Appendix F

Electro-Weak Theory

F.1 Fermi Interactions

Recall because of crossing symmetry, processes which differ in the grouping of incoming and outgoing particles are related to each other. In particular the matrix element of three body decay can be derived from that of the two-body scattering process. For example fig (F.1)

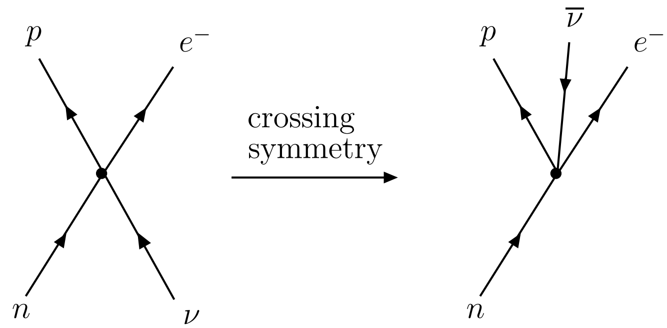


Figure F.1: .

From parity violatoin in nuclear β decay interaction was postulated to be of the form

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x [\overline{u}_p(x)\gamma_\mu(C_V + C_A)u_n(x)] \times [\overline{u}_e(x)\gamma^\mu(1-\gamma_5)u_\nu(x)] \tag{F.1}$$

where the leptonic contribution

$$\overline{u}_e(x)\gamma^\mu(1-\gamma_5)u_{\nu_e}(x) \tag{F.2}$$

resembles the electromagnetic transition current.

By analogy with the electromagnetic current, we therefore introduce the weak leptonic current:

$$\begin{aligned} J_\mu^{(L)}(x) &= J_\mu^{(e)}(x) + J_\mu^{(\mu)}(x) + J_\mu^{(\tau)}(x) \\ &= \overline{u}_e(x)\gamma_\mu(1-\gamma_5)u_{\nu_e}(x) + \overline{u}_\mu(x)\gamma_\mu(1-\gamma_5)u_{\nu_\mu}(x) \\ &\quad + \overline{u}_\tau(x)\gamma_\mu(1-\gamma_5)u_{\nu_\tau}(x) \end{aligned} \tag{F.3}$$

Motivated by (F.1)

$$H_{int} = \frac{G}{\sqrt{2}} \int d^3x J_\mu^{(L)\dagger}(x) J_{(L)}^\mu(x). \tag{F.4}$$

We consider purely leptonic processes.

F.2 Intermediate Vector Gauge Boson Theory

F.2.1 Free Massive Vector Boson

We construct wave functions which describe particles with spin 1 out of solutions of the Dirac equation and the wave equation it generates.

In the rest system we find the following linearly independent combinations

$$\begin{aligned} \omega_{\alpha\beta}^{(+)}(0, i=0) &= \delta_{\alpha 1}\delta_{\beta 1} \\ \omega_{\alpha\beta}^{(+)}(0, i=1) &= \delta_{\alpha 2}\delta_{\beta 1} + \delta_{\alpha 1}\delta_{\beta 2} \\ \omega_{\alpha\beta}^{(+)}(0, i=2) &= \delta_{\alpha 2}\delta_{\beta 2} \end{aligned} \tag{F.5}$$

Each of these spinors represents an eigenvector of the operator of total spin, which in the rest system is defined by

$$\frac{1}{2}\hbar\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'} = \frac{1}{2}\hbar\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\beta\beta'} + \frac{1}{2}\hbar\hat{\Sigma}^3_{\beta\beta'}\delta_{\alpha\alpha'} \tag{F.6}$$

where

$$\hat{\Sigma}^3_{\alpha\alpha'} = \begin{pmatrix} \hat{\sigma}_3 & 0 \\ 0 & \hat{\sigma}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{F.7}$$

We verify that the multispinors $\omega^{(+)}(0,4)$ fulfill the eigenvector equation

$$\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i) = \hbar(s-i)\omega^{(+)}(0,i)$$

Obviously we have

$$\begin{aligned} \hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'1} &= +\delta_{\alpha 1} \\ \hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2} &= -\delta_{\alpha 2} \end{aligned} \tag{F.8}$$

$$\begin{aligned} \left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=0)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\ \omega^{(+)}_{\alpha'\beta'}(0,i=0) \\ &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'1}\delta_{\beta 1} + \delta_{\alpha 1}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'1} \\ &= \frac{\hbar}{2}2(\delta_{\alpha 1}\delta_{\beta 1}) \\ &= \hbar(3-2)\omega^{(+)}_{\alpha\beta}(0,i=0) \end{aligned} \tag{F.9}$$

$$\begin{aligned} \left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=1)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\ \omega^{(+)}_{\alpha'\beta'}(0,i=1) \\ &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2}\delta_{\beta 1} + \delta_{\alpha 2}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'1} \\ &+ \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'1}\delta_{\beta 2} + \delta_{\alpha 1}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'2} \\ &= -\frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 1} + \frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 1} \\ &+ \frac{\hbar}{2}\delta_{\alpha 1}\delta_{\beta 2} - \frac{\hbar}{2}\delta_{\alpha 2}\delta_{\beta 2} = 0 \end{aligned} \tag{F.10}$$

$$\begin{aligned}
\left(\frac{1}{2}\hbar\hat{\Sigma}^3\omega^{(+)}(0,i=2)\right)_{\alpha\beta} &= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'\beta\beta'}\,\omega^{(+)}_{\alpha'\beta'}(0,i=2) \\
&= \frac{\hbar}{2}\hat{\Sigma}^3_{\alpha\alpha'}\delta_{\alpha'2}\delta_{\beta2} + \delta_{\alpha2}\frac{\hbar}{2}\hat{\Sigma}^3_{\beta\beta'}\delta_{\beta'2} \\
&= -\frac{\hbar}{2}2(\delta_{\alpha2}\delta_{\beta2}) \\
&= -\hbar(3-2)\omega^{(+)}_{\alpha\beta}(0,i=2) \qquad \text{(F.11)}
\end{aligned}$$

