Appendix B

Mathematics Glossary

• **absolute continuous functions:** These are functions of the form:

$$u(x) = \int_{x_0}^{x} v(y)dy + c$$

where $v(y)$ is considered as of u and the fundamental formula

$$u(x) = u(x_0) + \int_{x_0}^{x} \frac{d}{dy} u(y)dy + c$$

still holds. [from http://www.math.ku.dk/ grubb/distribution.htm]

• **absolute continuous measure:** We say that a Borel measure μ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}$ if there is a function, f, locally L^1 (that is, $\int_a^b |f(x)|dx < \infty$ for any finite interval (a, b)) so that

$$\int g d\mu = \int g f dx$$

for any Borel function $g \in L^1(\mathbb{R}, d\mu)$. This generalizes to to measures on topological spaces.

• **absolute continuous operator:**

• **absolute convergence:** The series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

• **algebra:** An algebra is simply a vector space over $\mathbb{C}$ (or more generally over a field k) in which there is defined a distributive and, (in a certain sense), associative multiplication:

(i) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$;

(ii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for every scalar α, (a complex number).

Note that an algebra need not be associative in the sense $x \cdot (y \cdot z) \neq (x \cdot y) \cdot z$ where x, y, z are elements of the algebra.

Example An example of a **non-associative algebra** is the vector space of 3-d Euclidean vectors with the multiplication relation taken to be the crossed product $\times$ of two vectors

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c}). \tag{B.1}$$

An **associative** algebra is an algebra whose multiplication also satisfies;

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \tag{B.2}$$

A **commutative** algebra is an algebra whose multiplication also satisfies the condition:

$$x \cdot y = y \cdot x. \tag{B.3}$$

It is unital if there is defined a unit **1** which satisfies

$$a\mathbf{I} = \mathbf{I}a = a, \quad \text{for all } a \in A. \tag{B.4}$$

Intro into $*$algebras.

A **$*$-algebra** if there is defined an **involution** satisfying

$$(xy)^* = y^* x^* \quad \text{and} \quad (x^*)^* = x \tag{B.5}$$

which reduces to complex conjugation on the scalars $\alpha \in \mathbb{C}$, i.e.,

$$(\alpha x)^* = \alpha^* x^*. \tag{B.6}$$

A **Banach algebra** is an algebra with norm $\|a\| \geq 0$ which satisfies the conditions $\|x+y\| \leq \|x\| + \|y\|$, $\|xy\| \leq \|x\| \ \|y\|$, $\|\alpha x\| = |\alpha| \ \|x\|$, $\|x\| = 0 \Leftrightarrow x = 0$ and with respect to which it is complete; it contains its limit points$\|x_n - x\| \to 0$.

A **C*-algebra** is a Banach $*$-algebra whose norm satisfies the C^*-property:

$$\|a * a\| = \|a\|^2, \quad \text{for all } a \in A. \tag{B.7}$$

A familiar example of a C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$.

Example An example would be an associative algebra of the space real-valued of continuous functions formed by ordinary pointwise addition $(f+g)(x) := f(x)+g(x)$ and multiplication $fg(x) := f(x)g(x)$. It is obvious that the addition of two continuous functions is continuous. The product of two continuous functions is continuous. To prove that $fg(x)$ is continuous we must show that given any $\epsilon > 0$ there exists $n > N$ such that

$$|fg(x) - fg(x_n)| < \epsilon. \tag{B.8}$$

Given any $\epsilon > 0$ define $\epsilon' := (|g(x)| + |f(x_n)|)/\epsilon$. As $f(x)$ and $g(x)$ are continuous for $\epsilon' > 0$ there exists $n > N$ such that

$$|f(x) - f(x_n)| < \frac{\epsilon'}{2} \quad \text{and} \quad |g(x) - g(x_n)| < \frac{\epsilon'}{2}. \tag{B.9}$$

continuity of $fg(x)$ follows from

$$\begin{aligned} |fg(x) - fg(x_n)| &= |f(x)g(x) - f(x_n)g(x) + f(x_n)g(x) - f(x_n)g(x_n)| \\ &\leq |f(x) - f(x_n)||g(x)| + |f(x_n)||g(x) - g(x_n)| \\ &< \epsilon'(|g(x)| + |f(x_n)|) \\ &= \epsilon \end{aligned} \tag{B.10}$$

• **adjoint representation:** $[t_a, t_b] = C^c_{ab} t_c$. The commutation constants C^c_{ab} form a representation of the group algebra.

• **algebraic dual** The algebraic dual $\mathcal{D}^*$ is the space of linear functionals on $\mathcal{D}$ without continuity assumptions. See dual spaces.

• **algebraic graph theory:** http://www.utm.edu/departments/math/graph/glossary.html

• **almost everywhere:**

• **almost periodic functions:** Almost periodic functions are functions that can be written as

$$f(x) = \sum_j f_j e^{\frac{\mu_j x}{2}} \tag{B.11}$$

where the sum contains a finite number of terms ($j = 1, 2, \ldots, N$ and $N < \infty$), $f_j \in \mathbb{C}$, and $\nu_j \in \mathbb{R}$.

For loop quantum cosmology models of spaces that have homogeneity and isotropy the holonomy algebra consists of the set of almost periodic functions.

The fundamental theorem for almost periodic functions is a generalisation of the Parseval identity for Fourier series. This result lead Bohr to a result on the uniform approximation to almost periodic functions by exponential functions.

• **analytic function:** A function is called real analytic at a point if it possesses derivatives of all orders and given by a convergent power series locally. For example, a function on the real line $\mathbb{R}$ is analytic at the point p if there exists an interval (a, b) containing p such that in this interval the function can be expanded as a convergent series

$$f(x) = a_0 + a_1(x-p) + a_2(x-p)^2 + a_3(x-p)^3 + \dots, \tag{B.12}$$

where

$$a_0 = f(p),\ a_1 = f'(p),\ a_2 = \frac{f''(p)}{2!},\ a_3 = \frac{f'''(p)}{3!}, \dots \tag{B.13}$$

A function is analytic if it is analytic at each point in its whole domain. The set of all analytic functions is contained in the set of smooth functions. Analytic functions are also referred to as C^ω-smooth functions.

• **analytic continuation:** If we have two complex functions $f(z)$ and $g(z)$ satisfying the following properties:

(a) $f(z)$ is defined on a set U of the z complex plane $\mathbb{C}$;

(b) $g(z)$ is analytic in the domain V containing U;

(c) $g(z)$ coincides with $f(z)$ on U;

then $g(z)$ is said to be the analytic continuation of $f(z)$ to the domain V.

• **analytic curve:** A curve in Euclidean space $\mathbb{R}^n$ is piecewise analytic if it can be expanded as a Taylor series locally. A curve in a manifold $\mathcal{M}$ is analytic if and only if its image under a chart is an analytic curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval (a, b) to $\mathbb{R}^n$ in Fig.(**??**) is a analytic map.

• **analytic structure:** a covering homeomorhic to open sets in a fixed Euclidean space, C^ω The coordinate transforms are analytic in both directions, i.e.,

$$\begin{aligned} (\phi_1 \circ \phi_2^{-1})(x_1, x_2, \dots, x_n) &= a_0 + a_1^{(i)} x_i + a_2^{(ij)} x_i x_j + a_3^{(ijk)} x^3 + \dots, \\ (\phi_2 \circ \phi_1^{-1})(y_1, y_2, \dots, y_n) &= b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \dots. \end{aligned} \tag{B.14}$$

in some interval containing p.

• **associated bundles:** Given a particular principal bundle $(P, \pi, \mathcal{M})$ with structure group G, we can form a fibre bundle F for each space F on which G acts as a group of transformations.

• **atlas:** Two charts ϕ_1, ϕ_2 are $C^\infty-$related if both the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and its inverse are $C^\infty-$related. A collection of related charts such that every point of $\mathcal{M}$ lies in the domain of at least one chart forms an atlas.

• automorphisms: An isomorphism $X \to X$ is called an automorphism of X. see group homomorphism.

[403] **Automorphisms of groups:** Let G be a group. An isomorphism $G \to G$ is called an automorphism of G. Let $x, y \in X$ and $g \in G$,

$$g(xy)g^{-1} = (gxg^{-1})(gyg^{-1}), \quad i_g(xy) = i_g(x)i_g(y) \tag{B.15}$$

The set $\mathrm{Aut}(G)$ of such automorphisms becomes a group under composition:

(i) the composite of two automorphisms is again an automorphism;

(ii) composition of maps is always associative;

(iii) the identity map $g \mapsto g$ is an identity element;

(iv) an automorphism is one-to-one and onto, and therefore has an inverse (we just change the direction of the arrows), which is again an automorphism.

Automorphisms of algebras: Let $\mathcal{A}$ be an algebra. An isomorphism $\mathcal{A} \to \mathcal{A}$ is called an automorphism of $\mathcal{A}$. Let $a, b \in G$

$$\begin{aligned} g(ab)g^{-1} &= (gag^{-1})(gbg^{-1}), \quad i_g(ab) = i_g(a)i_g(b) \\ g(a+b)g^{-1} &= gxg^{-1} + gyg^{-1}, \quad i_g(a+b) = i_g(a) + i_g(b). \end{aligned} \tag{B.16}$$

$*-$**automorphisms:**

Time automorphisms of operator algebras in quantum mechanics: $\hat{\mathcal{O}}_i \in \mathcal{A}$ where $\mathcal{A}$ is the quantum operator algebra of a system and $e^{i\hat{\mathcal{H}}t} \in G$ where G is the group of time-evolution operators of parameterized by time t, $-\infty < t < \infty$.

$$e^{-i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_i(t_0)e^{i\hat{\mathcal{H}}t} = \hat{\mathcal{O}}_i(t_0 + t) \tag{B.17}$$

it is an algebra homomorphism because

$$
\begin{aligned}
e^{i\hat{\mathcal{H}}t}(\hat{\mathcal{O}}_1\hat{\mathcal{O}}_2)e^{-i\hat{\mathcal{H}}t} &= (e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_1 e^{-i\hat{\mathcal{H}}t})(e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_2 e^{-i\hat{\mathcal{H}}t}) \\
e^{i\hat{\mathcal{H}}t}(\hat{\mathcal{O}}_1+\hat{\mathcal{O}}_2)e^{-i\hat{\mathcal{H}}t} &= e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_1 e^{-i\hat{\mathcal{H}}t} + e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_2 e^{-i\hat{\mathcal{H}}t}
\end{aligned} \tag{B.18}
$$

it is one-to-one and onto because

$$
e^{i\hat{\mathcal{H}}t}\hat{\mathcal{O}}_i(t_0+t)e^{-i\hat{\mathcal{H}}t} = \hat{\mathcal{O}}_i(t_0) \tag{B.19}
$$

is an inverse (obtained by simply replacing t with $-t$).

is an example of an inner automorphism of the algebra of observables. two equivalent ways of describing the time flow: either the flow instate space(Schrödinger picture), or (generalized Heisenberg picture). So we say time flow is as a one-parameter group of automorphisms of the algebra of observables. A mathematic theorem called the Tomita-Takikey that states in quantum field theory there is a unique one-parameter automorphism of the operator algebra of - the physical basis of time - this is the thermal time hypothesis [290] [291].

• **axiom of choice:** Say we have a family of nonempty sets $\{X_\alpha\}$. Then there is a set X which contains exactly one element from each set $\{X_\alpha\}$. For finite collection of sets, this is obvious and isn't really an axiom. It is when you It applies as an axiom when there are an infinite number (countable and uncountable) number of sets.

• **Banach algebra:** A Banach algebra is a complex Banach space which is also an algebra with identity 1, and in which

(i) $\|xy\| \leq \|x\| \, \|y\|$,

(ii) $\|1\| = 1$.

Also a normed vector ring.???????

If the norm satisfies the parallelogram law, its is also a Hilbert space.

A **Banach subalgebra** is a closed subalgebra of A which contains 1; they are precisely those subsets of A which themselves are Banach algebras with the same identity, and the same norm.

• **Banach space:** A Banach normed space complete normed space which is a complete metric space with respect to the metric induced by norm, $d(x-y) := \|x-y\|$.

A unital Banach space: A banach space containing the identity with respect to multiplication.

• **Bargmann transform:** The Bargmann transform is the unitary transformation that takes an $L^2(\mathbb{R}^n)$ function of the coordinates q_i to a holomorphic square integrable (with respect to a certain measure) function of n complex variables. It can be interpreted as the transform that takes

$$f \to \text{ Analytic continuation of } e^{-t\Delta/2} f,$$

that is, one obtains the representatioon by heat kernel evolution followed by analytic continuatiooon from the usual position space representation.

• **bijective:** A function is bijective if it is one-to-one and onto, that is injective and surjective.

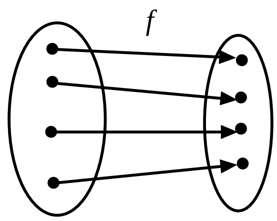


Figure B.1: bijective

See injective, surjective.

bilinear form: A bilinear form (or sesquilinear form) $F(u, v)$ on a Hilbert space H is an assignment of a scalar to each pair of vectors u, v of a subspace $D(F)$ of H in such a way that

(i) $F(\alpha u + \beta v, w) = \overline{\alpha} F(u, w) + \overline{\beta} F(v, w)$, and
(ii) $F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$

for $u, v, w \in D(F)$. The subspace $D(F)$ is called the domain of the bilinear form.

Bohr compactification: Bohr compactification $\overline{\mathbb{R}}$ of the real line $\mathbb{R}$ is the dual group of $\mathbb{R}$ equipped with the "discrete" topology in which the real line is totally atomized as if no point is near any other point.

The dual of the dual of a group G is the same group back again But suppose you take the dual of G and replace the usual topology with the discrete topology (the power set, same as saying that singletons are open) so you say H is equal to G' as a group but atomized as a topological space and all functions from it are continuous. *Then* you take the dual of H, and that is the Bohr compactification of G.

Bohr group: Dual group to the discrete line. Is compact.

Boolean algebra: A distributive lattice in which every element has a complement is a Boolean lattice or a *Boolean algebra*. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Boolean logic: Formulises in algebraic terms for statements that are either true or false.

See non-Boolean logic

Borel function:

Borel sets: One wants to define a probability distribution on the real line $\mathbb{R}$, that is, assign a real number to any subset Δ of $\mathbb{R}$ that is the probability of some quantity taking its value in Δ. It turns out that it is impossible to construct a probability measure defined on all subsets. one must restrict oneself to a particular family of sets - the Borel subsets, $\mathcal{B}(\mathbb{R})$. This is the smallest family of subsets of $\mathbb{R}$ that includes all the open sets and which is closed under complements and countable intersections.

Borel $\sigma-$algebra: Let X be a topological space. The smallest $\sigma-$algebra on X that contains all open sets of X is called Borel $\sigma-$algebra of X.

• **Borel-summable:** Borel-summation does not pick up on non-perturbative effects.

$$\sum_n a_n x^n \tag{B.20}$$

with $a_n \approx Cn!$

$$\int_0^\infty e^{-t/x}\, t^n dt = n!\; x^{n+1} \tag{B.21}$$

$$\int_0^\infty e^{-t/x} \left(\sum_n \frac{a_n}{n!} x^n \right) \frac{dt}{x} = \sum_n a_n x^n \tag{B.22}$$

$$g(t) := \sum_n \frac{a_n}{n!} x^n \tag{B.23}$$

• **bounded:**

(i) bounded linear operator:

$$\|A\| := \sup_{x\in X} \frac{\|Ax\|}{\|x\|} \equiv \sup_{\|x\|=1} \|Ax\| \tag{B.24}$$

then operator A is bounded if $\|A\| \leq C$. Note: boundedness and continuity are equivalent for linear operator.

- bounded set:

We call a nonempty subset $M \subset X$ a bounded set if its diameter

$$\delta(M) := sup_{x,y\in M}\, d(x,y) \tag{B.25}$$

is finite.

• **bounded contraction semigroup:** imaginary-time path integral has kernel

$$e^{\frac{-\hat{\mathcal{H}}t}{\hbar}} \tag{B.26}$$

• **bounded convergence theorem:**

If the sequence $\{f_n\}$ of measurable functions is uniformally bounded and if $f_n \to f$ in measure as $n \to \infty$, then

$$\lim_{n\to\infty} \int f_n d\mu = \int f d\mu.$$

• **bounded inverse theorem:** The range of a bounded operator A is closed in $\mathcal{H}$ implies that $A - \lambda\mathbb{I}$ has in inverse in $\mathcal{B}(\mathcal{H})$.

• **bounded linear functional:** A linear functional satisfying

$$|F(x)| \leq M\|x\|, \quad x \in X.$$

• **bounded linear operator:** An operator A is called bounded if its domain is the whole of X, i.e. $D(A) = X$, and there is a constant M such that

$$\|Ax\| \leq M\|x\|, \quad x \in X.$$

The norm of such an operator is defined by

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}.$$

• **bounded variation:** A function $f(x)$ defined on an interval $[a,b]$ is said to be of bounded variation if it satisfies

$$\sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < C \tag{B.27}$$

for any partition $a = x_0 < x_1 < \cdots < x_n = b$.

- **braid:**
- **braided moinodal category:** A 3-category with one object.
- **braid group B_n:** a restriction of the permutation group.

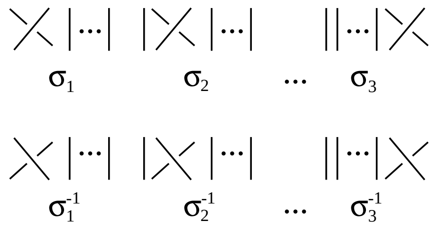


Figure B.2: Braid group generators

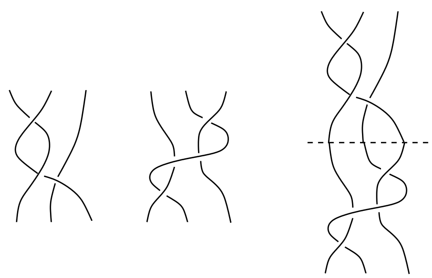


Figure B.3: Multiplication of braids.

$$
\begin{aligned}
\sigma_i \sigma_i^{-1} &= 1, & i &= 1, \ldots, n-1 \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_i . & i &= 1, \ldots, n-2 \\
[\sigma_i, \sigma_j] &= 0 & |i-j| &> 1.
\end{aligned} \tag{B.28}
$$

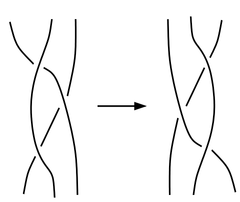


Figure B.4: A Reidermeister move on a three-string braid.

Quantum groups were invented largely to provide solutions for the Yang-Baxter equation and hence solvable models in 2-dimensional statistical mechanics and one-dimensional quantum mechanics. why i-d quantum???????????

- **Caley transform:**

$$a \mapsto u := \frac{a-i}{a+i}$$

• **canonical local trivialisation:** There are local sections $s_i : U_i \to \pi^{-1}(U_i)$ canonically associated to the trivialisation, eefined so that for every $p \in U_i$, $\phi_i(s_i(p)) = (p, e)$. In other words, the map from U_i to G is the constant function sending every point to the identity.

• **Cantor Set:** The Cantor set is

$$C = \bigcap_{k=1}^{\infty} C_k. \tag{B.29}$$

where

$$\begin{aligned} C_0 &= [0,1] \\ C_1 &= [0,\frac{1}{3}]\bigcup[\frac{2}{3},1]. \\ C_2 &= [0,\frac{1}{9}]\bigcup[\frac{2}{9},\frac{1}{3}]\bigcup[\frac{2}{3},\frac{7}{9}]\bigcup[\frac{8}{9},1], \end{aligned} \tag{B.30}$$

and in general C_{n+1} consists of the intervals of C_n with their open middle thirds removed.

• **cardinality:**

• **Cartan's structure equations:**

$$d\theta_i = -\frac{1}{2}c^i_{kl}\theta_k \wedge \theta_l. \tag{B.31}$$

• **category:** A category $\mathcal{C}$ consists of "objects", C, and "morhisms", f, between them, such that

(i) If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morhisms, then there exists a morphism $g \circ f : C_1 \to C_3$.

(ii) It is assumed that the identity map for C, $\mathrm{id} : C \to C$, is a morphism for every object C of $\mathcal{C}$.

The set of objects is usually denoted ob$\mathcal{C}$.

Examples:

The category of open sets in Euclidean spaces, where the morphisms are the smooth maps.

The category of abelian groups, where the morphisms are homomorphisms.

n-category:

n-categorical group: A is by definition a group-object of the category od groupoids.

• **category theory:** TAKEN DIRECTLY FROM mathworld.wolfram.com

Category Theory

The branch of mathematics which formalizes a number of algebraic properties of collections of transformations between mathematical objects (such as binary relations, groups, sets, topological spaces, etc.) of the same type, subject to the constraint that the collections contain the identity mapping and are closed with respect to compositions of mappings. The objects studied in category theory are called categories.

http://mathworld.wolfram.com/

an abstraction of many concrete concepts in diverse branches of mathematics.

• **CAR:** stands for canonical anti-commutation relations. CAR algebra....

• **Cat:** The category of all small categories.

• **Cauchy-Kowalewski:**

• **Cauchy sequence:** A sequence $\{a_l\}$ satisfying

$$\|a_n - a_m\| \to 0 \quad \text{as } m, n \to \infty.$$

• **CCR:** stands for canonical commutation relations.

• **Cech compactification:**

• **center:** An abelian subgroup of a ring R is the **center**

$$X = Z(R) \;\; X = \{a \in R : b \in R, \; ab = ba \;\text{ for all } b\} \tag{B.32}$$

The center of a Lie group is the set of commuting elements $\{x \in \mathbf{g} : [x, y] = 0$, for all $y \in \mathbf{g}\}$.

X is the centralizer of Y $X = C_R(Y)$

$$X = Z(R) \;\; X = \{a \in R : b \in Y, \; ab = ba \;\text{ for all } b\} \tag{B.33}$$

Alternative

The kernel of the $G-$action is the subgroup of G defined by

$$K := \{g \in G : gp = p \text{ for all } p \in \mathcal{M}\}.$$

The kernel measures the part of the group that is not represented at all in the $G-$action on $\mathcal{M}$. An example is given by the adjoint action of G on itself in which

$$\mathrm{Ad}_g(g') := gg'g^{-1}.$$

The kernel of this action is the centre $C(G)$ of G.

• **central extension:** A central extension of a Lie algebra $\mathrm{Lie}(G)$ is a Lie algebra E together with a homomorphism $\pi : E \to \mathrm{Lie}(G)$ such that $\mathrm{Ker}(\pi) \subset Z(E)$ where $Z(E) = \{A \in E : [A, B] = 0 \text{ for all } B \in E\}$ is the centre of E.

• **characters:**

Characters of a finite group:

$$\chi(A) = \mathrm{tr}A = \sum_i A_{ii}. \tag{B.34}$$

These characters are invariant under simultaneity transformations because of the cyclic symmetry of matrices

$$\chi(U^{-1}AU) = \mathrm{tr}(U^{-1}AU) = \mathrm{tr}A \tag{B.35}$$

Characters of algebras:

Let $\mathcal{U}$ be an abelian C^*-algebra. A **character** χ, of A, is a nonzero linear map, $\omega; A \in \mathcal{U} \mapsto \chi(A) \in \mathbb{C}$, of $\mathcal{U}$ into the complex numbers $\mathbb{C}$ such that

$$\chi(AB) = \chi(A)\chi(B) \tag{B.36}$$

for all $A, B \in \mathcal{U}$. As it preserves multiplication (B.36) it is a homomorphism. See the spectrum $\sigma(\mathcal{U})$.

