma$ is a closed spacelike two-surface surrounding the source.	659
A.18	Penrose process.	667
A.19	Sir Roger Penrose.	668
A.20	PonRegg.	669
A.21	Jorge Pullin.	670
A.22	radartimeF. Schematic of the definition of radar time $\tau(x)$.	672
A.23	ReggeGlos. Gluing flat simplices to get a surface with curvature.	673
A.24	Regge Lagrangian.	674
A.25		675
A.26	Carlo Rovelli.	676
A.27	Thomas Thiemann.	683
A.28	John Wheeler.	687
B.1	bijjective	693
B.2	DiffClass0. A chart on \mathcal{M} comprises an open set U of \mathcal{M} , called a coordinate patch, and a map $\phi : U \rightarrow \mathbb{R}^n$.	696
B.3	Computing the Gauss linking number.	706
B.4	Computing the Gauss linking number.	706

B.5	713
B.6 injective	714
B.7 knots Reidemeister moves.	716
B.8 knots Reidemeister moves.	716
B.9 Pachner move in d=3. (a) the $1 \rightarrow 4$ move subdivides.	723
B.10 pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$	727
B.11 pullbackDef. $\varphi_* _p : T_p\mathcal{M} \rightarrow T_{\varphi(p)}\mathcal{N}$	727
B.12 Simplices in 3d.	735
B.13 surjective	737
B.14 unicoverEx.	741
B.15 An upper semi-continuous function.	741
C.1 Coordinate induced basis.	751
C.2 coset. Suppose H is a subgroup of a finite group G , here the elements of H are listed first. The shaded box are the elements of the (left) coset of $g_r \in G$	761
C.3 infintesimal rotation.	776
C.4 infintesimal rotation.	776
C.5	793
C.6	793
C.7 Penrose diagram for Minkowskian spacetime.	795
C.8 Haarmeas1.	796
C.9 HaarmeasSO3.	796
C.10 TranRotGrF.	807
C.11	811
C.12	811
C.13 (a) (b) f is discontinuous at p_1	811
C.14 Open sets interior points (b)	812

C.15	(U, ϕ_1) and (V, ϕ_2) are two coordinate patches on X . Transition functions, $\phi_2 \circ \phi_1^{-1}$, are ordinary functions that go from points of one R^n space onto another, i.e. $\phi_2 \circ \phi_1^{-1} : R^n \mapsto R^n$. The domain and range of the transition function are the shaded regions in R^n	813
C.16	Möbius.	813
C.17	tangtoM.	814
C.18	connection.	816
C.19	coordbasevec. The vector ξ may be thought of as being composed of $\xi = \xi^1 \partial / \partial \mathbf{x}^1 + \xi^2 \partial / \partial \mathbf{x}^2$. ξ^1 and ξ^2 are the components of ξ in the (x^1, x^2) -coordinate system.	826
C.20	tangvector. maps the tangent spaces of \mathcal{M} <i>linearly</i> into the	828
C.21	Two curves $\lambda(t)$ and $\mu(t)$ are tangent at p if and only if their images are tangent at $\phi(p)$ in \mathbb{R}^n	830
C.22	activeDiffGeom. A pushforward of the tensor $T_{ab}(x)$, i.e. $T_{ab}(x) \rightarrow \tilde{T}_{ab}(y)$	832
C.23	activeDiffGeom1. The red dashed lines in (a) are the x -coordinate lines of the point P . We perform a coordinate transformation back to the original coordinate system. The pushed-forward tensor $\tilde{T}_{ab}(y)$ transforms to $\tilde{T}'_{ab}(x)$, i.e. $\tilde{T}_{ab}(y) \rightarrow \tilde{T}'_{ab}(x)$	833
C.24	The tangent vector field resulting from a congruence of curves.	834
C.25	The local congruence of curves resulting from vector field.	834
C.26	838
C.27	pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$	839
C.28	The push-forward map h_* that maps the tangent spaces of \mathcal{M} <i>linearly</i> into the tangent spaces of \mathcal{N}	839
C.29	The pullback map ϕ^* of a function f from \mathcal{N} to \mathcal{M} by a map $\phi : \mathcal{M} \rightarrow \mathcal{N}$ is the composition of ϕ with f	842
C.30	pushLie. The maps the co-tangent spaces of \mathcal{M} <i>linearly</i> into the co-tangent spaces of \mathcal{N}	843
C.31	The Killing vector field resulting from a congruence of curves.	845
C.32	Framefield or tetrad with one spatial dimension suppressed.	850
C.33	areaofPar.	854