Now we can transform these multispinors into an arbitrary frame of reference via the inverse Lorentz transform.

$$\hat{S}_{\alpha\alpha'\beta\beta'}\left(\frac{\mathbf{p}}{E}\right) = \hat{S}_{\alpha\alpha'}\left(\frac{\mathbf{p}}{E}\right)\hat{S}_{\beta\beta'}\left(\frac{\mathbf{p}}{E}\right) \qquad \text{(F.12)}$$

Applying this we get

$$\omega^{(+)}(x;p,i) = \hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^{(+)}(0,i), \qquad \omega^{(-)}(x;p,i) = \hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^{(-)}(0,i) \qquad \text{(F.13)}$$

Now every wave function can be written as a superposition of plane waves:

$$\begin{aligned}
\Psi^{(+)}_{\alpha\beta}(x;p,i) &= \omega^{(+)}_{\alpha\beta}(x;p,i)e^{-ip\cdot x/\hbar} \\
\Psi^{(-)}_{\alpha\beta}(x;p,i) &= \omega^{(-)}_{\alpha\beta}(x;p,i)e^{+ip\cdot x/\hbar} \qquad \text{(F.14)}
\end{aligned}$$

and thus

$$\begin{aligned}
\Psi_{\alpha\beta}(x) &= \sum_i \int c^{(+)}(p,i)\Psi^{(+)}_{\alpha\beta}(x;p,i)d^3p \\
&+ \sum_i \int c^{(-)}(p,i)\Psi^{(-)}_{\alpha\beta}(x;p,i)d^3p. \qquad \text{(F.15)}
\end{aligned}$$

It is easily seen that the plane wave satify Diracs equation. We consider one particular example

$$(i\hbar\gamma\cdot\partial - m_0c)(\omega^{(+)}(x;p,i=1)e^{-ip\cdot x/\hbar}).$$

first recall that

$$(i\hbar\gamma\cdot\partial - m_0 c)\left[\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^1(0)e^{-ip\cdot x/\hbar}\right] = 0$$

where

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now

$$\begin{aligned}
& (i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}(\omega^{(+)}_{\alpha'\beta}(x;p,i=1)e^{-ip\cdot x/\hbar}) \\
= \; & (i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\left(\hat{S}_{\alpha'\gamma}\left(\frac{\mathbf{p}}{E}\right)\hat{S}_{\beta\delta}\left(\frac{\mathbf{p}}{E}\right)\omega^{(+)}_{\gamma\delta}(0,i=1)e^{-ip\cdot x/\hbar}\right) \\
= \; & (i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\left(\hat{S}_{\alpha'1}\left(\frac{\mathbf{p}}{E}\right)e^{-ip\cdot x/\hbar}\right)\hat{S}_{\beta1}\left(\frac{\mathbf{p}}{E}\right) \\
= \; & \underbrace{\left[(i\hbar\gamma\cdot\partial - m_0 c)\hat{S}\left(\frac{\mathbf{p}}{E}\right)\omega^1(0)e^{-ip\cdot x/\hbar}\right]_\alpha}_{=0} \times\hat{S}_{\beta1}\left(\frac{\mathbf{p}}{E}\right) \qquad \text{(F.16)}
\end{aligned}$$

Let us apply the Dirac equation to this first index α

$$\begin{aligned}
(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) &= \sum_i\int c^{(+)}(p,i)(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi^{(+)}_{\alpha'\beta}(x;p,i)d^3p \\
&+ \sum_i\int c^{(-)}(p,i)(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi^{(-)}_{\alpha'\beta}(x;p,i)d^3p
\end{aligned} \qquad \text{(F.17)}$$

The multispinors therefore fulfill the following Dirac equations separately:

$$\begin{aligned}
(i\hbar\gamma\cdot\partial - m_0 c)_{\alpha\alpha'}\Psi_{\alpha'\beta}(x) &= 0 \\
(i\hbar\gamma\cdot\partial - m_0 c)_{\beta\beta'}\Psi_{\alpha\beta'}(x) &= 0
\end{aligned} \qquad \text{(F.18)}$$

These are the so-called Bargmann-Wigner equations. Each component is also a solution of the Klein-Gordon equation

$$\left(\Box + \frac{m_0^2c^2}{\hbar^2}\right)\Psi_{\alpha\beta}(x) = 0. \tag{F.19}$$

The two Dirac equations for the symmetric matrix $\Psi_{\alpha\beta}(x)$ may be written as

Since the 4×4 spinor is symmetric, it may be expanded in terms of a complete set of symmetric elements of the Clifford algebra representation

$$\gamma^\mu \hat{C}, \quad \hat{\sigma}^{\mu\nu}\hat{C} \tag{F.20}$$

We define

$$\Psi(x) = m_0\gamma_\mu \hat{C} W^\mu(x) + \frac{1}{2}\hat{\sigma}^{\mu\nu}\hat{C} G_{\mu\nu}(x) \tag{F.21}$$

where thye coefficients $W^\mu(x)$ and $G^{\mu\nu}(x)$ are generally complex and transform under Loretz transformations like a vector and an anti-symmetric tensor, respectively. The Bargmann-Winger equations now become

$$\begin{aligned}
(i\hbar \cdot \partial\gamma - m_0c)\left(\frac{1}{\hbar}m_0c\gamma_\mu W^\mu(x) + \frac{1}{2}\hat{\sigma}_{\mu\nu}G^{\mu\nu}(x)\right)\hat{C} &= 0 \\
\left(\frac{1}{\hbar}m_0c\gamma_\mu W^\mu(x) + \frac{1}{2}\hat{\sigma}_{\mu\nu}G^{\mu\nu}(x)\right)\hat{C}(i\hbar\gamma^T \cdot \overleftarrow{\partial} - m_0c) &= 0
\end{aligned} \tag{F.22}$$