• **chart:** Given a topological space $\mathcal{M}$, a chart on $\mathcal{M}$ is a one-to-one map ϕ from an open subset $U \subset \mathcal{M}$ to an open subset $\phi(U) \subset \mathbb{R}^n$, i.e., a map $\phi : \mathcal{M} \to \mathbb{R}^n$. A chart is often called a coordinate system.

• **Chern classes:** cohomology

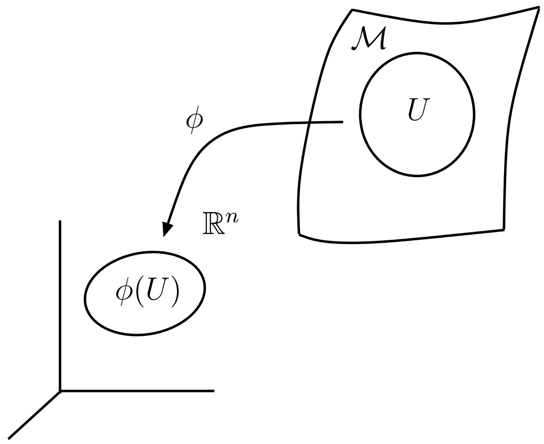


Figure B.5: DiffClass0. A chart on $\mathcal{M}$ comprises an open set U of $\mathcal{M}$, called a coordinate patch, and a map $\phi : U \to \mathbb{R}^n$.

$$\det(t\mathbf{I} + a_c T^c) \tag{B.37}$$

substituting

second *Chern class*

$$c_2(P) = \frac{1}{4\pi^2}\left(\frac{1}{2}(TrF) \wedge (TrF) - \frac{1}{2}Tr(F \wedge F)\right) \tag{B.38}$$

• **chromatic evaluation:** The chromatic evaluation of the spin network is equivalent to the Temperley-Lieb trace of the spin network, or, equivalent to the Kauffman bracket.

• **class function:** A function of elements $x, y, z, ..$ of a group G is said to be a class function if

$$f(x, y, \ldots, z) = f(g^{-1}xg, g^{-1}yg, g^{-1}\ldots zg) \tag{B.39}$$

where $g \in G$.

• **closable operator:** Let A be a linear operator from a normed vector space X to a normed vector space Y. It is called closable if for $\{x_k\} \subset D(A)$, $x_k \to 0$, $Ax_k \to y$ imply that $y = 0$.

• **closed bilinear form:** A bilinear form $F(u, v)$ is called closed if $\{u_n\} \subset D(F)$, $u_n \to u$ in H, $F(u_n - u_m) \to 0$ as $m, n \to \infty$ imply that $u \in D(F)$ and $F(u_n - u) \to 0$ as $n \to \infty$.

• **closed graph theorem:** The closed graph theorem implies that if $A : X \to Y$ is closed and has the domain of A is equal to X then A is bounded.

See bounded inverse theorem.

• **closed surface:** Examples of closed surface are the sphere, the torus, the Klein bottle. They are classified by the genus and their orientability. An examples of a non-closed surface is a disk which is a sphere with a puncture.

• **closure property:** If certain set of operations on sets in $\mathcal{F}$ again produce sets in $\mathcal{F}$, we say $\mathcal{F}$ is closed under these operations.

• **cobordism:** see Fig.(J.1) - referring to wrong figure for some reason!

• **co-cycle:**

• **cocycle Radon-Nikodym theorem:** The Cocycle Radon-Nikodym [] theorem states that two modular automorphisms defined by two states of a von Neumann algebra are inner-equivalent.

see rovelli thermal time hypothesis

• **codimension:** $\mathrm{codim}X = \mathrm{dim}X^{\perp}$.

• **cohomology:** Roughly speaking, the cohomology $H^p(M, \mathbb{R})$ counts the number of noncontractable p-dimensional surfaces in M.

we consider all closed forms modded out by exact forms. In other words two forms are said to be equivalent if

$$\lambda_1 = \lambda_2 + d\Phi \quad \rightarrow \quad [\lambda_1] = [\lambda_2] \tag{B.40}$$

for any Φ. two closed forms are called cohomology.

• **compact:** Every open cover has a finite subcover.

• **compactification:** The process of adding points to a given topological space in order to make it compact. The simplist compactifiction is adding just one point, for exmple the process of adding a point to a plane to make a sphere.

• **compact Lie group:**

Each group element is uniquely related by the parameters.

• **compact manifold:** A manifold is compact every open cover has a finite subcover. “A town is compact when it can be policed by a finite number of arbitrary short-sighted policemen”.

• **compact operators:** An operator A is said to be a compact operator if there exists a sequence of finite rank operators that converge to A.

• **comparable:** Let $\| \cdot \|_1$ and $\| \cdot \|_2$ be two norms defined on the same linear space Φ. These two norms are called **comparable** if if for every $\varphi \in \Phi$ there exists a constant $C > 0$ such that

$$\|\varphi\|_1 \leq \|C\varphi\|_2, \quad \text{for all } \varphi \in \Phi. \tag{B.41}$$

• **compatible:** Two norms are called compatible if and only if every sequence $(\varphi_n)_{n=1}^{\infty} \subset \Phi$ which is Cauchy with respect to both norms and which converges to 0with respect to one of them, also converges to 0 with respect to the other.

• **completion:** complicates matters: it would be like doing real analysis in the rational numbers instead of the real line R.

Completeness is an important property since it allows us to perform limit operations which arise frequently in our constructions.

• **completely regular topological space:** Let X be a topological space. If for each open neighbourhood $\mathcal{U}$ of x, there is a continuous function $0 \leq f(x) \leq 1$ such that $f(x) = 0$ and f is identically one on the complement $X - \mathcal{U}$ of $\mathcal{U}$ in X, i.e.

$$f(x) = \begin{cases} 0 & \text{for } x \in X \\ 1 & \text{for } x \in \mathcal{U} - X \end{cases} \tag{B.42}$$

(Completely regular spaces are also called Tychonoff spaces). are able to support sufficiently many continuous functions: for two distinct points x and y of a completely regular space X, there is a continuous function on X taking distinct values at x and y.

Third class of spaces contains, for example, all normal and all Hausdorff spaces.

• **complete set:** A collection f_i of functions on the symplectic space $(\mathcal{M}, \omega)$ which Poisson commute with each other (are in involution) is said to be complete if the vanishing of $\{f_i, g\}$ for all i implies that g is a function of the form $g(x) = h(f_1(x), \dots, f_n(x))$.

A collection of operators $\{A_j\}$ is said to be complete if any operator B which commutes with each A_j is a multiple of the identity. This condition is equivalent to the irreducibility of $\{A_j\}$, that is, there is no non-trivial subspace that is invariant under each A_j.

• **complete space:** all Cauchy sequences (defined wrt a norm??) converge to an element in the space.

• **complexification of a Lie group:** The complexification of a Lie group $\mathbf{g}$ denoted $\mathbf{g}_{\mathbb{C}}$.

Recall the defining properties of a Lie algebra: antisymmetric $[X, Y] = -[Y, X]$, bilinearity $[\alpha X, Y] = \alpha[X, Y] = [X, \alpha Y]$ for any real number α, and satisfies the Jacobi identity $[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] \equiv 0$. $X_1 + iX_2$ where $X_1, X_2 \in \mathbf{g}$

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1]). \tag{B.43}$$

It is clear that (B.43) is *real* bilnear and skew-symmetric. If we prove that it is *complex* linear in the first factor, it will be complex linear in the second because of the skew-symmetry. As we already know it is real linear in the first factor, it suffices to show that it is *imaginary* linear. This is not difficult to prove, all we need to do is verify that

$$[i(X_1 + iX_2), Y_1 + iY_2] = i[X_1 + iX_2, Y_1 + iY_2] \tag{B.44}$$

is true. This is easily done by expanding each side and seeing they are indeed equal.

It remains to check the Jacobi identity. First consider Y and Z to be in $\mathbf{g}$ but take X to be in $\mathbf{g}_{\mathbb{C}}$. Now, $X = X_1 + iX_2$ is linear in the Jacobi identity and the Jacobi identity holds separately for X_1 and X_2,

$$[[X_1, Y], Z] + [[Z, X_1], Y] + [[Y, Z], X_1] + i([[X_2, Y], Z] + [[Z, X_2], Y] + [[Y, Z], X_2]) \equiv 0, \tag{B.45}$$

and so the Jacobi identity holds for $X \in \mathbf{g}_{\mathbb{C}}$ and $Y, Z \in \mathbf{g}$. Similarly for Y and Z. Therefore we have shown that the elements of the complexification $\mathbf{g}_{\mathbb{C}}$ satisfy the Jacobi identity.

• **complexification:** - We can tensor a real vector space with the complex numbers and get a complex vector space; this process is called complexification. For example, we can complexify the tangent space at some point of a manifold, which amounts to forming the space of complex linear combinations of tangent vectors at that point. (from Geometric Quantization John Baez August 11, 2000).

• **complex manifold:** $\mathcal{M}$ is a complex manifold if

(i) $\mathcal{M}$ is a topological space;

(ii) $\mathcal{M}$ is provided with a family of pairs $\{(U_i, \varphi_i)\}$;

(iii) $\{U_i\}$ is a family of open sets which covers $\mathcal{M}$, φ_i is a homomorphism from U_i to an open subset U_i' of $\mathbb{C}^m$;

(iv) Given U_i and U_j such that $U_i \cap U_j \neq \emptyset$, the map $\psi_{ij} = \varphi_j \varphi_j^{-1}$ is holomorphic.

• **congruence:** A congruence is a set of curves which fill a manifold, or part of it, without intersecting. Through every point there passes one and only one curve.

• **congruence relation:**

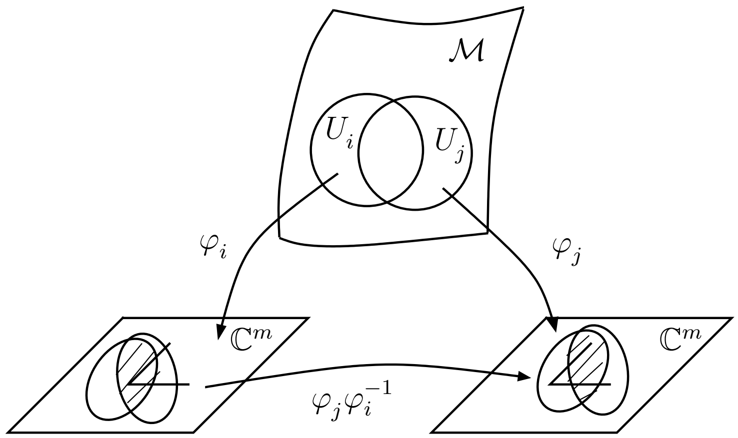


Figure B.6: Complex manifold.

• **connected:** In the topological sense: a topological set not able to be partitioned into non-empty open subsets each of which has no points in common with the closure of the other.

• **connection:** Roughly, comparison of objects in two different spaces is made by a prescribed mapping, and the mappings that connect the various spaces are called connections.

A connection on a principal bundle is an assignment to each local trivialisation $\phi_i : \pi^{-1}(U_i) \to U_i \times G$ (a choice of gauge in physics terms) a Lie algebra one-form ω_i on U_i which satisfies the following rule between different local trivialisations, (for simplicity here in the case of matrix groups G):

$$\omega_j = t_{ij}\omega_i t_{ij}^{-1} + t_{ij}^{-1} dt_{ij}.$$

A connection is an example of a gauge field.

• **continuity:**

− uniformaly continuous when for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x, x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \epsilon$

− pointwise continuous when for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \epsilon$. A function which is pointwise continuous at every point is pointwise continuous.

− sequentially continuous when it preserves limits of convergent sequences: if $(a_i)_{i\in N}$ converges to a in X then $(f(a_i))_{i\in N}$ converges to $f(a)$ in Y.

• **continuous function:** A function f is continuous at a point p if whenever we can force the distance between $f(x)$ and $f(p)$ to be as small as desired by taking the distance between x and p to be small enough.

A function is said to be absolutely continuous if for each $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon \tag{B.46}$$

whenever (a_i, b_i), $i = 1, 2, \ldots, n$, are non-overlapping subintervals of I with $\sum_{i=1}^{n} |b_i - a_i| < \delta$

• **contractible open cover:** A collection $\{U_i\}$ is a contractible open cover of M if each U_i and each non-empty finite intersection $U_i \cap U_j \cap \cdots$ is contractible to a point.

• **contraction:** Suppose there is a map A from the space M to itself, this is a contraction mapping if there is a positive constant $K < 1$ such that $\|Ax - Ay\| < K\|x - y\|$ for all $x, y \in M$.

• **convergence:**

(x_n) converges to x if and only if the sequence (x_n) is eventually in every neighbourhood of x.

(From Weak convergence of inner superposition operators) modes of convergence:

The precise definitions. Let $A_\nu : X \to Y$, $A : X \to Y$, where $\nu \in \mathbb{N}$ are the mappings between two Banach spaces X and Y . One says that the sequence A converges to A

(S) strongly (pointwise), if $A_\nu x \to Ax$ in Y for all $x \in X$;

(C) continuously, if $A_\nu x \to Ax$ in Y for any norm converging sequence $x_\nu \to x$ in X;

(W) weakly (pointwise), if $A_\nu x \rightharpoonup Ax$ in Y for all $x \in X$;

(CW) continuously weakly, if $Ax \rightharpoonup Ax$ in Y for any weakly converging sequence $x_\nu \rightharpoonup x$ in X.

• **countable:** (denumerable) A set is countably infinite if there is a one-to-one onto function from the set A to the set $\{1, 2, 3, \ldots, \}$.

• **countably Hilbert space:** is a complete linear topological space whose topology is defined by a countable family of Hilbert spaces, Φ_n. The topology. Countable set of norms $\|\varphi\|_n$ where the norms are associated with a scalar product - $\|\varphi\|_n := \sqrt{(\varphi, \varphi)_n}$. First property is that Φ_n is the Cauchy completion of Φ in the norm $\|.\|_n$???. Then for any m, n it is required that if (ϕ_k) is both a $\|.\|_m$ convergent sequence and an $\|.\|_n$ Cauchy sequence in Φ then is also $\|.\|_n$ convergent.

Second definition:

A space Φ is a countably Hilbert space (or a countably scalar product space) if an increasing denumerable number of scalar products

$$(\varphi, \varphi)_1 \leq (\varphi, \varphi)_2 \leq \cdots \leq (\varphi, \varphi)_p \leq \cdots \tag{B.47}$$

are defined on Φ such that the norms

$$\|\varphi\|_p := \sqrt{(\varphi, \varphi)_p}, \quad p = 0, 1, 2, \ldots \tag{B.48}$$

are comparable and compatible.

• **countably neighbourhood base:**

• **cover:** Given a set X with $A \subseteq X$ is said to be regular if, for a **cover** for A is a family of subsets $\mathcal{U}$ =$\{U_i : \in I\}$ of X such that $A \subseteq \cup_{i \in I} U_i$. A subcover $\mathcal{U}$ for A is a subfamily $\mathcal{V} \subset \mathcal{U}$ which still forms a cover for A.

• **cubulations:** Cubic triangulations of a four manifold are called cubulations. The first motivation for considering cubulations in spin foam models is that current semiclassical states used in LQG do not assign good classical behaviour to the volume operator of LQG (which plays a pivitol role in the dynamics) unlessthe underlying graph has cubic topology.

Cubulations also nicely fit in with the framework of Algebraic Quantum Gravity which in its minimal version is also formulated in terms of algebraic graphs of cubic topology only.

• **curve:** We refer to Fig.(**??**). A curve in a manifold $\mathcal{M}$ is a map λ of the open interval $I = (a, b) \in \mathbb{R} \to \mathcal{M}$ such that for every coordinate system of $\mathcal{M}$ $\phi \circ \lambda : I \to \mathbb{R}^n$. We say the curve smooth if $\phi \circ \lambda : I \to \mathbb{R}^n$ is smooth. The set of curves is denoted $\mathcal{C}^\infty$.

Alternative kinds of curves such as piecewise analytic, continuous, oriented, an embedding (does not come arbitrarily close to itself) are defined in the obvious way.

See piecewise-analytic

• **cyclic state:** GNS "vacuum state". For example for the SHO $|0>$ for which $\hat{\mathcal{H}}|0>= \frac{1}{2}\hbar|0>$. $\hat{a}|0>=0$. This, with $|n>:= (\hat{a}^\dagger)^n|0>$.

$$<0|[\hat{a}, \hat{a}^\dagger]|0>= \left(\frac{1}{2} + 1\right)\hbar \tag{B.49}$$

Any representation with a cyclic state, can be constructed from the "vacuum" expectation value of the algebra of operators.

the precise definition of a cyclic vector is: a vector Φ is cyclic for a C^*algebra A acting in a Hilbert space $\mathcal{H}$ if and only if the linear space $\mathcal{A}\Phi = \{A\Phi, A \in \mathcal{A}\}$ is dence in $\mathcal{H}$.

(In particular the vacuum) is cyclic and separating for the algebras $\mathcal{A}$, i.e. $\mathcal{A}\Phi$ is dense in $\mathcal{H}$ (Φ is cylic) and $A\Phi = 0$, $A \in \mathcal{A}$ implies $A = 0$ (Φ is separating).

More precisely, a vector Φ is separating for $\mathcal{A}$ acting in $\mathcal{H}$ if and only if it is cyclic for the commutant $\mathcal{A}'$ of $\mathcal{A}$; the commutant is the set of operators in $\mathcal{H}$ which commute with all $A \in \mathcal{A}$. If $[A, A'] = 0$ then

Let $\mathcal{R}$ be a von Neumann algebra generated by $\pi_\omega(\mathcal{A})$, i.e., $\mathcal{R} = ()''$ A cyclic and separating vector in the Hilbert space $\mathcal{H}_\omega$ (ξ is separating in $\mathcal{A}$ if it is cyclic in $\mathcal{A}'$).

• **cylindrical function:** Fake infinite functions: although they depend on a field, say a scalar field $\phi(x)$, they only really depend on a finite number of variables. One begins by introducing a space $\mathcal{S}$ of 'probes'. Elements of $\mathcal{S}$ probe the structure of the scalar field $\phi \in \mathcal{C}$, ($\mathcal{C}$ being the classical configuration space of scalar functions), through linear functions h_e on $\mathcal{C}$:

$$h_e(\phi) = \int_{\mathcal{M}} d^3x \; e(x)\phi(x)$$

which capture a small part of the scalar field. Given a set α of probes, $h_{e_1}, \ldots, h_{e_n}$ and a (suitably regular) complex-valued function ψ of n real variables, we can now define a more general function Ψ on $\mathcal{C}$,

$$\Psi_\alpha(\phi) := \psi(h_{e_1}(\phi), \ldots, h_{e_n}(\phi))$$

These are examples of cyclindrical functions. Their linear span is usually denoted Cyl_α.

???term cylindrical borrowed from free fields in Minkowski???

Kolmogorov used this special class of functions to define measures for infinite dimensional integration theory.

• **cylindrical measure theory:** The idea of cylindrical measure theory is to reduce the integration over infinite dimensional spaces to a series of finite dimensional subspaces by making use of a special class of function(-al)s - cylindrical function(al)s. However, because of nesting, overlapping, (as well as freedom in choice of basis) of the finite dimensional subspaces, integration over them are subject to certain non-trivial consistency conditions. When these consistency conditions are satisfied, we say the infinite dimensional function space is said to be equipped with a cylindrical measure.

• **Darboux theorem:** Let $\mathcal{Q}$ be a configuration space (manifold). In local coordinates $(q^1, \ldots, q^n)$ one may identify the canonical momentum variables $(p_1, \ldots, p_n)$ with the cotangent vectors in the coordinate basis of $(q^1, \ldots, q^n)$. The Darboux theorem asserts there is a sympletic form Ω on the cotangent space $T^*\mathcal{Q}$ (phase space Γ) given by

$$\Omega_{ab} = \sum_{\mu=1}^{n} 2\nabla_{[a} dp_\mu \nabla_{b]} dq^\mu \tag{B.50}$$

and the pair (Γ, Ω) is a symplectic manifold.

• **Dehn surgery:** Dehn surgery is a procedure by which one can construct all three-manifold topologies. One begins with drawing a knot or link (i.e., a set of knots) on a given manifold, then we thicken the knot into a tube. One then removes this tubular region, give it a twist and then glue it back.

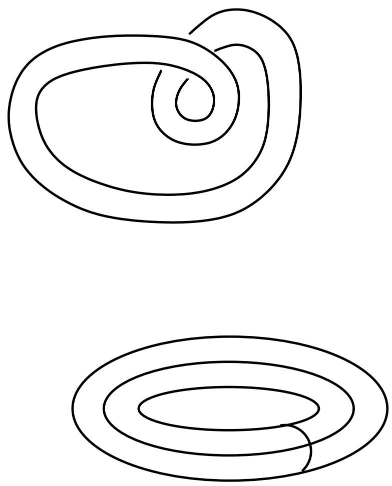


Figure B.7: Dehn surgery.

The three-sphere is two torus glued together. It can all be reduced to working with Chern-Simons on a torus. Once we solve Chern-Simons on a torus we have the exact closed solution for Chern-Simons on any three-manifold. Chern-Simons is related to $2+1$ quantum gravity, hence Witten showed that $2+1$ quantum gravity is exactly solvable, [387].

• **DeMorgan'a Laws:** Let A and B be subsets of X then

1) $X\backslash(A \cup B) = (X\backslash A) \cap (X\backslash B)$ and

2) $X\backslash(A \cap B) = (X\backslash A) \cup (X\backslash B)$.

• **dense:**

• **De'Ram cohomology:**

• **dihedral angles:** Angle subtended where to planes intersect???

• **diffeomorphic:** Two manifolds connected by a diffeomorphism are said to be diffeomorphic. From the differential-geometric point of view, diffeomorphic manifolds not distinguished by some othe structure (e.g. a metric) are effectively the same.

• **diffeomorphism:** A $f : M \to N$ from one manifold M to a manifold N is a smooth map whose inverse is also smooth.

• **differentiable manifold:** If $\mathcal{M}$ is a space and Φ its maximal atlas, the set $(\mathcal{M}, \Phi)$ is a differentiable manifold. We can have C^∞, C^k, analytic, and semianalytic manifolds.

• **differential forms:** A $p-$form is defined to be a completely antisymmetric tensor of type $\begin{pmatrix} 0 \\ p \end{pmatrix}$. A one-form is a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor and a scalar function is a zero form. The number p is the degree of the form.

• **Dirac operator:** a "square root" of the D'Alambert operator in flat Minkowski space-time

• **direct sum:**

$$\underline{\underline{A}} \oplus \underline{\underline{B}} = \begin{pmatrix} \underline{\underline{A}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}} \end{pmatrix}, \quad \vec{v} \oplus \vec{w} = \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \tag{B.51}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \oplus \begin{pmatrix} p & q \\ r & t \end{pmatrix} = \begin{pmatrix} a & b & c & 0 & 0 \\ d & e & f & 0 & 0 \\ g & h & i & 0 & 0 \\ 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & r & t \end{pmatrix} \tag{B.52}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ v \\ w \end{pmatrix} \tag{B.53}$$

• **discrete counting measure:**

• **discrete topology:** Let $\mathcal{T}$ be a topological space. A poit $p \in \mathcal{T}$ is called isolated if $\{p\}$ is open in $\mathcal{T}$. The unique topology in which every point is isolated is called the discrete topology. All functions are contiuous in the discrete topology.

• **distribution:** A distribution V on a manifold M is a choice of a subspace V_x of each tangent space $T_p(M)$, where the choice depends smoothly on x.

See integrable distribution.

• **division ring:** A ring with identity is called a *division ring* if all its non-zero elements are regular (invertible).

