Appendix E

Quantum Field Theory: Functional Integral and Canonical Approach

E.1 Lagrangian Field Theory

$$
\begin{aligned}
0 &= \delta S \\
&= \delta \int d^4 \mathcal{L} \\
&= \int d^4 \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) \right\}
\end{aligned} \tag{E.1}
$$

use

$$\delta(\partial_\mu \varphi) = \partial_\mu (\delta\varphi)$$

and apply integration by parts, the boundary terms vanish because the end points are fixed:

$$
\begin{aligned}
\int d^4 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta(\partial_\mu \varphi) &= \int d^4 \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\mu (\delta\varphi) \\
&= -\int d^4 \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi
\end{aligned} \tag{E.2}
$$

Altogether, the variation of the action is

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right\} \delta\varphi \tag{E.3}$$

Either the integrand takes positive and negative values or the integrand is zero over the domain of integration. As $\delta\varphi$ is arbitrary we know that the integrand must be zero. The terminside the braces vanishes. This gives the Euler-Lagrange equations for the field φ

$$0 = \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \tag{E.4}$$

$$\mathcal{H} = \pi(x)\dot{\varphi}(x) - \mathcal{L} \tag{E.5}$$

$$H = \int \mathcal{H} d^3x \tag{E.6}$$

following

$$p_r(t) = \frac{\partial \mathcal{L}}{\partial \dot{q}_r}$$

conjugate momentum defined in the usual way

$$\pi_r(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_r} \tag{E.7}$$

E.2 Bosonic Integration

E.2.1 N Real variables

$$Z(j) = \int \prod_{i=1}^{N} dx_i \exp\left(-\frac{1}{2} \sum_{i,j=1}^{N} x_i A_{ij} x_j + \sum_{i=1}^{N} j_i x_i \right) \tag{E.8}$$

where the matrix A_{ij} is symmetric and strictly positive.A more compact notation is to represent the column vectors $(x_i...x_N)$ and $(j_1...j_N)$ as x and j, then the row vectors would be x^T and j^T where T=transpose. We then have:

$$\sum_{i,j=1}^{N} x_i A_{ij} x_j = x^T A x, \qquad \sum_{i=1}^{N} j_i x_i = j^T x \tag{E.9}$$

Change variables to x' given by

$$x = x' + A^{-1} j, \tag{E.10}$$

(the matrix A^{-1} exists because A is assumed positive) then

$$-\frac{1}{2} x^T A X + J^T x = -\frac{1}{2} x'^T A x' + \frac{1}{2} j^T A^{-1} j, \tag{E.11}$$

The integral then becomes

$$Z(j) = \exp(\frac{1}{2} j^T A^{-1} j) Z(0) \tag{E.12}$$

In many cases equation () is all one needs (for example in calculating correlation functions, from which Z(0) cancels). Z(0) reads

$$Z(0) = \int \prod_{i=1}^{N} dx'_i \exp(-\frac{1}{2} x^T A x), \tag{E.13}$$

Let R be an orthogonal transformation ($RR^T = I$) diagonalising A,

$$A = \mathbf{R}^T \mathbf{D} \mathbf{R}, \qquad \mathbf{D} = \begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ 0 & & \ddots & \\ & & & d_N \end{pmatrix}, \qquad d_i > 0 \qquad \forall i. \tag{E.14}$$

Make the following change of variables with unit Jacobian:

$$x' = \mathbf{R} x \qquad (det \mathbf{R} = 1), \tag{E.15}$$

$$\int \prod_{i=1}^{N} dx_i \exp(\frac{1}{2} x^T \mathbf{A} x) = \int \prod_{i=1}^{N} dx'_i \exp(\frac{1}{2} x'^T \mathbf{D} x') \tag{E.16}$$

The last integral is the product of N independent gaussian integrals, and is given by

$$(2\pi)^{N/2}\prod_{i=1}^{N}(d_i)^{-1/2} = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{E.17}$$

$$Z(0) = \frac{(2\pi)^{N/2}}{(detA)^{1/2}} \tag{E.18}$$

$$\int \prod_{i=1}^{N} dx_i \exp(-x^T\mathbf{A}x + j^T x+) = \frac{\pi^{N/2}}{det\mathbf{A}^{1/2}}\exp((1/2)j^T A^{-1}j), \tag{E.19}$$

E.2.2 Complex variables

First consider the case of a single complex variable z=x+iy with integral

$$I = \int d^2 z e^{-z^* Az + z^* j + zj^*} = \int dxdy e^{-A(x^2+y^2)+2j_1 x + 2j_2 y} \tag{E.20}$$

where $j = j_1 + ij_2$ and $A = a_1 + ia_2$, with $a_1 > 0$. Then I follows immediately:

$$I = \frac{\pi}{A} a^{j^* A^{-1} j} \tag{E.21}$$

Next, consider the case of N complex variables z_i,

$$I = \int \prod_{i=1}^{N} d^2 z_i e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z}, \tag{E.22}$$

where transposes have been replaced by Hermitian conjugates. Assume that $\mathbf{A}$ can be diagonalized by a unitary transformation U,

$$A = U^\dagger D U, \tag{E.23}$$

where D is a diagonal matrix with elements d_i whose real parts are positive. Write

$$U = R + iS, \tag{E.24}$$

where R and S are real matrices; the relation $U^\dagger U = I$ entails

$$RR^T + SS^T = I, \quad RS^T - SR^T = 0. \tag{E.25}$$

The transformation $z' = Uz$ amounts to

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} R & -S \\ S & R \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{E.26}$$

and the matrix which transforms (x,y) into (x', y') is orthogonal so that the jacobian of the transformation is 1, which leads to the end result

$$\int \prod_{i=1}^{N} e^{-z^\dagger \mathbf{A} z + z^\dagger j + j^\dagger z} = \frac{\pi^N}{det\mathbf{A}} e^{j^\dagger \mathbf{A^{-1}} j} \tag{E.27}$$

E.3 Feynmann Rules for Scalar Quantum field theory

E.4 Perturbation Theory

E.4.1 Diagrammatic Perturbation Theory

In this section we investigate general rules for the perturbative calculation of correlation functions, rules designed to yield the result in the form of an expansion in powers of g,

$$G = G_0 + gG_1 + g^2 G_2 + g^3 G_3 + ... + g^n G_n + ... \tag{E.28}$$

Where G_0 is the correlation function of the Gaussian model, (non-interacting model). These rules are easily represented in diagrammatic form. These diagrams are the so-called *Feynman Diagrams*. As a simple example we examine the Ginzburg-Landau theory (see eq.(**??**)). It is impossible to find an exact closed formula for Z(0), but if g is small on can expand $\exp(-g \int d^d x \phi^4(x)/4!)$.