Using that $\hat{C}\gamma_\mu^T = -\gamma_\mu \hat{C}$

$$\begin{aligned}
&\Big(im_0c\partial_\alpha W_\mu(x)\gamma^\alpha\gamma^\mu - m_0^2c^2\gamma_\mu W^\mu(x) + \frac{1}{2}i\hbar\gamma_\alpha\hat{\sigma}^{\mu\nu}\partial^\alpha G_{\mu\nu}(x) \\
&\quad -\frac{1}{2}m_0c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\Big)\hat{C} = 0,
\end{aligned} \tag{F.23}$$

$$\begin{aligned}
&\Big(im_0c\partial_\alpha W_\mu(x)\gamma^\mu\gamma^\alpha - m_0^2c^2\gamma_\mu W^\mu(x) + \frac{1}{2}i\hbar\gamma_\alpha\partial^\alpha\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x) \\
&\quad +\frac{1}{2}m_0c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\Big)\hat{C} = 0,
\end{aligned} \tag{F.24}$$

Subtracting (F.24) from (F.23) gives

$$
\begin{aligned}
&i m_0 \partial_\alpha W_\mu(x)\{\gamma^\alpha\gamma^\mu - \gamma^\mu\gamma^\alpha\}\hat{C} - \frac{2m_0^2c^2}{\hbar}\gamma_\mu \hat{C} W_\mu(x) \\
&+\frac{1}{2} i\hbar\{\gamma_\alpha\hat{\sigma}^{\mu\nu} - \hat{\sigma}^{\mu\nu}\gamma_\alpha\}\partial_\alpha G_{\mu\nu}(x)\hat{C} - m_0 c\hat{\sigma}^{\mu\nu}G_{\mu\nu}(x)\hat{C} = 0. \qquad (F.25)
\end{aligned}
$$

Using

$$
\begin{aligned}
i m_0 \partial_\alpha W_\mu(x)\{\gamma^\alpha\gamma^\mu - \gamma^\mu\gamma^\alpha\}\hat{C} &= 2m_0\partial_\alpha W_\mu(x)\hat{\sigma}^{\alpha\mu}\hat{C} \\
&= m_0(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x))\hat{\sigma}_{\alpha\mu}\hat{C}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2} i\hbar\{\gamma_\alpha\hat{\sigma}^{\mu\nu} - \hat{\sigma}^{\mu\nu}\gamma^\alpha\}\partial^\alpha G_{\mu\nu}(x)\hat{C} &= \frac{1}{2} i\hbar 2i(\eta^{\alpha\mu}\gamma^\nu - \eta^{\alpha\nu}\gamma^\mu)\hat{C}\partial_\alpha G_{\mu\nu}(x) \\
&= -2\hbar\gamma_\mu\hat{C}\partial_\alpha G^{\alpha\mu}(x)
\end{aligned}
$$

It follows that

$$
m_0 c(\partial^\alpha W^\mu(x) - \partial^\mu W^\alpha(x) - G^{\alpha\mu})\hat{\sigma}_{\alpha\mu}\hat{C} - 2\gamma_\mu\hat{C}\left(\hbar\partial_\alpha G^{\alpha\mu} + \frac{m_0^2c^2}{\hbar}W^\mu\right) = 0 \qquad (F.26)
$$

The coefficients of the linearly independent matrices $\hat{C}$ must vanish separately. Hence, for $m_0 \neq 0$, this implies that

$$
G^{\mu\nu} = \partial^\mu W^\nu - \partial^\nu W^\mu, \qquad (F.27)
$$

$$
\partial_\mu G^{\mu\nu} = -\frac{m_0^2c^2}{\hbar^2}W^\nu. \qquad (F.28)
$$

Expressed in terms of the vector field W^μ

$$
\Box W^\mu(x) - \partial^\mu(\partial_\nu W^\nu(x)) + \frac{m_0^2c^2}{\hbar^2}\, W^\mu(x) = 0 \qquad (F.29)
$$

On taking the divergence of this equation, one automatically obtains the Lorentz condition

$$\frac{m_0^2 c^2}{\hbar^2}\,\partial_\mu W^\mu(x) = 0 \tag{F.30}$$

This is in contrast to the photon case, where the Lorentz condition must be imposed as a subsidiary condition. The Proca equation then reduces to

$$\Box W^\mu(x) + \frac{m_0^2 c^2}{\hbar^2}\,W^\mu(x) = 0 \tag{F.31}$$

The Propagator

$$\eta^{\nu\lambda}\Box W_\lambda(x) - \partial^\nu(\partial^\lambda W_\lambda) + \frac{m_0^2 c^2}{\hbar^2}\eta^{\nu\lambda}W_\lambda(x) = J^\nu \tag{F.32}$$

In momentum space the left hand side reads

$$[\eta^{\nu\lambda}(-q^2 + M^2) + q^\nu q^\lambda]W_\lambda \tag{F.33}$$

The inverse operator $D_{\lambda\mu}(q)$ will have the structure

$$A(q^2)\eta_{\lambda\mu} + B(q^2)q_\lambda q_\mu \tag{F.34}$$

because there are only two second-rank tensors that can be formed.

$$\begin{aligned} & [\eta^{\nu\lambda}(-q^2 + M^2) + q^\nu q^\lambda]\,[A(q^2)\eta_{\lambda\mu} + B(q^2)q_\lambda q_\mu] \\ = \;& A(q^2)\,[(-q^2 + M^2)\delta^\nu_\mu + q^\nu q_\mu] + B(q^2)\,[q^\nu q_\mu(-q^2 + M^2) + q^2 q^\nu q_\mu] \\ = \;& \delta^\nu_\mu . \end{aligned}$$