• **domain of an operator:** $\mathcal{D}(A) \subset \mathcal{H}$ such that $|\hat{|}\psi > |^2 < \infty$

• **dominating convergence theorem:** The (Lebesgue) dominating convergence theorem is concerned with when the integral of a limit functions is equal to the limit of integrals, i.e. , when

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n. \tag{B.54}$$

• **dual complex:** Let us fix a triangulation Δ of a $d-$dimensional spacetime manifold $\mathcal{M}$. This triangulation defines another decomposition of $\mathcal{M}$ into cells called a dual complex. There is a one-to-one correspondence between $k-$simplicies of the triangulation and the $(d-k)-$cells in the dual complex.

• **dual group:** The dual group $\hat{G}$ is the set of characters $\{\gamma\}$ of G, i.e., homorphisms of G into the circle group $\{z \in \mathbb{C},\ |z| = 1\}$, with group multiplication defined by

$$< \gamma_1\gamma_2, g >=< \gamma_1, g >< \gamma_2, g > \tag{B.55}$$

[157]

$$(\lambda_1 g_1 + \lambda_2 g_2) \cdot g_3 = \lambda_1 g_1 \cdot g_3 + \lambda_2 g_2 \cdot g_3 \tag{B.56}$$

group algebra $\mathbb{R}(G)$.

As this has the structure of a vector space we may form a dual vector space via some inner product Fun(G) of functions on G. This bilinear inner product will take values from $< .,. >$: $\mathbb{R}(G)\otimes$Fun(G) $\to$ **R**. A natural choice is simply as functions on the group space $< g, f >:= f(g)$.

given the inner product between these two vector spaces and given an operator on one of them we can define its dual action, that is its adjoint, acting on the other.

generators of the group vector space e_i and the generators of the dual space e^i.

• **dual space:**

the algebraic dual The set of all functionals defined on a vector space X is can itself be made into a vector space. This is called the alghebraic dual of X and is denoted X^*.

the topological dual The set of all linear bounded (that is, continuous) functionals on X is called the topological dual of X and is denoted X'. It is called topological dual because ... continuous transformation preserves the topology

the topological dual [208]:

Choose a dense and invariant domain Φ for a .is defined as the algebraic dual of Φ, i.e. the set of all linear functionals on Φ equipped with the weak *topology of pointwise convergence.

• **embedding:** doesn't come arbitrarily close to itself.

• **empty set:** Set which has no elements, denoted $\emptyset$.

• **entire function:** A complex function $f(z)$ analytic at all points of any open set of the complex plane.

OR

A complex function $f(z)$ analytic everywhere in the complex plane within a finite distance of the origin, it is an entire function. For example polynomials $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ are entire functions. They diverge at infinity...

• **enveloping algebra:** see universal enveloping algebra

If I is an ideal of a Lie algebra $\mathcal{G}$, the equivalence classes in $\mathcal{G}$

$$[A] = [A + I], \quad A \in \mathcal{G}, I \in I, \tag{B.57}$$

forms a Lie algebra - the quotient algebra, $\mathcal{G}/I$.

$$[[A], [B]] = [[A, B]] \tag{B.58}$$

$$[A + I_1, B + I_2] = [A, B] + [i_1, B] + [A, i_2] + [i_1, I_2] \tag{B.59}$$

$\psi : \mathcal{G}$

the sums and products in $im(\psi) \subset$ form an algebra called the **enveloping algebra** of $\mathcal{G}$

• **epimorphism:** A surjective group homomorphism.

• **epsilon net:** A finite or infinite number of points on a metric space such that each point on the space is with a distance of ϵ of some point on the net.

• **equivalence classes:** We transfer our attention from objects and products of objects to consideration of equivalence classes of objects and the induced multiplication between these classes.

• **essentially self-adjoint:** An operator which has a unique self-adjoint extension is said to be essentially self-adjoint, having a canonically defined Friedrichs extension and canonical functional calculus.

• **ergodic:**

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\omega(\tau^t(A)B)dt=\omega(A)\omega(B), \tag{B.60}$$

and weak-mixing

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}|\omega(\tau^t(A)B)-\omega(A)\omega(B)|^2dt=0. \tag{B.61}$$

• **Euler's theorem:**

$$F-E+V \tag{B.62}$$

• **extension of an operator:** An opeator B is an extension of an operator A if $D(A)\subset D(B)$ and $Bx=Ax$ for $x\in D(A)$.

• **extended diffeomorphisms:** Consider maps $\phi:\Sigma\to\Sigma$, that are continuous, invertible, and such that the map and its inverse are smooth everywhere, except, possibility, at a finite number of isolated points. The group formed by these maps is usually denoted $Diff^*$. (A strict diffeomorphism is a map that is smooth everywhere and whose inverse is also smooth everywhere).

See diffeomorphisms.

• **15-j symbols:** An elementary vertex of BF theory. A 4-simplex with 15 quantum numbers, the 10 irreducible representations associated with the 10 edges and the five interwiner labels of the 5 nodes (tetdrahedra) of the 4-simplex. See fig (B.8).

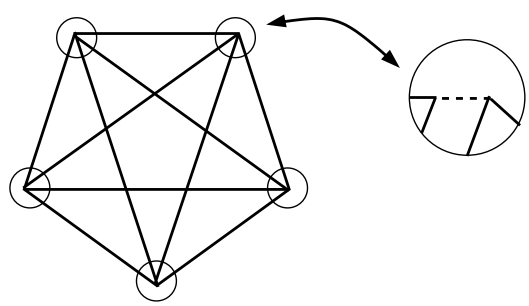


Figure B.8: 15JSymbol.

• **faithful:**

faithful representation: A representation is said to be faithful if Ker$\pi=\{0\}$ and non-degenerate if $\pi(a)\psi=0$ for all $a\in A$ implies $\psi=0$.

The representation () is said to **faithful** if and only if π is a $*$-morphism between $\mathcal{A}$ and $\pi(\mathcal{A})$, i.e., if and only if $\ker \pi = \{0\}$.

• **Fell's theorem:** From [11]: Let $(\mathcal{F}_1, \pi_1)$ and $(\mathcal{F}_2, \pi_2)$ be possibly unitary inequivalent representations of the Weyl algebra $\mathcal{A}$. Let $A_1, \ldots, A_n \in \mathcal{A}$ and let $\epsilon_1, \ldots, \epsilon_n > 0$. Let ω_1 be an algebraic state corresponding to a density matrix on $\mathcal{F}_1$. Then there exists a state ω_2 corresponding to a density matrix $\mathcal{F}_2$ such that for all $i = 1, \ldots, n$ we have

$$|\omega_1(A_i) - \omega_2(A_i)| < \epsilon_i. \tag{B.63}$$

See physics glossary.

• **fibre bundles:** A bundle is a triplet $(E, \pi, \mathcal{M})$, where E and $\mathcal{M}$ are manifolds of some differentiability class and $\pi : E \to \mathcal{M}$ is the projection map. The inverse image $\pi^{-1}(x)$, $x \in \mathcal{M}$, is the fibre, F_x, over x. Fibre bundles are those bundles whose fibres over all of $\mathcal{M}$ are homoemorphic to a common space F, the typical fibre. They are the proper mathematical notion for introducing internal symmetries in a field theory.

• **fibre metric:** Let s and s' be sections over U_I. The inner product between s and s' at p is defined by

$$(s, s')_p := h_{IJ}(p) s^I(p) s'^J(p)$$

if the fibre is in $\mathbb{R}^k$. If the fibre is $\mathbb{C}^k$ we define

$$(s, s')_p := h_{IJ}(p) \overline{s^I(p)} s'^J(p).$$

• **field:** (b) A commutative division ring. Are the "number systems" in maths. *over a field* means $\alpha\mathbf{x} + \beta\mathbf{v}$.

• **filter:** A filter in a set X is a system $\mathcal{F}$ of non-empty subsets of X satisfying the conditions:

i) $A \cap B \in \mathcal{F}$ for all A and B in $\mathcal{F}$,
ii) if $A \subset B$ and $A \in \mathcal{F}$, then $B \in \mathcal{F}$.

• **first countable:** A topological space is first countable space if it has a countable if it has a countable base at each point.

• **first homotopy group:**

• **foliation:** A foliation consists of an integrable sub-bundle of a tangent bundle.

• **folium:** A state's **folium** is the set of states ω_ρ on $\mathcal{U}$ defined by

$$\omega_\rho := \frac{\mathrm{tr}_{\mathcal{H}_\omega}(\rho\pi_\omega(a))}{\mathrm{tr}_{\mathcal{H}_\omega}(\rho)} \tag{B.64}$$

where ρ is a positive trace class operator on the GNS Hilbert space $\mathcal{H}_\omega$.

• **folium of states:** The folium of states is the set of states determined by the density matrices on the Hilbert space of the given representation.

• **forgetful functor:** MOSTLY TAKEN DIRECTLY FROM mathworld.wolfram.com

In many categories the objects are sets equipped with some kind of additional structure. For example the category of groups, where an object is a group, and an arrow is a group homomorphism $f : G_1 \to G_2$ from G_1 to G_2. However a category need not have structured sets as its objects.

In a mere set elements are either the same or different; and that's all.

A forgetful functor (also called underlying functor) is defined from a category of algebraic gadgets (groups, Abelian groups, modules, rings, vector spaces, etc.) to the category of sets. A forgetful functor leaves the objects and the arrows as they are, except for the fact they are finally considered only as sets and maps, regardless of their algebraic properties.

Other forgetful functors neglect only part of the algebraic properties, e.g., the commutative law when passing from Abelian groups to groups, or multiplication when passing from rings to Abelian groups.

• **free action:** The group action on a manifold (moving one point of the manifold to another in a way that has the structure of a group G) is said to be free if the every element that is not the identity of G has no fixed points.

• **free group:** see ???

• **free associative algebra:**

• **Fréchet space:** complete metric space???

• **Friedrichs extension:** quadratic form $Q_{\hat{\mathbf{M}}}$ whose closure is the quadratic form of a unique self-adjoint operator $\overline{\hat{\mathbf{M}}}$, called the Friedrichs extension of $\hat{\mathbf{M}}$.

Master constraint

The Friedrichs extension is a self-adjoint extension of a non-negative densely defined symmetric operator

Let A be a semi-bounded symmetric operator, that is,

$$q_A(\psi) = <\psi, A\psi> \geq \gamma\|\psi\|^2, \qquad \gamma \in \mathbb{R}.$$

Then there is a self-adjoint extension $\tilde{A}$ whihc is also bounded from below by γ and which satisfies $D(\tilde{A}) \subset= \mathcal{H}_{A-\gamma}$.

• **functional calculus:**

allows one to construct all kinds of operators.

finite dimensional Hilbert space $\mathcal{H}$

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

Define $\Psi : C(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ by

$$\Psi(f) = f(\lambda_1)P_1 + \cdots + f(\lambda_n)P_n.$$

It is not hard to see that Ψ is an isometric $*-$isomorphism into its range: i.e.,

$$\begin{array}{ll} (a) & \|\Psi(f)\| = \|f\|_\infty \\ (b) & \Psi(f+g) = \Psi(f) + \Psi(g) \\ (c) & \Psi(fg) = \Psi(f)\Psi(g) \\ (d) & \Psi(\overline{f}) = \Psi(f)^*. \end{array} \tag{B.65}$$

The process of passing from $f \in C(\sigma(T))$ to $f(T)$ is called functional calculus. It allows us to construct, for example, square roots, logs, and exponentials of operators.

• **functionals:**

Positive linear functionals correspond precisely to the set of density matrices.

• **functions:**

functions of compact support: function is non-zero only within a compact region - a function whose domain is a compact space. Interesting things about these are....

• **function spaces:**

Examples. Let $U \subset \mathbb{R}^n$ be an open set of $\mathbb{R}^n$.

(1) $P(U)$ is the space of all polynomials of n variables as functions on ω;

(2) $C(U)$ the space of all continuous functions on U;

(3) $C^k(U)$ the space of all functions with continuous partial derivatives of order k on U;

(4) $C^\infty(\omega)$ the space of all smooth (infinitely differentiable) functions on U.

• **functor:** Functors are the structure preserving assignments between categories.

A **contravariant functor** $F : \mathcal{C} \to \mathcal{V}$ between two categories

(i) maps every object $C \in \mathrm{ob}\mathcal{C}$ to an object $F(C) \in \mathrm{ob}\mathcal{V}$, and

(ii) every morphism $f : C_1 \to C_2$ in $\mathcal{C}$ to a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{V}$, such that

$$F(g \circ f) = F(f) \circ F(g), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}. \tag{B.66}$$

A **covariant functor** $F : \mathcal{C} \to \mathcal{V}$ that maps every morphism in $\mathcal{C}$ to $F(f) : F(C_1) \to F(C_2)$, and

$$F(g \circ f) = F(g) \circ F(f), \quad F(\mathrm{id}_C) = \mathrm{id}_{F(C)}. \tag{B.67}$$

Contravariant ones change the direction of arrows, the covariant ones preserve directions.

Examples:

Let A be a vector space and $F(C) = \mathrm{Hom}(C, A)$, the linear maps from C to A.

$$\begin{tikzcd} C_1 \arrow[rr, bend left, "\phi"] & & C_2 & \mathrm{Hom}\,(C_1,A) & & \mathrm{Hom}\,(C_2,A) \arrow[ll, bend right, "\mathrm{Hom}\,(\phi,A)"'] \end{tikzcd}$$

Figure B.9: A contravariant functor.

• **Gauss-Bonnet theorem:**

$$\int_{\mathcal{M}} K dS = 2\pi\chi(M) \tag{B.68}$$

• **Gauss-Codaza equations:** The torsion is the anti-symmetric part of the connection and the curvature in terms of the connection:

• **Gaussian curvature:** a disk D_ϵ centered at the point $\mathbf{p}$ with area $A(D_\epsilon)$

$$K(\mathbf{p}) = \frac{12}{\pi} \lim_{\epsilon \to 0} \frac{\pi\epsilon^2 - A(D_\epsilon)}{\epsilon^4} \tag{B.69}$$

• **Gauss linking number:**

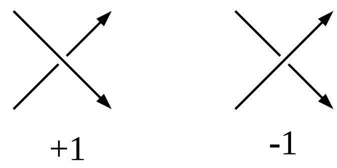


Figure B.10: Computing the Gauss linking number.

$$L = \frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\beta} \frac{(z^\beta - y^\beta)}{|z-y|^3} \tag{B.70}$$

Consider two closed loops γ and γ', as for example in Fig E.7 . If we think of first loop γ to be a wire carrying a current I, then by law it will generate a magnetic field $\mathbf{B}$ around the closed curve γ'.

$$B[\alpha] = \int_{S^\alpha} B^a d^2 S_a = \oint_\alpha A_a dl^a, \tag{B.71}$$

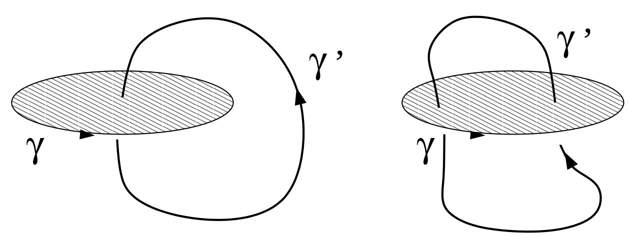


Figure B.11: Computing the Gauss linking number.

n is the number of times that the current passes through the closed loop. Let us set $I = 1$, we have

$$L = n = \frac{1}{4\pi} \oint_{\gamma'} \mathbf{B}(x') \cdot d\mathbf{x}. \tag{B.72}$$

We can calculate the magnetic field $\mathbf{B}(x')$ produced by wire γ with the Biot-Savart law, Eq(B.72). Substituting this into E.7, we get an explicit equation for the Gauss linking number:

$$L = \frac{1}{4\pi} \oint_{\gamma_i} dz^\mu \oint_{\gamma_j} dy^\nu \epsilon_{\mu\nu\beta} \frac{(z^\beta - y^\beta)}{|z-y|^3} \tag{B.73}$$

started off knot theory. knot theory of atoms. Irony that that the basic constituents of the fundamental physical description of nature put forward by LQG are knots (well actually links - a link being a set of knots).

We say that two spin networks are **isotopic**. So by abstract spin networks we mean the isotopic type rather than a particular way of realizing the spin network in "space".

• **Gauss self-linking number:**

$$\frac{1}{4\pi}\oint_{\gamma_i} dz^\mu \oint_{\gamma_i} dy^\nu \epsilon_{abc}\frac{(z^c - y^c)}{|\gamma(s)-\gamma(t)|} \tag{B.74}$$

Figure B.12: Gauss self-linking number.

• **Gel'fand map:** A map from the commutative $C*-$algebra $\mathcal{A}$ onto the space of continuous functions on the spectrum of the algebra, $\mathcal{C}(\Delta(\mathcal{A}))$.

$$\begin{array}{rcl} Alg & \rightarrow & C(\Delta) \\ a & \mapsto & \overset{\vee}{a} \end{array} \tag{B.75}$$

• **Gel'fand-Naimark-Segal construction:**

• **Gel'fand-Neimark theorem:** Every C^*-algebra with identity is isomorphic to the C^*-algebra of all continuous bounded functions on a compact Hausdorff space called the *spectrum* of the algebra.

• **Gel'fand spectral theorem:**

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with C.

• **Gel'fand topology:**

$$\begin{array}{rcl} Alg & \rightarrow & C(\Delta) \\ a & \mapsto & \overset{\vee}{a} \end{array} \tag{B.76}$$

• **Gel'fand spectral theorem:**

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with C.

• **Gel'fand topology:**

weak* convergence of a weak* convergence of a sequence (rather, a generalization of a sequence called a net) of functionals. Then weak* convergence of (f_n) means that there is an $f \in X'$ such that $f_n(x) \to f(x)$ for all $x \in X$.

Every character is a bounded linear functional on $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$. The Gel'fand topology on the spectrum of a unital, Abelean Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

• **Gel'fand transformation:** Let $\mathcal{A}$ be a Banach algebra?? Given any $A \in \mathcal{A}$ we can define a function $\hat{A} : \Delta \to \mathbb{C}$ by

$$\hat{A}(h) = h(A). \tag{B.77}$$

$\hat{A}$ is called the Gel'fand transform of A.

• **generalized functions:** distributions

• **generalized knot theory:** knots with intersections...

• **germs:** Let $\mathcal{M}$ and $\mathcal{N}$ be manifolds and $x \in \mathcal{M}$. Consider all smooth mappings $f : \mathcal{U}_f \to \mathcal{N}$, where $\mathcal{U}_f$ is some open neighbourhood of x in $\mathcal{M}$. We say two such functions f, g are equivalent and we put $f \sim_x g$ if there exists an open neighbourhood $\mathcal{V}$, which contains x, such that $f|\mathcal{V} = g|\mathcal{V}$. This is an equivalence relation on the set of mappings considered. The equivalence class of a mapping f is called the germ of f at x, sometimes denoted by $\mathrm{germ}_x f$. The set of all these germs is denoted by C.

We may also consider the composition of germs: $\mathrm{germ}_{f(x)} \circ \mathrm{germ}_x f = \mathrm{germ}_x (g \circ f)$.

Germ of an edge: Let $x \in \Sigma$ be given. The germ $[e]_x$ of an entire analytic edge e with $b(e) = e(0) = x$ is defined by the infinite number of Taylor coefficients $e^{(n)}(0)$ in some parameterisation. The germ $[e]_x$ encodes the orientation of e and its knowledge allows us to reconstruct $e(t)$ from x up to reparametrisation due to analyticity.

• **Gleason's theorem:** Let H have dimension greater than 2. Then every countably additive probability measure on the lattice $L(H)$ has the form $\mu(P) = Tr(WP)$, for a density operator W on H.

Stanford encyclopedia of philosophy.

• **GNS construction:**

• **graph**

For a linear operator $A : X \to Y$ the set of points $\{x \in X, Ax \in Y\}$ is called the graph of the operator A.

See partial function.

• **greatest lower bound:** the greatest lower bound is usually called its **infimum** and denoted $\inf A$.

• **Green's function:** Roughly, a Green's function $G(\vec{r},t)$ is the solution of a differential equation subject to the initial condition $G(\vec{r},t) = \delta(\vec{r})$.

• **Groenewold and van Howe theorem:**

• **group:** A collection G of objects g_i upon which we associate a 'multiplication' operation (techniqually know as a binary operation) which we write as " $\cdot$ ". By definition a group satisfies the following properties

(i) identity e such that $e \cdot g = g \cdot e = g$;
(ii) closed under multiplication i.e. for any $g_1, g_2 \in G$ their product is also an element of the group i.e., $g_1 \cdot g_2 \in G$;
(iii) every element has an inverse;
(iv) associativity $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$;

• **group homomorphism:** A homomorphism between two groups is, roughly speaking, a function between them which preserves the (respective) group operations. Let G_1, G_2 be groups. A function f that maps the elements of G_1 in to G_2 ($f : G_1 \to G_2$) is a homomorphism if and only if, for all $a, b \in G_1$ we have

$$f(ab) = f(a)f(b). \tag{B.78}$$

$f : G_1 \to G_2$ is an isomorphism if it is a bijection (one-to-one and onto). We write $G_1 \cong G_2$. An isomorphism from a group G to itself is called an automorphism.

• **groupoid:** closure applying the binary operator to two elements of a given set S returns a value which is itself a member of S. Associativity, existence of identity, and inverse of each element are not required. An example of a groupoid is for a, b positive definite real numbers

$$a \star b = \sqrt{(a^2 + b)} \tag{B.79}$$

There is no identity element because we require $a, b > 0$, and correspondingly no inverse.

• **group theory:**

• **Haag's theorem:** Two significant consequences of Haag's theorem are:

i) Scalar filed theories with different masses correspond to unitarily inequivalent representations of the Weyl algebra.

ii) Representations of interacting and free field theories are unitarily inequivalent and hence means that the interaction picture underlying perturbative QFT of Wightman fields strictly speaking does not exist, it only exists if there are no interactions.

The formal statement of Haag's theorem is:

Suppose that

(1) two weakly continuous and irreducible representations $(\pi_I, \mathcal{H}_I), I = 1, 2$ of the Weyl algebra $\mathfrak{A}$ of a scalar field theory are given,

(2) the Euclidean group E of spacial transaltions and rotations is implemented unitarily and weakly continuously by representations u_I on $\mathcal{H}_I$ such that $u_I(e)\pi_I(a)u_I^{-1}(e) = \pi_I(\alpha_e(a))$ for all $e \in E, a \in \mathfrak{A}$ and

(3) there is a unique Euclidean invariant state $\Omega_I \in \mathcal{H}_I$, that is, $u_I(e)\Omega_I = \Omega_I$.

If the two representations of the Weyl algebra are unitary equivalent, that is, there exista unitary operator $W : \mathcal{H}_1 \to \mathcal{H}_2$ such that $W\pi_1(a)W^{-1} = \pi_2(a)$ for all $a \in \mathfrak{A}$, then $Wu_1(a)W^{-1} = u_2(e)$ for all $e \in E$ and $V\Omega_1 = c\Omega_2$ where c is a complaex number of modulus one.

• **Haar measure:** For any compact group G the Haar measure is the (unique) measure dU that is group invariant.

Let G be a locally compact abelian group. A Haar measure on G is a positive regular Borel measure μ having the following two properties:

(1) $\mu(E) < \infty$ if E is compact;
(2) $\mu(E + x) = \mu(E)$ for all measurable $E \subset G$ and all $x \in G$.

One can prove that the Haar merasure always exists and that it is unique up to multiplication by a positive constant.

• **Hahn-Banach theorem:** The Hahn-Banach theorem is a central tool in functional analysis. It allows for the extending bounded linear functionals defined on a subspace of some vector space to the full space, in a norm-preserving manner, and shows that there are "enough" continuous linear functionals defined on every normed vector space to make the study of dual spaces interesting.

• **Hall transform:**

• **Harmonic polynomials:** homomgeneous polynomials P annihilated by the Laplace operator on $\mathbb{R}^d$:

$$\Delta P(x^1, \ldots, x^d) = 0, \tag{B.80}$$

for

$$\Delta = \partial_1^2 + \cdots + \partial_d^2 \tag{B.81}$$

where $\partial_i = \partial/\partial x_i$.

