Appendix C

Algebraic Quantum Gravity

C.1 Introduction

The natural Hilbert space representaion aquires the structure of an infinite tensor product (ITP) whose separable strong equivalence class Hilbert subspaces (sectors) are left invraint by the quantum dynamics.

M preserves all the strong equivalence class Hilbert spaces. This follows from the fact that **M** is a countable sum of operators each of which changes only a finite number of entries in a vector of the form $\otimes_f$, hence we get a countable sum of vectors in the same equivalence class (see lemma), which remains normalizable if $\otimes_f$ is in the domain of **M**.

and $t_e := \ell_p^2/a_e^2$ is the classicallity parameter.

C.2 The Infinite Tensor Product

We first consider the tensor product of a finite number of Hilbrt spaces. Say $f_k, g_k \in \mathcal{H}_k$ with inner product $(\cdot,\cdot)_k$ on $\mathcal{H}_k$. If an element of $\otimes_k \mathcal{H}_k$ is f, the inner product of the tensor product is defined as

$$(f, g) = \prod_{k=1}^{n} (f_k, g_k)_k$$

and the norm

$$\|f\| = \sqrt{\prod_{k=1}^{n} (f_k, f_k)_k} = \sqrt{\prod_{k=1}^{n} \|f_k\|_k^2} = \prod_{k=1}^{n} \|f_k\|_k.$$

We are lead to the consideration of the mathematics of products of arbitrary complex numbers.

Now when one forms the infinite tensor product of a collection of Hilbert spaces, a physical requirement is that this product must not depened on the order of the individual Hilbert spaces (whether the collection is countably or uncountably infinite).

Hence we are interested in the convergence properties of a countable or uncountable product of complex numbers which are independent of the ordering of the product. This was developed in the paper by von Neumann (availble at http://www.numdam.org/item?id=1939 0).

As we will see, convergence of products is related to convergence of corresponding summations. Now, it is a remarkable fact that whether an infinite series converges or not can depend on the ordering of the terms of that series. From which it follows that whether or not the coresponding product of complex numbers converges depends on the ordering of the terms of that product.

Absolutely and conditionally convergent series have completely different behaviours under rearrangement

Theorem C.2.1 *(***Riemann's Rearrangement Theorem***) Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. For each real number s, there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges and has sum s.*

Proof:

The nonnegative series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty}(-q_n)$ diverge. In fact, if both were to converge, it would follow that $\sum_{n=1}^{\infty} |a_n|$ converges, that is, $\sum_{n=1}^{\infty} a_n$ would be absolutely convergent. On the other hand, if one of these series converged and the other diverged, it would follow that the partial sums of $\sum_{n=1}^{\infty} a_n$ diverge to either $+\infty$ or to $-\infty$. The convergence of $\sum_{n=1}^{\infty} a_n$ itself implies that both $\{p_n\}$ and $\{q_n\}$ have limit zero.

Now we construct the rearrangement. Choose terms $p_1, p_2, \ldots$ up to the first index k_1 such that

$$p_1 + p_2 + \cdots + p_{k_1} > s.$$

This will occur , because $\sum_{n=1}^{\infty}$. Note

$$|p_1 + p_2 + \cdots + p_{k_1} - s| < p_{k_1}.$$

Next, we choose $q_1, q_2, \ldots$ up to the first index l_1

such that

$$(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) < s.$$

Note

$$|(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) - s| < \max\{p_{k_1}, |q_{l_1}|\}.$$

We then add just enough new p's to make the left hand side greater than s, followed by just enough q's to make it less than s, and continue. At each phase of the $2n-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_n}, |q_{l_n}|\}$, and at each phase of the $2n+1-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_{n+1}}, |q_{l_n}|\}$. As these have the limit 0, the rearranged series has sum s.

□

Corollary C.2.2 *Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. It has a rearrangement that diverges to $+\infty$.*

Proof:

□

Theorem C.2.3 *Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement. Then $\sum_{n=1}^{\infty} b_n$ converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

Proof: Let $\{s_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$, i.e,

$$s_n = \sum_{k=1}^{n} a_k,$$

and let s be the limit. Let $\{t_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$. Given $\epsilon > 0$, choose M so large that

$$\sum_{k=M+1}^{\infty} |a_k| < \epsilon/2. \tag{C.1}$$

It follows from this that $|s - s_M| < \epsilon/2$,

$$|s - s_M| = |\sum_{k=M+1}^{\infty} a_k| \leq \sum_{k=M+1}^{\infty} |a_k| < \epsilon/2.$$

Choose N so large that every one of the first M terms of $\{a_k\}$ occurs among the first N terms of $\{b_k\}$. So that for any $n \geq N$

$$|t_n - s_M| = |t_n - \sum_{k=1}^{M} a_k| \leq \sum_{k=M+1}^{\infty} |a_k|.$$

Therefore

$$|t_n - s| \leq |t_n - s_M| + |s_M - s| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

Theorem C.2.4 *If $b_n \geq 0$ for all n, then $\sum_{n=1}^{\infty} b_n$ converges if and only if the sequence of partial sums is a bounded sequence.it is bounded*

Proof:

□

C.2.1 Infinite Products of Complex Numbers

A sequence of numbers $a_1, a_2, a_3, \ldots$ the infinite product

$$\prod_{n=1}^{\infty} a_n = a_1 \; a_2 \; a_3 \cdots$$

is defined to be the limit of the partial products $a_1 a_2 \ldots a_n$ as $n \to \infty$. That is, let P_n be the partial product

$$P_n = \prod_{k=1}^{n} a_k$$

then

$$\prod_{n=1}^{\infty} a_n := \lim_{n\to\infty} P_n.$$

The product is said to converge when the limit exists. If the product converges, then the limit of the sequence a_n as $n \to \infty$ must be 1. Proof is easy. Assume that an infinite product $\prod a_n$ is convergent.

of this infinite product. Then for each a_n of the series we have $a_n = P_n/P_{n-1}$. Since the product is convergent, there exists a such that as $n \to \infty$, $P_n = a$ and $P_{n-1} = a$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} P_n/P_{n-1} = a/a = 1.$$

The contrary to this is in general not true. Therefore, the logarithm a_n will be defined for all but a finite number of n, and for those we have ($a_n \neq 0$)

$$\ln \prod_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \ln a_n$$

with the product on the left converging if and only if the sum on the right converges. This allows the translation of convergence criteria for infinite sums into convergence criteria for infinite products.

Lemma C.2.5 *By the defintion of convergence , if $\prod_n^{\infty} a_n$, $\prod_n^{\infty} a'_n$ converge to a, a' respectively then $\prod_n^{\infty} a_n a'_n$ converges to aa'.*

Proof: Note that if $\{P_n\}$ and $\{P'_n\}$ are converge sequences, converging to P, P' respectively, then

$$\lim_{n\to\infty} (P_n + P'_n) = \lim_{n\to\infty} P_n + \lim_{n\to\infty} P'_n$$

as

$$\begin{aligned} |(P_n + P'_n) - (P + P')| &= |(P_n - P) + (P'_n - P')| \\ &\leq |(P_n - P)| + |(P'_n - P')|. \end{aligned}$$

Therefore, with $P_n = \sum_k^n \ln a_k$, $P'_n = \sum_k^n \ln a'_k$, we have

$$
\begin{aligned}
\ln \prod_n^{\infty} (a_n a'_n) &= \sum_n^{\infty} \ln(a_n a'_n) \\
&= \sum_n^{\infty} (\ln a_n + \ln a'_n) \\
&= \sum_n^{\infty} \ln a_n + \sum_n^{\infty} \ln a'_n \\
&= \ln a + \ln a' \\
&= \ln(ab).
\end{aligned}
$$

□

A criterion for a product to converge.