Thus we get

$$A(q^2)\,(-q^2 + M^2) = 1$$

for $\mu = \nu$, and

$$A(q^2)\,[q^\nu q^\lambda] + B(q^2)\,[q^\nu q_\mu(-q^2 + M^2) + q^2 q^\nu q_\mu] = 0. \tag{F.35}$$

for $\mu \neq \nu$. Hence

$$A(q^2) = -\frac{1}{-q^2 + M^2} \tag{F.36}$$

and

$$B(q^2) = -\frac{1/M^2}{-q^2 + M^2} \tag{F.37}$$

The propagator

$$\begin{aligned} D^{\lambda\mu}(q) &= A(q^2)\eta^{\lambda\mu} + B(q^2)q^\lambda q^\mu \\ &= -\frac{\eta^{\lambda\mu}}{q^2 - M^2} + \frac{q^\lambda q^\mu/M^2}{q^2 - M^2} \\ &= \frac{-\eta^{\lambda\mu} + q^\lambda q^\mu/M^2}{q^2 - M^2} \end{aligned} \tag{F.38}$$

F.2.2 Interactions via Massive Bosons

vector meson exchange

$$\mathcal{L} = g_W^2(\overline{\Psi}_p\gamma^\mu\Psi_n)\frac{\eta_{\mu\nu} - q_\mu q_\nu/M_W^2}{q^2 - M_W^2 + i\epsilon}(\overline{\psi}_e\gamma^\nu\psi_\nu) \tag{F.39}$$

F.3 Lagrangian for Yang-Mills Theory

F.3.1 Fadeev-Popov Gauge Fixing

The line running upward represents the gauge orbit of the vector potential A_μ^Λ as the gauge transformation function Λ varies. By gauge invariance, all point along these lines are physically equivalent. Functionally integrating over all configurations overcounts the integrand repeatedly an infinite number of times giving us the non-sensical result

$$\int \mathcal{D}A_\mu \exp(iS[A]) = \infty. \tag{F.40}$$

This surface, the gauge slice, is the surface in function space that is described a gauge-fixing constraint. One might be tempted to solve the problem by simply inserting a gauge fixing factor

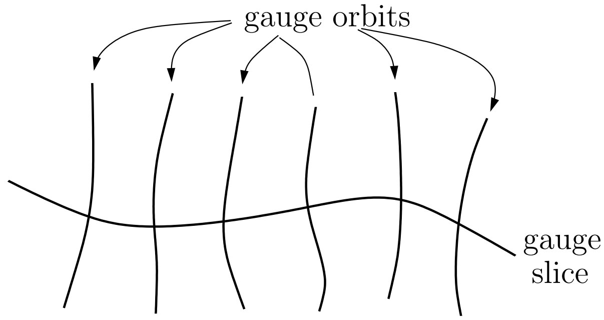


Figure F.2: Schematic representation of the vector potential gauge field configuration space.

$$\delta\left(F(A)\right).$$

into the functional integral, forcing it to respect the gauge choice. However, this is inconsistent. We know that the delta function changes when we make changes in it. For example, if we have a function $f(x)$ that has a zero at $x = a$, we recall that:

$$\delta(f(x)) = \frac{\delta(x-a)}{|f'(x)|}. \tag{F.41}$$

Consider the integral

$$1 = \int da\ \delta(g(a))\ \frac{dg}{da}. \tag{F.42}$$

$Z[J]$, in this form, is not particularly easy to calculate

$$1 = \int \mathcal{D}\Lambda(x)\ \delta(G(A^{\Lambda}))\ det\left(\frac{\delta G(A^{\Lambda})}{\delta\Lambda}\right) \tag{F.43}$$

This is the continuum generalisation of

$$1 = \left(\prod_i \int da_i\right) \delta^n(g(a))\ det\left(\frac{\partial g_i}{\partial a_j}\right) \tag{F.44}$$

which is a generalisation of

We choose

$$G(A) = F(A) - \omega(x), \tag{F.45}$$

where $\omega(x)$ can be any scalar function.

$$F[A] = \omega(x)$$

Substituting (F.43) into $Z[J]$

$$Z[J] = \int \mathcal{D}A \; det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \int \mathcal{D}\Lambda(x)\; \delta(G(A^\Lambda)) \;\exp(iS[A]) \tag{F.46}$$

Now we make the identification:

$$\begin{aligned} \mathcal{D}A &= \mathcal{D}A^\Lambda \\ S[A] &= S[A^\Lambda] \end{aligned} \tag{F.47}$$

Since A^Λ is just a dummy integration variable, we may rename it back to A.

The Fadeev-Popov determinant can be proven to be gauge independent:

$$\begin{aligned} \Delta_{FP}^{-1}(A_\mu^{\Lambda'}) &= \int \mathcal{D}\Lambda' \; \delta\left[G(A_\mu^{\Lambda'\Lambda})\right] \\ &= \int \mathcal{D}[\Lambda'\Lambda] \; \delta\left[G(A_\mu^{\Lambda'\Lambda})\right] \\ &= \int \mathcal{D}[\Lambda''] \; \delta\left[G(A_\mu^{\Lambda''})\right] \\ &= \Delta_{FP}^{-1}(A_\mu). \end{aligned} \tag{F.48}$$

Then we have

$$Z[J] = \left(\int \mathcal{D}\Lambda(x)\right) \int \mathcal{D}A \; \delta(G(A)) \; det\left(\frac{\delta G(A)}{\delta \Lambda}\right) \exp(iS[A]) \tag{F.49}$$

and so in (F.49) summation over gauge equivalent configurations has been factored put so that the divergent integral $\int \mathcal{D}\Lambda(x)$ gives a simply multiplicative factor.