• **Hausdorff (or T_2):** A topological space is called Hausdorff if and only if for any two distinct points x and y, $x \neq y$, there are open sets O_1, O_2 such that $x \in O_1$, $y \in O_2$, and the two open sets do not overlap, i.e., $O_1 \cap O_2 = \emptyset$.

• **heat kernal:**

the coherent states of the simple harmonic oscillator coherent states can be obtained as analytic continuation of the heat kernal on $\mathbb{R}^n$:

$$\psi_z^t(x) = e^{-t\Delta}\delta_{x'}(x)\Big|_{x' \to z} \quad x \in, z \in \mathbb{C}, \tag{B.82}$$

the Laplacian Δ playing the role of a *complexifier*???

It was shown by Hall [242] that coherent states on a connected compact Lie group G can analogously be defined as an analytic continuation of the heat kernel

$$\psi_g^t(x) = e^{-t\Delta_G}\delta_{h'}^{(G)}(h)\Big|_{h' \to u}, \tag{B.83}$$

to an element u of the complexification $G^{\mathbb{C}}$ of G.

• **heat kernal measure:**

$$\frac{d\mu}{dt} = \frac{1}{4}\Delta K_C \mu_t \tag{B.84}$$

let μ_t denote the associated heat kernel measure

$$d\mu_t(g) := \mu_t(g)dg. \tag{B.85}$$

• **Heisenberg group:** A matrix representation of the Heisenberg Lie algebra is

$$m(p, q, t) = \begin{pmatrix} 0 & p_1 & \cdots & p_n & t \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is easily verified that

$$m(p,q,t)m(p',q',t') = m(0,0,pq')$$

and so

$$[m(p,q,t), m(p',q',t')] = m(0,0,pq' - qp').$$

Using

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

we have

$$\exp m(p,q,t)\exp m(p',q',t') = \exp m\big(p+p', q+q', t+t'+\frac{1}{2}(pq'-qp')\big).$$

One identifies a point $X \in \mathbb{R}^{2n+1}$ with the matrix $e^{m(X)}$, and makes $\mathbb{R}^{2n+1}$ into a group with group law

$$(p,q,t)(p',q',t') = \big(p+p', q+q', t+t'+\frac{1}{2}(pq'-qp')\big).$$

This is called the Heisenberg group and is denoted $\mathbf{H_n}$. The element $(0,0,0)$ is the identity and the inverse of the element (p,q,t) is $(-p,-q,-t)$.

• **Hellinger-Toeplitz theorem:** If a is a self-adjoint operator whose domain of dependence $D(a)$ is the whole of the Hilbetr space $\mathcal{H}$, $D(a) = \mathcal{H}$, then a is bounded. This shows that unbounded operators are not everywhere defined on the Hilbert space.

• **Heyting algebra:** ??? [367] Internal observables satisfy a Heyting algebra, which is a weak version of the Boolean algebra of ordinary observables.

Open sets are the standard example of a Heyting algebra. Given an open set $U \subset O$, the complement of U, which we call U^c, is a closed set. If we want an algebra of open sets, we need, instead of U^c, to use the interior of the set complement of U, $Int(U^c)$. But then, clearly, $U \cap Int(U^c) \subset O$ since the closure of U has been left out. (example comes from [366]).

A poset $\mathbf{H}$ is a Heyting algebra if we have elements

$$T \in \mathbf{H}, \qquad \perp \in \mathbf{H} \tag{B.86}$$

and operations

$$T \in \mathbf{H} \tag{B.87}$$

See sections I.7.

• **higher-dimensional algebra:** [400]

• **Hilbert-Schmidt operators:** The space of Hilbert-Schmidt operators consists of those compact operators A such that the trace $\mathrm{Tr}(A^*A)$ exists. These operators naturally appear in Bogol'ubov transformations.

• **Hilbert space:** A complete inner product space which is a complete metric space with respect to the metric induced by its inner product (compare to a Banach space).

(see also a pre-Hilbert space).

Hilbert space completion: Consider the completion of an inner product space V as the metric space completion, $\mathcal{H}$, of V by taking equivalence classes of Cauchy sequences in V. It can be shown that the inner product structure of V naturally extends to $\mathcal{H}$ in such a way as to provide $\mathcal{H}$ with the structure of a Hilbert space, with V naturally identified with a dense subspace of $\mathcal{H}$. See Reed and Simon.

• **Hölder inequality:** integral (or sum) inequality

$$\left(\int |fg|\right) \leq \left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q} \tag{B.88}$$

where $1 \leq p, q \leq \infty$ and $1/p + 1/q = 1$.

$$\left(\sum_i |f_i g_i|\right) \leq \left(\sum_i |f_i|^p\right)^{1/p} \left(\sum_i |g_i|^q\right)^{1/q} . \tag{B.89}$$

• **homeomorphic:** Related by a homeomorphism.

• **homeomorphism:** A one-to-one correspondence that is continuous in both directions between the points of two topological spaces.

• **homogeneous function:** A function $f(x_1, \ldots, x_n)$ is a homogeneous function of degree D if

$$f(\rho x_1, \ldots, \rho x_n) = \rho^D f(x_1, \ldots, x_n) \tag{B.90}$$

Let us differentiate both sides of this with respect to ρ and noting $\partial f(\rho x)/\partial\rho = xf(\rho x)$. We find

$$\left[x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + \cdots + x_n\frac{\partial}{\partial x_n}\right] f(x_1,\ldots,x_n) = D\rho^{D-1} f(x_1,\ldots,x_n) \tag{B.91}$$

Setting $\rho = 1$ gives

$$\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} f(x_1,\ldots,x_n) = Df(x_1,\ldots,x_n). \tag{B.92}$$

• **homeomorphism:** A homeomorphism is a one-to-one correspondance that is continuous in both directions between two topological spaces. It is an equivalence relation and preserves topological properties. If it also preserves distances it is an isometry. See also diffeomorphism.

• **Hom(A,B):** $\mathrm{Hom}(A, B)$ denotes the collection of morphisms from A to B.

• **homogeneous simultaneous equations:**

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_1 &= 0 \\ a_3x + b_3y + c_3z + d_1 &= 0 \end{aligned} \tag{B.93}$$

$$\begin{aligned} a_1\tilde{x} + b_1\tilde{y} + c_1\tilde{z} &= 0 \\ a_2\tilde{x} + b_2\tilde{y} + c_2\tilde{z} &= 0 \\ a_3\tilde{x} + b_3\tilde{y} + c_3\tilde{z} &= 0 \end{aligned} \tag{B.94}$$

$$X = \frac{\tilde{x}}{\tilde{z}}, \quad Y = \frac{\tilde{y}}{\tilde{z}} \tag{B.95}$$

$$\begin{aligned} a_1X + b_1Y + c_1 &= 0 \\ a_2X + b_2Y + c_2 &= 0 \\ a_3X + b_3Y + c_3 &= 0 \end{aligned} \tag{B.96}$$

These have the trivial solution $X = 0, Y = 0, Z = 0$. If the condition

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \neq 0 \tag{B.97}$$

is met there exist non-trivial solutions to the homogeneous equations. In this case we can start with a solution to the non-homgeneous system of equations equations and obtain another one by adding arbitrary linear combination of the solutions to the homogeneous system of equations:

$$\begin{aligned} x &\rightarrow x + v_x X \\ y &\rightarrow y + v_y Y \\ z &\rightarrow z + v_Z Z \end{aligned} \tag{B.98}$$

• **homomorphism:** A homomorphism θ is a mapping from one algebraic structure to another under which the structual properties of its domain are preserved in its range in the sense that if $*$ is te operation on the domain, and $\circ$ is the operation on the range, then

$$\theta(x * y) = \theta(x) \circ \theta(y).$$

In particular, a group homomorphism is a mapping θ such that both domain and range are groups, and

$$\theta(xy) = \theta(x)\theta(y)$$

for all x and y in the domain. A ring homomorphism is a mapping θ from one ring to another such that

$$\theta(x + y) = \theta(x) + \theta(y) \quad \text{and} \quad \theta(xy) = \theta(x)\theta(y).$$

• **hoop group:** 'holonomy equivalence class of a loop based at x_0'.

• **Hopf algebra:** [190], [192], [191]

Algebra

multiplication $\mu : A \otimes A \rightarrow A$ $m(a \otimes b) = ab$. m is **associative**, $(ab)c = a(bc)$.

$$[m(m \otimes \mathbf{1})](a \otimes b \otimes c) = [m(\mathbf{1} \otimes m)](a \otimes b \otimes c) \tag{B.99}$$

mathematicians write this in the short-hand notation

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m) \tag{B.100}$$

and depict it as a diagram (I.2.4(a)).

unit η sends every number to the same number times the identity element, i.e.

$$\eta(k) := \mathbf{1}k, \quad \text{for all } k \in \mathbb{C}. \tag{B.101}$$

$$m(a \otimes \eta(k)) = ak = ka = m(\eta(k) \otimes a), \quad \text{for all } a \in A, \text{ and for all } k \in \mathbb{C}. \tag{B.102}$$

$$m \circ (\eta \otimes \mathrm{id})(a) = a = m \circ (\mathrm{id} \otimes \eta)(a) \tag{B.103}$$

Coalgebra

comultiplication $\Delta : C \to C \otimes C$. Δ is **coassociative**.

$$(\mathbf{1} \otimes \Delta)\Delta = (\Delta \otimes \mathbf{1})\Delta \tag{B.104}$$

counit

Coalgebra

A bialgebra structure on a vector space $A = C$ is a quadruple of objects $(\mu, \eta, \Delta, \epsilon)$ which satisfy all of the commutative diagrams as well as the following compatibility equations:

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1) = 1 \otimes 1, \quad \epsilon(hg) = \epsilon(h)\epsilon(g), \quad \epsilon(1) = 1, \quad \text{for all } g, h \in A. \tag{B.105}$$

linear antipode $S : H \to H$

$$\mu(S \otimes \mathrm{id}) \circ \Delta = \mu(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{B.106}$$

the antipode is a kind of inverse.

In component form using generators:

$$
\begin{aligned}
e_i e_j &= m^k_{ij} e_k \\
\Delta(e_k) &= \mu^{ij}_k e_i \otimes e_j
\end{aligned} \tag{B.107}
$$

$$\epsilon_i \mu^{ij}_k = \delta^j_k \tag{B.108}$$

Definition: Let C and D be coalgebras, with comultiplication Δ_C and Δ_D, and counits ϵ_C and ϵ_D respectively.

(i) A map $f : C \to D$ is a **coalgebra morphism** if $\Delta_D \circ f = (f \otimes f)\Delta_C$.

(ii) A subspace $I \subseteq C$ is a **coideal** if $\Delta \subseteq I \otimes C + C \otimes I$ and if $\epsilon(I) = 0$.

if I is a coideal then, the quotient C/I is a coalgebra with comultiplication induced from Δ. Proof in Appendix Q.

• **Hopf fibration:**

Hopf fibration of S^3. The base space is a 2-dimensional sphere S^2 and fibres are circles S^1

• **horizontal distribution:** Given a principal fibre bundle a horizontal distribution is the assignment of a subspace $V_p(P)$ to the tangent space $T_p(P)$ at each point p of P tat is tangent to the fibre.

• **horizontal vector fields:** Horizontal vector fields are fields whose flow lines move from one fibre into another.

• **hyperbolic differential equations:** wave equation. Consider the partial differential equation

$$g^{ab}\nabla_a \nabla_b \phi + A^a \nabla_a + B\phi + C = 0 \tag{B.109}$$

where A^a is an arbitrary smooth vector field, B and C are arbitrary smooth functions, and g_{ab} is an arbitrary smooth Lorentz metric such that the spacetime $(\mathcal{M}, g_{ab})$ is globally hyperbolic. A second order partial differential equation is said to be hyperbolic if and only if it can be expressed in the form

• **hyphs:** Hyphs are a type of collections of loops which have the advantage of being independent of the differentiablity category of the graphs under consideration and in particular includes the analytical and smooth category.

• **ideal:** An **ideal** a in A is a subset such that

(i) `a` is a subgroup of A regarded as a group under addition;

(ii) $a \in a,\ r \in A \Rightarrow ra \in A$. ??????????????? left-ideal

right-ideal

two-sided ideal

Ideals play the same role in Lie algebras as the normal subgroups play in Lie group theory.

The set of commutators of a Lie algebra, $\mathcal{G}$, denoted by $[\mathcal{G}, \mathcal{G}]$, is a subalgebra of $\mathcal{G}$.

It is also a two-sided ideal of $\mathcal{G}$, for any $A, A_1, A_2 \in \mathcal{G}$,

$$[[A_1, A_2], A] = [A_3, A] \in [\mathcal{G}, \mathcal{G}], \tag{B.110}$$

where $A_3 = [A_1, A_2]$.

$$[A, [A_1, A_2]] = [A, A_3] \in [\mathcal{G}, \mathcal{G}], \tag{B.111}$$

The set $[\Delta A, \Delta A_2]$. Two-sided coideal

The ideal of a Lie algebra $[h_i, g_k] = \sum a_{ikl} h_l$ for all $g_i \in \mathcal{L}(G)$

• **immersion:** A differentiable map $\phi : \mathcal{M} \to \mathcal{N}$ between finite-dimensional manifolds $\mathcal{M}, \mathcal{N}$ is called an immersion when ϕ has everywhere rank $\dim(\mathcal{M})$. An immersion need not be injective (se fig B.13) but when it is,it is called an embedding.

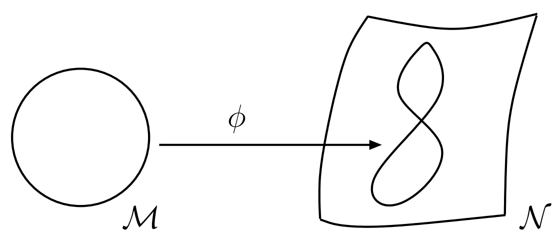


Figure B.13: .

• **implicit function theorem:** The simplest version of the implicit function theorem can be stated as follows. Let f be a continuous real-valued function on an open subset of $\mathbb{R}^2$ that contains the point (a, b), with $f(a, b) = 0$. Suppose that $\partial f/\partial y$ exists and is continuous on the given open subset and that $\partial f/\partial y(a, b) \neq 0$. Then there exist open intervals $U, V \in \mathbb{R}$, with $a \in U$ and $b \in V$, such that there exists a unique function $\rho : U \to V$ such that

$$f(x, \rho(x)) = 0$$

for all $x \in U$, and such that this function is continuous.

• **inclusion map:** IF $U \subseteq V$, the inclusion map i sends U to V, i.e., $i : U \rightarrow$.

• **if and only if:** This means necessary and sufficient. A statement is made of the form: "A is true if and only B is true". To establish A is true only if B is true, we assume A and then show it follows that B must be true. To establish A is true if B is true, we assume B and then show it follows that A must be true.

• **Infeld-van der Waerden symbols:**

• **infinite tensor product:**

• **injective:** A function that is one-to-one. Equivalently, a function is injective when no two distinct inputs give the same output.

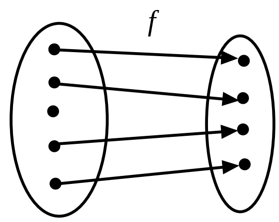


Figure B.14: injective

• **inner product space:** vector space equipped with an inner product

• **insertion operator:** Given an $r-$form

$$\omega = \frac{1}{r!}\omega_{\nu_1\nu_2\ldots\nu_r} x^{\nu_1} \wedge x^{\nu_2} \cdots \wedge x^{\nu_r}, \tag{B.112}$$

the insertion operator is defined by the operation

$$i_X\omega := \frac{1}{(r-1)!}\omega_{\nu\nu_2\ldots\nu_r} X^{\nu} \wedge x^{\nu_2} \cdots \wedge x^{\nu_r}. \tag{B.113}$$

• **integrable distribution:** A distribution on a manifold is said to be integrable if at least locally, there is a foliation of M by submanifolds such that V_x is the tangent space of the submanifold containing the point x.

• **interal subgroup:** Let G be a Lie group. An integral subgroup of G is a subgroup H with a connected Lie group structure such that the canonical injection of H into G is an immersion.

• **interchange of limit operations:** If a sequence of Riemann integral real-valued functions $f_1(x), f_2(x), \ldots$ converges to the function $f(x)$, can we assert that

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty}\int_a^b f_n(x)dx \tag{B.114}$$

is true?

If a series of

$$\frac{d}{dx}\lim_{n\to\infty}\sum_{n=1}^{N} e_n(x) = \lim_{n\to\infty}\sum_{n=1}^{N}\frac{d}{dx}e_n(x) \tag{B.115}$$

• **interwiner:** A trivalent interwiner is a map $I : V^i \otimes V^j \otimes V^k \to \mathbb{C}$ invarinat under the diagonal action of the group on the tensor product.

• **intuituinistic logic:** Comes from constructive mathematics: No proof by contradiction allowed, i.e. *"for all x, either x or not x*

• **invariant polynomial:** consider polynomials in n^2 variables. Let us label these n^2 variables as the entries of an $n-$dimensional matrix A_{ij}, We write for the polynomial: $P(A)$. The polynomial $P(A)$ is said to be **invariant** when

$$P(gAg^{-1}) = P(A) \tag{B.116}$$

for all $g \in GL_n(C)$.

For example

$$\det\left(\mathbf{1} + \frac{\lambda}{2\pi i}A\right) = \sum_{n=0}^{N}\lambda^n P_n(A) \tag{B.117}$$

the coefficients $P_n(A)$ are invariant polynomials of degree n in A.

$$\begin{aligned}
\sum_{n=0}^{N}\lambda^n P_n(gAg^{-1}) &= \det\left(\mathbf{1} + \frac{\lambda}{2\pi i}gAg^{-1}\right) = \det\left(g(\mathbf{1} + \frac{\lambda}{2\pi i}A)g^{-1}\right) \\
&= \det(g)\det\left(\mathbf{1} + \frac{\lambda}{2\pi i}A\right)\det(g^{-1}) \\
&= \sum_{n=0}^{N}\lambda^n P_n(A).
\end{aligned} \tag{B.118}$$

• **invariant subspace:** a subspace $\mathcal{M}$ is invariant under a set of operators if $A\psi$ is in $\mathcal{M}$ for every operator A in the set and every vector ψ in $\mathcal{M}$.

• **inverse function theorem:**

• **irreducible representation:** An representation of operator relations on $\mathcal{M}$, for example the Weyl relations, is irreducible if no proper subspace of $\mathcal{M}$ is invariant under a set of operator relations. Equivalently, given any $\Psi \in \mathcal{M}$, the span of all vectors under the operator relations forms a dense subspace of $\mathcal{M}$. The representation should be irreducible on physical grounds otherwise we have superselection sectors implying that the physically relevant information is already contained in a closed subspace.

• **ISO:**

• **isometrically isomorphic:** Let N and N' be normed linear spaces. These spaces are said to be isometrically isomorphic if the linear transformation T is a one-to-one from N to N' is a one-to-one such that

$$\|x\| = |\ T(x)\| \tag{B.119}$$

for all x in N. So that the normed linear space N' is essentially the same as N.

• **isometric operator:** An operator V defined on the whole of a Hilbert space $\mathcal{H}_1$ and mapping $\mathcal{H}_1$ on to the whole of another Hilbert space $\mathcal{H}_2$ is said to be isometric if, for all $f, g \in \mathcal{H}_1$,

$$(Vf, Vg)_2 = (f, g)_1,$$

where $(.\,,.\,)_1$ and $(.\,,.\,)_2$ denote the inner product on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively.

In particular, $\mathcal{H}_1$ and $\mathcal{H}_2$ may be subspaces of a single space (the subscrips on the inner products are then unnecessary).

A unitary operator is a particular case of an isometric operator; this case occurs if $\mathcal{H}_1$ and $\mathcal{H}_2$ coincide.

• **isomorphic:** Two groups G and G' are said to isomorphic if their elements can be put into one-to-one correspondence which is preserved under multiplication.

• **isomotopic:** If two objects can be deformed into each other they are said to be isomotopic. For example knots that can be deformed into each other are called isomotopic.

• **isomotopy:** There are two kinds regular and ambient isotopy.

• **Jones polynomial:** The Jones polynomial is defined by:

its value on the unknot,
its value on the disjoint union of a knot with the unknot,
and a "Skein relation".

value on the unknot is chosen to be 1:

$$p(U) = 1. \tag{B.120}$$

Let $U \sqcup K$ be the disjoint union of U and K

$$p(U \sqcup K) = -(x + x^{-1})p(K). \tag{B.121}$$

$$x^{-2}p(K^{+}) - x^{2}p(K^{-1}) + (x^{-1} - x)p(K^{0}) = 0. \tag{B.122}$$

• **Jones-Witten invariant:** won Witten his Fields medal.

• **Kauffman bracket:**

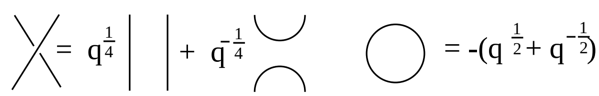


Figure B.15: Skien relation for the Kauffman bracket.

• **kernel:** The kernel of a group homomorphism $\varphi : G \to H$ is defined by

$$\ker\varphi := \varphi^{-1}(\{e_H\}) = \{x \in G : \varphi(x) = e_H\}.$$

• **Killing vector field:** isometries of the spacetime.

• **Kirby calculus:** must be related through sequence of operations. reminiscent of the Reiemeister moves.

• **knots and knot theory:** ... The formal statement of this intuitive idea is: a knot is a smooth embedding of a circle in three-dimensional Euclidean space. Two knots are said to be topologically equivalent if one can be deformed continuously into the other, without crossings.

Knots can be displayed as projections onto a 2-d plain surface.

Two knots or links are topologically equivalent if they can be transformed into each other by repeated use of the three Reidemeister moves.

The aim of knot theory is to characterize knots by a topological invariant.

• **lattice:** A lattice (a mathematical term) is a partially ordered set $(L, \leq)$ such that any two elements a and b possess a minimum in L, denoted $a \wedge b$ ("meet") and a maximum in L, denoted $a \vee b$ ("join"). That is, there exists respectively an element $a \wedge b$ satisfying

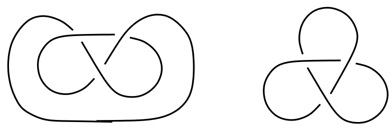


Figure B.16: knots Reidemeister moves.

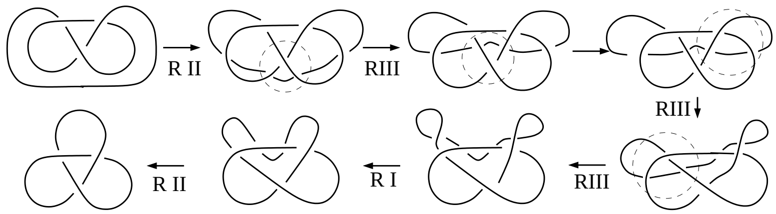
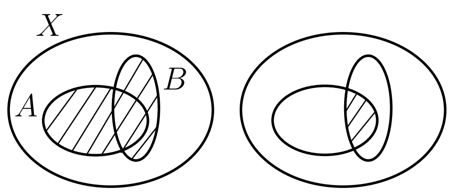


Figure B.17: knots Reidemeister moves.

$a \wedge b \leq a$, $a \wedge b \leq b$ such that any other element c which smaller than both a and b satisfies $c \leq a \wedge b$, and one element satisfying $a \vee b \geq a$, $a \vee b \geq b$ and any other element c with the same property is greater, i.e., $a \vee b \geq c$. For examples of lattices see section I.7.



