# Appendix I

# Algebraic Structures

overwhelmingly tested regimes of quantum theory and general relativity.

In loop quantum gravity, the geometry of space is built from qubits. Spacetime is like a parallel-processing quantum computer that constantly modifies its own topology.

Figure put how to simulate such systems more efficiently.

## I.1 Introduction

Lie algebras,
Tensor product algebras,
Universal enveloping algebras,
Hopf algebras,
Quasi-triangular Hopf algebras,
Quantum groups.

Boolean algebras,
Heyting algebras.

Braid group algebras,
Categories,
Topos.

C*-algebras,
von Neumann algebras.

Physical applications we simplify the situation considerably, leaving out significant details.

### I.1.1 Quantum Groups in Quantum Gravity

In three dimansions, Freidel and Livine have shown that it is possible to rigorously derive the effective quantum fields theory of a scalar self-gravitating field, starting from a spin foam model. Remarkably, the quantum properties of the graviatational field manifest itself in the effective QFT by a spaacetime non-commutativity. More precisely, there is a limit of the full theory leading to a QFT over a non-commutative spacetime. In the construction, a quantum group appears as a symmetry of an auxiliary formulation of the theory.

The situation is described by the partition function

$$\mathcal{Z} = \int \mathcal{D}g \int \mathcal{D}\phi \exp i(S[\phi, g] + S[g])$$

is the action for scalar filed $\phi$ (with non-derivative interactions) on a 3-dimensional spacetime with metric $g_{ab}$ and $S[g]$ is the Einstein-Hilbert action. We wish to integrate out the gravitational degrees of freedom to obtain an effective action $S_{eff}[\phi]$ for the matter field $\phi$:

$$\mathcal{Z} = \int \mathcal{D}\phi \exp iS_{eff}[\phi, g_0],$$

where $g_0$ is the flat metric. The standard Ableian products of fields in the original action $S[\phi, g]$ are just replaced by a non-Abelian $\star-$product in $S_{eff}[\phi, g_0]$.

While the action $S[\phi, g_0]$ is invariant under the six dimensional Poincare group, $S_{eff}[\phi, g_0]$ is invariant under the so called $\kappa$ deformation of Poincare group.

These particles can be regrded as moving in a 'dual' non-commutative position space coordiatized by $X_i$ satisfying the relations S

$$[X_i, X_j] = i\kappa\hbar\epsilon_{ijk}X_k. \tag{I.1}$$

The $\star-$algebra generated by the three operators $X_i$ subject to these commutation relations is referred to as the $\kappa-$Minkowski space.

#### Role of quantum groups

Quantizing a system of self-gravitating masive particles on $\mathbb{E}^3$ is equivalent to quantizing a system of non-gravitating massive point particles whose symmetry group is $DSU(2)$ instead of $ISU(2)$.

Spacetime non-commutativity is associated with the fact that the symmetry group of the theory is in fact a quantum group. It is $DSU(2)$, Drinfeld double of the classical group

$SU(2)$. It appears rather clearly that quantum groups play a role in LQG, although the precise meaning of this appearance is not yet clear.

### I.1.2 Quantum Deformation of Quantum Gravity

arXiv:gr-qc/9512020:

ABSTRACT

We describe a deformation of the observable algebra of quantum gravity in which the loop algebra is extended to framed loops. This allows an alternative nonperturbative quantization which is suitable for describing a phase of quantum gravity characterized by states which are normalizable in the measure of Chern-Simons theory. The deformation parameter, q, is

$$e^{i\hbar^2 G^2 \Lambda/6},$$

where $\Lambda$ is the cosmological constant. The Mandelstam identities are extended to a set of relations which are governed by the Kau man bracket so that the spin network basis is deformed to a basis of $SU(2)_q$ spin networks. Corrections to the actions of operators in non-perturbative quantum gravity may be readily computed using recoupling theory; the example of the area observable is treated here. Finally, eigenstates of the $q-$deformed Wilson loops are constructed, which may make possible the construction of a $q-$deformed connection representation through an inverse transform.

## I.2 Lie and Hopf Algebras

As an example, the reader may wish to keep ... in mind in what follows.

### I.2.1 Lie Algebra

The Pauli matrices and

Their algebra can be written

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y \tag{I.2}$$

We say this is an algebra over the complex numbers a vector space with commutator relations (I.2).

A field is a generalization or abstraction of scalars to more general mathematical structures "number systems of mathematics".