The essential point is that the factor Δ_{FP} gives the correct measure in the functional integral, the factor needed for so many years to cure previous attempts to quantise gauge theories.

We have deal with the delta function. Note that $Z[J]$ as originally defined, is completely independent of the arbitrary function $\omega(x)$. It is convenient computationally to add together the contribution from all slices labeled by $\omega(x)$, each slice weighted with a gaussian function centred on $\omega = 0$.

F.3.2 Example: QED

Photon propagator

We may write $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{2} A^\mu \left[\Box \eta_{\mu\nu} - \left(1 - \frac{1}{\alpha}\right) \partial_\mu \partial_\nu \right] A^\nu = \frac{1}{2} A^\mu \tilde{P}_{\mu\nu} A^\nu \tag{F.50}$$

In momentum space the operator $\tilde{P}_{\mu\nu}$ becomes

$$(-q^2 \eta^{\nu\lambda} + (1 - \alpha^{-1}) q^\nu q^\lambda) A_\lambda \tag{F.51}$$

The inverse operator $D_{\lambda\mu}(q)$ will have the structure

$$(A(q^2)\eta_{\lambda\mu} + B(q^2) q_\mu q_\lambda) \tag{F.52}$$

$$\begin{aligned} & [(-q^2 \eta^{\nu\lambda} + (1 - \alpha^{-1}) q^\nu q^\lambda)] \, [(A(q^2)\eta_{\lambda\mu} + B(q^2) q_\mu q_\lambda)] \\ = \; & A(q^2) \, [-q^2 \delta^\nu_\mu + (1 - \alpha^{-1}) q^\nu q_\mu] + B(q^2) \, [-\alpha^{-1} q^2 q^\nu q_\mu] \\ = \; & \delta^\nu_\mu. \end{aligned} \tag{F.53}$$

Thus we get

$$A(q^2) \cdot -q^2 = 1$$

for $\mu = \nu$, and

$$A(q^2) \, [(1 - \alpha^{-1}) q^\nu q_\mu] - \alpha^{-1} B(q^2) \, [q^2 q^\nu q_\mu] = 0 \tag{F.54}$$

for $\mu = \nu$. Hence

$$A(q^2) = -\frac{1}{q^2} \tag{F.55}$$

and

$$B(q^2) = -\frac{1-\alpha}{q^4} \tag{F.56}$$

The propagator is then

$$\begin{aligned}(A(q^2)\eta_{\lambda\mu} + B(q^2)q_\mu q_\lambda) &= \left(-\frac{\eta_{\lambda\mu}}{q^2} - \frac{(1-\alpha)q_\mu q_\lambda}{q^4}\right) \\ &= -\frac{\eta_{\lambda\mu} + (1-\alpha)q_\mu q_\lambda/q^2}{q^2 + i\epsilon} \end{aligned} \tag{F.57}$$

F.3.3 Example: Non-Abelian Case

In QED, the determinant was independent of A, so could be factorised out as another unimportant overall constant

$Z[J]$ then becomes

$$\begin{aligned} Z[J] &= N(\alpha)\int \mathcal{D}\omega \exp\left[-i\int d^4x \frac{Tr\omega^2(x)}{2\alpha}\right]\int \mathcal{D}\Lambda(x) \\ &\quad \times \int \mathcal{D}A\, \delta(F(A) - \omega(x))\, det\left(\frac{\delta F^a(A)}{\delta\Lambda^b}\right)\exp(iS[A]) \\ &= N(\alpha)\int \mathcal{D}\Lambda(x)\int \mathcal{D}A\, det\left(\frac{\delta F^a(A)}{\delta\Lambda^b}\right)\exp\left(iS[A] - i\int d^4x \frac{Tr(F[A])^2}{2\alpha}\right) \end{aligned} \tag{F.58}$$

where $N(\alpha)$ is an unimportant noralisation constant.

$$[T^a, T^b] = i\epsilon^{abc}T^c$$

$$A^\Lambda_\mu = UA_\mu U^\dagger + \frac{i}{g}(\partial_\mu U)U^\dagger, \tag{F.59}$$

where

$$U(x) = \exp(ig\Lambda^a(x)T^a) \tag{F.60}$$

We have

$$\begin{aligned}
A_\mu^{c\Lambda}T^c(x) &= (\mathbf{1} + ig\Lambda^a(x)T^a)A_\mu^b(x)T^b(\mathbf{1} - ig\Lambda^a(x)T^a) + \frac{i}{g}(ig\partial_\mu\Lambda^c(x)T^c)(\mathbf{1} - ig\Lambda^b(x)T^b) \\
&= A_\mu^c(x)T^c - \partial_\mu\Lambda^c(x)T^c + ig\Lambda^a(x)A_\mu^b(x)[T^a, T^b] + \mathcal{O}(\Lambda^2) \\
&= A_\mu^c(x)T^c - \partial_\mu\Lambda^c(x)T^c - g\Lambda^a(x)A_\mu^b(x)\epsilon^{abc}T^c + \mathcal{O}(\Lambda^2) \qquad \text{(F.61)}
\end{aligned}$$

or

$$A_\mu^{a\Lambda}(x) = A_\mu^a(x) - \partial_\mu\Lambda^a(x) \tag{F.62}$$

$$\begin{aligned}
M_{ab} = \frac{\delta F_a(x)}{\delta\Lambda^b(y)} &= \int d^4z \frac{\delta F_a(x)}{\delta A_c^\mu(z)}\frac{\delta A_c^\mu(z)}{\delta\Lambda^b(y)} \\
&= \int d^4z[-\partial_\mu\delta_{ac}\delta^4(x-z)][] \\
&= \\
&= \int d^4z\partial_\mu\delta_{ac}\delta^4(x-z)(D^\mu)_{cb}\delta^4(z-y) \\
&= (\partial_\mu D^\mu)_{ab}\delta^4(x-y) \qquad \text{(F.63)}
\end{aligned}$$

where

$$(D^\mu)_{ab} = \partial_\mu\delta_{ab} + ig\epsilon_{abc}A^{\mu c}$$

The determinant in terms of a fermionic Gaussian integral over complex anticommutationg functions of the operator put in between the fields,

$$det\ M = \int \mathcal{D}\eta^*\mathcal{D}\eta \exp\left(\int d^4x d^4x' \eta^*(x)M(x-x')\eta(x')\right) \tag{F.64}$$