The calculation of $\mathbf{G^{(2)}}$ to order g

First we calculate the 2-point greens function to order g. One must evaluate the integral

$$I(x,y) = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H} = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H_0} \left[1 - \frac{g}{4!} \int d^d z \phi^4(z) + \cdots \right]. \tag{E.29}$$

The first term in the square brackets merely yields

$$\mathcal{N} \left\langle \phi(x)\phi(y) \right\rangle_0 = \mathcal{N} G_0(x-y) \quad \text{where } \mathcal{N} = Z_0(j=0). \tag{E.30}$$

To evaluate the integral in the second term,

$$\int \mathcal{D}\phi\phi(x)\phi(y)e^{-H_0}\int d^dz\phi^4(z), \tag{E.31}$$

we use Wick's theorem (**??**). There are two types of result from the contractions:

$$(a)\quad \langle\phi(x)\phi(y)\rangle_0\left\langle\phi^4(z)\right\rangle \quad \text{and} \quad (b)\quad \langle\phi(x)\phi(z)\rangle_0\left\langle\phi^2(z)\right\rangle_0\langle\phi(y)\phi(z)\rangle_0 \tag{E.32}$$

in the wick expansion there were $4\times = 12$ terms of type (a) and 3 terms of type (b). It is convenient to represent these contractions as diagrams, by drawing two "external" points x and y ("external" means that they refer to the arguments of the correlation function), and "internal" point or "vertex" z, which stems from the expansion of exp(-V), and over which we integrate. Every contraction is represented by a line joining arguments of ϕ. The two types of terms possible in (E.32) are drawn

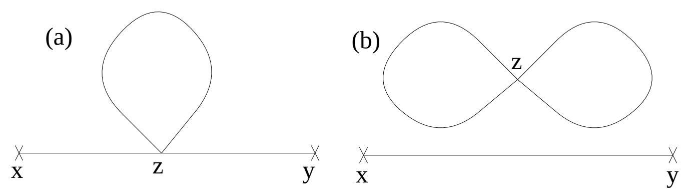

Figure E.1: The two diagrams of order g

These diagrams are called *Feynman diagrams (or graphs)*; one such diagram corresponds to every distinct group of terms of the perturbation expansion. The integral I reads

$$I(x,y) = \mathcal{N}\left[G_0(x-y) - \frac{1}{2}g\int d^dzG_0(x-z)G_0(0)G_0(z-y) - \frac{1}{8}gG_0(x-y)(G_0(0))^2\int d^dz\right] \tag{E.33}$$

In order to obtain the correlation function, we must divide by Z(0):

$$Z(0) = \int \mathcal{D}\phi e^{-H_0}\left(1 - \frac{g}{4!}\int d^dz\phi^4(z) + \cdots\right) = \mathcal{N}\left[1 - \frac{g}{8}(G_0(0))^2\int d^dz + \cdots\right]. \tag{E.34}$$

The second term in the square brackets is represented by the diagram.

Dividing (E.33) by (E.34) we obtain the correlation function to order g

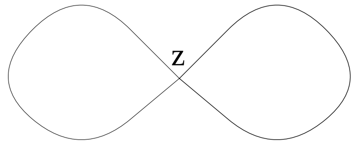

Figure E.2: The vacuum-fluctuation diagram

$$G^{(2)}(x-y) = \frac{I(x,y)}{Z(0)} = G_0(x-y) - \frac{1}{2}g \int d^d z G_0(x-z) G_0(0) G_0(z-y) + \mathcal{O}(g^2). \quad \text{(E.35)}$$

The graph(b) from fig.(E.1) does not feature in the perturbation expansion of G. Diagrams of this type are called "vacuum-fluctuation" (sub)diagrams, meaning a subgraph that is completely disconnected from the "external" points x and y. The sum of all vacuum-fluctuation diagrams is equal to $Z(0) = \mathcal{D}\phi e^{-H}$. **Division by $Z(0)$ cancels all graphs containing "vacuum-fluctuations" parts disconnected from the rest of the diagram**. A proof is given in citeBellac (p 160).

On taking the Fourier transform, eq.(E.35) becomes

$$G^{(2)}(k) = G_0(k) - \frac{1}{2}g G_0(k) \left[\int \frac{d^d q}{(2\pi)^d} G_0(q) \right] G_0(k). \quad \text{(E.36)}$$

The factor in front of the second term on the r.h.s. is called the *symmetry factor* of the diagram. To become familiar with the "Feynman rules", i.e. the rules for associating diagrams with the perturbation expansion, we move to the calculation of $G^{(2)}$ to order g^2.

The calculation of $\mathbf{G^{(2)}}$ to order $\mathbf{g^2}$

We use Wick's theorem to compute the expression

$$\left\langle \phi(x)\phi(y) \int d^d z d^d u \phi^4(z) \phi^4(u) \right\rangle_0 . \quad \text{(E.37)}$$

Eliminating the terms that contain vacuum-fluctuation parts, one finds three types of graphs shown in fig.(E.3), with their symmetry factors given in brackets:

The vertices z and u may be permuted, which yields a multiplicative factor 2!; however this is exactly cancelled by the factor 1/2! from the expansion of the exponential. This is the same kind of cancellation happens in the nth order.

We shall settle for examining the contribution $\bar{G}(x-y)$ to the correlation function from graph (a) in fig.(E.3). Thus

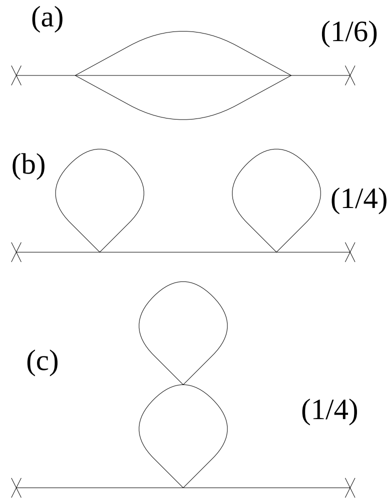

Figure E.3: The vacuum-fluctuation diagram

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^dz d^du G_0(x-z)[G_0(z-u)]^3 G_0(u-y). \tag{E.38}$$

Let us write $\bar{G}(x-y)$ as a Fourier transform, by replacing every factor G_0 by its Fourier representation

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^dz d^du \frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\prod_{l=1}^{3}\left\{\frac{d^dq_l}{(2\pi)^d}e^{i\sum_{l=1}^{3}q_l\cdot(z-u)}\right\}$$
$$\times e^{ik\cdot(x-z)}e^{ik'\cdot(u-y)}G_0(k)G_0(k')\prod_{l=1}^{3}G_0(q_l). \tag{E.39}$$

The integration over z and u yield a product of two delta functions

$$2\pi)^d\delta^d(k-q_1-q_2-q_3) \times (2\pi)^d\delta^d(k'-q_1-q_2-q_3) \tag{E.40}$$

which represent "momentum conservation" at the two vertices. Hence

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int \frac{d^dk}{(2\pi)^d}e^{ik\cdot(x-y)}[G_0(k)]^2 \times \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}G_0(q_1)G_0(q_2)G_0(k-q_1-q_2). \tag{E.41}$$

The last expression shows that $\bar{G}(x-y)$ is the Fourier transform of the function $\bar{G}(k)$,

$$\bar{G}(k) = \frac{1}{6} g^2 G_0(k) \left[\int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} G_0(q_1) G_0(q_2) G_0(k - q_1 - q_2) \right] G_0(-k), \tag{E.42}$$

(where we have used $G_0(k) = G_0(-k)$). (E.42) can be represented diagrammatically in fig.(E.4). The graph shown there has two external propagators $G_0(k)$ and $G_0(-k)$, and three internal propagators; because of the delta-functions $\delta^d(\ldots)$ ("momentum conservation"), only two of the three internal lines are independent. By following the internal propagators one can describe three different closed loops, but because of "momentum conservation" only two of these are independent; i.e. there are only two integration variables in (E.42).

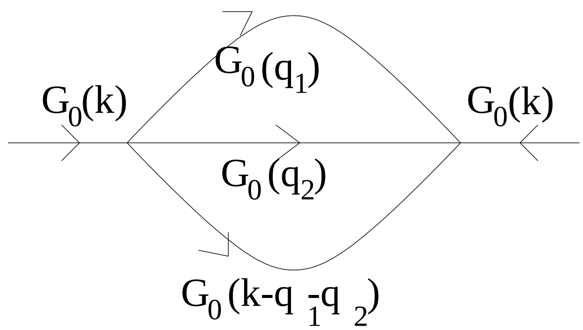

Figure E.4: Diagrammatic representation of (E.42)

Our experience with the previous examples suggest the following "Feynman rules" in k-space ("momentum space"):

1. We draw the Feynman diagram with a momentum assigned to each line. We must have overall momentum conservation and conservation at each vertex.