(a) (b)

Figure B.18: If $A \subseteq B$ then $A \leq B$. partially ordered set of subsets of X is a lattice. (a) Their union $A \cap B$ is the l.u.b. (b) $A \cup B$ is the g.l.b.

a lattice is called complete if and

Orthomodular lattice

$$(a \vee (a^{\perp} \wedge (a \vee b))) = (a \vee b) \tag{B.123}$$

• **least upper bound:** 3 is an upper bound and its least upper bound the least upper bound is usually called its **supremum** and denoted sup A.

• **Lebesgue integral:** The Reimann integral doesn't deal with all functions we need for our purposes in formulating quantum mechanics and quantum field theory. Disturbing fact that the limit function $f(x) := \lim n \to \infty f_n(x)$ of a sequence of continuous functions $\{f_n(x)\}$ can be discontinuous. A function $\mu(x)$ we would hope has the following properties:

$$
\begin{aligned}
\mu((a,b]) &= b-a \\
&=
\end{aligned} \tag{B.124}
$$

Instead of splitting the integration domain into small parts, we decompose the range of the function (see Fig(**??**)).

$$\sum_{n}^{N} c_n \mu(f^{-1}(J_n)) \tag{B.125}$$

maximal invariant subspace:

• **left invariant vector field:**

$$g(t+s) = g(t)g(s) \tag{B.126}$$

Differentiating with respect to s and setting $s=0$,

$$g'(t) = g(t)g'(0) \tag{B.127}$$

$$g(t) := L_g \exp(tX_e) = g\exp(tX_e) \tag{B.128}$$

Or the push-forward of the vector $X(g_e)$ at g_e by multiplication on the left by any g produces a vector $g_*[X(g_e)]$ at gg_e that coincides with the vector $X(gg_e)$ already at that point. So it is the natural definition of a 'constant' vector field on G.

• **left invariant form:** A differential $p-$form ω is called left invariant provided

$$L_x^* \omega = \omega.$$

If ω_0 is any given $p-$form at e, then a left invariant form is defined by

$$\omega_x = L_{x^{-1}}^* \omega_0.$$

One of the most important applications of left invariant forms is in the theory of connections in fibre bundles - featuring in theoretical physics in relation to the general mathematical framework for Yang-Mills theories.

• **Lie derivative:** Generates infinitesimal active difffeomorphisms.

• **Lie Group:**

one definition: A **Lie group** G is a group which is also a smooth manifold such that the multiplication $G \times G \to G$, $(a, b) \mapsto ab$,and the inverse $G \to G$, $a \mapsto a^{-1}$, are smooth.

• **limit curve lemma:** Introduce a background Riemannian (positive definite) metric on spacetime $\mathcal{M}$. A future inextendible causal curve will have infinite length to the future, as measured in the metric h, parameterization defined on the interval $[0, \infty)$. The limit curve lemma can then be stated:

Let $\gamma_n : [0, \infty) \to \mathcal{M}$ be a sequence of future inextendible causal curves, parameterized with respect to h-arc length, and suppose that $p \in \mathcal{M}$ is an accumulation point of the sequence $\{\gamma_n(0)\}$. Then there exists a future inextendible C^0 causal curve $\gamma : [0, \infty) \to \mathcal{M}$ such that $\gamma(0) = p$ and a subsequence $\{\gamma_m\}$ which converges to γ uniformaly with respect to h on compact subsets of $[0, \infty)$.

There are analogous versions of the limit curve lemma for past inextendible, and (past and future) inextendible causal curves.

• **linear:**

(i) $\alpha(x + y) = \alpha x + \alpha y$;

(ii) $(\alpha + \beta)x = \alpha x + \beta x$;

(iii) $(\alpha\beta)x = \alpha(\beta x)$;

(iv) $1 \cdot x = x$

A linear space is called a **real** linear space or a complex linear space according to whether the scalars are the real numbers or complex numbers.

Some jargon: A complex vector space will often be referred to as a vector space over *over* the complex numbers. This phrase isn't just reserved for vector fields, and will also be used in reference to say rings, groups. Also, the scalars may not just refer to numbers, there are more general "number systems" used in mathematics called *fields*. The real and complex numbers are special cases of fields.

see physics glossary.

• **linear analysis:**

Uniform boundedness

interior mapping principle

Hahn-Banach theorem

• **line bundle:** A line bundle L is a complex vector bundle with one-dimensional fibres.

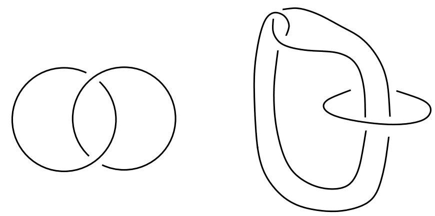


Figure B.19: links Whitehead link.

• **links:**

• **link invariants:**

invariants for three-manifolds Edward Witten topological field theory Chern-Simons

$$\int \mathcal{D}A e^{\int \mathcal{L}_{CS}} W_{\alpha}[A] \tag{B.129}$$

Knots and links can be obtained by the closure of braids (Alexander theorem)

• **linear independence:**

$$\sum_{i=1}^{N} a_i \mathbf{e_i} = 0 \implies \tag{B.130}$$

• **Lioville's theorem:** A bounded entire function is constant.

• **locally compact:** A topological space is said to be *locally compact* if every point $x \in X$ has an open neighbourhood whose closure is compact.

• **locally finite:**

• **locally trivial:** Given a fibre bundle P with typical fibre F and base space $\mathcal{M}$. Locally trivial means that for each $x \in \mathcal{M}$, there is a neighbourhood U of x and an isomorphism,

$$\phi : \pi^{-1}(U) \to U \times F,$$

sending each fibre $\pi^{-1}(x)$ to $\{x\} \times F$. Intuitively, a bundle looks locally as a product of the base manifold and the fibre. We call ϕ a local trivialisation. If the bundle space E is globally $\mathcal{M} \times F$ the bundle is said to be trivial.

• **logic:**

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

$$\frac{p}{p \vee q} \qquad \frac{q}{p \vee q} \qquad \frac{q \vee p \qquad p \to r \qquad q \to r}{r} \tag{B.131}$$

Addition

$$\begin{array}{l} p \\ \hline p \vee q \end{array}$$

Simplification

$$\begin{array}{l} p \wedge q \\ \hline p \end{array}$$

Conjunction

$$\begin{array}{l} p \\ q \\ \hline p \wedge q \end{array}$$

$$\frac{p \qquad q}{p \wedge q} \tag{B.132}$$

Modus ponens

$$\begin{array}{l} p \\ p \to q \\ \hline q \end{array}$$

Modus tollens

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \neg p \end{array}$$

Hypothetical syllogism

$$\begin{array}{l} p \to q \\ q \to r \\ \hline p \to r \end{array}$$

Disjunctive syllogism

$$\begin{array}{l} p \vee q \\ \neg p \\ \hline q \end{array}$$

See intuituinistic logic. See sections I.7.1 and J.2.

• **Mandelstam identity:** $SL(2, \mathbb{C})$ A satisfies $\det A = 1$. for $A, B \in SL(2, \mathbb{C})$

$$\mathrm{tr}A\mathrm{tr}B = \mathrm{tr}(AB) + \mathrm{tr}(AB^{-1}) \tag{B.133}$$

$SU(2) \subset SL(2, \mathbb{C})$ so the identity. Holonomies H_α are $SU(2)$ matrices.

• **manifold:** In simply terms, a manifold is a space $\mathcal{M}$ which locally looks like an n-dimensional Euclidean space $\mathbb{R}^n$.

Formally, a topological space $\mathcal{M}$ is an $n-$dimensional manifold if there is a locally finite open cover, $\{U_\lambda : \lambda \in \Lambda\}$, of $\mathcal{M}$ such that, for each λ, there is a map ϕ_λ that maps U_λ homeomorphically onto an open subset of $\mathbb{R}^n$.

(i) differentiable (or smooth manifolds, on which one can do calcus;

(ii) Riemannian manifolds, on which distances and angles can be defined;

(iii) symplectic manifolds, which serve as the phase space of dynamical systems;

(iv) 4D pseudo-Riemanniam manifolds which are used in general relativity .

• **maximal atlas:** We take as an example the definition of a maximal atlas with the $C^\infty-$property. Two charts ϕ_1, ϕ_2 are $C^\infty-$related if both the map

$$\phi_2 \circ \phi_1^{-1} : \phi_1(U_1 \cap U_2) \to \phi_2(U_1 \cap U_2)$$

and its inverse are $C^\infty-$related. A collection of related charts such that every point of $\mathcal{M}$ lies in the domain of at least one chart forms an atlas. The collection of all such $C^\infty-$related charts forms a maximal atlas.

• **maximal ideal:** A *maximal left ideal* in A is a proper left ideal which is not properly contained in any other proper left ideal.

• **measure:** A measure

• **measurable functions:** Let X, Y be metric spaces, and let $f : X \to Y$ be a function.

A function $f : E \to \mathbb{R}$ is (Lebesgue-)measurable if for any interval $I \subseteq \mathbb{R}$

$$f^{-1}(I) = \{x \in \mathbb{R} \ : \ f(x) \in I\} \in \text{ collection of measurable sets.????} \tag{B.134}$$

• **measure space:** A measure space $(\Omega, \mathcal{B}, \mu)$ is a set Ω together with a $\sigma-$algebra $\mathcal{B}$ of subsets of Ω and μ is $\sigma-$additive, that is,

$$\mu\left(\bigcup_{n=1}^{\infty} U_n\right) = \sum_{n=1}^{\infty} \mu(U_n) \tag{B.135}$$

for all disjoint measurable sets U_n.

• **minimal loop:** Given a graph γ and a vertex v of this graph, and two different edges e and e' outgoing from v, a loop $\alpha(\gamma, v, e, e')$ within γ with outgoing along e and incoming along e' is said to be minimal if there is no other loop within γ with the same properties and fewer edges tranaversed.

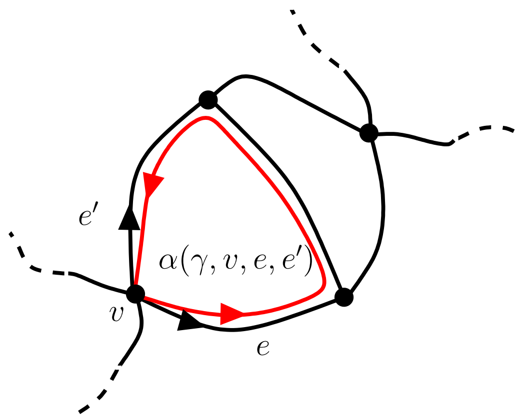


Figure B.20: minimal loop.

• **Minkowski's inequality:**

$$\left(\sum_i |x_i + y_i|^2\right)^{1/2} \leq \left(\sum_i |x_i^2|\right)^{1/2} \left(\sum_i |y_i|^2\right)^{1/2} \tag{B.136}$$

• **module:** Modules are also referred to as representations: for instance, representations of a group are essentially the same as modules over the group algebra. Even if you have not come across the term "module" you surely have come across some examples. Vector spaces are (rather simple) examples, as are abelian groups. We elaborate on some of the examples below after giving the formal definition of a module.

A commutative group on which there is defined an *exterior multiplication* (left or right) by elements of a *ring* R, such that multiplication is associative and distributive, and a group element multiplied by an element of the ring is a group element.

1) $(u + v) + w = u + (v + w)$ for all $u, v, w \in M$

2) $u + v = v + u$ for all $u, v \in M$

3) There exists an element $0 \in M$ such that $u + 0 = u$ for all $u \in M$

4) For any $u \in M$, there exists an element $v \in M$ such that $u + v = 0$

5) $a \cdot (b \cdot u) = (a \cdot b) \cdot u$ for all $a, b \in R$ and $u \in M$

6) $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ for all $a \in R$ and $u, v \in M$

7) $(a + b) \cdot u = (a \cdot u) + (b \cdot u)$ for all $a, b \in R$ and $u \in M$

A vector space is a module where complex numbers (so the ring is the field of complex numbers). A more complex example would be a set of operators acting on a vector space. This is a left module where the ring is the collection of operators.

$$Lu = x_1 L_{1i} + x_2 L_{2i} + \cdots + x_2 L_{ni} = \sum_{m=1}^{n} x_m L_{mi}$$

The action of L on a basis can be represented by a matrix L_{ij}. Any vector, u say, can be represented in the basis

$$u = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{m=1}^{n} a_m x_m$$

$$Lv = L(a_1 x_1 + a_2 x_2 + \cdots + a_n x_n) = a_1 L x_1 + a_2 L x_2 + \cdots + a_n L x_n$$

so the coefficients of Lv are

$$(Lv)_m = \sum_{m'=1}^{n} L_{mm'} a_{m'}$$

• **moniod:** A set M with a binary operation $\cdot$ that is

(i) associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$, and

(ii) there is a unit element $\mathbf{1}$ in M such that $a\mathbf{1} = \mathbf{1}a = a$ for all $a \in M$.

A moniod is a semigroup that has an *identity*.

A monoid is a category with one object.

• **moniodal category:** A moniodal category is a 2-category with one object.

• **morphism:**

A $*$**-morphism** between two $*$-algebras $\mathcal{A}$ and $\mathcal{B}$ is a mapping $\pi : A \in \mathcal{A} \in \mathcal{A} \mapsto \pi(A) \in \mathcal{B}$, defined for all $A \in \mathcal{A}$ such that

$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$,

$\pi(AB) = \pi(A)\pi(B)$,

$\pi(A^*) = \pi(A)^*$.

• **neighbourhood:** A set N is a neighbourhood of a point x in a topological space $(X, \mathcal{T})$ if and only if there is a open set U in $\mathcal{T}$ such that $x \in U \subseteq N$. Note that N need not be open itself.

• **neighbourhood base:** a collection of neighbourhoods, so that every open set of the topology can be expressed as a union of some of these neighbourhoods.

• **net:** Generalization of a the concept of a sequence to permit talk of convergence in non-meterizable topological spaces. Nets are often called Moore-Smith sequences.

• **normal operator:** A normal operator commutes with its self-adjoint.

If N is normal then,

$$N = \sum_{i=1}^{N} \lambda_i |f_i >< f_i|,$$

where $\lambda_1, \ldots, \lambda_N$ are igenvalues and $\{|f_i >\}$ are orthonormal eigenvectors of N. This is known as a strectral decomposition.

• **normal topological space:** A topological space is normal when it is T_1 (see separation conditions), and given any two disjoint closed sets V_1 and V_2, there are disjoint open sets U_1 and U_2 such that $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$.

• **non-Boolean logic:** Non-Boolean logic does not assume every statement can be judged true or false, there are some statements upon which one cannot decide. An example of a physical situation where the underlying logic would be non-Boolean is in cosmology where observers can only make judgements upon statements that have to do with their backwards light cone.

relational type quantum mechanics.

see topos , Heyting algeras

• **norm:**

(i) $\|zx\| = |z|\|x\|$

(ii) $\|x + y\| \leq \|x\| + \|y\|$

• **normed space:** A normed linear space in which every vector x there corresponds a **norm** $\|x\|$ of x, such that

(i) $\|x\| \geq 0$, and $\|x\| = 0 \;\Rightarrow\; x = 0$,

(ii) $\|x + y\| \leq \|x\| + \|y\|$,

(iii) $\|x\| = |\alpha|\|x\|$.

• **normed space:** A vector space endowed with a norm.

• **nuclear spectral theorem:**

• **nuclear topology:**

• **open algebras:**

The smeared Hamiltonian Poisson brackets are an example, see Eq(E.7). Very little is known about the representation theory of open algebras.

• **open mapping theorem:** A fundamental result of functional analysis which states that if a continuous linear operator between Banach spaces is surjective (onto) then it is an open map (i.e. if U is an open set in the intial Banach space then $A(U)$ open set in the Banach space which the operator A maps to).

• **open sets:** Intuitively a set U is open if any point x in U can be moved in any "direction" and still be in the set. For example the set (a, b) of the real line $\mathbb{R}$ is open in $\mathbb{R}$, any point in (a, b) is arbitrarily close to the boundaries a or b, but cannot be a or b, as such if moved a sufficiently small amount are still contained in (a, b).

Subets understood with the properties

(1) Let A be a topological space and $\{\mathcal{U}_\alpha\}$, be any collection (possibly infinite) of open sets in A.

Then

$$\cup_\alpha \mathcal{U}_\alpha \tag{B.137}$$

is open

(2) Let be a finite collection of open sets in A

$$\mathcal{U}_1 \cap \mathcal{U}_2 \cap \cdots \cap \mathcal{U}_n \tag{B.138}$$

is open.

The concept of an open set is fundamental to amny areas of mathemetics, we see them in operator theory as well as in point-set topolgy (singularity theorems for example).

• **operator topologies:** The set of all bounded, linear operators acting on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. A subset $S \subset \mathcal{B}$ is called an algebra if $\alpha A + \beta B \in S$ and $AB \in S$

Topology needs a definition of what we mean by a neighbourhood of an element.

The norm topology: the ϵ-neighbourhood of A is the set of operators B with $\|A - B\|$. **norm topology** or **uniform topology** in $\mathcal{B}(\mathcal{H})$.

Other important topologies in $\mathcal{B}(\mathcal{H})$ are defined by means of seminorms.

For topologies that which are defined by seminorms it is not enough to consider only the convergence of sequences. The closure of a set is obtained by adding the limit points of em nets.

generally, if a subset is closed in one topology it is automatically closed in every stronger topology one gets even more limit points.

weak operator topology: topology such that are all continuous.

the weak topology is the topology such that all functionals on X^{**} are continuous.

weak star operator topology: The weak star topology is obtained if we use the absolute values of $| < \Phi|A|\Psi > |$ between arbitrary state vectors as a system of seminorms. Thus a sequence of operators converges weakly if all matrix elements converge.

• **oriented manifold:** Intuitively, in the case of a 2-manifolds, a surface is oriented if it is two-sided, and non-oriented if it is 1-sided. The cylinder is oriented, but the Möbius strip is not.

• **orbit:** Let G act on a set X. A subset $\subset X$ is said to be **stable** under the action of G if

$$g \in G \quad x \in S \Rightarrow gx \in S. \tag{B.139}$$

• **outer measure:** Let a be an algebra of subsets of X and μ a measure on it. For $A \subset X$ is defined by

$$\mu^*(A) = \inf \sum_{i=1}^{\infty} \mu(E_i) \tag{B.140}$$

where the infimum is taken over all $\mathcal{U}$-coverings of the set A all collections (E_i), $E_i \in \mathcal{U}$ with $\cup_i E_i \supset A$. to extend μ to as many elements of the powerset as possible.

• **Pachner moves:** Two finite triangulations related by a finite sequence of local modifications, called Pachner moves. Sequences of Pachner moves are the combinatorial equivalent of applying spacetime diffeomorphisms.

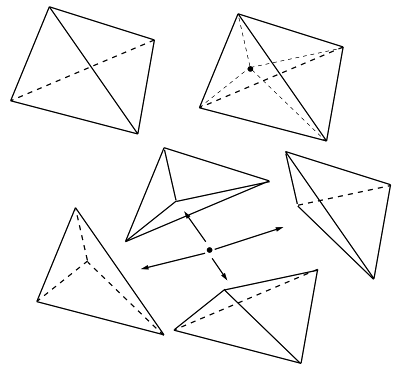


Figure B.21: Pachner move in d=3. (a) the $1 \rightarrow 4$ move subdivides.

?????????is the lattice as being embedded in some continuum manifold or whether one regards the lattice itself as existing independently of any background embedding.????????

• **paracompact:** A topological space is said to be paracompact if evry open cover admits an open locally finite refinement.

• **parallelogram law:** A normed space

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{B.141}$$

• **parallel transport:** Parallel transport of a vector is defined as transport without change.

• **parseval's formula:** The generalized Fourier series

$$f(x) = \sum_{-\infty}^{\infty} f_n e_n(x)$$

with Fourier coefficients

$$f_n = \int_a^b e_n^*(x) F(x) dx$$

$$
\begin{aligned}
E_{min} &= \int_a^b |F(x) - f(x)|^2 dx \\
&= \int_a^b F^2(x)dx - \sum_{-\infty}^{\infty} f_n^2 \geq 0. \qquad \text{(B.142)}
\end{aligned}
$$

gives an average measure of convegence. The Fourier series $f(x)$ of a function $F(x)$ is said to converge in the mean to $F(x)$ if $E_{min} = 0$. When this happens, the Bessel inequality becomes the Parseval equation

$$
\int_a^b F^2(x)dx = \sum_{-\infty}^{\infty} f_n^2. \qquad \text{(B.143)}
$$

• **partial function:** A partial function is the triple $f = (A, G, B)$ where A and B are sets (possibly empty) and G is a functional relation (possibly empty) between them, called the **graph** of f.

$f : A \to B$ is a **total function** if and only if domain$(f) = A$. A total function is often just referred to as a function.

• **partial isometry:** An operator U in the space of linear operators $\mathcal{L}(\mathcal{H})$ from a separible Hilbert space $\mathcal{H}$ to itself is called a partial isometry if $\|Ux\| = \|x\|$ for all $x \in \mathcal{H}$ when restricted to the closed subspace $(\text{Ker}U)^\perp$ (Ker$U$ being the set of $x \in \mathcal{H}$ for which $Ux = 0$).

See polar decomposition.

• **partially ordered set:** Let P be a non-empty set. A partial order relation in P is a relation denoted by $\leq$ which has the following properties:

(i) $a \leq a$ for every a (reflexivity);

(ii) $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry);

(iii) $a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

• **partition of unity:** Take an open covering $\{U_i\}$ of $\mathcal{M}$ such that each point of $\mathcal{M}$ is covered by a finite number of U_i (if this is always possible, $\mathcal{M}$ is called paracompact, which we assume to be the case). If a family of differentiable functions $\epsilon_i(p)$ satisfies

(i) $0 \leq \epsilon_i(p) \leq 1$

(ii) $\epsilon_i(p) = 0$ if $p \neq U_i$

(iii) $\epsilon_1(p) + \epsilon_2(p) + \cdots = 1$ for any point $p \in \mathcal{M}$

the family $\{\epsilon_i(p)\}$ is called a partition of unity subordinate to the covering $\{U_i\}$. It follows from (iii) that

$$f(p) = \sum_i f(p)\epsilon_i(p) = \sum_i f_i(p)$$

where $f_i(p) \equiv f(p)\epsilon_i(p)$ vanishes outside of U_I by (ii).

• **path ordered:** $\mathcal{P}\left(A(x(s_2))\ A(x(s_1))\ A(x(s_4))\ldots\right) = A(x(s_1))\ A(x(s_2))\ \ldots\ A(x(s_n))$, where $s_1 \geq s_2 \geq \cdots \geq s_n$.

• **Penrose's abstract index notation:** In this notation, the index 'a' of a vector v^a is to be seen as a label indicating that v is a vector (very much like the arrow in $\vec{v}$), and it does not take values in any set. That is, 'a' is not the component of v on any basis.

• **permutation group:**

• **Peter-Weyl Theorem:** The Peter-Weyl theorm asserts that the representaion matries, $D^J_{mn}(g)$, form a complete set of orthogonal functions on the group manifold.

$$f(g) = f(UgU^{-1}) \Rightarrow \quad f(g) = \sum_i a_i\chi_i(g) \tag{B.144}$$

where

$$a_i = \sum \tag{B.145}$$

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{B.146}$$

where $f(\theta) \in L^2(U(1))$.

• **p-form:** An anti-symmetric covariant tensor field over a manifold $\mathcal{M}$.

• **piecewise:** We say something piecewise, with respect to some property, if it is made up of a finite number of pieces, each of which shares this property.

• **piecewise-analytic curve:** A curve is piecewise analytic if it is made up of a finite number of pieces, each of which is analytic.

See semianalytic curve.

piecewise-smooth curve: A curve is piecewise smooth if it is made up of a finite number of pieces, each of which is smooth.