C.34 areaofPar2.	855
C.35 surfElement.	856
C.36 stokeExam.	871
C.37 stokeExam2.	871
C.38 boundary.	873
C.39 boundarychain. Boundary of a chain.	874
C.40 A curve through e under the map $h \mapsto ghg^{-1}$, first a right action $R_{g^{-1}}$ as $h \mapsto hg^{-1}$ followed by the left action L_g as $hg^{-1} \mapsto ghg^{-1}$. The identity e is mapped to itself but points h and f near it are generally changed, so that a tangent vector at e , in $T_e(G)$, is mapped to another one in $T_e(G)$	877
C.41 A vector $X \in T_e(G)$ is mapped to another one in $Ad_g(X) \in T_e(G)$. Written formally as $(ad_g)_* : T_e(G) \rightarrow T_e(G)$	878
C.42 If a Lie group is a direct product of the proper subgroup and some discrete subgroup then each connected component G_i is obtained from the proper subgroup G_1 by applying some discrete transformation γ_i of a discrete subgroup Γ	879
C.43 leftTran. The left translation along g maps a neighbourhood of e onto one of g . There is a natural map of a vector at e to one at g	882
C.44 Fibre bundle, $TS^1 = S^1 \times R$. The base manifold \mathcal{M} (the real line R^1). The circle is the fibre. The fibre bundle consists of a manifold and a projection map π . $\pi^{-1}(U)$ is the local product space.	888
C.45 The inverse map $\pi^{-1}(U)$ is the local product space.	888
C.46 tangent bundle, $TS^1 = S^1 \times R$. The base manifold \mathcal{M} (the circle S^1) consists of a manifold and a projection map π . $\pi^{-1}(U)$ is the local product space.	889
C.47 Möbius.	889
C.48 tangent bundle, $TS^1 = S^1 \times R$. The curved line is a section.	890
C.49 Travelling up fibre.	891
C.50 Defintion of the right action of G on the principal fibre bundle P	895
C.51 The horizontal subspace $H_{gu}P$, defining by the connection in definition (1), is obtained from H_uP by the left action.	897
C.52	898

C.53	901
D.1 Legendre transform relating Lagrangian formulation to the Hamiltonian formulation.	916
D.2 Legendre transform relating Lagrangian formulation to the Hamiltonian formulation.	917
D.3 We cannot solve to get p from...	918
D.4 Hamilton.	940
D.5 Hamilton.	941
D.6 Here a smooth family of 2-planes coincides with the tangent spaces of a nonintersecting space filling surfaces. Suprisingly this is not always the case.	967
D.7 partComptDitt1.	986
D.8 partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^t(x))$ assumes the value τ . (b) The function $F_{[f,T]}(\tau, x)$ gives the value that the function $f(\alpha_C^t(x))$ assumes if the function $T(\alpha_C^t(x))$ assumes the value τ . $F_{[f,T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$	987
D.9 comptasGIE.	987
D.10 partComptDitt4.	990

Terminology and Notation

Here is a list of symbols.

$[,]$	commutator
$\{ , \}$	Poisson bracket
\dagger	Hermitian conjugation
$:=$	definition
\equiv	identity
$\stackrel{*}{=}$	only true in a special coordinate system
iff	If and only if
η_{ab}	Minkowski metric
$\eta(x)$	test function of a variation of action
\mathcal{A}	space of gauge fields or area
$A_\mu(x)$	Yang-Mills connection
D_μ	covariant derivative
\mathcal{M}	spacetime manifold
\mathbf{M}	The Master constraint
$\hat{\mathbf{M}}$	The Master constraint operator
$\omega_{\mu\beta}^\alpha$	spin connection
\mathcal{C}	constraint surface in phase space
S	labells spin-network
s	equivalent class of spin-networks under the action of Diff denoted s - knots
$s(S)$	denotes equivalent class S to which belongs
g_{ab}	spacetime metric
K_{ab}	extrinsic curvature of Σ
G_{ab}	Einstein tensor
T_{ab}	The energy-momentum tensor
e_I^a, E_i^a	tetrad and triad
\mathcal{L}_t	Lie derivative with respect to t
n_a	unit normal to Σ_t
$N, (\tilde{N})$	lapse function (density)
N^a	shift vector on Σ
$\Omega_{\alpha\beta}$	symplectic form
\mathcal{A}/\mathcal{G}	space of gauge fields moduli gauge transformations
$[A]$	gauge equivalence classe of the connection A
\mathcal{HA}	the holonomy algebra
$\overline{\mathcal{HA}}$	the completion of the holonomy algebra in the norm $\ f\ := \sup_{[A] \in \mathcal{A}/\mathcal{G}} f([A]) $
$\overline{\mathcal{A}/\mathcal{G}}$	spectrum of $\overline{\mathcal{HA}}$

Preface

Acknowledgments

Discussions with Tong,Pun Wai on parts of the proof of the singularity theorems.

Introduction

the beginning of the revolutionary contributions to physics by Einstein,

Paths through the report

Introductory book on general relativity [?]