Theorem C.2.6 *For $\rho_k > 0$, if $\prod_{n=1}^{\infty} |\rho_n - 1|$ converges then $\prod_{n=1}^{\infty} \rho_n$ converges.*

Proof: Set

$$
P_n = \prod_{k=1}^{n} (1 + a_k), \qquad \tilde{P}_n = \prod_{k=1}^{n} (1 + |a_k|)
$$

Then

$$
\tilde{P}_n \leq \exp(|a_1| + \ldots |a_n|)
$$

If $P_n - 1 < 0$ then, expanding the product, we see

$$
-(\tilde{P}_n - 1) < P_n - 1
$$

and if $P_n - 1 \geq 0$ then

$$
P_n - 1 \leq (\tilde{P}_n - 1),
$$

therefore

$$
|P_n - 1| \leq \tilde{P}_n - 1
$$

Hence if the product

$$\prod_{n=1}^{\infty}(1+|a_n|)$$

converges, then so does

$$\prod_{n=1}^{\infty}(1+a_n).$$

If $\sum_{n=1}^{\infty}|a_n|<\infty$ then

$$\prod_{n=1}^{\infty}(1+a_n)$$

converges. Putting $\rho_n = 1+a_n$ $(\rho_n > 0)$ we have $\sum_{n=1}^{\infty}|\rho_n - 1|$ converges then $\prod_{n=1}^{\infty}\rho_n$ converges.

$\square$

In fact the implication can be reversed. To prove this we need to first establish a couple of lemmas.

Lemma C.2.7 *For $r_n > 0$, $\sum_{n=0}^{\infty} r_n$ converges if and only if $\prod_{n=0}^{\infty}(1+r_n)$ converges.*

Proof:

$\square$

Lemma C.2.8 *If $0 < r_n < 1$, then $\sum_{n=0}^{\infty} r_n$ converges if and only if $\prod_{n=0}^{\infty}(1-r_n)$ converges.*

Proof: Note the product obviously is finite, it only diverges if it equals zero. First say the product $\prod_{n}^{\infty} r_n$ converges. As $0 < x <$ then

$$\ln(1-x) = -x - x^2/2 - \cdots < -x,$$

so

$$x < -\ln(1-x).$$

This implies

$$\sum_n^\infty r_n < -\sum_n^\infty \ln(1-r_n) = \ln\left(1/\prod_{n=0}^\infty (1-r_n)\right) < \infty,$$

i.e., the summation converges.

Now say the summation $\sum_{n=0}^\infty r_n$ converges. Then there exists finite N such that $r_n < \delta$ for all $n \geq N$. Since $\ln(1-x)$ is convex, it follows that for $x < \delta$

$$\ln(1-x) > -kx$$

where $k = -(\ln(1-\delta))/\delta$. To see this more clearly draw the function $\ln(1-x)$. Replacing x by r_n, summing over n and then exponentiating gives

$$\prod_{n=N}^\infty (1-r_n) > \exp(-k\sum_{n=N}^\infty r_n) > 0,$$

i.e., the product is non-zero, and so by definition convergent.

$\square$

Theorem C.2.9 *For $\rho_n > 0$, $\sum_{n=0}^\infty |\rho_n - 1|$ converges if and only if $\prod_{n=0}^\infty \rho_n$ converges.*

Proof: Consider the product

$$\prod_{n=0}^\infty \rho_n = \prod_{n=0}^\infty [(1 + (\rho_n - 1)]$$

for $\rho_n > 1$. It follows from lemma (C.2.7) that $\sum_{n=0}^\infty |\rho_n - 1|$ converges if and only if $\prod_{n=0}^\infty (1 + |\rho_n - 1|)$ converges. This proves the theorem for the case of $\rho_n > 1$ for all n.

Consider the product

$$\prod_{n=0}^\infty \rho_n = \prod_{n=0}^\infty [(1 - (1 - \rho_n)]$$

for $0 < \rho < 1$. It follows from lemma (C.2.8), by replacing r_n by $1 - \rho_n$, the theorem is proved for the case of $\rho_n < 1$ for all n.

Now we consider the general case. We factor the partial product

$$P_N = \prod_{n=1}^{N} \rho_n$$

into the product of terms for which $0 < \rho < 1$ and into the product of terms for which $\rho > 1$, which we denote respectively as Q_N and R_N (obviously we can ignore terms for which $\rho_n = 1$). Then

$$P_N = Q_N \cdot R_N.$$

We then use the fact that if $\prod_n^\infty \rho_n$, $\prod_n^\infty \rho'_n$ converge to ρ, ρ' respectively then $\prod_n^\infty \rho_n\, \rho'_n$ converges to $\rho\rho'$ to complete the proof of the theorem.

□

Now we consider products of complex numbers.

Definition The infinite product of complex numbers

$$\prod_{n=0}^{\infty} z_n \tag{C.2}$$

is said to **converge** to the number z provided that for each positive number $\delta > 0$ there exists $N < \infty$ such that for all $n \geq N$

$$|z - \prod_{k=0}^{n} z_k| < \delta.$$

□

Lemma C.2.10 *If z_n are arbitrary complex numbers, then $\sum_n^\infty z_n$ converges if and only if $\sum_n^\infty |z_n|$ converges.*

Proof: Convegrence of $\sum_n^\infty z_n$ is equivalent to the convegrence of both $\sum_n^\infty \mathrm{Re} z_n$ and $\sum_n^\infty \mathrm{Im} z_n$. Similarly, as

$$|\mathrm{Re} z_n|,\ |\mathrm{Im} z_n| \leq |z_n| \leq |\mathrm{Re} z_n| + |\mathrm{Im} z_n|.$$

the convergence of $\sum_n^\infty |z_n|$ is equivalent to the convegrence of both $\sum_n^\infty |\mathrm{Re} z_n|$ and $\sum_n^\infty |\mathrm{Im} z_n|$.

Sufficiency. As we have for real series

$$\left| \sum_k^n a_k \right| \leq \sum_k^n |a_k|,$$

$|\sum_k^n a_k|$ will be bounded as $\sum_k^n |a_k|$ is bounded by hypothesis, this proves sufficiency.

Now we prove necessity. Suppose $\sum_n^\infty z_n$ converges, and let a be its value.

$\square$

in general it is not guatanteed we can write

$$\lim_{n\to\infty} (\prod_k^n |z_k| \, e^{i\varphi_k}) = \lim_{n\to\infty} \prod_k^n |z_k| \, \lim_{n\to\infty} \prod_k^n e^{i\varphi_k}$$

However we have: if $\{\xi_n\}$ and $\{\zeta_n\}$ are two complex convergent sequences with limits a and b, then

$$\lim_{n\to\infty} (\xi_n \zeta_n) = \xi\zeta$$

Proof:

$$\begin{aligned} |\xi_n\zeta_n - \xi\zeta| &= |(\xi_n\zeta_n - \xi\zeta_n) + (\xi\zeta_n - \xi\zeta)| \\ &\leq |(\xi_n\zeta_n - \xi\zeta_n)| + |(\xi\zeta_n - \xi\zeta)| \\ &= |\xi_n - \xi| \, |\zeta_n| + |\xi| \, |\zeta_n - \zeta| \end{aligned}$$

For $n \geq$ to some N, $|\zeta_n| = |\zeta + (\zeta_n - \zeta)| \leq |\zeta| + |(\zeta_n - \zeta)| \leq |\zeta| + 1$. Given $\epsilon > 0$ we can choose N so large that $n \geq N$ implies $|\zeta_n| \leq |\zeta| + 1$ and also

$$|\xi_n - \xi| < \frac{\epsilon}{2|\zeta| + 2}, \qquad |\zeta_n - \zeta| < \frac{\epsilon}{2|\xi|}.$$

It follows that $n \geq N$ implies $|\xi_n\zeta_n - \xi\zeta| < \epsilon$.