2. To every vertex we assign a factor -g

3. To every line we assign a factor $G_0(k)$

4. To every independent loop there corresponds an integration $\int d^d q/(2\pi)^d$.

5. Finally, every graph is multiplied by a symmetry factor.

E.4.2 The Generating Functional of Connected Diagrams

We start with an example, by investigating the correlation function $G^{(4)}$. It subdivides into one connected and three disconnected diagrams,

$$G^{(4)}(1,2,3,4) = G_c^{(4)}(1,2,3,4) + \{G_c^{(2)}(1,2)G_c^{(2)}(3,4) + permutations\}, \tag{E.43}$$

where G_c denotes a connected correlation function. (note $G_c^{(2)} = G^{(2)}$. In terms of graphs this is represented as in fig.(refbubble0)

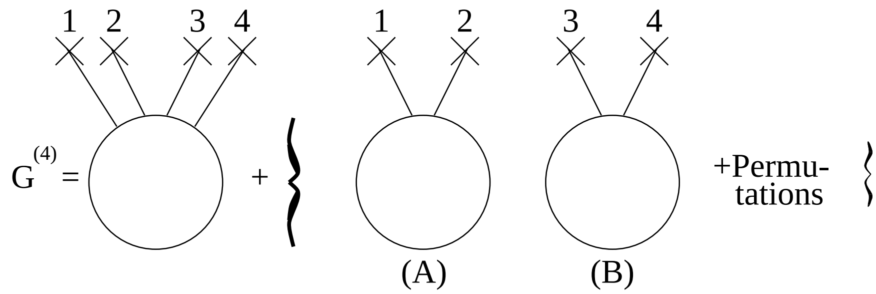

Figure E.5:

The number of disconnected terms is $3 = 4!/[(2!)^2 \times (2!)]$. 4! is the number of permutations of the external points (1,2,3,4); but the result is unaffected by permuting (1,2), or (3,4), or the two bubbles (A) and (B), hence a factor $(2!)^2 \times 2!$.

We have been only considering theories where the n-point correlation functions with n odd vanish: $G^{(2k+1)} = 0$. For more generality, we shall assume that the interaction contains terms inφ^{2p+1}. Consider a disconnected diagram of $G^{(N)}$ corresponding to the subdivision into connected diagrams (fig E.6):

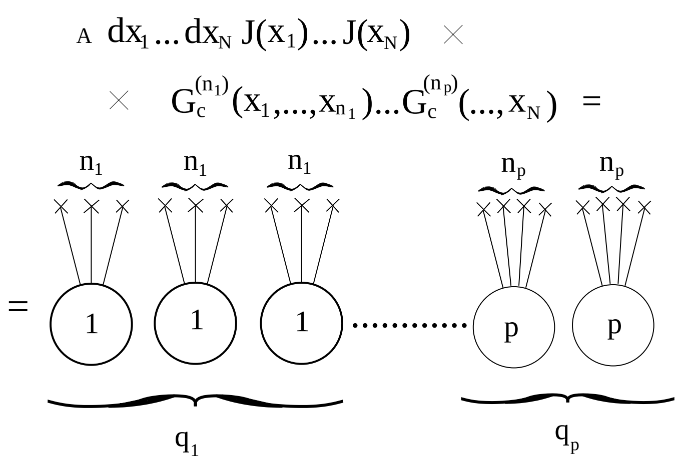

Figure E.6:

There are q_l bubbles connected to n_l external points,...., q_p bubbles connected to n_p external points, with

$$q_1 n_1 + \cdots + q_p n_p = N. \tag{E.44}$$

The number of independent terms is

$$\frac{N!}{[(n_1!)^{q_1} q_1!] \ldots [(n_p!)^{q_p} q_p!]} \tag{E.45}$$

It is found that the Functional that generates just connected diagrams is the logarithm of the normalised Generating functional. Hence, consider the exponential of the generating functional of connected diagrams:

$$\exp \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) G_c^N(x_1 \ldots x_N) \tag{E.46}$$

This should give the expansion for the generating function of all possible diagrams. When the exponential is expanded it is obvious that the amplitude for every possible disconnected diagram will be produced. To complete the proof that this is the correct Generating Functional we need to check each diagram comes with the correct prefactor.(i.e. equation (E.45). So expanding equation (E.46)

$$\sum_{q=0}^{\infty} \frac{1}{q!} \left(\sum_{n=1}^{\infty} \int dx_1 \ldots dx_n j(x_1) \ldots j(x_n) G_c^{(n)}(x_1 \ldots x_n) \right)^q \tag{E.47}$$

We convert this sum into a summation over N, the number of legs of the disconnected diagrams (figure).

$$\sum_{N=0}^{\infty} \sum_{q_1 n_1 + \ldots + q_p n_p = N} \prod_{i=1}^{p} \frac{1}{q_i!} \left[\frac{\int dx_1 \ldots dx_{n_i} j(x_1) \ldots j(x_{n_i}) G_c^{n_i}(x_1 \ldots x_{n_i})}{n_i!} \right]^{q_i} \tag{E.48}$$

Now we use (E.45) and the symmetry of G_c with respect to its arguments to rewrite the above equation as

$$\sum_{N=0}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) \sum_{q_1 n_1 + \ldots + q_p n_p = N} G_c^{n_i}(x_1 \ldots x_{n_i}) \ldots G_c^{n_p}(\ldots x_N) \tag{E.49}$$

Which is the correct form for the generating functional. Thus we have found that the generating functional of connected diagrams W(j) is indeed ln[Z(j)/Z(0)],

$$W(j) = ln \frac{Z(j)}{Z(0)} = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) G_c^N(x_1 \ldots x_N) \tag{E.50}$$

E.4.3 Generating functional of proper vertices

We have seen that to find $\log Z$ we only need consider the connected diagrams. The number of diagrams that need to be calculated can be reduced further. This is possible because some connected diagrams are made up of two or more connected diagrams joined together in a simple way. The following examples from ϕ^4 theory illustrates this redundancy connected diagrams can still have:

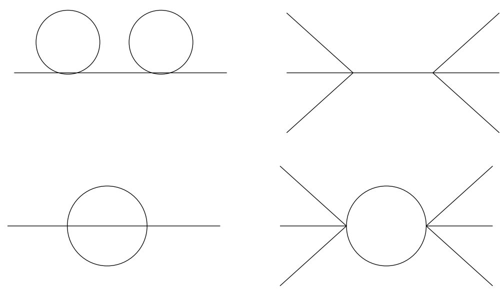

Figure E.7: Examples of reducible and irreducible diagrams from ϕ^4 theory.