$X \otimes Y$ spanned by elements $x \otimes y$

$$\begin{aligned}(x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2 \\ \alpha(x \otimes y) &= (\alpha x) \otimes y = x \otimes (\alpha y)\end{aligned} \tag{I.3}$$

A vector basis of $X$ is $x_i$ and of $Y$ is $y_i$ provide a basis for the tensor product space $X \otimes Y$. An arbitrary element of $X \otimes Y$ can be written

$$\sum_{i,j} a_{ij} x_i \otimes y_j \tag{I.4}$$

where $a_{ij}$ are scalars. If $X$ has dimension $m$ and $Y$ has dimension $n$ then $X \otimes Y$ has dimension $mn$.

We now introduce some formal mathematical notation used in the literature:

$$m \circ (a \otimes b) = a \cdot b \tag{I.5}$$

$\imath$ is an operation which takes a given value in $k$ and gives you an element of $A$, we write $\imath : k \to A$ and say $\imath(k) \mapsto a$. The relationship between the identity $1_A$ and the unit 1 is $\imath(1) = 1_A$.

(a) $m$ is associative, i.e., $(ab)c = a(bc)$

(b) $1_A \cdot a = a \cdot 1_A = a$, for all $a \in A$.

Equivalently, an algebra over $k$

(a) $m$ is associative,

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (id_A \otimes m) \tag{I.6}$$

(b) unit condition

$$m \circ (\imath \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes \imath) = \mathrm{id}_A \tag{I.7}$$

An algebra over $k$ is a vector space $A$ over $k$ with multiplication

**Modules** Let $G$ be a group and let the field $k$ (e.g. $\mathbb{R}$, $\mathbb{C}$). suppose is a representation of $G$. Write

Modules are also referred to as representations: for instance, representations of a group are essentially the same as modules over the group algebra. the term "module" you surely have come across some examples. Vector spaces are (rather simple) examples, as are abelian groups. Representations of algebras and of groups all are modules, as are representations of Lie algebras.

**Example** The identity element the over the complex numbers is an (unital) algebra $A$. Complex linear combinations of the $2 \times 2$ identity matrix together with the Pauli matrices

$$1_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{I.8}$$

acting (or "with action") on a 2-dimensional vector space $V$ with elements

$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad x, y \in \mathbb{C} \tag{I.9}$$

is an example of a A-module.

A matrix acts on any 2-component vector and gives back some vector in $V$, we write $A \otimes V \to V$ and $a \cdot v = av$.

From the properties of matrices we have that

$$(a_i a_j)v = a_i(a_j v), \quad \text{for all } a_i, a_j \in A \text{ and for all } v \in V. \tag{I.10}$$

and that

$$1_A \cdot v = v, \quad \text{for all } v \in V. \tag{I.11}$$

Motivated by the above observations, we now define a $k-$module.

**Definition** modules

Let $A$ be an algebra over $k$. An $A$-*module* is a vector space $V$ over $k$ with an A-action

$$\begin{aligned} A \otimes V &\rightarrow V \\ a \otimes v &\mapsto a \cdot v = av \end{aligned} \tag{I.12}$$

such that

(a) $(ab)v = a(bv)$ , for all $a, b \in A$ and $v \in V$, and
(b) $1_A v = v$ , for all $v \in V$.

**Non-associativity of Lie Algebras**

**Definition** Lie bracket multiplication

$$a \circ b := [a, b] \tag{I.13}$$

Lie bracket multiplication rule obviously satisfies the distributive law

$$\begin{aligned} [(\alpha a + \beta b)c] &= \alpha[a, c] + \beta[b, c] \\ (\alpha a + \beta b) \circ c &= \alpha a \circ \beta b \circ c \end{aligned} \tag{I.14}$$

$$\begin{aligned} [a, [b, c]] + [b, [c, a]] + [c, [a, b]] &= 0 \\ a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b) &= 0 \end{aligned} \tag{I.15}$$

$$a \circ (b \circ c) = (a \circ b) \circ c + b \circ (a \circ c) \tag{I.16}$$

$$a \circ (b \circ c) \neq (a \circ b) \circ c \tag{I.17}$$

Instead of associativity we have the Jacobi identity

associative algebras are easier to handle. the associative quotient algebra

## I.2.2 Tensor Product Algebra

Let $k$ be the complex numbers $\mathbb{C}$ or the reals $\mathbb{R}$, and let $A, B$ be vector spaces over $k$. It is the direct product $\otimes$ that is the binary operation or product that make the tensor algebra an algebra,

(i) Distributive. left distributive

$$\begin{aligned} (x_1 \otimes \cdots \otimes x_r) \quad &\otimes \quad (y_1 \otimes \cdots \otimes y_s + z_1 \otimes \cdots \otimes z_s) \\ &= x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_s \\ &\quad + x_1 \otimes \cdots \otimes x_r \otimes z_1 \otimes \cdots \otimes z_s \end{aligned} \tag{I.18}$$

right distributive property demonstrated similarly.