Theorem C.2.11 *Let $z_n = \rho_n e^{i\varphi_n} \in \mathbb{C}$ where $\rho_n = |z_n|$, $\varphi_n \in [-\pi, \pi]$. Then $\prod_\alpha z_n$ converges if and only if*

i) either $\prod_n \rho_n$ converges to zero in which case $\prod_n z_n = 0$, prod ii) or $\prod_n \rho_n$ converges to $\rho > 0$ and $\sum_n |\varphi_n|$ converges in which case

$$\prod_n z_n = \rho e^{i \sum_n \varphi_n}.$$

Proof: the partial product can be written

$$P_n = \prod_{k=1}^{n} a_k = (\prod_{k=1}^{n} \rho_k)\ e^{i \sum_{k=1}^{n} \varphi_k}$$

$$\lim_{n\to\infty} e^{i \sum_{k=1}^{n} \varphi_k}$$

$$|z - e^{i \sum_{k=1}^{n} \varphi_k}| < \delta.$$

We require

$$\lim_{n\to\infty} P_n/P_{n-1} = 1$$

that is

$$\varphi_n \to 0$$

We show that when $\prod_n^\infty |z_n| > 0$, then it implies the convergence of $\sum_n^\infty |\varphi_n|$ also.

As $\prod_n^\infty z_n$, $\prod_n^\infty |z_n|$ converge, with the latter > 0

$$\prod_n^\infty \frac{z_n}{|z_n|} = \prod_n^\infty e^{i\varphi_n}$$

converges also.

□

Corollary C.2.12 *Quasi-convergence of $\prod_n z_n$ to a non-vanishing value $\neq 0$ is equivalent with convergence with such a value (this is by definition of quasi-convergence). It holds if and only if all $z_n \neq 0$, and $\sum_n^\infty |z_n - 1|$*

Proof: As it is quasi-convergent we have that $\sum_n^\infty \big|\, |z_n| - 1 \big|$

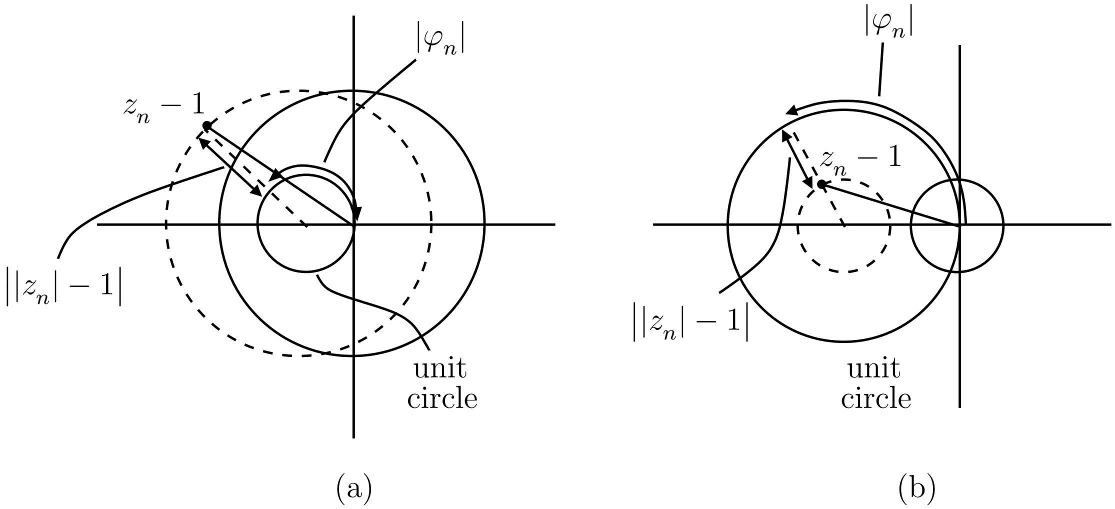

Figure C.1: Proof of the inequalities $\big||z_n| - 1\big| \leq |z_n - 1| \leq \big||z_n| - 1\big| + |\varphi_n|$. (a) Case where $|z_n| > 1$. (b) Case where $|z_n| < 1$.

$$\big||z_n| - 1\big|,\ \frac{1}{\pi}|\varphi_n| \leq |z_n - 1| \leq \big||z_n| - 1\big| + |\varphi_n|$$

Hence the convergence of the sum $\sum_n^\infty |z_n - 1|$ is equivalent to the convergence of both $\sum_n^\infty \big||z_n| - 1\big|$ and $\sum_n^\infty |\varphi_n|$.

Lemma C.2.13 *By the defintion of convergence, if*

1) $\prod_n^\infty z_n$, $\prod_n^\infty z'_n$ *converge to* z, z' *respectively then* $\prod_n z_n z'_n$ *converges to* zz'*,*

2) if $\prod_n^\infty z_n$ *converges to* z*, then* $\prod_n^\infty z_n^*$ *converges to* z^**.*

Proof: 1)

$$\begin{aligned}
\ln \prod_n^\infty (z_n z'_n) &= \sum_n^\infty (\ln \rho_n + \ln \rho'_n + i\varphi_n + i\varphi'_n) \\
&= \sum_n^\infty (\ln \rho_n + i\varphi_n) + \sum_n^\infty (\ln \rho'_n + i\varphi'_n) \\
&= \ln \prod_n^\infty z_n + \ln \prod_n^\infty z'_n \\
&= \ln(zz')
\end{aligned}$$

$\square$

Lemma C.2.14 *For any complex numbers, the convergence of one of* $\sum_{n=0}^{\infty} |\, |z_n| - 1|$, $\sum_{n=0}^{\infty} |\, |z_n|^2 - 1|$ *implies the convergence of the other.*

Proof: If one or other of the series converges, then $\lim_{n\to\infty} |z_n| = 1$. So

$$\begin{aligned} \lim_{n\to\infty} \frac{|\, |z_n|^2 - 1|}{|\, |z_n| - 1|} &= \lim_{n\to\infty} |\, |z_n| + 1| \\ &= 2 > 0. \end{aligned}$$

Hence there exists finite N such that

$$1 < \frac{|\, |z_n|^2 - 1|}{|\, |z_n| - 1|} < 3 \qquad \text{for all } n \geq N.$$

Equivalently

$$|\, |z_n| - 1| < |\, |z_n|^2 - 1| < 3|\, |z_n| - 1| \qquad \text{for all } n \geq N.$$

If the series $\sum_{n=0}^{\infty} |\, |z_n|^2 - 1|$ converges then $\sum_{n=0}^{\infty} |\, |z_n| - 1|$ also converges as

$$\sum_{n=N}^{\infty} |\, |z_n| - 1| < \sum_{n=N}^{\infty} |\, |z_n|^2 - 1|.$$

If the series $\sum_{n=0}^{\infty} |\, |z_n| - 1|$ converges then $\sum_{n=0}^{\infty} |\, |z_n|^2 - 1|$ also converges as

$$\sum_{n=N}^{\infty} |\, |z_n|^2 - 1| < \sum_{n=N}^{\infty} 3|\, |z_n| - 1|.$$

The result also follows by applying theorem C.2.9: $\sum_{n=0}^{\infty} |\, |z_n| - 1|$ converges if and only if $\prod_{n=0}^{\infty} |z_n|$ converges, which converges if and only if $\prod_{n=0}^{\infty} |z_n|^2$ converges, which in turn converges if and only if $\sum_{n=0}^{\infty} |\, |z_n|^2 - 1|$ converges.