The diagrams in the first row can be split into two disjoint parts by cutting a single line, thus they are called *reducible*. The examples on the second row are called *one-particle irreducible*, 1PI, since they cannot be dissected in this way. The irreducible diagrams will play an important role later because they are closely related to the parameters of the theory and also they play an important role for the systematic construction of diagrams in higher loop orders. We define an *irreducible vertex function* which is an n-point function $\Gamma^{(n)}(x_1, \ldots, x_n)$. It is possible to construct a generating functional from which the irreducible vertex functions can be obtained. This functional is defined by a Legendre transformation:

$$\Gamma[\langle\phi\rangle] = W[J] - \int d^dx J(x) \langle\phi(x)\rangle \tag{E.51}$$

where

$$\langle\phi(x)\rangle = \frac{\delta W}{\delta J(x)} \tag{E.52}$$

This functional Γ has no explicit dependence on $j(x)$. It is a functional of the expectation value of the operator $\phi(x)$ in the presence of the source J denoted $\langle\phi\rangle$. It is easy to find from Eq.(E.51) that:

$$J(x) = -\frac{\delta\Gamma}{\delta\langle\phi(x)\rangle} \tag{E.53}$$

If we functionally differentiate equation E.52 by $J(y)$ and use the chain rule we get the identity

$$\int d^d z \frac{\delta^2 W}{\delta j(x) \delta j(z)} \frac{\delta^2 \Gamma}{\delta \langle \phi(z) \rangle \delta \langle \phi(y) \rangle} = \int d^d z G^{(2)}(x-z) \Gamma^{(2)}(z-y) = -\delta^{(d)}(x-y) \quad \text{(E.54)}$$

Which shows that $-\Gamma^{(2)}(x-y)$ is the inverse of the connected Green's function $G^{(2)}(x-y)$

The self-energy $\Sigma(k)$ is defined as the sum of all two point 1PI diagrams shorn of their external lines. The correlation function $G^{(2)}(k)$ can be written in terms of $\Sigma(k)$ as

$$G^{(2)}(k) = G_0(k) + G_0(k)\Sigma(k)G_0(k) + \cdots = \left[G_0^{-1}(k) - \Sigma(k)\right]^{-1} \quad \text{(E.55)}$$

We have the relationships in ϕ^4 theory:

$$G^{(2)}(k) = \frac{1}{k^2 + m^2 - \Sigma(k)}, \quad \Gamma^{(2)}(k) = -(k^2 + m^2 - \Sigma(k)) \quad \text{(E.56)}$$

We have a new effective mass $-\Gamma^{(2)}(0) = m^2 - \Sigma(0)$.

One can begin to see how the functions $\Gamma^{(n)}$ are related to the running parameters of the model.

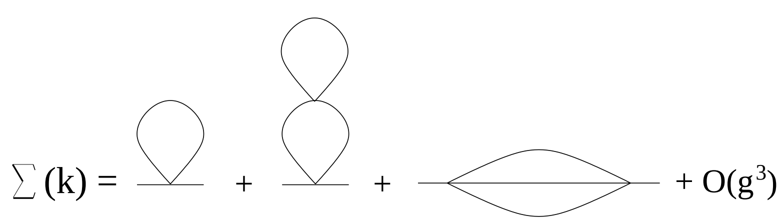

Figure E.8: The self-energy in ϕ^4 theory to order g^2

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To ease the notation in the proof of what follows, we use

$$\frac{\delta}{\delta J(x)} \to \frac{\delta}{\delta j_i}, \quad \int d^d x \to \sum_i \quad \text{(E.57)}$$

We have just met the identity

$$\sum_l \frac{\delta^2 W}{\delta j_i \delta j_l} \frac{\delta^2 \Gamma}{\delta \bar{\phi}_l \delta \bar{\phi}_k} = \sum_l G^{(2)}_{il} \Gamma^{(2)}_{lk} = -\delta_{ik} \quad \text{(E.58)}$$

which shows that $-\Gamma^{(2)}_{lk}$ is the inverse of the(connected) Green's function $G^{(2)}_{kl}$.

We now proceed to Green's functions of higher order, by differentiating the identity Eq.(E.58) with respect to j_m:

$$\sum_l \frac{\delta^3 W}{\delta j_i \delta j_l \delta j_m} \frac{\delta^2 \Gamma}{\delta \langle\phi\rangle_l \delta \langle\phi\rangle_k} + \sum_l \frac{\delta^2 W}{\delta j_i \delta j_l} \frac{\delta^3 \Gamma}{\delta j_m \delta \langle\phi\rangle_l \delta \langle\phi\rangle_k} = 0 \tag{E.59}$$

Since Γ is a function of the $\langle\phi\rangle_i$, we must transform the second derivative in Eq.(E.59); we do this for the general case ($\Gamma^{(N)}_{i_1 \ldots i_N} = \delta^{(N)}\Gamma/\delta\langle\phi\rangle_{i_1} \ldots \delta\langle\phi\rangle_{i_N}$):

$$\frac{\delta}{\delta j_m} \Gamma^{(N)}_{i_1 \ldots i_N} = \sum_n \frac{\delta\langle\phi\rangle_n}{\delta j_m} \frac{\delta\Gamma^{(N)}_{i_1 \ldots i_N}}{\delta\langle\phi\rangle_n} = \sum_n G^{(2)}_{mn} \Gamma^{(N+1)}_{n i_1 \ldots i_N}. \tag{E.60}$$

Equations (E.59) and (E.60) can be put into a graphical form if we represent the $\Gamma^{(N)}$ by shaded bubbles Fig.(E.9) (Here we have used Eq.(E.60) in order to transform Eq.(E.59).)

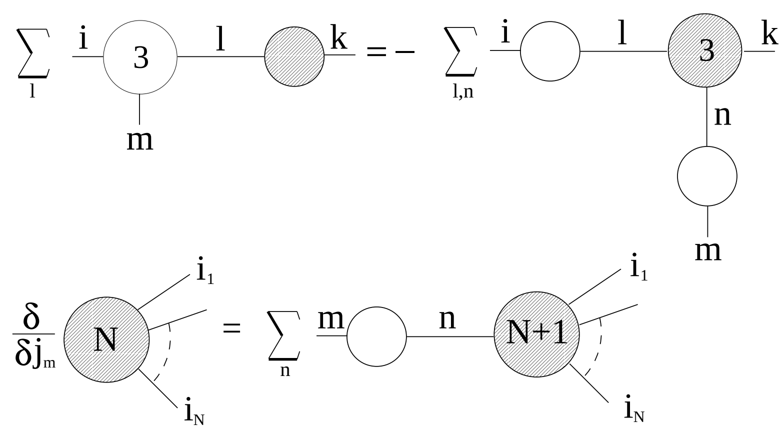

Figure E.9: Graphical representation of Eq.(E.59) and Eq.(E.60).

Multiplying the two terms in Eq.(E.59) from the right by G_{kp} and summing over k we find the relation between $G^{(3)}_c$ and $\Gamma^{(3)}$,

$$G^{(3)}_{ijk} = \sum_{l,m,n} G_{il} G_{jm} G_{kn} \Gamma^{(3)}_{lmn} \tag{E.61}$$

see Fig(E.10). Clearly, $\Gamma^{(3)}$ represents the connected, truncated 3-point function since $G^{(3)}$ is connected, and each $\Gamma^{(2)}$ above chops off one of the external legs. $\Gamma^{(3)}$ is automatically 1PI, because it is a 3-point function.

We continue the process by differentiating Eq.(E.61) once more with respect to j_l. Using Eq.(E.59) and Eq.(E.60) we obtain the relation between $G^{(4)}$ and $\Gamma^{(4)}$, Fig.(E.11)

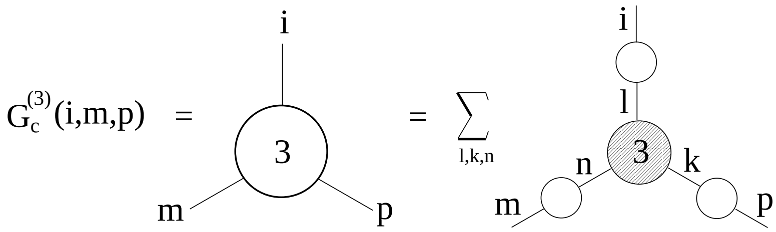

Figure E.10: 3-point Green's function, $G^{(3)}$, written in terms of the proper 3-point vertex and 2-point Green's functions $G^{(2)}$