The Jacobian, Δ^F, may be represented as

$$\begin{aligned}
&det\ |M_{ab}|_{F=0} \\
&= \int \mathcal{D}\eta^{*a}\mathcal{D}\eta^a \exp\left(\int d^4x d^4x' \eta^{*a}(x)M_{ab}(x-x')\eta^b(x')\right) \qquad \text{(F.65)}
\end{aligned}$$

So far we have worked only with the integral $\int \mathcal{D}A \exp(iS[A])$. Say we wanted to calculate the quantity

$$\frac{\int \mathcal{D}A\mathcal{O}(A)\exp(iS[A])}{\int \mathcal{D}A \exp(iS[A])} \tag{F.66}$$

where the operator is gauge invariant. The same procedure goes through with the numerator as the replacement of A by A^Λ works. We find for the correlation function

$$\frac{\int \mathcal{D}A \int \mathcal{D}\eta^*\mathcal{D}\eta\ \mathcal{O}(A)\exp(iS[A] - \frac{1}{2\alpha}(F[A])^2 + iS_g[\eta,\eta^*;A])}{\int \mathcal{D}A \int \mathcal{D}\eta^*\mathcal{D}\eta \exp(iS[A] - \frac{1}{2\alpha}(F[A])^2 + iS_g[\eta,\eta^*;A])} \tag{F.67}$$

where the awkard constant factors have canceled.

F.3.4 Final Lagrangian for Yang-Mills

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 - \frac{1}{2\alpha}(\partial^\mu A^a_\mu)^2 + \eta^\dagger_a \partial^2 \eta_a \tag{F.68}$$

$$\begin{aligned}\mathcal{L}_{int} &= -\frac{1}{2}g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)\epsilon^{abc}A^{b\mu}A^{c\nu} \\ &\quad +\frac{1}{4}g^2\epsilon^{abc}\epsilon^{ade}A^b_\mu A^c_\nu A^{d\mu}A^{e\nu} \\ &\quad -ig\eta^{a\dagger}\epsilon^{abc}\partial^\mu A^c_\mu \eta^b.\end{aligned} \tag{F.69}$$

Boson propagator

$$(-q^2\eta^{\nu\lambda} + (1-\alpha^{-1})q^\nu q^\lambda)\delta^{bc}A^c_\lambda \tag{F.70}$$

The inverse operator $D^{ab}_{\lambda\mu}(q)$ will have the structure

$$(A(q^2)\eta_{\lambda\mu} + B(q^2)q_\mu q_\lambda)\delta^{ca} \tag{F.71}$$

The propagator is

$$-\frac{\eta_{\lambda\mu} + (1-\alpha)q_\mu q_\lambda/q^2}{q^2 + i\epsilon} \tag{F.72}$$

Ghost Propagator

The free ghost lagrangian is given by

$$\mathcal{L}_{FP} = -\eta_a^\dagger \Box \eta_a \tag{F.73}$$

The ghost field propagates like a masslees spin-zero field:

$$\Delta^{ab}(p) = \frac{\delta^{ab}}{p^2 + i\epsilon}. \tag{F.74}$$

As we saw the functional integral method allows us to read off the Feynmann rules for vertices directly from the interacting field theory.

F.3.5 Interaction Vertices of Gauge Fields

The fermion and Yang-Mills vertex

In QED the interaction term of the Lagrangian is given by

$$\mathcal{L}_{int}^{EM} = -e\overline{\Psi}\gamma^\mu \Psi A_\mu \tag{F.75}$$

The vertex function is just

$$\Gamma_{EM}^\mu = -e\gamma^\mu. \tag{F.76}$$

In the same way we can read off from the interaction Lagrangian the vertex function for the coupling of fermions and Yang-Mills field

In general the Feynman rules are obtained by varying the corresponding action integral in momentum space.

Triple vertex

$$S_{int}^{trip} = \int d^4x \mathcal{L}_{int}^{trip} \tag{F.77}$$

$$\begin{aligned}\mathcal{L}_{int}^{quad} &= -\frac{1}{2}g(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)\epsilon_{abc}A_b^\mu A_c^\nu \\ &= -g\partial_\mu A_\nu^a \epsilon_{abc} A_b^\mu A_c^\nu \end{aligned} \tag{F.78}$$

$$A_\mu^a(x) = \int \frac{d^4p}{(2\pi)^4} A_\mu^a(p) e^{-ip\cdot x} \tag{F.79}$$

in momentum space

$$\begin{aligned} S_{int}^{trip} &= \int d^4x \mathcal{L}_{int}^{trip} \\ &= -g\epsilon_{abc}\int d^4x \partial_\mu A_\nu^a A_b^\mu A_c^\nu \\ &= -g\epsilon_{abc}\int d^4x \frac{d^4p d^4k d^4q}{(2\pi)^{12}}(-ip_\mu)A_\nu^a(p)A_b^\mu(q)A_c^\nu(k)e^{-i(p+q+k)\cdot x} \\ &= ig\epsilon_{abc}\int \frac{d^4p\ d^4k\ d^4q}{(2\pi)^8} p_\mu A_\nu^a(p)A_b^\mu(q)A_c^\nu(k)\delta(p+q+k) \\ &= ig\epsilon_{abc}\int \frac{d^4p\ d^4k\ d^4q}{(2\pi)^8}\ \delta(p+q+k) \\ &\quad \frac{1}{3!}(p_\mu A_\nu^a(p)A_b^\mu(q)A_c^\nu(k) + \\ &\quad\ p_\mu A_\nu^a(p)A_b^\mu(k)A_c^\nu(q) + \\ &\quad\ q_\mu A_\nu^a(q)A_b^\mu(k)A_c^\nu(p) + \\ &\quad\ q_\mu A_\nu^a(q)A_b^\mu(p)A_c^\nu(k) + \\ &\quad\ k_\mu A_\nu^a(k)A_b^\mu(p)A_c^\nu(q) + \\ &\quad\ k_\mu A_\nu^a(k)A_b^\mu(q)A_c^\nu(p)) \end{aligned} \tag{F.80}$$