□

It is understood that the sum is meaniful only if at most a countable number of terms are different from zero.

Recall that for a series $\sum_{\alpha} z_{\alpha}$ to converge absolutely it is necessary that $z_{\alpha} = 0$ for all but countably infinitely many $\alpha \in \mathcal{I}$.

Definition Let $\{z_\alpha\}_{\alpha\in\mathcal{I}}$ be a collection of complex numbers. The infinite product

$$\prod_{\alpha\in\mathcal{I}} z_\alpha \tag{C.3}$$

is said to **converge** to the number z provided that for each positive number $\delta > 0$ there exists a finite set

$$\mathcal{I}_0 \subset \mathcal{I}$$

such that for any other finite $\mathcal{J}$ with

$$\mathcal{I}_0(\delta) \subset \mathcal{J} \subset \mathcal{I}$$

it holds that

$$|z - \prod_{\alpha\in\mathcal{J}} z_\alpha| < \delta.$$

□

In contrast to the case of an infinite series, absolute converge of an infinite product does not imply converge, the phases of the factors could fluctuate wildly. This motivates the following definition.

Definition We say that $\prod_{\alpha\in\mathcal{I}} z_\alpha$ is **quasi-convergent** if $\prod_{\alpha\in\mathcal{I}} |z_\alpha|$ converges. If $\prod_{\alpha\in\mathcal{I}} z_\alpha$ is quasi-convergent but not convergent we define $\prod_{\alpha\in\mathcal{I}} z_\alpha := 0$.

□

This definition assigns a value to the infinite product of numbers which converge absolutely but not necessarily non-absolutely.

Theorem C.2.15 *1) Let $\rho_\alpha \geq 0$.*

i) If for every $\alpha_0 \in \mathcal{I}$ it holds that $\rho_{\alpha_0} = 0$ then $\prod_\alpha \rho_\alpha = 0$.

ii) If for $\rho_\alpha > 0$ for all α then $\prod_\alpha \rho_\alpha$ converges if and only if

$$\sum_\alpha \max(\rho_\alpha - 1, 0)$$

converges.

iii) If for $\rho_\alpha > 0$ *for all* α *then* $\prod_\alpha \rho_\alpha$ *converges if and only if*

$$\sum_\alpha |\rho_\alpha - 1|$$

converges.

2) Let $z_\alpha = \rho_\alpha e^{i\varphi_\alpha} \in \mathbb{C}$ *where* $\rho_\alpha = |z_\alpha|$, $\varphi_\alpha \in [-\pi, \pi]$. *Then* $\prod_\alpha z_\alpha$ *converges if and only if*

i) either $\prod_\alpha \rho_\alpha$ *converges to zero in which case* $\prod_\alpha z_\alpha = 0$,

ii) or $\prod_\alpha \rho_\alpha$ *converges to* $\rho > 0$ *and* $\sum_\alpha |\varphi_\alpha|$ *converges in which case*

$$\prod_\alpha z_\alpha = \rho e^{i \sum_\alpha \varphi_\alpha}.$$

Proof:

□

Corollary C.2.16 *Quasi-convergence*

Proof:

□

After having introduced convergence for infinite products of complex numbers we can now turn to ITP Hilbert spaces.

C.2.2 $C-$vectors of the ITP

Definition Let $\mathcal{H}_\alpha$, $\alpha \in \mathcal{I}$ be an arbitrary collection of Hilbert spaces. For a sequence $f := \{f_\alpha\}_{\alpha\in\mathcal{I}}$, $f_\alpha \in \mathcal{H}_\alpha$ the object

$$\otimes_f := \otimes_\alpha f_\alpha \tag{C.4}$$

is called a $C-$**vector** provided that $\prod_\alpha \|f_\alpha\|_\alpha$ converges, where $\|\cdot\|_\alpha$ denotes the Hilbert norm of $\mathcal{H}_\alpha$. The set of vectors will be denoted V_C.

$\square$

Lemma C.2.17 *For two* $C-$*vectors* $\otimes_f = \otimes_\alpha f_\alpha$, $\otimes_g = \otimes_\alpha g_\alpha$ *the inner product*

$$< \otimes_f, \otimes_g > := \prod_\alpha < f_\alpha, g_\alpha >_\alpha \tag{C.5}$$

is a quasi-convergent product of the individual inner products $< f_\alpha, g_\alpha >$ *on* $\mathcal{H}_\alpha$.

Proof: We prove that if $f_\alpha, g_\alpha \in \mathcal{H}_\alpha$ for all $\alpha \in \mathcal{I}$, and if $\prod_\alpha \|f_\alpha\|_\alpha$, $\prod_\alpha \|g_\alpha\|_\alpha$ are convergent, then so is $\prod_a |(f_\alpha, g_\alpha)_\alpha|$, that is, $\prod_a (f_\alpha, g_\alpha)_\alpha$ is quasi-convergent.

Since $\|f_\alpha\|_\alpha = 0$ or $\|g_\alpha\|_\alpha = 0$ implies $(f_\alpha, g_\alpha)_\alpha = 0$

by theorem C.2.15 we need only show that $\sum_\alpha \max(|(f_\alpha, g_\alpha)_\alpha| - 1, 0)$ converges as a consequence of the convergence of $\sum_\alpha \max(\|f_\alpha\|_\alpha^2 - 1, 0)$, $\sum_\alpha \max(\|g_\alpha\|_\alpha^2 - 1, 0)$.

Now as $|(f_\alpha, g_\alpha)_\alpha| \leq \frac{1}{2}\|f_\alpha\|_\alpha^2 + \frac{1}{2}\|g_\alpha\|_\alpha^2$,

$$|(f_\alpha, g_\alpha)_\alpha| - 1 \leq \frac{1}{2}(\|f_\alpha\|_\alpha^2 - 1) + \frac{1}{2}(\|g_\alpha\|_\alpha^2 - 1)$$

and hence

$$\max(|(f_\alpha, g_\alpha)_\alpha| - 1, 0) \leq \frac{1}{2}\max(\|f_\alpha\|_\alpha^2 - 1, 0) + \frac{1}{2}\max(\|g_\alpha\|_\alpha^2 - 1, 0).$$

$\square$

There are $C-$vectors $\otimes_f$ such that $\prod_\alpha \|f_\alpha\|_\alpha = 0$ although $\|f_\alpha\|_\alpha > 0$ for all $\alpha \in \mathcal{I}$. Thus, it is conceivable that it happens that $< \otimes_f, \otimes_g > \neq 0$ for some $C-$vector $\otimes_g$. If that were the case, the Schwarz inequality is violated for the inner product (C.5) for $C-$vectors.

Definition We say that for $f_\alpha \in \mathcal{H}_\alpha$ the ITP $\otimes_f := \otimes_\alpha f_\alpha$ is a C_0-**vector** (and $f = \{f_\alpha\}$ a C_0 sequence) if $\| \otimes_f \| := \prod_{\alpha \in \mathcal{I}} \|f_\alpha\|_\alpha$ converges to a non-vanishing number. The set of C_0-vectors will be denoted V_0.

$\square$

To distinguish trivial $C-$vectors from non-trivial ones we define

Definition A sequence $\{f_\alpha\}$ defines a C_0-**vector** $\otimes_f = \otimes_\alpha f_\alpha$ if and only if

$$\sum_\alpha | \ \|f_\alpha\|_\alpha - 1| \tag{C.6}$$

converges. The set of C_0-vectors will be denoted V_0.