Property involving scalar multiplication is easy to see,

$$\alpha(x_1 \otimes x_2 \otimes \cdots \otimes x_r) \tag{I.19}$$

It is an associative algebra

$$\begin{aligned} & x_1 \otimes \cdots \otimes x_r \otimes (y_1 \otimes \cdots \otimes y_s \otimes z_1 \otimes \cdots \otimes z_s) \\ = \; & x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_s \otimes z_1 \otimes \cdots \otimes z_s \\ = \; & (x_1 \otimes \cdots \otimes x_r \otimes y_1 \otimes \cdots \otimes y_s) \otimes z_1 \otimes \cdots \otimes z_s \end{aligned} \tag{I.20}$$

The tensor product of any number of elements can be taken as many times as one likes The tensor algebra $T(\mathcal{L})$ is the collection of (including up to an infinite number) collectively as direct summation:

$$T(\mathcal{L}) = \mathbb{C} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \dots. \tag{I.21}$$

### I.2.3 Universal Enveloping Algebra

we must take note of the commutators, or Lie brackets $[A, B] = AB - BA$. The Lie algebra is then constructed by means of the repeated application of the operations of $+$, its inverse $-$ and the bracket operation $[\ ,\ ]$, where we also allow the multiplication by real or complex numbers.

http://www.math.uni-bielefeld.de/ philfahr/enveloping/node1.html

$\mathfrak{A}$ $\mathfrak{su}(2, \mathbb{C})$

consider the set of elements generated by

$$u_{x,y} = (x \otimes y - y \otimes x) - [x, y] \in T. \tag{I.22}$$

that is, any element of $T(\mathcal{L})$ that can be written can be expressed in the form

$$v \otimes (x \otimes y - y \otimes x) \otimes w - v \otimes [x, y] \otimes w \tag{I.23}$$

we say the set of elements generated by $u_{x,y}$.Let $\mathcal{I}$ be a ideal in $T$ generated by the set of all elements $u_{x,y}$.

Any element of $T(\mathcal{L})$ that can be written can be expressed in the form

$$v \otimes (x \otimes y - y \otimes x) \otimes w - v \otimes [x, y] \otimes w \tag{I.24}$$

Recall the notion of an ideal, but now in the context of tensor algebra: say $\mathcal{I}$ is a linear subspace of $T(\mathcal{L})$. If $\mathcal{I}$ satisfies the conditions $x \otimes i \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ and if $i \otimes x \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ then $\mathcal{I}$ is a two-sided ideal (or simply an ideal). It is trivial to see that the set of elements generated by (I.22) form an ideal of $T(\mathcal{L})$:

$$\frac{T(\mathcal{L})}{\mathcal{I}} =: U(\mathcal{L}) \tag{I.25}$$

is the *universal enveloping algebra*

$$x_{i_j} x_{i_j+1} - x_{i_j+1} x_{i_j} - [x_{i_j}, x_{i_j+1}] \in \mathcal{I} \tag{I.26}$$

It may be helpful to have to hand the flow diagram fig (I.4)

Recall the definition of a two-sided ideal, (or simply an ideal): say that $\mathcal{I}$ is a linear subspace of $\mathcal{A}$. If $\mathcal{I}$ satisfies the conditions $ai \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ and if $ia \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ then $\mathcal{I}$ is a two-sided ideal.