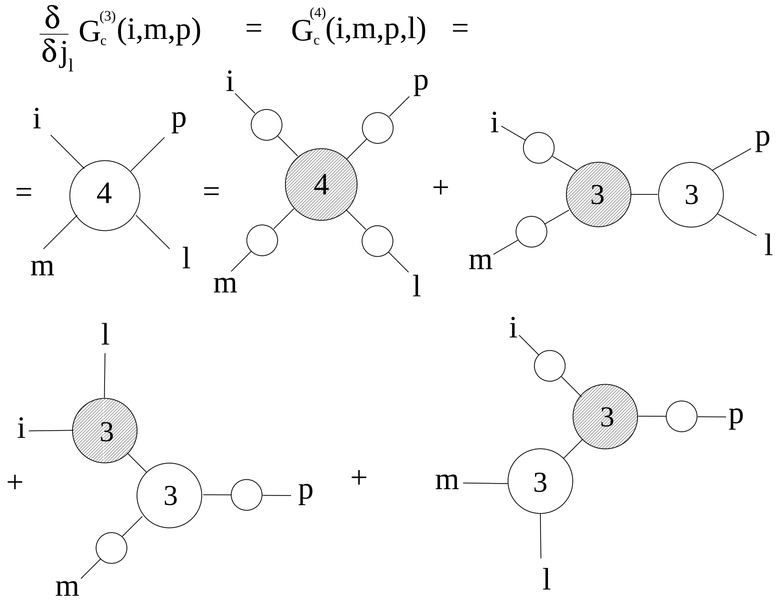

Figure E.11: Relation between $G^{(4)}$ and $\Gamma^{(4)}$, $\Gamma^{(3)}$ and $G^{(2)}$.

As we know that $\Gamma^{(3)}$ is a sum of all 3-point 1PI diagrams, the last three terms of Fig.(E.11) supply all the diagrams of the 4-point Green's functions that can rendered disconnected by cutting one internal line. So the first term in Fig.(E.11) must be the summation over those diagrams of $G^{(4)}$ that don't become disconnected by cutting an internal line. Hence, we can identify the fourth derivative of the Effective action, $\Gamma^{(4)}$, as the summation over all 4-point 1PI diagrams.

It transpires that the higher order derivatives of Γ are the summation over 1PI diagrams. These can be joined together with 2-point Green's functions to construct the four and higher point connected Green's functions. The special cases N=3 and N=4 which we have just studied enable us to see how one can set out to prove the above statement.

One can prove by induction, assuming that an equation like that of Fig(E.11) can be written to order N, and that the N-th order proper vertex can be identified with the Nth derivative of the generating functional. Differentiating this equation with respect to j_i,

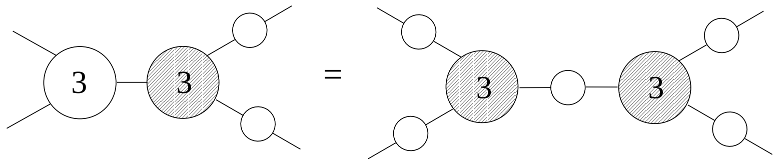

Figure E.12: We can replace each $G^{(3)}$ for $\Gamma^{(3)}$ in Fig.(E.11) using Fig.(E.10).

one finds: see Fig.(E.13)

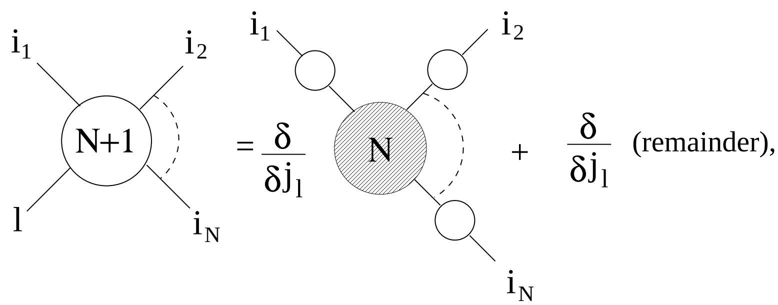

Figure E.13: Differentiating $G^{(N)}$ with respect to source j_i.

where the "remainder" does not contain $\Gamma^{(N)}$, but only $\Gamma^{(N)}$ $\Gamma^{(N-2)}$, etc. (see Fig.(E.11)). Hence we obtain

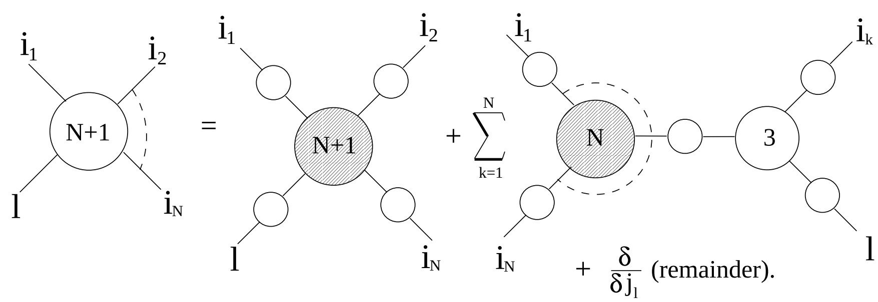

Figure E.14: All possible Green's functions with N external legs.

On removing the full external propagators from both sides, we can identify $\Gamma^{(N+1)}$ as the N+1-point 1PI diagram.

There is an algebraic proof that by cutting one line in all possible ways in a diagram contributing to $\Gamma[\langle\phi\rangle]$ does not produce disconnected diagrams that is given in [**?**] (p. 135). This proof is easily generalsied to models that are more complicated than ϕ^4. Their proof is presented in Appendix C.

E.5 Grassmann Integration

Grassmann Algebra

We consider a set of anticommutating Grassmann variables $\{\zeta_i\}_{i=1,...,n}$, with complex linear coefficients, where n is the dimension of the algebra. The decisive relation defining the structure of the algebra is the anticommutation relation

$$\zeta_i \zeta_j + \zeta_j \zeta_i = 0 \tag{E.62}$$

for all i and j. As a particular consequence of this condition the square and all higher powers of a variable vanish,

$$\zeta_i^2 = 0 \tag{E.63}$$

The Grassmann algebra generate a Grassmann algebra of functions which have the form

$$f(\zeta) = f^{(0)} + \sum_i f_i^{(1)} + \sum_{i_1<i_2} f_{i_1 i_2}^{(2)} \zeta_{i_1} \zeta_{i_2} + ... + f^{(n)} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_n} \tag{E.64}$$

where the coefficients $f^{(k)}$ are ordinary complex numbers.

On this algebra we will need to define the derivative. We first consider a simple Grassmann algebra of order $n = 2$ with the variables ζ_1 and ζ_2.

$$f(\zeta_1, \zeta_2) = f^{(0)} + f_1^{(1)} \zeta_1 + f_2^{(1)} \zeta_2 + f^{(2)} \zeta_1 \zeta_2$$

$$\frac{\partial f}{\partial \zeta_1} = f_1^{(1)} + f^{(2)} \zeta_2, \qquad \frac{\partial f}{\partial \zeta_2} = f_2^{(1)} - f^{(2)} \zeta_1. \tag{E.65}$$

Note the minus sign in the last equation of (E.65),

$$\frac{\partial}{\partial \zeta_j} \zeta_1 \zeta_2 = \delta_{j1} \zeta_2 - \delta_{j2} \zeta_1.$$

The general rule for differentiation of a product is given by

$$\frac{\partial}{\partial \zeta_j} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_m} = \delta_{j i_1} \zeta_{i_2} ... \zeta_{i_m} - \delta_{j i_2} \zeta_{i_1} \zeta_{i_3} ... \zeta_{i_m} + ... + (-1)^{m-1} \delta_{j i_2} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_{m-1}} \tag{E.66}$$

The respective factor ζ_{i_k} is anticommuted to the left until the derivative operator can be directly applied. We may prove the following properties of the derivatives

$$\frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial \zeta_j}\frac{\partial}{\partial \zeta_i} = 0 \tag{E.67}$$

$$\frac{\partial}{\partial \zeta_i}\zeta_j + \zeta_j\frac{\partial}{\partial \zeta_i} = 0 \tag{E.68}$$

Grassmann integration

An attempt to to introduce an indefinite integral as the inverse of differentiation is bound to fail. This illustrated by the fact that according to (E.67) the second derivative of any Grassmann function vanishes, so that the inverse operation does not exist, for if there was an inverse to $\frac{\partial^2 F}{\partial \zeta^2}$ it should give

$$\int d\zeta \frac{\partial^2 F}{\partial \zeta^2} = \frac{\partial F}{\partial \zeta}$$

However as

$$0 = \int d\zeta 0 = \int d\zeta \frac{\partial^2 F}{\partial \zeta^2}$$

this would imply we always have

$$\frac{\partial F}{\partial \zeta} = 0$$

which is not true in general.

