Appendix H

Perturbative Quantum Gravity

An open problem in quantum gravity is to compute particle scattering amplitudes from the background-independent theory and recover the low energy physics.

Calculations should agree with low energy conventional field theory. Here we introduce conventional scattering theory.

H.0.11 Renormalisability

In higher orders, when we allow bubbles and loops in the diagrams we infinities which can be accounted for by any kind of physical argument.

H.1 Coupling of Scalar Matter and Gravity

$$-\frac{1}{2\kappa^2}\int d^4x\sqrt{-g}R+\frac{1}{2}\int d^4x\sqrt{-g}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}-m^2\phi^2\right) \tag{H.1}$$

H.2 Expansion

Let $\hat{A}$ and $\hat{B}$ be matrices

$$\frac{1}{\hat{A}+\lambda\hat{B}}=\frac{1}{\hat{A}}-\lambda\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}+\lambda^2\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}-\dots \tag{H.2}$$

$$
\begin{aligned}
g^{\mu\nu} &= (\eta_{\mu\nu} + \lambda h_{\mu\nu})^{-1} \\
&= \eta^{\mu\nu} - \lambda \eta^{\mu\alpha} h_{\alpha\beta} \eta^{\beta\nu} + \lambda^2 \eta^{\mu\alpha} h_{\alpha\beta} \eta^{\beta\gamma} h_{\gamma\delta} \eta^{\delta\nu} - \dots \\
&= \eta^{\mu\nu} - \lambda h^{\mu\nu} + \lambda^2 h^{\mu}{}_{\alpha} h^{\alpha\nu} - \lambda^3 h^{\mu}{}_{\alpha} h_{\alpha\beta} h^{\beta\nu} + \dots
\end{aligned}
\tag{H.3}
$$

$$
\begin{aligned}
&\sqrt{-det\ g_{\mu\nu}} \\
&= \sqrt{-det\ \eta_{\mu\alpha}} \exp\left[\frac{1}{2} Tr\ \ln(\delta^{\alpha}_{\nu} + \lambda h^{\alpha}{}_{\nu})\right] \\
&= \exp\left[\frac{1}{2} Tr \left(\lambda h^{\alpha}{}_{\nu} - \frac{1}{2}\lambda^2 h^{\alpha}{}_{\tau} h^{\tau}{}_{\nu} + \frac{1}{3}\lambda^3 h^{\alpha}{}_{\tau} h^{\tau}{}_{\sigma} h^{\sigma}{}_{\nu} - \dots\right)\right] \\
&= 1 + \frac{1}{2}\lambda h^{\alpha}{}_{\alpha} - \frac{\lambda^2}{4} h^{\alpha}{}_{\rho}(h^{\rho}{}_{\alpha} - \frac{1}{2}\delta^{\rho}{}_{\alpha} h^{\sigma}{}_{\sigma}) + \dots
\end{aligned}
\tag{H.4}
$$

$$
\begin{aligned}
&S_m \\
&= \frac{1}{2}\int d^4x \sqrt{-g}\left(g^{\mu\nu}\phi,_{\mu}\phi,_{\nu} - m^2\phi^2\right) \\
&= \frac{1}{2}\int d^4x (1 + \lambda h^{\rho}{}_{\rho} - \lambda^2 h^{\sigma}{}_{\rho}(h^{\rho}{}_{\alpha} - \frac{1}{2}\delta^{\rho}{}_{\alpha} h^{\sigma}{}_{\sigma}) + \dots) \\
&\quad \times \left[(\eta^{\mu\nu} - \lambda h^{\mu\nu} + \lambda^2 h^{\mu}{}_{\alpha} h^{\alpha\nu} + \dots)\phi,_{\mu}\phi,_{\nu} - m^2\phi^2\right] \\
&= \frac{1}{2}\int d^4x(\phi,_{\mu}\phi,_{\nu} - m^2\phi^2) \\
&\quad - \frac{\lambda}{2}\int d^4x h^{\mu\nu}\left[\phi,_{\mu}\phi,_{\nu} - \frac{1}{2}\eta_{\mu\nu}\phi,^{\sigma}\phi,_{\sigma} + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}\right] \\
&\quad - \frac{\lambda^2}{4}\int d^4x\left[\frac{1}{2}h^{\lambda}{}_{\rho}(h^{\rho}{}_{\lambda} - \frac{1}{2}\delta^{\rho}{}_{\lambda}h^{\sigma}{}_{\sigma})(\phi,^{\mu}\phi,_{\mu} - m^2\phi^2) - 2h^{\mu\rho}(h_{\rho}{}^{\nu} - \frac{1}{2}\delta_{\rho}{}^{\nu}h^{\sigma}{}_{\sigma})\phi,_{\mu}\phi,_{\nu}\right]
\end{aligned}
\tag{H.5}
$$

H.3 Graviton Propagator

The graviton propagator is obtained from the quadratic term in the field $h_{\mu\nu}$.