$[\mathcal{G}, \mathcal{G}]$ form an ideal of a Lie algebra, with ordinary multiplication $a \cdot b$.

$$[[a, b], c] = [d, c] \in \mathcal{I} \equiv [\mathcal{G}, \mathcal{G}] \tag{I.27}$$

$$T(\mathcal{L}) = \mathcal{F} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots \tag{I.28}$$

$$x \in T(\mathcal{L}) \tag{I.29}$$

Say $\mathcal{I}$ is a linear subspace of $T(\mathcal{L})$. If $\mathcal{I}$ satisfies the conditions $x \otimes i \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ and if $i \otimes x \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in T(\mathcal{L})$ then $\mathcal{I}$ is a two-sided ideal.

$$[a, b] - (a \otimes b - b \otimes a) \tag{I.30}$$

so a two-sided ideal $\mathcal{I}$.

$$\frac{T(\mathcal{L})}{\mathcal{I}} =: U(\mathcal{L}) \tag{I.31}$$

is the *universal enveloping algebra*

The quotient

$$v \otimes (x \otimes y - y \otimes x - [x, y]) \otimes w \sim v \otimes w \tag{I.32}$$

universal enveloping algebra of $sl(2)$

### I.2.4 Hopf Algebras

nice introduction to Hopf algebras [356] (in quantum field theory)

an algebra whose multiplication is associative, $(ab)c = a(bc)$. $m(a \otimes b) = ab$.

**1. Algebra**

multiplication $\mu : A \otimes A \to A$ $m(a \otimes b) = ab$. $m$ is **associative**, $(ab)c = a(bc)$.

$$m(m \otimes \mathbf{1})(a \otimes b \otimes c) = m(\mathbf{1} \otimes m)(a \otimes b \otimes c) \tag{I.33}$$

mathematicians write this in the short-hand notation

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m) \tag{I.34}$$

and depict it as a diagram (I.2.4(a)).

**unit** $\eta$ sends every number to the same number times the identity element, i.e.

$$\eta(k) := \mathbf{1}k, \quad \text{for all } k \in \mathbb{C}. \tag{I.35}$$

$$m(a \otimes \eta(k)) = ak = ka = m(\eta(k) \otimes a), \quad \text{for all } a \in A, \text{ and for all } k \in \mathbb{C}. \tag{I.36}$$

$$m \circ (\eta \otimes \mathrm{id})(a) = a = m \circ (\mathrm{id} \otimes \eta)(a) \tag{I.37}$$

**2. Coalgebra**

A coalgebra is a vector space with the structure obtained by reversing the arrows in the diagrams Fig.(I.2.4) characterizing an algebra. A comultiplication $\Delta : C \to C \otimes C$. $\Delta$ is **coassociative**.

$$
\begin{tikzcd}[column sep=small]
& A \otimes A \otimes A \arrow[dl, "m \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes m"] & \\
A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\
& A &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
A \otimes A \arrow[dr] & \\
k \otimes A \arrow[u, "\eta \otimes \mathrm{id}"] = & A
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
A \otimes A \arrow[dr] & \\
A \otimes k \arrow[u, "\mathrm{id} \otimes \eta"] = & A
\end{tikzcd}
$$

(a) (b)

Figure I.1: (a) Assiciativity condition diagram commutes. (b) Unit condition.

$$(\mathrm{id}_C \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id}_C)\Delta \tag{I.38}$$

seems a little too abstract can put in matrix form (exercise)

**Sweedler notation for comultiplication**

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)} \tag{I.39}$$

$$(\mathrm{id}_C \otimes \Delta)\Delta(a) = (\mathrm{id} \otimes \Delta) \sum_a a_{(1)} \otimes a_{(2)} = \sum_a a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \tag{I.40}$$

$$\sum_a a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)} = \sum_a a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} \tag{I.41}$$

**counit condition,** $\epsilon : C \to k$

$$m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta(a) = m \circ (\mathrm{id}_A \otimes \epsilon) \circ \sum_a a_{(1)} \otimes a_{(2)} = \sum_a a_{(1)}\epsilon(a_{(2)}) \tag{I.42}$$

$$m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta(a) = m \circ (\epsilon \otimes \mathrm{id}_A) \circ \Delta(a) = \mathrm{id}_A(a) = a, \tag{I.43}$$

$$\sum_a a_{(1)}\epsilon(a_{(2)}) \quad = \quad a \tag{I.44}$$

$$\sum_a \epsilon(a_{(1)})a_{(2)} \quad = \quad a \tag{I.45}$$

$$\begin{tikzcd}
 & C \otimes C \otimes C \arrow[dl, "\Delta \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \Delta"] & \\
C \otimes C \arrow[dr, "\Delta"'] & & C \otimes C \arrow[dl, "\Delta"] \\
 & C &
\end{tikzcd}$$

(a)

$$\begin{tikzcd}
C \otimes C \arrow[d, "\epsilon \otimes \mathrm{id}"'] \\
k \otimes C = C \arrow[u, bend right, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"'] \\
C \otimes k = C \arrow[u, bend right, "\Delta"']
\end{tikzcd}$$

(b)

Figure I.2: A coalgebra has the structure obtained by reversing the arrows in the diagrams Fig.(I.2.4) characterizing an algebra. (a) Coassiciativity condition diagram commutes. (b) Counit condition.