We must be content with some formal definition. One way to arrive at it is to require that integration be translationally invariant. For an arbitrary function $g(\zeta) = g_1 + g_2\zeta$ we have

$$\begin{aligned}\int d\zeta g(\zeta+\eta) &= \int d\zeta \left[g_1 + g_2(\zeta+\eta)\right] = \int d\zeta \left[g_1 + g_2\zeta\right] + \int d\zeta g_2\eta \\ &= \int d\zeta g(\zeta) + \left[\int d\zeta 1\right] g_2\eta = \int d\zeta g(\zeta)\end{aligned} \tag{E.69}$$

The translational invariance requires the integral of 1 is zero. The following postulates uniquely fix the value of any integral.

$$\int d\zeta 1 = 0, \tag{E.70}$$

$$\int d\zeta\zeta = 1. \tag{E.71}$$

Eq. (E.70) comes from the condition of translational invariance. The sole non-vanishing integral $\int d\zeta\zeta$ arbitrarily is assigned the value 1. This is a convenient normalisation condition.

We see that integration is equivalent to differentiation. Generalising integration rules to higher dimensions straightforward

$$\int d\zeta_i 1 = 0, \tag{E.72}$$

$$\int d\zeta_i\zeta_j = \delta_{ij}. \tag{E.73}$$

Note that the differentials $d\zeta_i$ must anticommute with all other Grassmann variables as integration is equivalent to differentiation. In order to obtain analog results of conventional integration we introduce complex Grassmann variables. Let us start with two disjoint sets of Grassmann variables $\zeta_1^*, ..., \zeta_n^*$ and $\zeta_1, ..., \zeta_n$, which are all mutually anticommutating

$$\{\zeta_i, \zeta_j\} = \{\zeta_i^*, \zeta_j^*\} = \{\zeta_i, \zeta_j^*\} = 0 \tag{E.74}$$

The two sets are related, using complex cunjugation, according to

$$\begin{aligned}
(\zeta_i)^* &= \zeta_i^*, \\
(\zeta_i^*)^* &= -\zeta_i, \\
(\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m})^* &= \zeta_{i_m}^*...\zeta_{i_2}^*\zeta_{i_1}^* \\
(\lambda\zeta_i)^* &= \lambda^*\zeta_i^*
\end{aligned} \tag{E.75}$$

where λ is a complex number.

In order to develop functional integral formalism for Grassmann fields we need to solve *Gaussian integrals.*

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp\left\{ - \sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l \right\} \tag{E.76}$$

To simplify the notation, let us write this as

$$I = \int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} \tag{E.77}$$

The calculation in principle is very simply because grassmann functions can at worst be linear in each variable, causing the series expansion of the exponential function to terminate. On the other hand , according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes.

Let us consider the case where we have two pairs of variables. The exponential then reads

$$e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2 - \frac{1}{3!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^3 + \ldots \tag{E.78}$$

Obviously this series terminates beyond second order, so we have

$$e^{-\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b} = 1 - \sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b + \frac{1}{2!} (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b)^2. \tag{E.79}$$

Let us consider the integration of the first two terms, obviously we have

$$\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; 1 = 0$$

$$\int d\zeta_1^* d\zeta_2^* \int d\zeta_1 d\zeta_2 \; (\sum_{a,b=1}^{2} \zeta_a^* M_{ab} \zeta_b) = 0$$

as the number of variables integrated over is greater than the number of variables appearing in the integrand. For the case of two pairs of variables one effectively has

$$\begin{aligned} e^{-\zeta^* M \zeta} &\rightarrow \frac{1}{2!} (\zeta^* M \zeta)^2 \\ &= \frac{1}{2!} (\zeta_1^* M_{11} \zeta_1 + \zeta_1^* M_{12} \zeta_2 + \zeta_2^* M_{21} \zeta_1 + \zeta_2^* M_{22} \zeta_2)^2 \\ &= (M_{11} M_{22} - M_{12} M_{21}) \zeta_1^* \zeta_1 \zeta_2^* \zeta_2 \end{aligned} \tag{E.80}$$

where the last line follows from the anticommutating character of the Grassmann numbers. The integration of $\zeta_1^*\zeta_1\zeta_2^*\zeta_2$, gives unity, and so for this case

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = detM \tag{E.81}$$

One should suspect that this result holds in general. For the case of N pairs of variables, only the term of order $(\zeta^* M\zeta)^N$ survives in the expansion of the exponential and contributes to the integral:

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = \frac{(-1)^N}{N!} \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k)(\zeta^* M\zeta)^N. \tag{E.82}$$

In view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than go through this combinatorial exercise we will follow the method given in (Brown QFT).(page83) which is presented in Appendix(B). The integral is some function $I(M)$ of the matrix M, let us derive a differential equation for this function. Since $\zeta^* M\zeta$ contains the product of two anti commutating variables and thus a commutating variable itself,

$$\begin{aligned} \delta_M(\zeta^* M\zeta)^N &= (\zeta^*(M+\delta M)\zeta)^N - (\zeta^* M\zeta)^N \\ &= n(\zeta^*\delta M\zeta)(\zeta^* M\zeta)^{n-1} \end{aligned} \tag{E.83}$$

$$\begin{aligned} \delta I &= \int [d\zeta^* d\zeta]\, (e^{-\zeta^*(M+\delta M)\zeta} - e^{-\zeta^* M\zeta}) \\ &= \int [d\zeta^* d\zeta] \sum_{n=0} \frac{(-1)^n}{n!} [(\zeta^*(M+\delta M)\zeta)^n - (\zeta^* M\zeta)^n] \\ &= -\int [d\zeta^* d\zeta]\, (\zeta^*\delta M\zeta) \sum_{n=0} \frac{(-1)^{n-1}}{(n-1)!} (\zeta^* M\zeta)^{n-1} \\ &= -\int [d\zeta^* d\zeta]\, \zeta^*\delta M\zeta\, e^{-\zeta^* M\zeta} \end{aligned} \tag{E.84}$$

Since $\zeta^* M\zeta$ commute, the derivative of $e^{-\zeta^* M\zeta}$ is given by

$$
\begin{aligned}
\frac{\partial}{\partial\zeta_k^*}e^{-\zeta^* M\zeta} &= \frac{\partial}{\partial\zeta_k^*}\sum_{n=0}\frac{(-1)^n}{n!}(\zeta^* M\zeta)^n \\
&= \sum_{n=1}\frac{(-1)^n}{n!}\Bigg([\frac{\partial}{\partial\zeta_k^*}(\zeta^* M\zeta)](\zeta^* M\zeta)^{n-1}+ \\
&\quad + (\zeta^* M\zeta)[\frac{\partial}{\partial\zeta_k^*}(\zeta^* M\zeta)](\zeta^* M\zeta)^{n-2}+\cdots+(\zeta^* M\zeta)^{n-1}[\frac{\partial}{\partial\zeta_k^*}(\zeta^* M\zeta)]\Bigg) \\
&= -\sum_{m=1}M_{km}\zeta_m\sum_{n=1}\frac{(-1)^{n-1}}{(n-1)!}(\zeta^* M\zeta)^{n-1} \\
&= -\sum_{m=1}M_{km}\zeta_m e^{-\zeta^* M\zeta}.
\end{aligned}
\tag{E.85}
$$