$\square$

Lemma C.2.18 *For any complex numbers,* $\sum_\alpha | \ |z_\alpha| - 1|$ *converges if and only if* $\sum_\alpha | \ |z_\alpha|^2 - 1|$.

Proof:

$\square$

For a C_0-vector $\|f_\alpha\|_\alpha \neq 0$.

The norm of a C_0-vector does not zanish. By the previous lemma C_0-vector can be defined equivalently as a $\otimes_f = \otimes_\alpha f_\alpha$ such that

$$\sum_\alpha | \ \|f_\alpha\|^2_\alpha - 1|$$

converges. By theorem C.2.15 this implies that

$$\prod_\alpha \|f_\alpha\|^2_\alpha = \ (\otimes_f, \otimes_f) \ = \| \otimes_f \|^2$$

converges to a non-zero number, i.e. $\| \otimes_f \| > 0$.

C.2.3 Strong Equivalence Classes

Definition We will call two C_0-sequences $f = \{f_\alpha\}, g = \{g_\alpha\}$ **strongly equivalent**, denoted $f \approx g$, provided that

$$\sum_\alpha |(f_\alpha, g_\alpha)_\alpha - 1| \tag{C.7}$$

converges.

$\square$

Lemma C.2.19 *Strong equivalence of C_0-sequences is an equivalence relation, i.e., a relation such that*

1) (Reflexivity) for any $\otimes_f \in V_0$, we have $\otimes_f \approx \otimes_f$.

2) (Symmetry) for all $\otimes_f, \otimes_g \in V_0$, if $\otimes_f \approx \otimes_g$ then $\otimes_g \approx \otimes_f$.

3) (Transitivity) for all $\otimes_f, \otimes_g, \otimes_h \in V_0$, if $\otimes_f \approx \otimes_g$ and $\otimes_g \approx \otimes_h$ then $\otimes_g \approx \otimes_h$.

Proof: Condition 1) follows from the fact that a C_0-vector is defined by the condition that $\sum_\alpha | \, \|f_\alpha\|_\alpha^2 - 1|$ converges.

$$|(f_\alpha, g_\alpha)_\alpha - 1| = |\overline{(f_\alpha, g_\alpha)_\alpha} - 1| = |\overline{(f_\alpha, g_\alpha)_\alpha - 1}| = |(g_\alpha, f_\alpha)_\alpha - 1|.$$

Condition 3) We prove that, apart from a possible finite number of exceptions, the following

$$|(f_\alpha, h_\alpha)_\alpha - 1| \leq D\left(\left|\|f_\alpha\|_\alpha - 1\right| + \left|\|g_\alpha\|_\alpha - 1\right| + |(f_\alpha, g_\alpha)_\alpha - 1| + |(g_\alpha, h_\alpha)_\alpha - 1|\right)$$

holds for some constant D.

We put

$$\begin{gathered} \|f_\alpha\|_\alpha = 1 + \lambda_f, \quad \|g_\alpha\|_\alpha = 1 + \lambda_g, \quad \|h_\alpha\|_\alpha = 1 + \lambda_h, \\ (f_\alpha, g_\alpha) = 1 + \lambda_{fg}, \quad (g_\alpha, h_\alpha) = 1 + \lambda_{gh} \end{gathered}$$

So $|\lambda_f|$, $|\lambda_g|$, $|\lambda_h|$, $|\lambda_{fg}|$, $|\lambda_{gh}| \leq C$, and except for a finite number of α's, $|\lambda_g| \leq \frac{1}{2}$ (or $-1/2 \leq \|g_\alpha - 1\|_\alpha \leq 1/2$).

We express $g_\alpha, f_\alpha, h_\alpha \in \mathcal{H}_\alpha$ as:

$$\begin{aligned} g_\alpha &= a_{11} x_\alpha, \\ f_\alpha &= a_{21} x_\alpha + a_{22} x'_\alpha, \\ h_\alpha &= a_{31} x_\alpha + a_{32} x''_\alpha + a_{33} x'''_\alpha, \end{aligned}$$

where

$$\|x_\alpha\|_\alpha = \|x'_\alpha\|_\alpha = \|x''_\alpha\|_\alpha = 1, \qquad (x_\alpha, x'_\alpha) = (x_\alpha, x''_\alpha) = (x'_\alpha, x''_\alpha) = 0.$$

Then

$$\begin{array}{rcl} \|g_\alpha\|_\alpha^2 &=& |a_{11}|^2, \\ \|f_\alpha\|_\alpha^2 &=& |a_{21}|^2 + |a_{22}|^2, \\ \|h_\alpha\|_\alpha^2 &=& |a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2, \\ \\ (f_\alpha, g_\alpha)_\alpha &=& a_{21}\overline{a_{11}} \\ (g_\alpha, h_\alpha)_\alpha &=& a_{11}\overline{a_{31}} \\ (f_\alpha, h_\alpha)_\alpha &=& a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} \end{array}$$

Now

$$\begin{array}{rcl} |(f_\alpha, g_\alpha)_\alpha - 1| &=& |a_{21}\overline{a_{31}} + a_{22}\overline{a_{32}} - 1| \\ &=& \left|(a_{21}\overline{a_{11}} \cdot a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1) + a_{22}\overline{a_{32}}\right| \\ &\leq& \left|a_{21}\overline{a_{11}}a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1\right| + |a_{22}\overline{a_{32}}|. \end{array}$$

$$\begin{array}{rcl} \left|a_{21}\overline{a_{11}}a_{11}\overline{a_{31}}|a_{11}|^{-2} - 1\right| &\leq& (1 + |\lambda_{fg}|)\,(1 + |\lambda_{gh}|)\,(1 - |\lambda_g|)^{-2} - 1 \\ &\leq& (1 + |\lambda_{fg}| + |\lambda_{gh}| + |\lambda_{fg}|\,|\lambda_{gh}|)\,(1 + 2|\lambda_g|) - 1 \\ &\leq& \left(1 + |\lambda_{fg}| + |\lambda_{gh}| + \dfrac{C}{2}(|\lambda_{gh}| + |\lambda_{fg}|)\right)(1 + 2|\lambda_g|) - 1 \\ &\leq& D_1(|\lambda_g| + |\lambda_{fg}| + |\lambda_{gh}|) \end{array}$$

where in the second line we used that $1/(1-x)^2$ is convex.

$$|a_{22}|^2 = (|a_{21}|^2 + |a_{22}|^2) - \frac{|a_{21}\overline{a_{11}}|^2}{|a_{11}|^2} \leq (1 + |\lambda_f|)^2 - \frac{(1 - |\lambda_{fg}|)^2}{(1 + |\lambda_g|)^2}$$

$$|a_{32}|^2 = (|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2) - \frac{|a_{11}\overline{a_{31}}|^2}{|a_{11}|^2} \leq (1 + |\lambda_h|)^2 - \frac{(1 - |\lambda_{gh}|)^2}{(1 + |\lambda_g|)^2}$$

$$\begin{aligned}(1+|\lambda_f|)^2 - \frac{(1-|\lambda_{fg}|)^2}{(1+|\lambda_g|)^2} &= 1+2|\lambda_f|+|\lambda_f|^2 - \frac{1-2|\lambda_{fg}|}{(1+|\lambda_g|)^2} - \frac{|\lambda_{fg}|^2}{(1+|\lambda_g|)^2} \\ &\leq (2+C)|\lambda_f| - (1-2|\lambda_{fg}|)[1-2|\lambda_g|] \\ &\leq (2+C)|\lambda_f| + 2|\lambda_{fg}| + 2|\lambda_g| - 4|\lambda_{fg}|\,|\lambda_g| \\ &\leq (2+C)|\lambda_f| + 2|\lambda_{fg}| + 2|\lambda_g| \\ &\leq (2+C)(|\lambda_f|+|\lambda_g|+|\lambda_{fg}|)\end{aligned}$$

where in the second line we used $1/(1+x)^2 \geq 1-2x$ for $x \geq 0$. Obviously we also have

$$|a_{32}|^2 \leq (2+C)(|\lambda_g|+|\lambda_h|+|\lambda_{gh}|)$$

Thus

$$|a_{22}a_{32}| \leq D_2(|\lambda_f|+|\lambda_g|+|\lambda_h|+|\lambda_{fg}|+|\lambda_{gh}|).$$