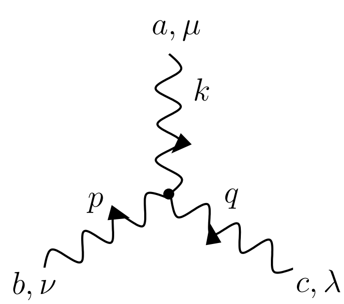


Figure F.3: .

Variation yields

$$\frac{\delta A^a_\mu}{\delta A^b_\nu} = \eta^\alpha_\beta \; \delta_{ab} \tag{F.81}$$

in momentum space

Quadruple vertex

$$S^{quad}_{int} \;\; = \;\; \int d^4x \mathcal{L}^{quad}_{int} \tag{F.82}$$

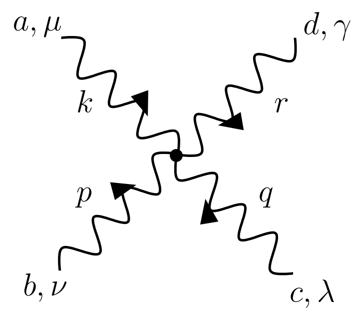
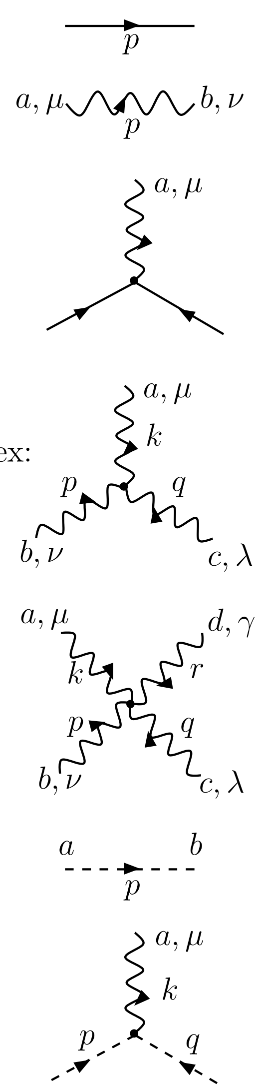


Figure F.4: .

F.3.6 Ghosts and Coupling to the Gauge Field

$$\mathcal{L}_g = \eta^\dagger_a \partial^2 \eta_a + g\eta^{a\dagger} \epsilon^{abc} \partial^\mu A^c_\mu \eta^b \tag{F.83}$$

F.3.7 Feynmann Rules for Yang-Mills Theory



Fermion:	$S_F(p) = \frac{\gamma^\mu p_\mu + M}{p^2 - M^2 + i\epsilon}$
Boson:	$(D_F)^{ab}_{\mu\nu} = \frac{-iP_{\mu\nu}\delta^{ab}}{p^2 + i\epsilon}$
Fermion vertex:	$ig\gamma^\mu T^a$
triple vertex:	$ig\epsilon_{abc}[\eta_{\mu\nu}(k-p)_\lambda + \eta_{\nu\lambda}(p-q)_\mu + \eta_{\lambda\mu}(q-k)_\nu]$
quadruple vertex:	$-ig^2[\epsilon_{abe}\epsilon_{cde}(\eta_{\mu\lambda}\eta_{\nu\gamma} - \eta_{\mu\gamma}\eta_{\nu\lambda}) + \epsilon_{ace}\epsilon_{bde}(\eta_{\mu\nu}\eta_{\lambda\gamma} - \eta_{\mu\gamma}\eta_{\mu\lambda}) + \epsilon_{ade}\epsilon_{cbe}(\eta_{\mu\lambda}\eta_{\mu\gamma} - \eta_{\mu\nu}\eta_{\lambda\gamma})]$
Ghost:	$\Delta_{ab}(p) = \frac{\delta_{ab}}{p^2 + i\epsilon}$
Ghost-vertex:	$g\epsilon_{abc}q^\mu$

Figure F.5: .

F.4 Electro-Weak Theory

Veltman: I do not care what or how, but what we must have is at least one renormalisable theory with massive charged bosons, and whether that looks like Nature is of no concern, those are details that will be fixed later by some model freak...

’t Hooft: I can do that.

Veltman: What do you say?

’t Hooft: I can do that.

F.4.1 Introduction

The Higgs field is introduced into the model causing spontaneous symmetry breaking. This leads to the electron gaining mass.