Hence

$$
\begin{aligned}
\delta I &= \int[d\zeta^* d\zeta]\zeta^*\delta M(M^{-1}M\zeta\ e^{-\zeta^* M\zeta}) \\
&= \int[d\zeta^* d\zeta]\zeta^*\delta MM^{-1}\frac{\partial}{\partial\zeta^*}e^{-\zeta^* M\zeta}
\end{aligned}
\tag{E.86}
$$

From () we have

$$
\frac{\partial}{\partial\zeta_k^*}(\zeta_m^* F)=\delta_{km}F-\zeta_m^*\frac{\partial}{\partial\zeta_k^*}F. \tag{E.87}
$$

Therefore,

$$
\delta I=\int[d\zeta^* d\zeta]\sum_{k,m}(\delta MM^{-1})_{mk}\left\{\delta_{km}e^{-\zeta^* M\zeta}-\frac{\partial}{\partial\zeta_k^*}(\zeta_m^* e^{-\zeta^* M\zeta})\right\} \tag{E.88}
$$

Since the Grassmann integral of a derivative vanishes,

$$
\delta I=(Tr\delta MM^{-1})I, \tag{E.89}
$$

which gives

$$
\delta\ln I=\frac{\delta I}{I}=Tr(\delta MM^{-1}), \tag{E.90}
$$

It turns out that

$$Tr(\delta M M^{-1}) = \delta(\ln\ detM) \tag{E.91}$$

which we digress to prove.

$$detM = \sum_n M_{kn} C_{nk} \tag{E.92}$$

Taking the derivative of this, noting that C_{mk} is independent of the element M_{km},

$$\frac{\partial}{\partial M_{km}} detM = C_{mk} \tag{E.93}$$

Now the well known formul for the inverse matrix M^{-1} is

$$(M^{-1})_{kl} = \frac{C_{kl}}{detM} \tag{E.94}$$

So that we have

$$\frac{\partial}{\partial M_{km}} detM = (M^{-1})_{km} detM \tag{E.95}$$

or

$$\frac{\partial}{\partial M_{km}} \ln detM = (M^{-1})_{km}. \tag{E.96}$$

Accodingly,

$$\begin{aligned} \delta \ln detM &= \sum_{k,m} \delta M_{km} \frac{\partial \ln detM}{\partial M_{km}} \\ &= \sum_{k,m} \delta M_{km} (M^{-1})_{mk} \\ &= Tr(\delta M M^{-1}) \\ &= Tr(M^{-1} \delta M). \end{aligned} \tag{E.97}$$

We have thus established (E.91). Therefore the equation (E.90) for $I(M)$ becomes

$$\delta \ln I = \delta(\ln\ detM), \tag{E.98}$$

with the solution

$$I(M) = Const.\ detM. \tag{E.99}$$

Treating the problem as a differential equation for $I(M)$, we set $M = 1$ in order to determine the proportionalitiy constant,

$$\begin{aligned}
I(1) &= Const. = \left[\int d\zeta^* d\zeta e^{-\zeta^*\zeta}\right]^N \\
&= \prod_{k=1}^{N} \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^{N} \zeta_i^* \zeta_i)^N \\
&= \frac{1}{N!} \prod_{k=1}^{N} \int d\zeta_k^* d\zeta_k (-\sum_{i=1}^{N} \zeta_i^* \zeta_i)^N \\
&= (-1)^N \int d\zeta_N^* d\zeta_N \ldots d\zeta_1^* d\zeta_1 (\zeta_N^* \zeta_N \ldots \zeta_1^* \zeta_1) \\
&= 1^N = 1.
\end{aligned} \tag{E.100}$$

Hence the constant is unity and we do obtain the expected result:

$$I = \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) e^{-\zeta^* M \zeta} = detM \tag{E.101}$$

This should be compared to the ordinary integration where the corresponding integral gives $\det M^{-1}$.

Grassmann generating Functional

It is not surprising that the Gaussian integral formula (E.101) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp -\zeta^* M \zeta = detM e^{-\frac{1}{2}\rho^T A^{-1} \rho}. \tag{E.102}$$

Here ρ is an n-component vector of Grassmann variables. Equation (E.102) is obtained by translating the integration variable, $\zeta' = \zeta + A^{-1}\rho$.

The construction of functional integration in section (4.1.2) did not make use of any special properties of the integration over field variables which might restrict the validity to ordinary c-numbers.

$$\boxed{\begin{array}{c}\int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[-\int d^dx'\, d^dx\bar{\chi}(x')A(x',x)\chi(x) + \int d^dx(\bar{\rho}(x)\chi(x) + \bar{\chi}(x)\rho(x))\right] \\ = detA \exp\left[\int d^dx'\, d^dx\bar{\rho}(x')A^{-1}(x',x)\rho(x)\right].\end{array}} \tag{E.103}$$

in which the measure is $\propto \prod_{\mathbf{r}} d\bar{\varphi}(r)d\varphi(r)$ and $Z(\rho = 0) = detA$. Note that to normalise the functional we divide by $detA$ as apposed to $det(A^{-1})$ in the bosonic case (**??**).

It is rather straightforward to extend the results of esction 4.1 to the fermionic case: The Grassmann functional derivative is defined

$$\frac{\delta G[\chi(y)]}{\delta\chi(x)} = \lim_{\Delta V_i \to 0} \frac{\partial G}{\partial \chi_i} \quad \text{where } \mathbf{x} \text{ is located in cell } \Delta V_i \tag{E.104}$$

The (2n)-point correlators

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \langle \chi(y_n), \ldots, \chi(y_1); \bar{\chi}(x_1), \ldots, \bar{\chi}(x_n)\rangle \tag{E.105}$$

can now be obtained by forming derivatives of the generating functional [1]

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \left.\frac{\delta^{2n} Z[\rho, \bar{\rho}]}{\delta\rho(x_n)\cdots\delta\rho(x_1)\delta\bar{\rho}(y_1)\cdots\delta\bar{\rho}(y_n)}\right|_{\rho=\bar{\rho}=0}. \tag{E.106}$$

E.6 QED from a Functional Integral

E.6.1 Photon Propagator in Different Gauges

The form of the propagator we used in our QED calculations was (in momentum space)

$$D_{F\mu\nu} = -\frac{4\pi}{k^2 + i\varepsilon}\eta_{\mu\nu} \tag{E.107}$$

which is only one of many choices of defining the photon propagator.

[1] The order of the derivatives was chosen in such that we get agreement with the bosionic case. This is not a trivial matter as the Grassmann derivatives $\delta/\delta\rho(x)$ and $\delta/\delta\rho(x)$ anticommute with the field variables $\chi(x)$ and $\bar{\chi}(x)$. One can show, however, that there is an even number of commutations when we carry out the differentiations of (E.106) and write the result in the form (E.105).

When we construct S-matrix elemenet, e.g.

$$j^{\mu}_{43}(p_4, p_3) D_{F\mu\nu}(k) j^{\nu}_{21}(p_2, p_1) \tag{E.108}$$

where $p_2 = p_1 - k$ and $p_4 = p_3 + k$. Now the transition currents obey the equation of continuity. Their four divergencies vanish, i.e. in momentum space

$$\begin{aligned} k_\nu j^{\nu}_{21}(p_1 - k, p_1) &= 0 \\ k_\nu j^{\nu}_{43}(p_3 + k, p_3) &= 0. \end{aligned} \tag{E.109}$$

Therefore one can add to $D_{F\mu\nu}$ the expression $k_\mu f_\nu(k) + k_\nu g_\mu(k)$ with arbitrary functions $f_\nu(k)$ and $g_\mu(k)$ without changing the result of the calculation.