Combining all the inequalities, we obtain

$$|(f_\alpha, h_\alpha)-1| \leq D(|\lambda_f|+|\lambda_g|+|\lambda_h|+|\lambda_{fg}|+|\lambda_{gh}|).$$

with a finite number of exceptions.

□

An equivalence relation will split the set V_0 into disjoint subsets, the equivalence classes.

Definition The strong equivalence class of a C_0-sequence f is denoted by $[f]$. The set of strong equivalence classes of C_0-sequences will be called $\mathcal{S}$.

□

Theorem C.2.20 *If two C_0-sequences f_α, g_α belong to two different equivalence classes, then*

$$(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = 0.$$

If they belong to the same equivalnce class, then $(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = 0$ if and only if some $(f_\alpha, g_\alpha) = 0$.

Proof: Clearly

$$(\prod_\alpha f_\alpha, \prod_\alpha g_\alpha) = \prod_\alpha (f_\alpha, g_\alpha)$$

in the sence of quasi-convergence,

□

C_0-vectors from different strong equivalence classes are always orthogonal and those from the same strong equivalence class are orthogonal if and only if they are orthogonal in at least one tensor product factor.

Theorem C.2.21 *$[f] = [g]$ if and only if both $\sum_\alpha \|f_\alpha^0 - g_\alpha^0\|_\alpha^2$ and $\sum_\alpha |Im(f_\alpha^0, g_\alpha^0)_\alpha|$ converge*

Proof: In othere words, the combined convergence is equivalent to that of $\sum_\alpha |(f_\alpha^0, g_\alpha^0) - 1|$. Note that if $|z_{\alpha_1}| + |z_{\alpha_2}| + \cdots + |z_{\alpha_n}|$ is bounded then so is $|\zeta_{\alpha_1}| + |\zeta_{\alpha_2}| + \cdots + |\zeta_{\alpha_n}|$ if $|\zeta_{\alpha_1} - z_{\alpha_1}| + |\zeta_{\alpha_2} - z_{\alpha_2}| + \cdots + |\zeta_{\alpha_n} - z_{\alpha_n}|$ is bounded.

Now as $\sum_\alpha \big|\|f_\alpha\| - 1\big|$ converges if and only if $\sum_\alpha \big|\|f_\alpha\|^2 - 1\big|$, we can prove the theorem by demonstrating the convergence of

$$\sum_\alpha |(f_\alpha^0, g_\alpha^0) - \frac{1}{2}\|f_\alpha^0\|^2 + \frac{1}{2}\|g_\alpha^0\|^2|.$$

Now

$$\begin{aligned} \mathrm{Re}\big((f_\alpha^0, g_\alpha^0) - \frac{1}{2}\|f_\alpha^0\|^2 + \frac{1}{2}\|g_\alpha^0\|^2\big) &= -\frac{1}{2}\big(\|f_\alpha^0\|^2 + \|g_\alpha^0\|^2 - 2\mathrm{Re}(f_\alpha^0, g_\alpha^0)\big) \\ &= -\frac{1}{2}\|f_\alpha^0 - g_\alpha^0\|_\alpha^2 \end{aligned}$$

and

$$\mathrm{Im}\big((f_\alpha^0, g_\alpha^0) - \frac{1}{2}\|f_\alpha^0\|^2 + \frac{1}{2}\|g_\alpha^0\|^2\big) = \mathrm{Im}(f_\alpha^0, g_\alpha^0)$$

Recall that the convergence of $\sum_\alpha z_\alpha$ is equivalent to the combined convergence of $\sum_\alpha |\mathrm{Re} z_\alpha|$, $\sum_\alpha |\mathrm{Im} z_\alpha|$. This completes the proof.

□

Lemma C.2.22 *for each* $[f] \in \mathcal{S}$ *there exists* $f^0 \approx f$ *such that* $\|f^0_\alpha\|_\alpha = 1$ *for all* $\alpha \in \mathcal{I}$.

Proof: Choose $f \in \mathcal{S}$. As $\sum_\alpha \big|\|f'_\alpha\|_\alpha - 1\big|$ converges, apart from a finite number of α's, we have

$$\big|\|f_\alpha\|_\alpha - 1\big| \leq \frac{1}{2}, \qquad \|f_\alpha\|_\alpha \geq \frac{1}{2}.$$

Recall that $f \approx f'$ if $f_\alpha \neq f'_\alpha$ for a finite number of α's. So we replace f_α by any f'_α such that $1/2 \leq \|f'_\alpha\|_\alpha \leq 3/2$ for the exceptional α's.

As $1/x$ is convex, for $1/2 \leq x \leq 1$ we have $(1/x) - 1 \leq 2(1-x)$ and for $1 \leq x \leq 3/2$ we have $1 - (1/x) \leq 2(x-1)$, that is,

$$\left|\frac{1}{x} - 1\right| \leq 2|x-1|$$

for $1/2 \leq x \leq 3/2$. So we have

$$\left|\frac{1}{\|f'_\alpha\|_\alpha} - 1\right| \leq 2\big|\|f'_\alpha\|_\alpha - 1\big|,$$

therefore

$$\sum_\alpha \left|\frac{1}{\|f'_\alpha\|_\alpha} - 1\right|$$

converges. By theorem C.2.15 ii), replace $|z_\alpha|$ by $1/\|f'_\alpha\|_\alpha$, as f'_α is a C_0-sequence so is

$$f^0_\alpha = \frac{1}{\|f'_\alpha\|_\alpha} f'_\alpha.$$

By theorem C.2.15 iv), replace z_α by $1/\|f'_\alpha\|_\alpha$, the C_0-sequences f'_α and f^0_α are equivalent. $\|f^0_\alpha\|_\alpha = 1$ is obvious, therefore the proof is complete.

□

By $\otimes_\alpha f_\alpha$ we denote the functional on X defined by

$$\otimes_\alpha f_\alpha(\{x_\alpha\}) = \prod_\alpha (f_\alpha, x_\alpha)_\alpha$$

We need to prove

$$\otimes_\alpha f_\alpha \neq 0 \quad \Leftrightarrow \quad \prod_\alpha \|f_\alpha\|_\alpha \neq 0$$

But $\otimes_\alpha f_\alpha \neq 0$ implies $\prod_\alpha \|f_\alpha\|_\alpha \neq 0$.

C.2.4 ITP Hilbert Space

1) All the $\mathcal{H}^{\otimes}_{[f]}$ are isomorphic and mutually orthogonal.

2) Every $\mathcal{H}^{\otimes}_{(f)}$ is the closed direct sum of all the $\mathcal{H}^{\otimes}_{[f']}$ with $[f'] \in \mathcal{S} \cap (f)$.