The Lagrangian, in this approximation, is

$$
\mathcal{L}_0 = \frac{1}{4}[-(\partial_{\nu} h_{\beta\gamma})^2 + (\partial_{\mu} h^{\beta}_{\beta})^2 - 2\partial_{\gamma} h^{\beta}_{\beta}\partial_{\mu} h^{\gamma\mu} + 2\partial_{\beta} h_{\nu\gamma}\partial^{\nu} h^{\beta\gamma}]
\tag{H.6}
$$

We can make a gauge chooice by adding the term

$$-\frac{1}{2}C_\mu^2; \qquad C_\mu = \partial_\nu h_\mu^\nu - \frac{1}{2}\partial_\mu h_\nu^\nu \tag{H.7}$$

to the action.

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\beta\gamma} V^{\beta\gamma\mu\nu}\partial^\lambda h_{\mu\nu} \tag{H.8}$$

where

$$V_{\alpha\beta\mu\nu} = \frac{1}{2}\delta_{\alpha\mu}\delta_{\beta\nu} - \frac{1}{4}\delta_{\alpha\beta}\delta_{\mu\nu} \tag{H.9}$$

We derive the source equations

$$-V^{(\alpha\beta)\mu\nu}\Box h_{\mu\nu}(x) = S^{\alpha\beta}(x) \tag{H.10}$$

$$h_{\mu\nu}(x) = \int d^4y D_{\mu\nu\lambda\gamma}(x-y)S^{\lambda\gamma}(y) \tag{H.11}$$

We must have

$$\begin{aligned}
-V^{(\alpha\beta)\mu\nu}\Box h_{\mu\nu}(x) &= -\int d^4y V^{(\alpha\beta)\mu\nu}\Box D_{\mu\nu\lambda\gamma}(x-y)S^{\lambda\gamma}(y) \\
&= S^{\alpha\beta}(x)
\end{aligned} \tag{H.12}$$

implying

$$-V^{(\alpha\beta)\mu\nu}\Box D_{\mu\nu\lambda\gamma}(x-y) = \delta^4(x-y)\delta_\lambda^{(\alpha}\delta_\gamma^{\beta)} \tag{H.13}$$

from

$$D_{\mu\nu\lambda\gamma}(x-y) = \int \frac{d^4q}{(2\pi)^4}\exp[-iq\cdot(x-y)]D_{\mu\nu\lambda\gamma}(q^2) \tag{H.14}$$

in momentum space

$$q^2 V^{(\alpha\beta)\mu\nu} D_{\mu\nu\lambda\gamma}(q) = \delta^{(\alpha}_{\lambda}\delta^{\beta)}_{\gamma} \tag{H.15}$$

The propagator should have the form

$$D_{\mu\nu\lambda\gamma}(q) \;=\; A(q^2)\delta_{\mu\nu}\delta_{\lambda\gamma} + B(q^2)\delta_{\mu\lambda}\delta_{\nu\gamma} + C(q^2)\delta_{\mu\gamma}\delta_{\nu\lambda} \tag{H.16}$$

Now

$$V^{(\alpha\beta)\mu\nu} = \frac{1}{4}\delta^{\alpha\mu}\delta^{\beta\nu} + \frac{1}{4}\delta^{\beta\mu}\delta^{\alpha\nu} - \frac{1}{4}\delta^{\alpha\beta}\delta^{\mu\nu} \tag{H.17}$$

so

$$\begin{aligned}
q^2 V^{(\alpha\beta)\mu\nu} D_{\mu\nu\lambda\gamma}(q) &= q^2 A(q^2) V^{(\alpha\beta)\mu\nu}\delta_{\mu\nu}\delta_{\lambda\gamma} q^2 B(q^2) V^{(\alpha\beta)\mu\nu}\delta_{\mu\lambda}\delta_{\nu\gamma} \\
&\quad q^2 C(q^2) V^{(\alpha\beta)\mu\nu}\delta_{\mu\gamma}\delta_{\nu\lambda} + \ldots \\
&= q^2 A(q^2)\frac{1}{4}(\delta^{\alpha\beta}\delta_{\lambda\gamma} + \delta^{\beta\alpha}\delta_{\lambda\gamma} - 4\delta^{\alpha\beta}\delta_{\lambda\gamma}) \\
&\quad q^2 B(q^2)\frac{1}{4}(\delta^{\alpha}_{\lambda}\delta^{\beta}_{\gamma} + \delta^{\beta}_{\lambda}\delta^{\alpha}_{\gamma} - \delta^{\alpha\beta}\delta_{\lambda\gamma}) \\
&\quad q^2 C(q^2)\frac{1}{4}(\delta^{\alpha}_{\gamma}\delta^{\beta}_{\lambda} + \delta^{\beta}_{\gamma}\delta^{\alpha}_{\lambda} - \delta^{\alpha\beta}\delta_{\gamma\lambda}) + \ldots \\
&= \delta^{(\alpha}_{\lambda}\delta^{\beta)}_{\gamma}
\end{aligned} \tag{H.18}$$