Keeping the symmetry between the transition currents, we generalise () to

$$D_{F\mu\nu} = -\frac{4\pi}{k^2 + i\varepsilon}\eta_{\mu\nu} + k_\mu f_\nu(k) + k_\nu f_\mu(k). \tag{E.110}$$

The origin of this amniguity of the photon propagator is the gauge degrees of freedon of the electromagnetic field:

$$A_\mu(x) \to A_\mu(x) + \frac{\partial}{\partial x^\mu}\chi(x). \tag{E.111}$$

The Coulomb gauge

The propagator takes the form

$$\begin{aligned} D_{Cij}(k) &= \frac{4\pi}{k^2 + i\varepsilon}\left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) \\ D_{C0j}(k) &= D_{Ci0}(k) = 0 \\ D_{C00}(k) &= \frac{4\pi}{\mathbf{k}^2} \end{aligned} \tag{E.112}$$

The component $D_{C00}(k)$ is just the fourier transform of the electrostatic potential $1/r$.

E.6.2 The Generating Functional Integral of QED

We include coupling of the electromagnetic field to the Dirac field. Since our previous work is valid in any sort of current coupling to the photon field

$$Z[J, \eta, \overline{\eta}] = \int [\mathcal{D}A][\mathcal{D}\psi][\mathcal{D}\overline{\psi}] \exp\left\{ i \int \mathcal{L}_{QED} \right\}, \tag{E.113}$$

where

$$\begin{aligned} \mathcal{L}_{QED} &= -\frac{1}{4e^2} F_{\mu\nu}^2 + \frac{1}{2\xi}(\partial_\mu A^\mu)^2 - \overline{\psi}\left[\gamma_\mu \left(\frac{1}{i}\partial^\mu - A^\mu\right) + m_0\right]\psi + \\ & \quad J^\mu A_\mu + \overline{\psi}\eta_0 + \overline{\eta_0}\psi. \end{aligned} \tag{E.114}$$

E.7 Canonical Quantisation of Scalar Field

We promote fields to operators, and impose equal time commutation relations on the fields and their conjugate momenta.

Position and momentum p are not operrators, instead they are just numbers.

The fields $\varphi(x,t)$ and their conjugate momentum fields $\pi(x,t)$ are operators

We have states as we do in ordinary quantum mechanics, but these are states of the field

$$\varphi = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\varphi(\vec{k})e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} + \varphi^*(\vec{k})e^{i(\omega_k t - \vec{k}\cdot\vec{x})}\right] \tag{E.115}$$

We promote φ to an operator by promoting the coefficients $\varphi(\vec{k})$ and $\varphi^*(\vec{k})$:

$$\begin{aligned} \varphi(\vec{k}) &= \hat{a}(\vec{k}) \\ \varphi^*(\vec{k}) &= \hat{a}^\dagger(\vec{k}) \end{aligned} \tag{E.116}$$

Conjugate momentum

$$
\begin{aligned}
\partial_t \hat{\varphi} &= \partial_t \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\vec{k})e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \\
&= \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_k}} \left[\hat{a}(\vec{k})(-i\omega_k)e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} + \hat{a}^\dagger(\vec{k})(+i\omega_k)e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right] \\
&= -i \int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega_k}{2}} \left[\hat{a}(\vec{k})e^{-i(\omega_k t - \mathbf{k}\cdot\mathbf{x})} - \hat{a}^\dagger(\vec{k})e^{i(\omega_k t - \mathbf{k}\cdot\mathbf{x})}\right]
\end{aligned}
\tag{E.117}
$$

E.7.1 Commutatin Relations

To quantise the scalar field we postulate the standard commutatin relations

$$
\begin{aligned}
[\hat{\varphi}(x), \hat{\pi}(y)] &= i\delta^3(\mathbf{x} - \mathbf{y}) \\
[\hat{\varphi}(x), \hat{\varphi}(y)] &= 0 \\
[\hat{\pi}(x), \hat{\pi}(y)] &= 0
\end{aligned}
\tag{E.118}
$$

E.7.2 Bose Statistics

$$
\begin{aligned}
|\vec{k}_1, \vec{k}_2 > &= \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)|0> \\
&= \hat{a}^\dagger(\vec{k}_2)\hat{a}^\dagger(\vec{k}_1)|0> \\
&= |\vec{k}_2, \vec{k}_1 >
\end{aligned}
\tag{E.119}
$$

which implies

$$
|\vec{k}_1, \vec{k}_2 >= |\vec{k}_2, \vec{k}_1 > \tag{E.120}
$$

E.8 Perturbation Theory in Canonical Approach

Any physical process is a transition from an initial state $|i >= |A(t_0) >$ to a final state $|f >= |A(t) >$

E.8.1 Pictures

The Schrodinger picture

In the Schrodinger picture the time dependence is carried by the states according to Schrodinger's equation

$$i\hbar\frac{d}{dt}|A(t)>_S = \hat{H}|A(t)> \tag{E.121}$$

This can be formally solved in terms of the state of the system at an arbitrary initial time t_0

$$|A(t)>_S = \hat{U}|A(t_0)>_S \tag{E.122}$$

where $\hat{U}$ is the unitary operator:

$$\hat{U} := \hat{U}(t, t_0) = e^{-i\hat{H}(t-t_0)/\hbar} \tag{E.123}$$

The Heisenberg picture

Using $\hat{U}$ we perform the following transformations, defining the state $|A(t)>_H$ and operator $O^H(t)$:

$$|A(t)>_H = \hat{U}^\dagger|A(t)>_S = |A(t_0)>_S \tag{E.124}$$

and

$$O^H(t) = \hat{U}^\dagger O^S \hat{U} \tag{E.125}$$

The H indicates that this is the Heisenberg picture. At $t = t_0$ states and operators in the two pictures are the same. We see that in the Heisenberg picture states are constant in time.

Differentiation of (E.125) gives the Heisenberg equation of motion

$$\begin{aligned} i\hbar\frac{d}{dt}O^H(t) &= (i\hbar\frac{d}{dt}\hat{U}^\dagger)O^S\hat{U} + \hat{U}^\dagger O^S(i\hbar\frac{d}{dt}\hat{U}) \\ &= \hat{U}^\dagger\hat{O}^S\hat{U}\hat{H} - \hat{H}\hat{U}^\dagger\hat{O}^S\hat{U} \\ &= [\hat{O}^H, \hat{H}] \end{aligned} \tag{E.126}$$

The Interaction picture

The Interaction picture arises if the Hamiltonian is split into two parts

$$\hat{H} = \hat{H}_0 + \hat{H}_{int} \tag{E.127}$$

The interaction picture is related to the Schrodinger picture by the unitary transformation

$$\hat{U}_0 = \hat{U}_0(t, t_0) = e^{-i\hat{H}_0(t-t_0)/\hbar} \tag{E.128}$$

i.e.

$$|A(t) >_I = \hat{U}_0^\dagger |A(t) >_S \tag{E.129}$$

and

$$O^I(t) = \hat{U}_0^\dagger O^S \hat{U}_0 \tag{E.130}$$

Differentiation of (E.130) gives the equation of motion in the interaction picture

E.8.2 Perturbation Theory

E.9 QED from Interaction Picture

$$\mathcal{H}_{int} = -ie\overline{\psi}\gamma^\mu \psi A_\mu. \tag{E.131}$$

There are various factrs of $\mathcal{H}_{int}$ in the expansion of thr interacting Lagrangian. We use Wick's theorem to pair off the various fermion photon lines to for propagators and vertices.