• **piecewise-linear (PL-) manifolds:** The transition functions are maps between polyhedra which map simplices to simplices.

• **Poincare lemma:** Say we have a $p-$form ω. Recall the exterior derivative, denoted d. If $d\omega = 0$, then ω is said to be closed. If $\omega = d\alpha$, then ω is said to be exact. Exactness implies closure, since $\omega = d\alpha \Rightarrow d\omega = d^2\alpha = 0$. The converse is in general not true. The Poincare lemma states that every closed form is locally exact. That is, if $d\omega = 0$, then $\omega = d\alpha$ in some local region. In general, this will not hold globally.

• **Poincare algebra in** $(2+1)$ **dimensions** $iso(2,1)$**:** Poincare algebra $iso(2,1)$ $(a, b, c = 0, 1, 2)$:

$$[J_a, J_b] = \epsilon^c{}_{ab} J_c, \quad [J_a, P_a] = \epsilon^c{}_{ab} P_c, \quad [P_a, P_b] = 0. \tag{B.147}$$

• **Poisson resummation formula:** As

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is periodic in x we may write

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} a_k \exp(2\pi i k x)$$

then via the inverse Fourier transform we have

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{k=-\infty}^{\infty} \exp(2\pi k x) \int_{-\infty}^{\infty} dy f(y) \exp(-2\pi i k y). \tag{B.148}$$

$$\sum_n e^{-\epsilon(n-N)^2} f(n) = \sum_n e^{-\epsilon(y-N)^2} f(y) e^{2\pi I y n} \tag{B.149}$$

• **polar decomposition:** A polar decomposition is the analogue to the decomposition $z = |z|e^{iargz}$ for positive operators on a Hilbert space. Let $\mathcal{H}$ be a Hilbert space. If $A \in \mathcal{B}(\mathcal{H})$ then (by the polar decomposition theorem) it can be uniquely expressed as $A = W|A|$ where $|A|$ is the positive square root of A^*A and W is a partial isometry with initial space equal to the closure of the range of $|A|$ and final space equal to the range of A. The expression $W|A|$ is called the polar decomposition of A.

left polar decomposition

• **polyhedron:** A subset $X \subseteq \mathbb{R}^n$ is said to be a polyhedron if eveery point $x \in X$ has a neighbourhood in X of the form

$$\{\alpha x + \beta y : \alpha, \beta \geq 0, \alpha + \beta = 1, y \in Y\}$$

where $Y \subseteq X$ is compact.

• **Pontyagin duality:**

• **positive elements:** An element of a C^*algebra $\mathcal{A}$ is said to be positive if $A = A^*$ and $\sigma(A) \geq 0$, or equivalently, $A = B^*B$ for some $B \in \mathcal{A}$.

• **positive operator:** An operator $B \in \mathcal{L}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is called positive if $(Bx, x) \geq 0$ for all $x \in \mathcal{H}$

• **power set:** Let X be a set. The power set $\mathcal{P}(X)$ is the collection of all subsets of X.

• **presheaf:** TAKEN DIRECTLY FROM mathworld.wolfram.com

For X a topological space, the presheaf $\mathcal{F}$ of Abelian groups (rings, ...) on X is defined such that

1. For every open subset $U \subseteq X$, an Abelian group (ring, ...) $\mathcal{F}(U)$, and

2. For every inclusion $V \subseteq U$ of open subsets of X, a morphism of Abelian groups (rings, ...) $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

subject to the conditions:

1. If $\emptyset$ denotes the empty set, then $\mathcal{F}(\emptyset) = 0$,

2. ρ_{UU} is the identity map $\mathcal{F}(U) \to \mathcal{F}(U)$, and

3. If $W \subseteq V \subseteq U$ are three open subsets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

In the language of categories, let $\mathbf{Top}(X)$ be the category whose objects are the open subsets of X and the only morphisms are the inclusion maps (these map V to U if $V \subseteq U$). Thus, $\mathbf{Hom}(V, U)$ is empty if $V \not\subseteq U$ and $\mathbf{Hom}(V, U)$ has just one element if $V \subseteq U$. Then a presheaf is a contravariant functor from the category $\mathbf{Top}(X)$ to the category $\mathbf{Ab}$ of Abelian groups (Ring of rings, ...).

As a terminology, if $\mathcal{F}$ is a presheaf on X, then $\mathcal{F}(U)$ are called the sections of the presheaf over the open set U, sometimes denoted as $\Gamma(U, \mathcal{F})$. The maps ρ_{UV} are called the restriction maps. If $s \in \mathcal{F}(U)$, then the notation $\rho_{UV}(s)$ is usually used instead of $s|_V$.

Sheaf: Important in twistor theory and other applications of algebraic geometry and topology in physics. A presheaf $\mathcal{F}$ is called a *sheaf* if for every collection U_i of open subsets of X with $U = \cap_{i\in I} U_i$ the following conditions hold:

1. If $s, t \in \mathcal{F}(U)$ and $\rho_{UU_i}(s) = \rho_{UU_i}(t)$ for all i, then $s = t$;

2. If $s_i \in \mathcal{F}(U_i)$ and if for $U_i \cap U_j \neq \emptyset$ we have

$$\rho_{U_i, U_i\cap U_j}(s_i) = \rho_{U_j, U_i\cap U_j}(s_j) \tag{B.150}$$

for all i, then there exists an $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = s_i$ for all i.

The example of germs of functions on a differentiable manifold is a familiar example of a sheaf.

(definition taken from Topi and computation)

patching condition: Let an open set U be covered by open subsets U_i; then any given sections over the U_i's, such that the sections over U_i and U_j have the same restriction to $U_i \cap U_j$, are restrictions of a single section over U. (for example letting $\mathcal{F}(U)$ be the set of all continuous real-valued functions on U). The crucial thing about this definition is that the space X enters the picture only through its poset of open subsets and the notion of a cover.

• **primative elements of a Hopf algebra:**

• **principal bundle:** A principal bundle is a fibre bundle $\pi : P \to E$ with fibre F equal to the structure group G and having the property that for all U_a and U_b with $U_a \cap U_b \neq \emptyset$,

$$\varphi_{ba} : U_a \cap U_b \to Left(F) \subset Diff(F),$$

where $Left(F) = \{L_g | L_g(h) = gh, \text{ for all } h \in G, g \in G\}$. In other words, changing coordinates corresponds to multiplying the fibre on the left by some element of G.

As an example consider the frame bundle $B(\mathcal{M})$. The total space consists of the set of basis vectors v_i^a of the tangent space for all points over the manifold $\mathcal{M}$. Here i is an index $i = 1, \ldots, n$ labeling the n basis vectors v^a. There is a natural free right action by $GL(n, \mathbb{R})$ on $B(\mathcal{M})$: If we have a basis $\{v_i^a, i = 1, \ldots, n\}$, we know that

$$(v_1^a, v_2^a, \ldots, v_n^a)g := (v_j^a g^j{}_1, v_j^a g^j{}_2, \ldots, v_j^a g^j{}_n),$$

where g in $GL(n, \mathbb{R})$. We can also see the array of vectors $(v_1^a, v_2^a, \ldots, v_n^a)$ as a $n \times n$ matrix with non-zero determinant.

The frame bundle is an example of a principal bundle. In a principal bundle $(P, \pi, \mathcal{M}, G)$ each fibre is diffeomorphic to the structure group G. Principal bundles are in a sense more fundamental than vector bundles, since one can always regard vector bundles as associated bundles to a particular principal bundle. In this example, the tangent bundle $T\mathcal{M}$ is the associated vector bundle to the frame bundle $B(\mathcal{M})$.

• **product spaces:**

• **product topology:** Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces, and form the product $\mathcal{S} \times \mathcal{T} = \{(u, v) : u \in \mathcal{S} \text{ and } v \in \mathcal{T}\}$ of the two sets $\mathcal{S}$ and $\mathcal{T}$. The product topology on $\mathcal{S} \times \mathcal{T}$ consists of all subsets that are unions of sets of the form $U \times V$, where U is open in $\mathcal{S}$ and V is open in $\mathcal{T}$. Thus these open rectangles form a basis for the product topology.

• **projective family:** A *projective* family consists of sets χ_S, together with a family of onto *projections*,

$$P_{SS'}\, \chi_{S'} \rightarrow \chi_S, \tag{B.151}$$

such that

$$P_{SS'} \ \circ p_{s's''} = p_{ss''}. \tag{B.152}$$

• **projective limit:** Unfortunately the projective family itself does not have a largest element from which one can project to any other. However, such an element can in fact be obtained by a standard procedure called the "projective limit".

• **projective limit construction:**

A graph is a collection of edges such that if two edges meet. Consider the space A_γ, each element of which assigns to every edge in γ

The projective limit of these spaces are precisely the spaces $\mathcal{A}_\gamma$, $\mathcal{G}_\gamma$ and $\mathcal{A}_\gamma/\mathcal{G}_\gamma$.

What about operators that act on this Hilbert space? All operators that are well defined on that Hilbert space arise from consistent family of operators. These operators on each of these individual finite Hilbert spaces fit together in a certain way. If it is well defined on here then it can be shown that that they come from something that fits together in this way.

• **projection mappings:**

$$\hat{P}_k^2 = \hat{P}_k \tag{B.153}$$

$$\sum_{k=1}^{N} \hat{P}_k = \mathbf{I}. \tag{B.154}$$

group averaging Rovelli projection onto physical states not strictly projection operators.

• **pull-back:** The diffeomorphism ϕ maps points in $\mathcal{M}$ to points in $\mathcal{N}$. The push-forward $\phi^*|_p$ is the natural map between the co-tangent spaces $T_p^*\mathcal{M}$ and $T_{\phi(p)}^*\mathcal{N}$ induced by the diffeomorphism ϕ.

$$[\phi^*\omega](X_1, X_2, \ldots, X_p) = \omega(\phi_* X_1, \phi_* X_2, \ldots, \phi_* X_p). \tag{B.155}$$

• **pure point spectrum:** Let a be a self-adjoint operator. The pure point spectrum $\sigma_{pp}(a)$ is the set of eigenvalues of a.

• **pure states:** Those states which cannot be written as convex linear combinations of other states, i.e. states which cannot be written as $\alpha a + (1-\alpha)b$ for some pair of states a, b.

• **push-forward:** map head-to-head and tail-to-tail. If the vector has components X^μ and the map takes the point with coordinates x^μ to one with coordinates $\xi(x)$, the vector ϕ_* has components

$$(\phi_* X)^\mu = \frac{\partial \xi^\mu}{\partial x^\nu} X^\nu. \tag{B.156}$$

This looks like the transformation formula for contravariant vector components under a coordinate transformation. We are doing an active transformation, changing a vector into a different one.

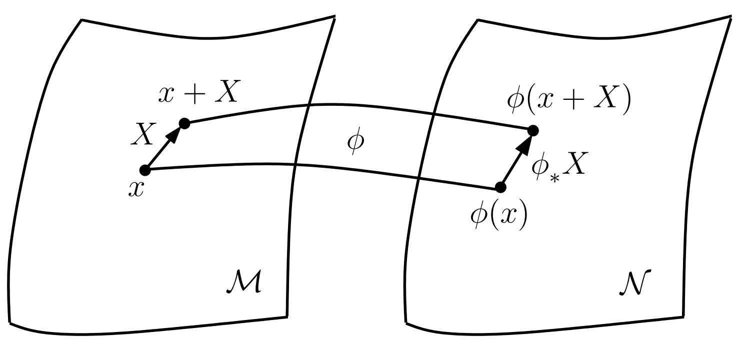


Figure B.22: pullbackDef0. Pushing forward a vector X from $T\mathcal{M}_x$ to $T\mathcal{N}_{\phi(x)}$.

The diffeomorphism φ maps points in $\mathcal{M}$ to points in $\mathcal{N}$. The push-forward $\varphi_*|_p$ is the natural map between the tangent spaces $T_p\mathcal{M}$ and $T_{\varphi(p)}\mathcal{N}$ induced by the diffeomorphism φ.

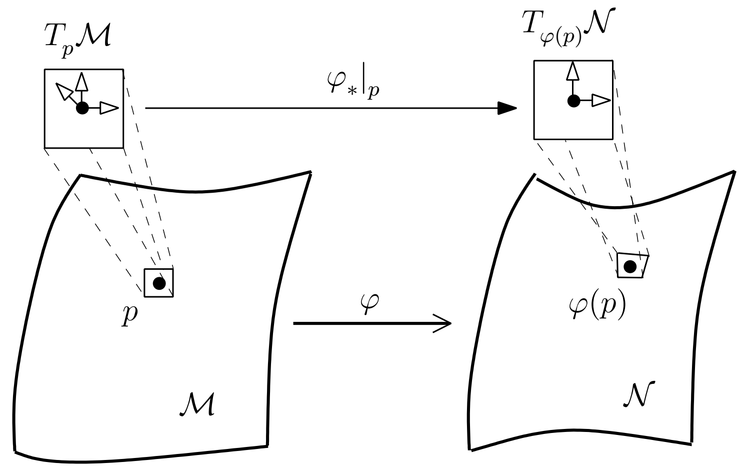


Figure B.23: pullbackDef. $\varphi_*|_p : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$

• **q-deformation:** Introduces a non-commutative to a mathematical structure that was absent in the first place. The non-commutativity is measured by a deformation q parameter. The limiting case gives back the results afforded by the ordinary situation. Some examples are $su_q(2)$ which defines a q-analogue of angular momentum.

• **q-series:**

by L'Hopital's rule

$$\lim q \to 1 \frac{1-q^a}{1-q} = a \tag{B.157}$$

q-shifted factorial

$$(a;q)_n = \begin{cases} 1, & n=0, \\ (1-a)(1-aq)(1-aq^{n-1}), & n=1,2,\ldots \end{cases} \tag{B.158}$$

q-binimial expansion

$$\sum_{m=0}^{n} y^m q^{\frac{m(m+1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix} \tag{B.159}$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\ldots(1-q^{n-m+1})}{(1-q)(1-q^2)\ldots(1-q^m)}, \tag{B.160}$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1. \tag{B.161}$$

• **quantum groups:** non-co-commutative quasi-triangular Hopf algebras. deform any classical Lie algebra. The initial Hopf algebra is the universal enveloping algebra of the Borel subalgebra of a Lie algebra.

The limiting case - as q goes to 1, the quantum algebra $su_q(2)$ Quantum $SL(2)$ group and quantum $SU(2)$ group.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{B.162}$$

$$U\tilde{\epsilon}U^T = \tilde{\epsilon}, \quad \text{where} \;\; \tilde{\epsilon} = \begin{pmatrix} 0 & -A \\ A^{-1} & 0 \end{pmatrix} \tag{B.163}$$

Figure B.24: links.

$$\begin{aligned} & ba = qab \;\; dc = qcd \\ & ca = qac \;\; db = qbd \\ & bc - cb = 0 \\ & ad - da = -(q - q^{-1})bc \\ & ad - q^{-1}bc = 1 \end{aligned} \tag{B.164}$$

where $q = aA^2$. complex non-commuting components a, b, c, d. $q = e^{in}$

$su_q(2)$ algebra

$$\begin{aligned} [j_3, j_\pm] &= \pm j_\pm, \\ \left[j_+, j_-\right] &= \frac{q^{2j_3} - q^{-2j_3}}{q - q^{-1}} \end{aligned} \tag{B.165}$$

In a similar way we can define the quantum group $GL_q(2)$,

$$UQU^T = z_1 Q, \;\; U^T Q U = z_2 Q, \quad z_1, z_2 \in \mathbf{C}^* \tag{B.166}$$

$z_1 = z_2 = z$

• **quasilinear differential equations:** Differential equations linear in the highest derivative terms. For second order quasilinear differential equations, many of the results on linear systems apply locally.

• **quasi-triangular Hopf algebras:** See Hopf algebras. A non-co-commutative quasi-triangular Hopf algebra is called a quantum group.

$$\Delta'(a) = \sigma \circ \Delta(a) \tag{B.167}$$

It is non-co-commutative, but the non-co-commutativity is controlled by a matrix $\mathcal{R}$,

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad \text{for all } a \in A. \tag{B.168}$$

and

$$\begin{aligned}
(\mathbf{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} = \sum_{i,j} A_i A_j \otimes B_j \otimes B_i, \\
(\mathbf{1} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} = \sum_{i,j} A_i \otimes A_j \otimes B_i B_j
\end{aligned} \tag{B.169}$$

and

$$(\gamma \otimes \mathbf{1})\mathcal{R} = (\mathbf{1} \otimes \gamma^{-1})\mathcal{R} = \mathcal{R}^{-1} \tag{B.170}$$

Where is called the universal $\mathcal{R}$-matrix. We write $\mathcal{R} = \sum_i A_i \otimes B_i$ and let

$$\mathcal{R}_{12} = \sum_i A_i \otimes B_i \otimes \mathbf{1},$$

$$\mathcal{R}_{13} = \sum_i A_i \otimes \mathbf{1} \otimes B_i, \tag{B.171}$$

$$\mathcal{R}_{23} = \mathbf{1} \otimes \sum_i A_i \otimes B_i. \tag{B.172}$$

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \tag{B.173}$$

• **quotient map:**

• **quotient topology:** Let

$$\mathcal{T}' = \{V : V \subseteq Y \text{ and } p^{-1}(V) \text{ is open in } X\}.$$

It is immediate that $\emptyset, Y \in \mathcal{T}'$ as $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(Y) = X$.

• **radical:** A *radical* R of an algebra A is the intersection of all its maximal left ideals. R itself is obviously a proper left ideal.

• **Randon-Nikodym derivative:**

• **Randon-Nikodym theorem:**

• **(real infinite) sequence** is a map a $a : N \to R$

Of course if is more usual to call a function f rather than a; and in fact we usually start labelling a sequence from 1 rather than 0; it doesn't really matter. What the definition is saying is that we can lay out the members of a sequence in a list with a first member, second member and so on. If $a : N \to R$, we usually write a_1, a_2 and so on, instead of the more formal $a(1)$, $a(2)$, even though we usually write functions in this way.

• **reconstruction Theorem:** Does it? We say it *separates* points in $\mathcal{A}/\mathcal{G}$ in that , if A_1 and A_2 are not related by a gauge transformation, there exists a loop γ such that: $T_\gamma(A_1) \neq T_\gamma(A_2)$.

Suppose Σ is a connected manifold with basepoint x_0 and the map $H : \Omega_{x_0} \to G$ satisfies the following conditions:

(i) H is a homomorphism of the composition law of loops, $H(\gamma_1 \circ \gamma_2) = H(\gamma_1)H(\gamma_2)$,

(ii) H takes the same values on thinly equivalent loops: $\gamma_1 \sim \gamma_2$ if $\gamma_1 \circ \gamma_2^{-1}$ is thin??,

(iii) For any smooth finite-dimensionale family of loops $\bar{\psi} : U \to \Omega_{x_0}\Sigma$, the composite map $H\psi : U \to \Omega_{x_0}\Sigma \to G$ is smooth.

Then there exists a differentiable principle fibre bundle ..R. Lool hep-th/9309056

• **regular Borel measure:** A non-negative countably additive set function μ defined on $\mathcal{B}$ is called a *regular Borel measure* if for every Borel set B we have:

$$\begin{aligned} \mu(B) &= \inf\{\mu(O) : O \text{ open}, O \supset B\}, \\ \mu(B) &= \sup\{\mu(F) : F \text{ closed}, F \subset B\}. \end{aligned} \tag{B.174}$$

taken from Measure, Integral and Probability, M Capiński and E. Kopp, [**?**].

Notice that regularity of μ on a compact Hausdorff space X reduces to the fact that the measure of every measurable set can be approximated arbitrarily well open or compact (and hence closed since in a Hausdorff space every compact subset is closed, see ??) sets respectively.

• **regular measure:**

• **regular embedding:** For an embedding $\phi : \mathcal{M} \to \mathcal{N}$, the map $\phi : \mathcal{M} \to \phi(\mathcal{M})$ is a bijection and the manifold structure induced by ϕ on $\phi(\mathcal{M})$ is given by the atlas $\{\phi(U_I), \varphi_I \circ \phi^{-1}\}$ where $\{U_I, \varphi_I\}$ is an atlas of $\mathcal{M}$. This structure need not be equivalent to the submanifold structure of $\phi(\mathcal{N})$ which is given by the atlas $\{V_J \cap \phi(\mathcal{M}), \phi_J\}$ where $\{V_I, \phi_I\}$ is an atlas of $\mathcal{N}$. When both differential structures are equivalent the embedding is called regular.

• **regular representation of a C^*-algebra:** If we wish to require that a Weyl algebra is represented weakly continuously, states whose GNS representation have this property are said to be regular.

• **regular representation of a finite group:** It is a matrix representation of the group. We construct the matrices as follows:

	e	a	b	c
e	e			
a^{-1}		e		
b^{-1}			e	
c^{-1}				e

To establish that this is a representation we must prove that

$$A^{(i)}_{jk} = \begin{cases} 1 \text{ if } a_j^{-1} a_k = a_i \\ 0 \text{ otherwise.} \end{cases} \tag{B.175}$$

Proof: It is easily established that

$$\sum_k A^{(i)}_{jk} A^{(m)}_{kl} = \begin{cases} 1 \text{ if and only if } a_k = a_j a_i = a_l a_m^{-1} \\ 0 \text{ otherwise.} \end{cases} \tag{B.176}$$

• **regular representation of a Lie group:** Provides a systematic procedure to construct irreducible representations of a group.

$$\int d\mu(c) \mathcal{D}^{\text{reg}}(a; b, c) f(c) \tag{B.177}$$

$$\mathcal{D}^{\text{reg}}(a; b, c) = \frac{\delta^p(c - \phi(a, b))}{\rho(c)}, \tag{B.178}$$

• **Reidermeister Moves: ??**

• **Reiz lemma:** A vector in the normed space uniquely defines a continuous functional via

$$F_f(V) =< f|V > \tag{B.179}$$

• **Riesz representation theorem:**

Application: [96] - since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the Riesz representation theorem ensures that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \tag{B.180}$$

• **Reiz representation theorem:**

An application ([96]): Now, since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, the ensues that there is a unique regular Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ such that

$$\Gamma(f) = \int_{\overline{\mathcal{A}/\mathcal{G}}} d\mu([A]) \; \tilde{f}([A]) \tag{B.181}$$

where $\tilde{f} \in C^0(\overline{\mathcal{A}/\mathcal{G}})$ corresponds to f in $\overline{\mathcal{HA}}$.

• **relative topology:** (or induced topology) Let X be a topological space, and let Y be a non-empty subset of X. The relative topology on Y is defined to be the class of all intersections with Y of open sets in X.

• **repeated bisection argument:**

• **representation:** A representation of a group G over a field k (often. $\mathbb{R}$ or $\mathbb{C}$) is a homomorphism $\pi : G \rightarrow GL(V)$, where V is a (usually finite dimension) vector space over k. That is, a representation π of G "compares" the "abstract" group G with a concrete group $GL(V)$. There are similar defintions for C^*-algebras and other such ab.

• **representation theory:** Representation theory studies how any given abstract group can be realized as a group of matrices.

operators on the vector space??? concrete example often matrices and operators on a Hilbert space.

• **resolvent set:** The resolvent set $\rho(A)$ is defined as the subset of $\mathbb{C}$ defined as $\{z \in \mathbb{C} : A - z\mathbf{1}$ has bounded inverse $\}$.

• **Riemann integral:**

$$\overline{\int} f d\alpha = \underline{\int} f d\alpha \tag{B.182}$$

• **Reimann's crition:** We first give the crition of the most elementary case of a real function f of a closed interval, $[a,b]$, of the real, $f:[a,b]\rightarrow\mathbb{R}$. Such Reimann-integrable if and only if for every $\epsilon>0$ there exists a partition P_ϵ such that $U(P_\epsilon,f)-L(P_\epsilon,f)<\epsilon$.