F.4.2 Massless Dirac Lagrangian

The Dirac Lagrangian with zero mas is given by

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \tag{F.84}$$

$$\begin{aligned}
\mathcal{L} &= i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \\
&= i(\overline{\psi}_L + \overline{\psi}_R)\gamma^{\mu}\partial_{\mu}(\psi_L + \psi_R) \\
&= i\overline{\psi}_L\gamma^{\mu}\partial_{\mu}\psi_L + i\overline{\psi}_R\gamma^{\mu}\partial_{\mu}\psi_R + i(\overline{\psi}_L\gamma^{\mu}\partial_{\mu}\psi_R + \overline{\psi}_R\gamma^{\mu}\partial_{\mu}\psi_L)
\end{aligned} \tag{F.85}$$

The last term vanishes as

$$\begin{aligned}
\overline{\psi}_L\gamma^{\mu}\partial_{\mu}\psi_R &= \left(\frac{1-\gamma_5}{2}\right)\overline{\psi}\gamma^{\mu}\partial_{\mu}\left(\frac{1+\gamma_5}{2}\right)\psi \\
&= \frac{1}{4}(1-\gamma_5+\gamma_5-\gamma_5^2)\,\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi \\
&= 0
\end{aligned} \tag{F.86}$$

so the mixed terms vanish and we are left

$$\mathcal{L} = i\overline{\psi}_L\gamma^{\mu}\partial_{\mu}\psi_L + i\overline{\psi}_R\gamma^{\mu}\partial_{\mu}\psi_R. \tag{F.87}$$

We see that the Lagrangian splits up into left- and right-handed parts.

F.4.3 Leptonic Fields in Electro-Weak Theory

$$\Psi_L = \begin{pmatrix} \nu_e \\ e_L \end{pmatrix} \tag{F.88}$$

where μ_e is the electron neutrino and e_L is the left-handed electron field.

$$e_L = \left(\frac{1-\gamma_5}{2}\right) e, \qquad e_R = \left(\frac{1+\gamma_5}{2}\right) e \tag{F.89}$$

If we take the nuetrino be massless, the there is only the left-handed component of the neutrino field. Since the field is entirely left-handed, it satifies the equation

$$\left(\frac{1-\gamma_5}{2}\right) \nu_e = \nu_e . \tag{F.90}$$

With no right-handed component of the neutrino field, we define

$$\Psi_R = \begin{pmatrix} 0 \\ e_R \end{pmatrix} \tag{F.91}$$

F.4.4 Charges of the Electroweak Interaction

F.4.5 Higgs Field

In the standard model of particle physics, which obviusly contains electroweak theory, the massess of all the particles are zero. an extra field, the so-called Higgs field, is inserted by hand to give the particles mass.

F.4.6 Feynmann Rules for Electroweak Theory

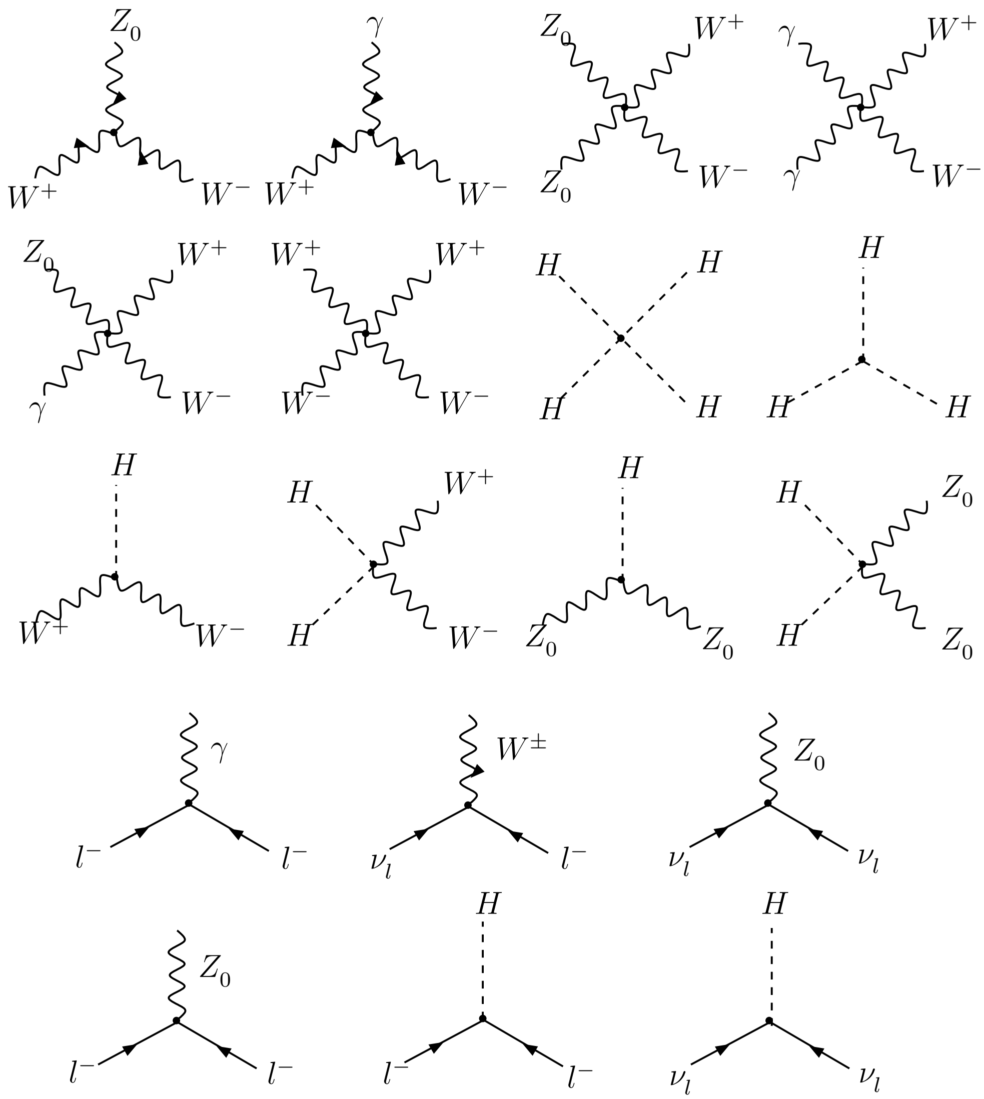


Figure F.6: The 18 vertices of standard electroweak theory.