3) The ITP $\mathcal{H}^{\otimes}$ is the closed direct sum of all the $\mathcal{H}^{\otimes}_{(f)}$ with $(f) \in \mathcal{W}$.

4) Every $\mathcal{H}^{\otimes}_{[f]}$ has an explicitly known orthonormal von Neumann basis.

5) If s, s' are two different strong equivalence classes in the same weak equivalence class then there exists a unitary operator on $\mathcal{H}^{\otimes}$ that maps $\mathcal{H}^{\otimes}_s$ to $\mathcal{H}^{\otimes}_{s'}$, otherwise such an operator does not exist, the two Hilbert spaces are unitarily inequivalent subspaces of $\mathcal{H}^{\otimes}$.

Notice that two isomorphic Hilbert spaces can always be mapped into each other such that scalar products are preserved (just map some orthonormal bases) but here the question is whether this map can be extended unitarily to all of $\mathcal{H}^{\otimes}$. Intuitively then, strong classes within the same weak classes describe the same physics, those in different weak classes describe different physics such as an infinite difference in energy, magnetization, volume etc.

Definition By $\mathcal{H}_C$ we denote the completion of the complex vector space of finite linear combinations of elements from V_C equipped with the sesquilinear form $< \cdot, \cdot >$ obtained by extending (C.5) from V_C to $\mathcal{H}_C$ by sesquilinearity,

$$\begin{aligned} < \alpha\xi + \beta\chi, \zeta > &= \overline{\alpha} < \xi, \zeta > + \overline{\beta} < \chi, \zeta > \\ < \xi, \alpha\chi + \beta\zeta > &= \alpha < \xi, \chi > + \beta < \xi, \zeta > \end{aligned}$$

for all $\xi, \chi, \zeta \in \mathcal{H}_C$ and $\alpha, \beta \in \mathbb{C}$.

□

We can now give the defintion of the ITP.

Definition We denote by

$$\mathcal{H}^{\otimes} := \otimes_{\alpha}\mathcal{H}_{\alpha} \tag{C.8}$$

the Cauchy completion of the pre-Hilbert space $\mathcal{H}_C$. It is called the ITP of the $\mathcal{H}_{\alpha}$.

□

The strong equivalence classes provide the basic tool to analyze the structure of $\mathcal{H}^{\otimes}$.

Definition For a strong equivalence class $[f] \in \mathcal{S}$ we define the closed subspace $\mathcal{H}_{[f]}$ of $\mathcal{H}^{\otimes}$ by the closure of the finite linear combinations of $\otimes_{f'}$'s with $f' \in [f]$, i.e., the closure of

$$\left\{\sum_{k=1}^{N} z_k \otimes_{f^k} : z_k \in \mathbb{C},\ f^k \in [f],\ N < \infty\right\}. \tag{C.9}$$

It is called the $[f]-$**adic incomplete ITP** of the $\mathcal{H}_{\alpha}$'s.

□

Theorem C.2.23 *The complete ITP decomposes as the direct sum over strong equivalence classes $[f]$ of $[f]-$adic ITP's,*

$$\mathcal{H}^{\otimes} = \overline{\otimes_{[f]\in\mathcal{S}}\mathcal{H}_{[f]}} \tag{C.10}$$

Proof:

□

C.2.5 Non-Associativity of ITPs

The associative law of tensor products is false. By this we mean the following: Let us subdivide the index set $\mathcal{I}$ into mutually disjoint index sets

$$\mathcal{I} = \cup_{\beta}\mathcal{I}_{\beta}$$

where β runs over some other index set $\mathcal{L}$. One can now form the different ITP

$$\mathcal{H}^{\otimes\prime} = \otimes_\beta \mathcal{H}^{\otimes}_\beta, \qquad \mathcal{H}^{\otimes}_\beta = \otimes_{\gamma\in\mathcal{I}_\beta}\mathcal{H}_\gamma.$$

Unless the index set $\mathcal{L}$ is finite, a generic C_0-vector of $\mathcal{H}'^{\otimes}$ is orthogonal to all of $\mathcal{H}^{\otimes}$.

Scalar multiplication is not multi-linear. That is, if f and $z\cdot f$ are C_0-sequences where

$$(z\cdot f)_\alpha = z_\alpha f_\alpha$$

for some complex numbers z_α then $\otimes_{z\cdot f} = (\prod_\alpha z_\alpha)\otimes_f$.

Lemma C.2.24 *Let $\prod_\alpha z_\alpha$ be quasi-convergent. Then*

i) If f is a $C-$sequence, so is $z\cdot f$ with $(z\cdot f)_\alpha := z_\alpha f_\alpha$.

ii) If moreover $\sum_\alpha \big||z_\alpha| - 1\big|$ converges and f is a C_0-sequence, so is $z\cdot f$.

iii) The product formula

$$\otimes_{z\cdot f} = [\prod_\alpha z_\alpha]\otimes_f \tag{C.11}$$

fails to hold only if

1) $\prod_\alpha z_\alpha$ is (quasi-convergent but) not convergent and

2) $(\otimes_f, \cdot\,) \neq 0$ considered as a linear functional over $C-$vectors.

In this case, $\{z_\alpha\}$, f satisfy the assumptions of ii), and all $z_\alpha \neq 0$.

iv) If $\{z_\alpha\}$, f satisfy the assumptions of ii), then $[z\cdot f] = [f]$ if and only if $\sum_\alpha |z_\alpha - 1|$ converges. If all $z_\alpha \neq 0$, then this is equivalent to the convergence of $\prod_\alpha z_\alpha$ (beyond mere quasi-convergence).

Proof: i): As both $\prod_\alpha |z_\alpha|$, $\prod_\alpha \|f_\alpha\|_\alpha$ converge, so does

$$\prod_\alpha \|z_\alpha f_\alpha\|_\alpha = \prod_\alpha |z_\alpha| \cdot \prod_\alpha \|f_\alpha\|_\alpha$$

ii) $|z_\alpha| \leq C$

$$\begin{aligned}
\big|\|z_\alpha f_\alpha\|_\alpha - 1\big| &= \big||z_\alpha|\ \|f_\alpha\|_\alpha - 1\big| \\
&= \big|(|z_\alpha| - 1) + |z_\alpha|(\|f_\alpha\|_\alpha - 1)\big| \\
&\leq \big|(|z_\alpha| - 1) + C(\|f_\alpha\|_\alpha - 1)\big|
\end{aligned}$$

and thus $\sum_\alpha \left| \|z_\alpha f_\alpha\|_\alpha - 1 \right|$ converges.

□

C.2.6 Von-Neumann Algebras on ITPs

A von Neumann algebra over a Hilbert space is weakly (equivalently strongly) closed sub$-^*$ algebra of the algebra of bounded operators on that Hilbert space.

Given a bounded operator a_α on $\mathcal{H}_\alpha$ (notice that closed unbounded operators have a polar decomposition into a unitary and self-adjoint piece and that the self-adjoint operator is completely determined by its bounded spectral projections so that restriction to bounded operators is no loss of generality) we can extend it in the natural way to $\mathcal{H}^\otimes$ by defining $\hat{a}_\alpha$ densely on C_0-vectors through

$$\hat{a}_\alpha \otimes_f = \otimes_{f'}$$

with

$$f'_{\alpha'} = f_{\alpha'} \quad \text{for} \quad \alpha' \neq \alpha$$

and

$$f'_\alpha = a_\alpha f_\alpha$$

As we will see, it turns out that the algebra of these extended operators for a given label is automatically a von Neumann algebra for $\mathcal{H}^\otimes$ and we call the weak closure of all these algebras the von Neumann algebra $\mathcal{R}^\otimes$ of local operators.