The Reimann integral doesn't deal with all functions we need for our purposes in formulating quantum mechanics and quantum field theory in a mathematically rigorous way.

• **Riemann-Stieltjes integral:**

$$s(P_n)=\sum_{j=1}^{n}x(t_j)[w(tj)-w(t_{j-1})] \tag{B.183}$$

$$\int_a^b x(t)dw(t) \tag{B.184}$$

• **ring:**

In the set algebraic sense:

If groups are roughly thought of as collections of elements that can be added together, then rings are collections of elements for which there is addition and multiplication. To be more specific, a ring is an additive abelian group whose elements can be multiplied as well as added, and in which multiplication is

(i) associative, that is, if $x,\ y,\ z$ are any three elements in R, then $x(yz)=(xy)z$; and

(ii) is distributive, that is, if $x,\ y,\ z$ are any three elements in R, then $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$.

If an element x of R has an inverse, then x is said to be **regular** (or invertible).

a division ring: A ring with identity is called a division ring if all its non-zero elements are regular.

In the set theoretical sense:

Let X be a set, let $R\subseteq\mathcal{P}(X)$. Then R is a ring of subsets if

(i) $\emptyset\in R$;

(ii) if $A,B\in R$ then $A\cap B$, $A\cup B$ and $A\backslash B$ are all in R.

An an alternative definition of a ring, equivalent to the above, R is a ring if

(i) $\emptyset\in R$;

(ii) if $A,B\in R$ then $A\cap B$ is in R, and $A\triangle B\in R$, (where $A\triangle B$ is the symmetric difference $(A\backslash B)\cup(B\backslash A)$).

With operations $\cap$ as multiplication, $\triangle$ as addition, R is a ring in the algebraic sense.

• **right action of a group:**

$$R_g(g') := g'g. \tag{B.185}$$

$g = e^{tX_g}\big|_{t=1}$

$$\frac{d}{dt}(g'e^{tX_g}) \tag{B.186}$$

Vectors of the tangent space at g, $T_g(G)$, are related to vectors in the tangent space at e, $T_e(G)$. This map is denoted $(R_{g^{-1}})_*$ and is called the **pull-back** of the right action $R_g(G)$, the notation and name will be clarified in a moment. We can find this relationship. For clarity we will derive the matrix element of this map

There is a map which acts on the tangent space $T_g(G)$ and takes it to $T_e(G)$

$$(R_{g^{-1}})_* : T_g(G) \to T_e(G) \tag{B.187}$$

• **rooted tress:**

• **saddle-point approximation:** The idea of the saddle point method (or method of steepest descent) is to deform the contour in such a way that the main contribution to the integral comes from a neighborhood of a single point, (or finite number of points).

Let C be a contour in the complex plane and the functions g and S are holomorphic in a neighborhood of this contour. We will consider the asymptotics as $\lambda \to \infty$ of the Laplace integrals

$$G(\lambda) = \int_C g(z)\exp[\lambda S(z)]dz : \tag{B.188}$$

$$f(z) = \int d\tau g(\tau)e^{f(\tau)} \tag{B.189}$$

$$f(z) \approx \sum_i g(\tau_i)e^{f(\tau_i)} \tag{B.190}$$

τ_i determined by

$$\left.\frac{df(\tau)}{d\tau}\right|_{\tau=\tau_i} = 0. \tag{B.191}$$

• **Schwartz space** $\mathbb{S}(\mathbb{R}^n)$**:** - also called the space of **test functions**. The set of functions that are infinitely differentiable and whose derivatives

$$\frac{\partial^k \varphi(x)}{\partial x^k} \to 0 \ \text{ as } x \to \infty \tag{B.192}$$

faster than any power of $1/|x|$, for $k = 0, 1, 2, \ldots$.

natural duality between Schwartz space and space of tempered distributions??

• **Schwarz's inequality:** For two square-integrable functions $f(x), g(x) \in L^2[dx]$

$$\int |f(x)g(x)|^2 dx \leq \int |f(x)|^2 dx \ \int |g(x)|^2 dx \tag{B.193}$$

holds. Easily follows from

$$\int |f(x) + \alpha g(x)|^2 dx \geq 0 \tag{B.194}$$

where α is real, as this implies (when $\int |g(x)|^2 dx \neq 0$)

$$\left(\alpha + \frac{\int |f(x)||g(x)|dx}{\int |g(x)|^2 dx}\right)^2 + \frac{\int |f(x)|^2 dx}{\int g(x)^2 dx} - \frac{\int |f(x)g(x)|dx}{(\int |g(x)|^2 dx)^2} \geq 0. \tag{B.195}$$

Used in derivation of the uncertainty principle. For a Banach space X with norm $\|\cdot\|$ for any $A, B \in \mathcal{B}(X)$

$$\|AB\| \leq \|A\| \cdot \|B\|. \tag{B.196}$$

• **section:** It is a smooth assignment to each point in the base space of a point in the fibre over it. As an example, the section of the tangent bundle of a manifold $\mathcal{M}$ is a vector field. Note that a section is globally defined and will not alway exist. In the case of a principal fibre bundle, such as a frame bundle, it has a section if and only if it is trivial. This is not necessarily the case for general fibre bundles, such as a tangent fibre bundle. Notice, however, that local sections always exist as bundle spaces are locally trivial.

• **Segal-Bargmann transformation:** The Segal-Bargmann transformation B_t in ordinary quantum mechanics on the phase space $\mathbb{R}^2$ gives a representation in which wave functions are holomorphic, square integrable (with respect to the Liouville measure) functions of the complex variable $z = q - ip \in \mathbb{C}$, given by the convolution of f with a Gaussian,

$$B_t f(z) = \frac{1}{(2\pi t)^{-d/2}} \int_{\mathbf{R}^d} e^{-(z-x)^2/2t} f(x) dx, \quad z \in \mathbb{C}. \tag{B.197}$$

• **self-adjoint:**

An Unbounded operator are self-adjoint if it is a densely defined operator a with domain $D(A)$ is called self-adjoint if $a^\dagger = a$ and $D(a^\dagger) = D(a)$ where

$$D(a^\dagger) := \{\psi \in \mathcal{H}; \sup_{0 \neq \psi' \in D(a)} | < \psi, a\psi' > | / \|\psi'\| < \infty\} \tag{B.198}$$

..... definition not finished this taken from Thiemann Intro to Modern...

• **self-adjoint extensions:**

• **semianalytic function:** A function $f : U \to \mathbb{R}^m$, where U is some open subset of $\mathbb{R}^n$, is called semianlytic

• **semianalytic partition:** Let $U \subset \mathbb{R}^n$ be open and $h := \{h_1, \ldots, h_N\}$ be a system of real-valued analytic functions h_k defined on a neighbourhood of U. The subsets defined by the 3^N sequences of inequalities/inequalities

$$\begin{aligned}
&h_1(x) > 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0;\\
&h_1(x) < 0, \quad h_2(x) > 0, \ldots, h_N(x) > 0;\\
&\\
&h_1(x) > 0, \quad h_2(x) = 0, \ldots, h_N(x) < 0;\\
&\\
&h_1(x) = 0, \quad h_2(x) = 0, \ldots, h_N(x) = 0
\end{aligned} \tag{B.199}$$

satisfy the conditions of a partition (i.e. the union of these subsets covers U and the intersection between any two such distinct subsets is empty), this is called the semianalytic partition of U subordinate to h. These subsets can vary in dimension from $n-$dimensional to zero-dimensional.

• **semianalytic structures:** Semianalytic structures are objects such as paths or surfaces which are analytic except possibly on lower-dimensional subsets. On these subsets there is again a notion of semianalyticity. These objects, unlike analytic ones, have a local character since knowing the object on an arbitrary open subset only determines it up to where it fails to be analytic.

• **semi-continuous function:**

• **semi-direct product:** A group G is said to be a semidirect product of the subgroups N and Q, denoted $N \otimes_S Q$, if

(i) $N \triangleleft G$ ($\triangleleft$ denoting that N is a normal subgroup);

(ii) $NQ = G$; (meaning that every element of G can be written as a product nq where n is some element of N and q is some element of Q);

(iii) $N \cap Q = \mathbf{1}$. (Milne Group theory)

For example the translation-rotation group G, for Q as the rotation group and N the translation group. As

$$\hat{R}\hat{T}\hat{R}^{-1} = \hat{T}',$$

where $\hat{R}$ is a rotation and $\hat{T}$ a translation, is again a translation. Hence, translations form an (abelean) normal subgroup of G. (iii) is obvious.

Equivalent condition: and $G \to G/N$ induces an isomorphism $Q \xrightarrow{\approx} G/N$.

+ + + + + + + ++

When a group B acts on another group A as a subgroup of the automorphisms of A, a larger group $A \triangleleft B$ can be constructed, whose elements are all pairs $\{(a, b) \ : \ a \in A, b \in B\}$,

• **semi-group:** elements with an associative multiplication, which is closed under multiplication.

• **semi-norm:** same as norm except that we do not demand that only the zero element has zero norm.

in the study of topology defined by a system of semi-norms requires nets instead of sequences??

• **semi-semianalytic partition:** A semi-semianalytic partition of an open set is analagous to a semianalytic partition, except that the functions h are not required to be analytic, just semianalytic.

• **semisimple group:**

Any compact Lie algebra is semisimple.

• **separating:** A collection of functions $\mathcal{C}$ on a (topological) space X is said to separate its points if and only if for any $x_1 \neq x_2$ we can find $f \in \mathcal{C}$ such that $f(x_1) \neq f(x_2)$.

The only if part of the definition says given the values assumed by each and every function in the collection $\mathcal{C}$ exists a unique point $p \in X$ for which the functions take their given values.

They encode all the information about the (topological) space X. This is the starting point for non-commutative geometry...

An important example is that of gauge theory when connection representation to the loop representation. It separates points of $\mathcal{A}/\mathcal{G}$ in the sense that, if $[A_1] \neq [A_2]$, there exists a loop α such that: $T_\alpha(A_1) \neq T_\alpha(A_2)$. The set of configuration variables is sufficiently large in that they encode all the information that is contained in a connection

these functionals form a separating set on $\mathcal{A}/\mathcal{G}$: if all the T_α assume the same values at two connections, they are necessarily gauge related.

• **separation conditions:** There is a whole hierarchy of separation conditions, here we mention a few of them. Let $\mathcal{T}$ be a topological space, and let P and Q be two distinct points of $\mathcal{T}$. $\mathcal{T}$ is called

(i) T_0 if at least one of the points has a neigbourhood excluding the other,

(ii) T_1 if each point has a neigbourhood excluding the other,

(iii) T_2 is the Hausdorff condition holds.

One more separation condition of note is a normal topological space.

• **sesquileanr forms:**

$(i)\ F(\alpha u + \beta v, w) = \overline{\alpha} F(u, w) + \overline{\beta} F(v, w)$, and
$(ii)\ F(u, \alpha v + \beta w) = \alpha F(u, v) + \beta F(u, w)$.

sesquileanr form with....

• **sets:** a **set** is a collection of "things".

Standard notation for often-used sets

$\emptyset = \{\}$ = set with no elements

$\mathbb{Z}$ = the integers

$\mathbb{Q}$ = the rational numbers

$\mathbb{R}$ = the real numbers

$\mathbb{C}$ = the complex numbers

• **shadow state:** A shadow state $|\Psi_\gamma^{shad}>$ is an element $(\Psi|$ of Cyl^* projected to the subspace Cyl_γ by the projection operator $\hat{p}_\gamma$:

$$(\Psi|\hat{p}_\gamma : \sum_{\vec{x}_i \in \gamma} \Psi(\vec{x}_i)|\vec{x}_i> \equiv |\Psi_\gamma^{shad}>, \quad \text{(B.200)}$$

with $(\Psi| = \sum_{\vec{x}} \overline{\Psi}(x_i)(\vec{x}|$.

See physics glossary.

• **sheaf:** See presheaf.

• **sieve:** closed under post composition.

• $\sigma-$**additive:** the measure of a *countable* union of non-intersecting measurable sets is equal to the sum of their measures:

$$\mu\left(\bigcup_i^\infty A_i\right) = \sum_i^\infty \mu(A_i) \quad \text{where} \;\; A_i \cap A_j = \emptyset \text{ for any} i \neq j. \tag{B.201}$$

• $\sigma-$**algebra:**

The word "sigma" refers to sum, meaning union, while the word "algebra" indicates that $\mathcal{M}$ is defined in terms of certain operations, in this case unions and complements of sets

• **simple functions:** Let X be a non-empty set. Then a *simple function* s is a mapping from X to the real line, i.e. $s :\rightarrow R$, such that s only takes finitely many different values.

• **simple representations:** The representation π of G in the vector space V over k is said to be simple if no proper subspace of V is stable under G. That is, π is simple if the following property holds: if U is a subspace of V such that

$$g \in G, \quad u \in U \Rightarrow gu \in U$$

then either $U = 0$ or $U = V$.

• **simplex:** the most elementary geometric figure of a given dimension. For zero dimension it is a point, in two dimensions it is the line, in three it is the tetrahedron in, the 4-simplex in four dimensions, etc. B

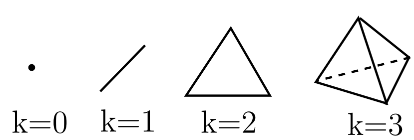


Figure B.25: Simplices in 3d.

• **simplicial complex:**

• **Skien relation:** A recursive relation which allows us to assign a polynomial to a knot which is a topological invariant of the knot. A skien relation is an equation that relates the polynomial of links obtained by changing the crossings in the projection of the original link. Skien relations allows us to calculate the invariants by decomposing the link step by step to a union of unkotted, unlikned loops.

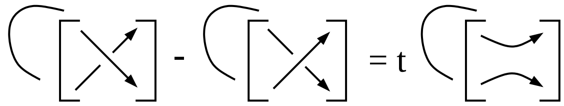


Figure B.26: Skien relation for the Conway polynomial.

If two knots have different polynomials then they are topologically inequivalent. Two different knots can have the same polynomial.

We want a way of obtaining the bracket polynomial of a link in terms of the bracket polynomials of simpler links. Skein relations.

The holy grail of knot theory is to have a recursive relation that distinguishes between all topologically inequivalent knots. The standard example is evaluation of the Conway polynomial for the Trefoil knot

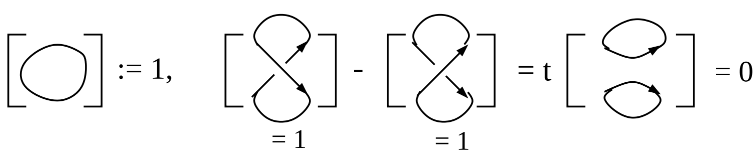


Figure B.27: Skien relation for the Conway polynomial.

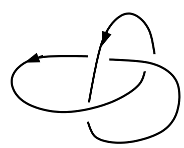


Figure B.28: The Trefoil knot.

(see knot invariants, Kauffman bracket, knot polynomials,...)

• **smooth curve:** A curve in Euclidean space $\mathbb{R}^n$ is smooth if and only if it is infinitely differentiable. A curve in a manifold $\mathcal{M}$ is smooth if and only if its image under a chart is a smooth curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval (a, b) to $\mathbb{R}^n$ in Fig.(**??**) is a analytic map. These curves are denoted as $C^\infty-$curves.

Finite differentiable curves (C^r-curves) are defined in the obvious way.

• **smooth function:** A smooth function is a function that is infinitely differentiable, that is, it does not matter how many times you differentiate the function, the resulting functions are always continuous. Such functions are denoted C^∞.

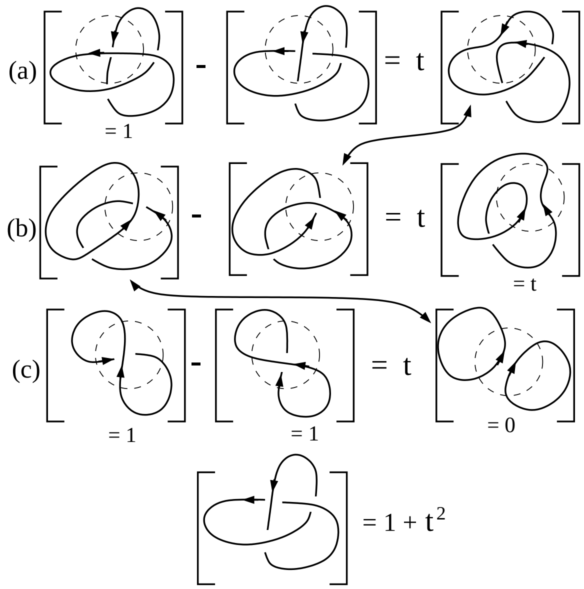


Figure B.29: Evaluation of the Conway polynomial for the Trefoil knot $1 + t^2$.

• **Sobolev embedding theorem:** It shows that $C^k(\omega) \subset H^s(\omega)$ (with continuous inclusion - i.e. the C^k-norm is bounded in terms of the H^s-norm) provided that $s > k + n/2$.

• **Sobolev inequalities:** One can relate the Sobolev norm to more usual norm via Sobolev inequalities.

• **Sobolev norm:** The Sobolev norm is used in the standard formulation of well-posed initial value problems in a general globally hyperbolic spacetime [7] - stability of closed trapped surfaces away from spherical symmetry (I think). Energy inequalities. Relate initial conditions to the solution at a time t later. Can relate the Sobolev norm to more usual norm via Sobolev inequalities.

E.g. Consider the Klein-Gordon equation in $(1+1)$- dimensional Minkowski spacetime, $(-\partial_t^2 + \partial_x^2)f = 0$ defined on a suitable region of spacetime. (Who's Afraid of Naked Singularities? gr-qc/9907009)

$$\|f\| := \left(\frac{q^2}{2} \int dx |f|^2 + \frac{1}{2} \int dx \left| \frac{df}{dx} \right|^2 \right)^{\frac{1}{2}}, \tag{B.202}$$

where q^2 is a positive constant.

$$\|\psi, \mathcal{N}\|_m = \left[\sum_{i=0}^{m} \int_{\mathcal{N}} |D^p \psi|^2 d\sigma \right]^{1/2} \tag{B.203}$$

$$\|\psi\| < Const \times \|\psi, \tilde{\mathcal{N}\|}_m \tag{B.204}$$

• **Sobolev space:** The **Sobolev space or** H^1 is the function space $\mathcal{H} = \{f \| f\| < \infty\}$. with inner product

$$(f, g) := \left(\frac{q^2}{2} \int dx f^* g + \frac{1}{2} \int dx \frac{df^*}{dx} \frac{dg}{dx} \right)^{\frac{1}{2}}, \tag{B.205}$$

so that $\|f\|^2 = (f, f)$.

• **span:** For a nonempty subset $M \subset X$ the set of all linear combinations of vectors of $x_i \in M$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, \tag{B.206}$$

is called the span of M, written span M. The completion of span M is denoted $\overline{\text{span } M}$.

• **spectral mapping theorem:** If a $\mathcal{A}$ is a C^*-algebra and if $A \in \mathcal{A}$ is normal then $\sigma_{\mathcal{A}}(f(A)) = f(\sigma_{\mathcal{A}}(A))$ for all continuous functions f. This result is known as the spectral mapping theorem. Can be generalized to f measurable function.

• **spectral measure:**

• **spectral radius:**

$$\sigma(A) = \lim_{n \to \infty} \|A^n\|^{1/n} \tag{B.207}$$

• **spectral representation:** A self-adjoint operator in a (finite) Hilbert space $\mathcal{H}$ has a spectral representation - it's eigenstates form a complete orthonormal basis in $\mathcal{H}$. We can express a self-adjoint operator $\mathbf{A}$ as

$$\mathbf{A} = \sum_n a_n \mathbf{P_n},$$

where a_n is an eigenvalue of $\mathbf{A}$ and $\mathbf{P_n}$ is the corresponding orthogonal projection onto the space of eigenvectors with eigenvalues a_n.

• **spectral theorem:** Every Hertitian matrix is unitarily equivalent to a diagonal one. The spectral theorem is the generalisation of this assertion to operators on Hilbert space.

We first review the spectral theorem for $\mathcal{H}$ finite dimensional:

Suppose that $\mathcal{H}$ is finite dimensional. If $T \in \mathcal{B}(\mathcal{H})$, then $\sigma(T) \neq \emptyset$. Furthermore, $\lambda \in \sigma(T)$ if and only if λ is an eigenvalue of T. Let $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$. Let M_i the corresponding eigenspaces and P_i be the orthogonal projections on these eigenspaces. The spectral theorem then states that the following three conditions are equivalent:

(a) The M_i's are pairwise orthogonal and span $\mathcal{H}$;

(b) The P_i's are pairwise orthogonal

$$I = P_1 + \cdots + P_n.$$

$$T = \lambda_1 P_1 + \cdots + \lambda_n P_n;$$

(c) T is normal.

This decomposition is usually called the Spectral Theorem in finite dimensions.

In general, the classical Spectral Theorem says that each normal $T \in \mathcal{B}(\mathcal{H})$ is associated to a (unique) $\mathcal{H}-$projection valued measure on $\sigma(T)$, so that

$$T = \int_{\sigma(T)} d\lambda P(\lambda).$$

• **spectrum:** Infinite dimensional spaces there are operators that have no eigenvalues. For example, $(x_1, x_2, x_3, \dots)$ the operator that translates the components by one place - $(0, x_1, x_2, \dots)$ has no eigenvectors. The next best thing that is useful... is the spectrum of an operator A. The spectrum is defined as the set of values λ for which the operator $(A - \lambda \hat{I})$ is not invertible.

Let $\mathcal{U}$ be a C^*-algebra. The spectrum $\sigma(\mathcal{U})$, of $\mathcal{U}$ is the set of all characters on $\mathcal{U}$.

Any Abilian C*-algebra with identity is naturally isomorphic with the C*-algebra of all continuous functions on a compact, Hausdorff space, called the spectrum of the algebra.

• **spinor group:** The spinor group Spin(n) is a particular double cover of the rotation group $SO(n)$.

$Sp(2n)$ is generated by $2n \times 2n$ matrices X that satify

$$XG + GX^T = 0, \tag{B.208}$$

where $G = -G^T$ is some non-degenerate antisymmeteric matrix. We can write G, and the generators X, as tensor products 2×2 and $n \times n$ matrices. One takes

$$G = \sigma_2 \otimes \mathbf{1}, \tag{B.209}$$

where $\mathbf{1}$ is the $n \times n$ unit matrix, and σ_2 the second Pauli matrix. Explicitly G is

$$G = \begin{pmatrix} 0 & -i\mathbf{1} \\ i\mathbf{1} & 0 \end{pmatrix}$$

The generators X are Hermitian matrices, and in addition they must satisfy (B.208). With G given by (B.209), the set of all X can be obtained from the following sets of matrices:

$$\mathbf{1} \otimes A, \quad \sigma_1 \otimes S_1, \quad \sigma_2 \otimes S_2, \quad \sigma_3 \otimes S_3.$$

Here A is an arbitrary $n \times n$ imaginary antisymmetric matrices, S_1, S_2 and denote S_3 denote arbitrary $n \times n$ real symmetric matrices. Explicitly one has

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \begin{pmatrix} 0 & S_1 \\ S_1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -iS_2 \\ iS_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} S_3 & 0 \\ 0 & -S_3 \end{pmatrix}.$$

• **stable:** Let G act on a set X. A subset $S \subset X$ is said to be **stable** under the action of G if

$$g \in G \quad x \in S \Rightarrow gx \in S. \tag{B.210}$$

• **stablizer:** The **stabalizer** of an element $x \in X$ is

$$Stab\ (x) = \{g \in G : gx = x\}. \tag{B.211}$$

It is a group - It's obviously associative and closed. $\mathbf{1} \in Stab\ (x)$ as $\mathbf{1}x = x$. If $gx = x$ then $g^{-1}gx = g^{-1}x$ so that $g^{-1}x = x$, hence $g^{-1} \in Stab\ (x)$.