## 3. Bialgebra

A bialgebra is a vector space $A$ with the structure of an algebra $(m, \eta)$ and the structure of a coalgebra $(\Delta, \epsilon)$ which are compatible with each other, i.e.,

$$\Delta(ab) = \Delta(a)\Delta(b), \qquad \Delta(\mathrm{id}_A) = \mathrm{id}_A \otimes \mathrm{id}_A, \tag{I.46}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b), \qquad \epsilon(\mathrm{id}_A) = 1_k, \tag{I.47}$$

for all $g, h \in A$. These ensure that the comultplication $\Delta$ and counit $\epsilon$ are consistent with the multiplication $m$:

### 3.1. Algebra homomorphism condition for $\Delta$:

$\Delta(ab) = \Delta(a)\Delta(b)$

Firstly $\Delta(ab)$

$$\Delta(ab) = \Delta \circ m \circ (a \otimes b) \tag{I.48}$$

Next in Sweedler notation $\Delta(a)\Delta(b) = \sum_{a,b} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$

$$(\Delta \otimes \Delta) \circ (a \otimes b) = (\sum_a a_{(1)} \otimes a_{(2)}) \otimes (\sum_b b_{(1)} \otimes b_{(2)}) \tag{I.49}$$

$$\begin{aligned}(\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \quad \circ \quad & (\sum_a a_{(1)} \otimes a_{(2)}) \otimes (\sum_b b_{(1)} \otimes b_{(2)}) \\ = \quad & \sum_a \sum_b (a_{(1)} \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)})\end{aligned} \tag{I.50}$$

$$(m\otimes m)\circ\sum_a\sum_b(a_{(1)}b_{(1)}\otimes a_{(2)}b_{(2)})=\sum_a\sum_b(a_{(1)}a_{(2)})\otimes(a_{(2)}b_{(2)})) \tag{I.51}$$

$$\Delta(a)\Delta(b)=(m\otimes m)\circ(\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_A)\circ(\Delta\otimes\Delta)\circ(a\otimes b) \tag{I.52}$$

The algebra homomorphism condition (I.46) in terms of $m$ and $\Delta$ is

$$\Delta\circ m\circ(a\otimes b)=(m\otimes m)\circ(\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_A)\circ(\Delta\otimes\Delta)\circ(a\otimes b). \tag{I.53}$$

### 3.2. Algebra homomorphism condition for $\epsilon$:

The algebra homomorphism condition (I.47) $\epsilon(ab)=\epsilon(a)\epsilon(b)$ in terms of $m$ and $\epsilon$ is straightforward

$$\epsilon\circ m\circ(a\otimes b)=(\epsilon\otimes\epsilon)\circ(a\otimes b). \tag{I.54}$$

Summerizing, a bialgebra is a quadraple of objects $(m,\eta,\Delta,\epsilon)$ which satisfy all of the commutative diagrams as well as the compatibility conditions (I.46) and (I.47).

## 4. Hopf algebra

There is no inverse operation

$$\mathrm{Inv}(a)\cdot a=a\cdot\mathrm{Inv}(a)=1 \tag{I.55}$$

Replace $a\otimes a$ with the coproduct on $a$, and 1 with the counit, $\eta(a)$. We name $S$ as the generalization of the inverse, Inv. So we have

$$m(S\otimes\mathrm{id})\Delta(a)=\eta(a). \tag{I.56}$$

An operation $S:A\to A$ satisfying the above equation, and also

$$m(\mathrm{id}\otimes S)\Delta(a)=\eta(a), \tag{I.57}$$

is called an *antipode*.

• **Hopf algebra.** Hopf algebra has an antipode $S$.

linear antipode $S:H\to H$

$$m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon. \tag{I.58}$$

the antipode is a kind of inverse.

In terms of its elements

$$\sum a_{(1)} S a_{(2)} = \eta(a) \text{ and} \tag{I.59}$$

$$\sum S a_{(1)} a_{(2)} = \eta(a). \tag{I.60}$$

$$\begin{tikzcd} H \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & \mathbf{M} \arrow[r, "\eta"] & H \\ H \bigotimes H \arrow[rr, "\mathrm{id} \otimes S,\ S \otimes \mathrm{id}"] & & H \bigotimes H \arrow[u, "m"'] \end{tikzcd}$$

Figure I.3: links.