Definition We denote by $\mathcal{B}(\mathcal{H}_\alpha)$ the set of bounded operators on $\mathcal{H}_\alpha$ and by $\mathcal{B}^\otimes := \mathcal{B}(\mathcal{H}^\otimes)$ the set of bounded operators on the ITP $\mathcal{H}^\otimes$.

□

Definition We denote by $\mathcal{B}_\alpha$ the extension of $\mathcal{B}(\mathcal{H}_\alpha)$ to the ITP, that is,

$$\mathcal{B}_\alpha = \{\hat{A}_\alpha : A_\alpha \in \mathcal{B}(\mathcal{H}_\alpha)\} \tag{C.12}$$

□

Lemma C.2.25 *For all $\alpha \in \mathcal{I}$, the algebra $\mathcal{B}_\alpha$ is a von Neumann algebra over $\mathcal{H}^\otimes$.*

Proof: To prove the theorem one uses the von Neumann density theorem, proved in the next chapter. This states that a set $\mathcal{U}$ of the space of bounded operators of a Hilbert space $\mathcal{H}$ is a von Neuman algrba if it is equal to its double commutant, the commutant of a set $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ being

$$\mathcal{U}' = \{B \in \mathcal{B}(\mathcal{H}) : [A, B] = 0 \text{ for all } A \in \mathcal{U}\},$$

Let us write

$$\mathcal{B}^\otimes = \mathcal{B}(\mathcal{H}_\alpha \otimes \mathcal{H}_{\overline{\alpha}})$$

where $\overline{\alpha} = \mathcal{I} - \alpha$.

$$\mathcal{B}'_{\overline{\alpha}} = \{\hat{B} \in \mathcal{B}^\otimes : [\hat{A}, \hat{B}] = 0 \text{ for all } \hat{A} \in \mathcal{B}_{\overline{\alpha}}\},$$

the commutant of $\mathcal{B}_{\overline{\alpha}}$.

□

Definition Two C_0−sequences f, g are said to be weakly equivalent, denoted $f \sim g$ provided that there are complex numbers z_α such that $z \cdot f$ and g are strongly equivalent, that is, $z \cdot f \approx g$.

□

Lemma C.2.26 *The definition of weak equivalenceis unchanged if we restrict to complex numbers with $|z_\alpha| = 1$.*

Proof: We prove that if f and $z \cdot f$ are C_0−sequences we can find z'_α with $|z'_\alpha| = 1$ such that $(z \cdot f) \approx (z' \cdot f)$.

As $\sum_\alpha \big| \|z_\alpha f_\alpha\|_\alpha - 1 \big|$ converges, $z_\alpha f_\alpha = 0$ can occur only for a finite number of α's. For these we may replace z_α by 1 and f_α by non-zero f^0_α. As this so constructed C_0−sequence differes from f by only a finite number of terms, it is srtongly equivalent to it (this follows from theorem C.2.21). So we may assume that $z_\alpha f_\alpha \neq 0$ for all α.

As $\sum_\alpha \big| \|f_\alpha\|_\alpha - 1 \big|$, $\sum_\alpha \big| \|z_\alpha f_\alpha\|_\alpha - 1 \big|$ converge, and all $\|f_\alpha\|_\alpha, \|z_\alpha f_\alpha\|_\alpha \neq 0$, by theorem C.2.15, $\prod_\alpha \|f_\alpha\|_\alpha$, $\prod_\alpha \|z_\alpha f_\alpha\|_\alpha$ converge and have non-zero values. Thus

$$\prod_\alpha \frac{1}{|z_\alpha|}$$

converges too, and has a non-zero value, as

$$\frac{\|f_\alpha\|_\alpha}{\|z_\alpha f_\alpha\|_\alpha} = \frac{1}{|z_\alpha|}.$$

Therefore

$$\sum_\alpha \left|\frac{1}{|z_\alpha|} - 1\right|$$

converges by theorem C.2.15. Now it follows from lemma, (ii), (iv), that $(z \cdot f)$ and $(\frac{z}{|z|} \cdot f)$ are equivalent. Thus

$$z'_\alpha = \frac{z_\alpha}{|z_\alpha|}.$$

Lemma C.2.27 *Weak equivalence is an equivalence relation.*

Proof: Reflextivity: $f \sim f$ as $(1 \cdot f) \approx f$.

Symmetry: If $|z_\alpha| = 1$ we have

$$(f_\alpha, (1/z_\alpha) g_\alpha)_\alpha = z_\alpha (f_\alpha, g_\alpha)_\alpha = (z_\alpha f_\alpha, g_\alpha)_\alpha$$

and $\|(1/z_\alpha) g_\alpha\|_\alpha = \|g_\alpha\|_\alpha$. These together with the symmetry property of strong equivalence implies that if $(z \cdot f) \approx g$ with $|z_\alpha| = 1$ then $(1/z) \cdot g \approx f$ with $(1/z_\alpha) g_\alpha$ a C_0-sequence, that is, $g \sim f$.

Transitivity: If $z \cdot f \approx g$ and $z' \cdot g \approx h$, $|z_\alpha|, |z'_\alpha| = 1$, then $z \cdot f \approx (1/z') \cdot h$ and $z' \cdot z \cdot f \approx h$. So $f \sim g$.

□

Lemma C.2.28 *f and g are weakly equivalent if and only if*

$$\sum_\alpha \big||(f_\alpha, g_\alpha)_\alpha| - 1\big|$$

converges.

Proof: By definition, $f \sim g$ if we can find complex numbers z_α with $|z_\alpha| = 1$ such that

$$\sum_\alpha \left|(z_\alpha f_\alpha, g_\alpha)_\alpha - 1\right|$$

converges. If

$$\sum_\alpha \left||(f_\alpha, g_\alpha)_\alpha| - 1\right|$$

converges, then $f \sim g$ with $z_\alpha = (f_\alpha, g_\alpha)^*_\alpha / |(f_\alpha, g_\alpha)_\alpha|$.

Now, from fig (), it is easy to see that the minimum value of $|(z_\alpha f_\alpha, g_\alpha) - 1| = |z_\alpha (f_\alpha, g_\alpha)_\alpha - 1|$ is $\left||(f_\alpha, g_\alpha)_\alpha| - 1\right|$. Thus $\sum_\alpha \left|(z_\alpha f_\alpha, g_\alpha)_\alpha - 1\right|$ converges for some complex numbers z_α, when $f \sim g$, only if

$$\sum_\alpha \left||(f_\alpha, g_\alpha)_\alpha| - 1\right|$$

converges.

□

Lemma C.2.29 *Assume that a z_α with $|z_\alpha| = 1$ is given for each $\alpha \in \mathcal{I}$. Then there exists one and only one closed, linear operator U, such that*

$$U \otimes_f = \otimes_{z \cdot f}$$

for every C_0-vector f. The operator U is unitary.

Proof:

Proof of uniqueness. If U' is another closed, linear operator meeting the requirements, then

□

C.3 Outline of Semiclassical Analysis

The $U(1)^3$ coherent states are given by

$$\Psi^t_{\alpha,(A_0,E_0)} = \prod_{e\in E(\alpha)} \prod_{j=1,2,3} \Psi^{t_e}_{g_e(A_0,E_0)}, \tag{C.13}$$

where

$$\Psi^{t_e}_{g_e(A_0,E_0)} = \sum_{n\in\mathbb{Z}} e^{-t_e n^2/2} \left(g_e h^{-1}\right)^n \tag{C.14}$$