• **states:** A state on a $^*-$algebra is a linear functional $\omega : \mathbb{U} \to \mathbb{C}$ which is positive, that is, $\omega(A^*A) \geq 0$ for all $A \in \mathbb{U}$. The states that physicists are most familiar with are vector states, that is, if we are given a representation $(\mathcal{H}, \pi)$ and an element ψ in the common domain of all the $A \in \mathbb{U}$ then $a \mapsto < \psi, \pi(A)\psi >_{\mathcal{H}}$ evidently defines a state.

• **stratification:** [90] or [102] Let $\mathbb{X}$ be a manifold of differentiability p, and U be a subset of $\mathbb{X}$. Then:

$\mathcal{M}$ is called a **stratification** of $\mathbb{X}$ is a locally finite, disjoint decomposition of $\mathbb{X}$ into connected embedded C^p manifolds $\mathbb{X}_i$ of $\mathbb{X}$, such that:

$$\mathbb{X}_i \cap \mathbb{X} \neq \emptyset \Rightarrow \mathbb{X}_i \subseteq \partial \mathbb{X}_j \text{ and } dim\mathbb{X}_i < dim\mathbb{X}_j. \tag{B.212}$$

for all $\mathbb{X}_i, \mathbb{X}_j \in \mathcal{M}$.

The elements of the decomposition are called **strata**. $\mathcal{M}$ is called a stratification of U, if and only if U is the union of some elements of $\mathcal{M}$.

• **stratified diffeomorphism:** a stratified map f is a stratified diffeommorphism if and only if $f|_{\mathbb{X}_i}$ is a injective and the restriction of each f_i to the restriction of each f_i to the respective U_i is a C^p-diffeomorphism.

• **stratified isomorphism:** A map f is a stratified isomorphism if and only if in addition to being a stratified mononmorphism, f is a homeomorphism and each $f_i : U_i \to f_i(U_i)$ is a C^p diffeomorphism.

• **stratified map:** Let f be a continuous map from C^p-manifold $\mathbb{X}$ to a $\mathbb{Y}$-manifold. The map f is called a stratified map if and only if:

There is a pair of stratifications $\mathcal{M}$,$\mathcal{N}$ of $\mathbb{X}$ respectively $\mathbb{Y}$, and for each stratum $\mathbb{X}_i$ there exists an open neighbourhood U_i and a C^p map $f_i : \mathbb{X}_i \subseteq U_i \to \mathbb{X}$ with:

$$\overline{\mathbb{X}_i} \subseteq U_i, \quad f_i|_{\mathbb{X}_i} = f|_{\mathbb{X}_i}, \quad \mathbb{X}_i \in \mathcal{N} \quad \text{and} \quad rankf|_{\mathbb{X}_i} = dimf(\mathbb{X}_i).$$

• **stratified monomorphism:** A map f is a stratified monomorphism if and only if in addition to being a stratified map, $f|_{\mathbb{X}_i}$ is injective.

• **strongly continuous one-parameter unitary group:** An operator valued function satisfying

(i) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$.

(ii) If $\varphi \in$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$,

is called a strongly continuous one-parameter unitary group.

• **subcover:** See cover.

• **subgroup:** A subgroup of G is a subset $H \subset G$ such that

(a) $E_G \in H$

(b) $xy \in H$ for all $x, y \in H$

(c) $x^{-1} \in H$ for every $x \in H$.

• **submanifold:**

This class of subsets of a manifold should exclude anything with sharp corners, such as the surface of a cube or of a cone, or anything whose dimensioon could be said to vary.

• **subobject:** A subobject is the analogue of the set-theoretical idea of a subset.

• **subobject classifier:** [360] encodes the possible answers to natural multiple choice question one can ask of the 'elements' of which objects in the topos are built.

While the category **Set** has just the two truth-values , $\{0, 1\}$, so also in any topos the collection of truth-values is an object in the topos - the subobject classifier, written Ω.

• **subring:** Let A be a ring. A **subring** of A is a subset containing I that is closed under addition, multiplication, and the formation of negatives.

• **sujective:** A function is surjective if it is onto.

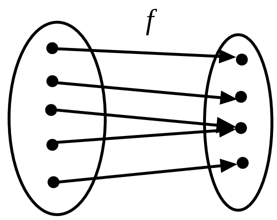


Figure B.30: surjective

See injective, bijective.

• **symmetric operator:** An operator A is called symmetric if its domain of dependence $D(A)$ is contained in the domain of dependence, $D(A^\dagger)$, of its adjoint operator $A^\dagger$, i.e. $D(A) \subset D(A^\dagger)$, and if $A^\dagger = A$ for all $A \in D(A)$.

• **sympletic manifold:** A sympletic manifold (M, ω) is a smooth real $n-$dimensional manifold M without boundary, equipped with a closed non-degenerate two-form ω. The most familiar example of a sympletic manifold is a cotangent bundle $M = T^*Q$. This is nothing but the traditional phase space, Q being the configuration space.

• **tangles:**

• **Temperly-Lieb algebra:**

• **tempered distributions:** The tempered distributions are continuous linear maps from the Schwarz space to complex numbers.

• $10j$ **symbol:** The $10j$ symbol depends on ten spins and appears as the vertex amplitude in the Barrett-Crane model.

• **tensor:** covariance etc.

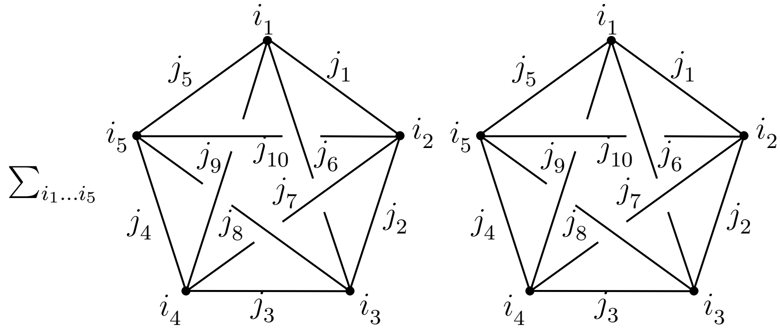


Figure B.31: tenjsymbol. This depends on ten spins and is called the $10j$ symbol.

• **tensor algebra:** It is the direct product $\otimes$ that is the binary operation or product that make the tensor algebra an algebra,

The tensor product of any number of elements can be taken as many times as one likes The tensor algebra $T(\mathcal{L})$ is the collection of (including up to an infinite number) collectively as direct summation:

$$T(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots . \tag{B.213}$$

• **Theta function:**

$$\theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x} \tag{B.214}$$

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right). \tag{B.215}$$

• **three-manifolds:** All three-dimensional topologies can be constructed by starting with a three-sphere S^3, drawing all possible links on it, and performing all possible surgeries on them, (although one may need to perform an infinite number of surgeries to construct some manifolds). The classification of three-manifolds is related to the classification of all links.

Theorem (Lickorish and Wallace): Every closed, orientable, connected three-manifold, M^3 can be obtained by surgery on an unoriented framed knot or link $[L, f]$ in S^3. [hep-th/99071119] topological quantum field theories - a meeting ground for physicists and mathematicians

• **Tomita theorem:** Given any state ρ over a von Neumann algebra, there is always a flow α_t, called the Tomita flow of ρ, such that

$$\rho_0[\alpha_t(A)B] = \rho_0[\alpha_{(-t-i\beta)}(B)A] \tag{B.216}$$

• **topological dual:** See dual spaces.

• **topological vector space:** A topological vector space is a vector space with a topology defined on it such that the operations of addition and scalar multiplication are continuous.

• **topological structure:** In order to have a notion of convergence of points in a space X and a notion of continuity for functions $f : X \to Y$ into some space Y, one has to give X a topological structure.

• **topology:** A very far from being a comprehensive account of the subject of topology.

topology as determined by which functions are continuous. Weakest topology so that a given set of mappings are continuous.

Example: projection mappings $p_\gamma \to$ Hausdorff topology

• **topology of pointwise convergence:** Topology which arises from the seminorm given by

$$\|f\|_x = |f(x)|.$$

The space of functions with this topology is called the space of pointwise convergence.

• **topological space:** A topological space X is a set, with a specified family of subsets...

• **topos:** Very roughly, a topos is a category which behaves much like the category of sets; in fact this category, which is denoted **Set**, is itself a topos.

OR: topos is a categorical model of constructive set theory.

a branch of category theory.

applications to interpretive problems in quantum theory and quantum gravity, see [361].

Non-Boolean logic - that does not assume every statement can be judged true or false, there are statements upon which one cannot decide. Instead a Boolean algebra one has a Heyting algebra. The notion of a topos plays provides a good framework for making sense of the idea of partial truth.

Fotini Markopoulou should be modified to take into account the fact that observers can only give truth values to observables that have to do with their backwards light cone.

topos: To find out more about the subject to John Baez homepage (http://math.ucr.edu/home/baez/README.html) "fun stuff" and then to the link topos.

1. an initial object (an object like the empty set)
2. a terminal object (an an object like a set with one element)
3. binary coproducts (something like the disjoint union of two sets)
4. binary products (something like the Cartesian product of two sets)
5. equalizers (something like the subset of X consisting of all elements x such that $f(x) = g(x)$, where $f, g : X \to Y$)
6. coequalizers (something like the quotient set of X where two elements $f(y)$ and $g(y)$ are identified, where $f, g : Y \to X$)

(3) analogy of $\{0, 1\}$ denoted Ω, intuitively, the elements of Ω are the answers to a natural 'multiple-choice questions' about objects in the objects in the topos, just as "$x \in X$" is a natural for sets.

coming from [361]

See section J.2.

• **total function:** See partial function.

• **totally bounded:** A set X for which all $\epsilon > 0$ there are a finite number of points $x_1, \ldots, x_n$ such that

$$X \subset \bigcup_{i=1}^{n} B_\epsilon(x_i). \tag{B.217}$$

• **trace:** Let $\{e_n\}_{n\in\mathbb{N}}$ be a basis for a Hilbert space $\mathcal{H}$ and let A be an operator on $\mathcal{H}$. The trace $\mathrm{Tr}(A)$ is defined as $\sum_{n=1}^{\infty} < Ae_n, e_n >$, whnever this limit exists.

• **trace class operators:** Let $\mathcal{L}_1$ denote the space of, necessarily compact, operators A such that $\mathrm{Tr}(|A|)$ exists. Then $A \mapsto \mathrm{Tr}(|A|)$ defines a norm on $\mathcal{L}_1$. The space $\mathcal{L}_1$ is called the space of trace class operators. The name trace class comes from its property that if A is trace class, then for any orthononormal basis $\{\varphi_n\}$

$$tr(A) = \sum_n < e_n, Ae_n > \tag{B.218}$$

is finite and independent of the orthononormal basis.

• **transition functions:** Whenever we define an object by use of local coordinates, it must have the same meanig in all coordinate systems. For it to have a basis-free significance, one must require the object's coefficients to have a special transformation property under a chgange of basis, this is acheived by the transition functions. As an example, the components of a vector field in one coordinate system must be related to those in another overlapping coordinate system by the Jacobian matrix which is the corresponding transition function. In practical situations transition functions are the gauge transformations required for pasting local charts together.

• **transitively:** given any two points x, y, in a group G there is at least one $g \in G$ that takes x to y, i.e. $gx = y$.

• **triangulable:** A space X is said to be triangulable if there exists a simplicial complex K_X that is exactly homeomorphic to X.

• **triangulation:** When a space X is triangulable the pair (X, K_X) is called a triangulation of X. The triangulation of a space is not unique.

• **trivialisation:** Say P is a fibre bundle with typical fibre F and base space $\mathcal{M}$. Trivialisations are global maps and only apply to trivial bundles. For each $x \in \mathcal{M}$, a trivialisation is an isomorphism,

$$\phi : P \to \mathcal{M} \times F$$

sending each fibre $\pi^{-1}(x) \in P$ to $\{x\} \times F$.

• **twists and writhe:**

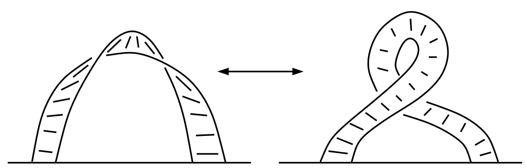


Figure B.32: Twists going to writes.

$$L = Tw + Wr \tag{B.219}$$

• **Tychonov topology:** The Tychonov topology on the direct product $X_\infty = \prod_{l \in \mathcal{L}} X_l$ of topological spaces X_l is the weakest topology such that all projections

$$p_l : X_\infty \to X_i;\ (x_{l'})_{l' \in \mathcal{L}} \to x_l \tag{B.220}$$

are continuous, that is, a net $x^\alpha = (x_l^\alpha)_{l \in \mathcal{L}}$ converges to $x = (x_l)_{l \in \mathcal{L}}$ if and only if $x_l^\alpha \to x_l$ for every $l \in \mathcal{L}$ pointwise (not necessarily uniformally) in $\mathcal{L}$.

• **uniform boundedness theorem:** A fundamental theorem of functional analysis. Let X be a Banach space and Y a normed space. Let $\Phi \subseteq B(X, Y)$ be the set of bounded operators from X to Y which is pointwise bounded, in the sense that, for each $x \in X$ there is some $c \in \mathbb{R}$ so that

$$\|Tx\| \leq c$$

for all $T \in \Phi$. Then Φ is uniformaly bounded: There is some constant C with

$$\|T\| \leq C$$

for all $T \in \Phi$.

• **uniform convergence:** A series of functions $\{f_n(x)\}$ is said to converge uniformally to $f(x)$ if when we put an $\epsilon-$tube around the function $f(x)$, the functions $f_n(x)$ eventually fit inside this.

• **uniform Rovelli-Smolin topology:** The Hamiltonian constraint operator $\hat{S}^\epsilon(N)$ does not converge with respect to the weak operator topology in $\mathcal{H}_{kin}$ when $\epsilon \to 0$. The convergence of $\hat{S}^\epsilon(N)$ holds with respect to the uniform Rovelli-Smolin topology, where one defines $\hat{S}^\epsilon(N)$ to converge if and only if $\Psi_{Diff}[\hat{S}^\epsilon(N)\phi]$ converge for all $\Psi_{Diff} \in Cyl^*_{Diff}$ and $\phi \in Cly(\overline{\mathcal{A}/\mathcal{G}})$

• **uniqueness theorems:**

Some important examples:

Stone-von Neumann theorem

• **unitary transformations:** We must make a distinction between unitary and isometric transformations which do not arise in finite-dimensional vector spaces. An isometry is defined in vector spaces of any dimension as a linear transformation U satisfying

$$(Uf, Ug) = (f, g)$$

for all in the vector space. This implies that $U^\dagger$ is a left inverse of U:

$$U^\dagger U = I.$$

now in finite-dimensional space, the existence of a left inverse guarantees the existence of a right inverse, but on an infinite-dimensional space this is not always the case. If a right inverse also exists, U is said to be unitary. This right inverse must be equal to the left inverse, since if $BA = I = AC$, then $B = C$. Thus for unitary transformations, we write

$$U^\dagger = U^{-1}.$$

• **universal cover:**

• **universal enveloping algebra:**

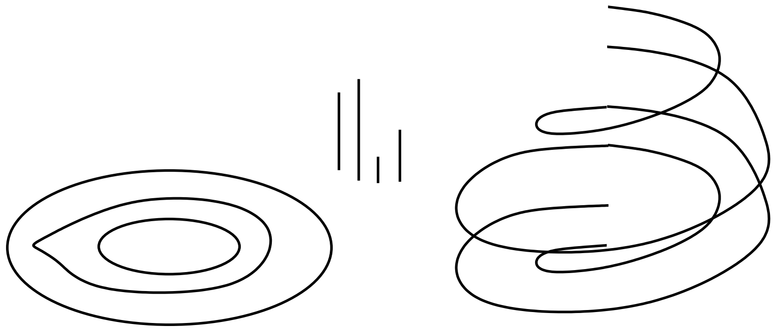


Figure B.33: unicoverEx.

$U_q(\mathbf{g})$ denotes the universal enveloping algebra - i.e., all formal powers and linear combinations of the elements of the deformed algebra modulo the standard Lie algebra relations.

Let g be a Lie algebra.

We know $T(\mathcal{L})$ is an associative algebra.

$$[x, y] - (x \otimes y - y \otimes x) \tag{B.221}$$

this collection forms a two-sided ideal I. $z \in I$

described as “universal” because any Lie algebra homomorphism $\psi\mathbf{g} \rightarrow A$, where A is a *unital* associative algebra, extends uniquely to a unital algebra homomorphism.

$$z(x \otimes y - y \otimes x) \tag{B.222}$$

$$T(\mathcal{L})/I \equiv U(\mathcal{L}) \tag{B.223}$$

$$\begin{aligned} \Delta\xi &= \xi \otimes 1 + 1 \otimes \xi \quad \text{coroduct} \\ \epsilon\xi &= 0 \quad \text{counit} \\ S\xi &= -\xi \quad \text{antipode} \end{aligned} \tag{B.224}$$

It is a **biideal** The quotient B/I is a bialgebra. An element of a general Hopf algebra which has this linear form

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi \tag{B.225}$$

is called **primitive**.

$$\Delta(a) = a \otimes a + b \otimes c$$
$$\Delta(b) = a \otimes b + b \otimes d$$
$$\Delta(c) = c \otimes a + d \otimes c$$
$$\Delta(d) = c \otimes b + d \otimes d \qquad \text{(B.226)}$$

• **universal home:** a certain completion, $\overline{\mathcal{A}/\mathcal{G}}$, of $\mathcal{A}/\mathcal{G}$ the universal home for measures

• **universal net:**

• **upper semi-continuous:** A function $f(x)$ is said to be upper semi-continuous if for any x in the domain of f and for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$f(x) - f(x_0) < \epsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

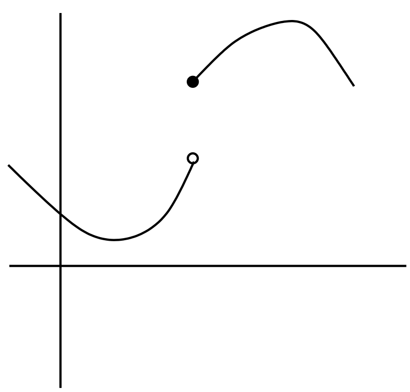


Figure B.34: An upper semi-continuous function.

• **Vassiliev Invariants:**

• **weakly continuous:**

$$(\dot{f}, x) \qquad \text{(B.227)}$$

• **weakness of a topology:**

Roughly, one topology is weaker that another if it has fewer open sets, and stronger than another if it has more open sets. Let X be a non-empty set. $\emptyset, X$ is the weakest topology and the discrete topology is the strongest topology on X. The more open sets there are, the more continuous functions the space has.

• **weak operator topology:** The *weak operator topology* on $\mathcal{B}(H)$ is the weak topology generated by all functions of the form $T \to (Tx, y)$; that is, it is the weakest topology with respect to which all these functions are continuous. It is easy to see from the inequality $|(Tx, y) - (T_0x, y)| \leq \|T - T_0\| \, \|x\| \, \|y\|$ that this topology is weaker than the usual norm topology, so that its closed sets are also closed in the usual sense. A C^*-algebra with the further property of being closed in the weak operator topology is called a W^*-algebra. Algebras of this kind are also called *rings of operators, or von Neumann algebras.*

(from intro topology and modern analysis)

• **weak topology:** The weak topology on X^* is the topology such that all functionals on X^{**} are continuous.

• **weak star operator topology:** The weak $*$topology with respect to a Hilbert space $Y = X$: this is similar to the weak topology, however, insted of $X' = X$ we now take a subspace $\mathcal{D}$ of X equipped with a finer topology and as $\mathcal{D}'$ teh topological dual of that topological space. Physical applications are the topology in which the Hamitonian constraint converges and Refined algebraic quantization (RAQ).

The weak star topology is obtained if we use the absolute values of $| < \Phi|A|\Psi > |$ between arbitrary state vectors as a system of seminorms. Thus a sequence of operators converges weakly if all matrix elements converge.

• **webs:** From [arXiv:math-ph/0304002]: For technical purposes, one assumed in the very beginning that these graphs are formed by piecewise analytic paths only. Namely, only in this case two finite graphs are always both contained in some third, bigger graph being again finite. This restriction has the drawback that only analyticity preserving diffeomorphisms can be implemented into that framework. In order to guarantee the inclusion of all diffeomorphisms, at least, piecewise smooth and immersive paths have to be considered as well. For the first time, this has been done by Baez and Sawin [q-alg/9507023] introducing so-called webs. A web is defined as a piecewise smooth graph determined by the union of a finite number of smooth curves that intersect in a controlled way, albeit possibly a countably infinite number of times.

• **Weierstrass approximation theorem:** Any real-valued continuous function f on $[a, b]$ can be arbitrary well approximated by a finite polynomial: given any $\epsilon > 0$, there is a polynomial P such that $\|f - P\| < \epsilon$:

$$|f(x) - P(x)| < \epsilon, \quad \text{for all } x, \; a \leq x \leq b. \tag{B.228}$$

We can restate the theorem as: polynomials are dense in the space of real-valued continuous functions on $[a, b]$ (dense in the same kind of way that the rationals $\mathbf{Q}$ are dense in $\mathbb{R}$).

• **well-posedness of initial value problem:** An initial value problem compose by a differential equation together with initial conditions on an suitable boundary. Well-posedness of an initial value problem requires

(i) existence of solutions,

(ii) uniqueness of solutions,

(iii) continuous dependence of solutions on initial conditions.

• **Whitehead's theorem:** a smoothing. For each smooth manifold M, there exists a PL-manifold M_{PL}, called its Whitehead triangulation, so that M is diffeomorphic to a smoothing of M_{PL}. [396]

• **Whitehead's triagulation:** Whitehead's triagulation provide us with a way of "discretizing" spacetime which is not merely some approximation nor introduces a physical cut-off, but [396]

• **Wigner's $6-j$ symbol:**

$$W(j) = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \tag{B.229}$$

• **Wigner transform:**

• **Von-Neumann algebra:** A von Neumann algebra is a *-algebra of bounded operators on a Hilbert space which is closed in the weak operator topology, or more explicitly:

A bounded operator B is a **weak limit** of a set of bounded operators if for each choice of positive number ϵ, positive integer n, and vectors $\psi_1, \psi_2, \ldots \psi_n$ and $\phi_1, \phi_2, \ldots \phi_n$ there is an operator A in the set such that

$$|(\phi_k, A\psi_k) - (\phi_k, B\psi_k)| < \epsilon \tag{B.230}$$

for $k = 1, 2, \ldots, n$. The extension of a set of bounded operators to its weak limits is its **weak closure**. A set of bounded operators is **weakly closed** if it contains its weak limits. A symmetric ring of bounded operators which is weakly closed is a von Neumann algebra.

Applications are:

(i) in Algebraic Quantum Field theory. In algebraic Quantum Field theory associates a von Neumann algebra to each causally complete region of spacetime.

(ii) Tomita-Takesaki theorem - thermal states and thermal time hypothesis.

(iii) Quantum causal histories

(iv) Noiseless subsystems

(v) Infinite tensor product Hilbert spaces.

• **Voronoi Diagram:**

http://www.lepp.cornell.edu/spr/2001-06/msg0033516.html

• **Zorn's lemma** Let $X \neq \emptyset$ be a partially ordered set with the property that every linearly ordered subset $Y \subset X$ (i.e., $y \leq y'$ or $y' \leq y$ for all $y, y' \in Y$) has an upper bound $x_Y \in X$ (i.e., $y \leq x_Y$ for all $y \in Y$). Then X has a maximal element $m \in X$ (i.e., $m \leq x$ for $x \in X$ implies $x = m$) which is a common upper bound for all linearly ordered subsets.