Implying

$$A(q^2) = -\frac{1}{q^2}, \quad B(q^2) = \frac{1}{q^2}, \quad C(q^2) = \frac{1}{q^2}. \tag{H.19}$$

The propagator is then

$$D_{\mu\nu\lambda\gamma}(q) = \frac{\delta_{\mu\lambda}\delta_{\nu\gamma} + \delta_{\mu\gamma}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\gamma}}{q^2 + i\epsilon} \tag{H.20}$$

H.3.1 Lowest Order Vertex

$$S_m = \frac{1}{2}\int d^4x(\phi^{,\mu}\phi_{,\mu} - m^2\phi^2) - \frac{\lambda}{2}\int h^{\mu\nu}[\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\phi^{,\rho}\phi_{,\rho} + \frac{1}{2}m^2\phi^2\eta_{\mu\nu}] + \mathcal{O}(\lambda^2) \tag{H.21}$$

This leads to the equations of motion

$$\begin{aligned}(\partial^\mu\partial_\mu + m_0^2)\phi(x) &= \lambda\partial_\mu[h^{\mu\nu}(x)\partial_\nu\phi(x)] \\ &\quad -\lambda\frac{1}{2}\partial_\mu[h^\rho{}_\rho(x)\eta^{\mu\nu}\partial_\nu\phi(x)] \\ &\quad -\lambda\frac{1}{2}m_0^2 h^\rho{}_\rho(x)\phi(x) + \mathcal{O}(\lambda^2) \\ &= -\hat{V}\phi(x) \end{aligned} \tag{H.22}$$

$$\hat{V}\phi(x) = -\lambda\left\{\partial_\mu[h^{\mu\nu}(x)\partial_\nu - \frac{1}{2}h^\rho{}_\rho(x)\eta^{\mu\nu}\partial_\nu] - \frac{1}{2}m_0^2 h^\rho{}_\rho(x)\right\}\phi(x)$$

$$\begin{aligned} S_{fi} &= \lambda Const. \int d^4x e^{+ip_f\cdot x}\left\{\partial_\mu[h^{\mu\nu}(x)\partial_\nu - \frac{1}{2}h^\rho{}_\rho(x)\eta^{\mu\nu}\partial_\nu] - \frac{1}{2}m_0^2 h^\rho{}_\rho(x)\right\}e^{-ip_i\cdot x} \\ &= \lambda Const. \int d^4x e^{+ip_f\cdot x}\left\{[-(ip_{f\mu})][h^{\mu\nu}(x) - \frac{1}{2}h^\rho{}_\rho(x)\eta^{\mu\nu}](-ip_{i\nu}) - \frac{1}{2}m_0^2 h^\rho{}_\rho(x)\right\}e^{-ip_i\cdot x} \\ &= -\lambda Const. \int d^4x \left[h^{\mu\nu}(x)p_{f\mu}p_{i\nu} - \frac{1}{2}h^\rho{}_\rho(x)(p_{f\tau}p_i^\tau - m_0^2)\right]e^{-i(p_f-p_i)\cdot x} \\ &= \lambda Const. \left[h^{\mu\nu}(p_f-p_i)p_{f\mu}p_{i\nu} - \frac{1}{2}h^\rho{}_\rho(p_f-p_i)(p_{f\tau}p_i^\tau - m_0^2)\right] \end{aligned} \tag{H.23}$$

If

$$h^{\mu\nu}(x) = e_{\mu\nu}e^{iq\cdot x}$$

then

$$h^{\mu\nu}(p_f - p_i) = (2\pi)^4\delta^4(q - p_i + p_f)$$

H.3.2 Harmonic Gauge

We choose the Harmonic gauge, where:

$$\frac{1}{2\alpha}(\partial_\mu h^{\mu\nu})^2 \tag{H.24}$$

is added to the action for arbitrary α. We calculate the graviton propagator, and the Fadeev-Popov ghost term.

$$-\lambda[p_{f\mu}p_{i\nu} - \tfrac{1}{2}\eta_{\mu\nu}(p_{f\tau}p_i^{\tau} - m_0^2)]$$

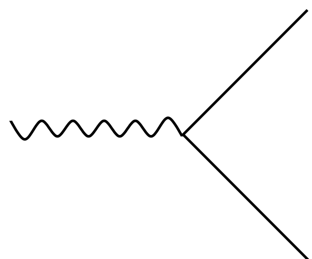

Figure H.1: interaction.