Uniqueness of the antipode. Let $S, S'$ be two antipodes on a bialgebra, we show the antipode is unique by demonstrating that $S'a = Sa$ for all $a \in A$.

$$\begin{aligned} S'a &= S'a_{(1)}\eta(a_{(2)}) && \text{by counity } (I.44) \\ &= S'a_{(1)}a_{(2)(1)}Sa_{(2)(2)} && \text{by antipode } (I.59) \\ &= S'a_{(1)(1)}a_{(2)(1)}Sa_{(2)} && \text{by } m(m \otimes S) \text{ applied to both sides of } (I.41) \\ &= \eta(a_{(1)})Sa_{(2)} && \text{by antipode } (I.60) \text{ used on } S' \\ &= Sa && \text{by counity } (I.45). \end{aligned} \tag{I.61}$$

We now collect all the conditions (I.6), (I.7), (I.53)

**Definition of Hopf algebras**

A Hopf algebra is a vector space $A$ over $k$ with antipode $S$

| | |
|---|---|
| a multiplication, | $m : A \otimes A \to A$, |
| a comultiplication, | $\Delta : A \to A \otimes A$, |
| a unit, | $\imath : k \to A$, |
| a counit, | $\epsilon : A \to k$, |
| an antipode, | $S : A \to A$, |

such that

(1) $m$ is associative,
$$m \circ (\mathrm{id}_A \otimes m) = m \circ (m \otimes \mathrm{id}_A), \tag{I.62}$$
(2) $\Delta$ is coassociative,
$$(\mathrm{id}_A \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}_A) \circ \Delta, \tag{I.63}$$
(3) unit condition,
$$m \circ (\mathrm{id}_A \otimes \imath) = m \circ (\imath \otimes \mathrm{id}_A) = \mathrm{id}_A, \tag{I.64}$$
(4) counit condition,
$$m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta = m \circ (\epsilon \otimes \mathrm{id}_A) \circ \Delta = \mathrm{id}_A, \tag{I.65}$$
(5) $\Delta$ is an algebra homomorphism,
$$\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta), \tag{I.66}$$
(6) $\epsilon$ is an algebra homomorphism,
$$\epsilon \circ m = \epsilon \otimes \epsilon, \tag{I.67}$$
(7) antipode condition
$$m \circ (\mathrm{id}_A \otimes S) \circ \Delta = m(S \otimes \mathrm{id}_A) \circ \Delta = \imath \circ \epsilon. \tag{I.68}$$

### I.2.5 Hopf Algebras from Groups

Let $G$ be a finite group, and $A = Fun(G)$ be a set of functions from $G$ to the complex numbers $\mathbb{C}$. $A = Fun(G)$ is an algebra over $\mathbb{C}$ with the usual sum and product
$$(f+h)(g) = f(g) + h(g), \quad (f \cdot h)(g) = f(g)h(g), \quad (\lambda f)(g) = \lambda f(g),$$
for $f, h \in Fun(G)$, $g \in G$, $\lambda \in \mathbb{C}$. The unit of this algebra is $I$, defined by $I(g) = 1$, for all $g \in G$. Using the group structure of $G$ (multiplication map, existence of the unit element and inverse element), we can introduce on $Fun(G)$ three other linear maps, the coproduct $\Delta$, the counit $\epsilon$, and the coinverse (antipode) $S$:
$$\begin{aligned}
\Delta(f)(g, g') &\equiv f(gg'), \quad \Delta : Fun(G) \to Fun(G) \otimes Fun(G) && \text{(I.69)}\\
\epsilon(f) &\equiv f(1_G), \quad \epsilon : Fun(G) \to \mathbb{C} && \text{(I.70)}\\
(Sf)(g) &\equiv f(g^{-1}), \quad S : Fun(G) \to Fun(G) && \text{(I.71)}
\end{aligned}$$
where $1_G$ is the unit of $G$.

In general a coproduct can be expanded on $Fun(G) \to Fun(G) \otimes Fun(G)$ as:

$$\Delta(f) = \sum_i f_1^i \otimes f_2^i \equiv f_1 \otimes f_2, \tag{I.72}$$

where $f_1^i, f_2^i \in A = Fun(G)$ and $f_1 \otimes f_2$ is shorthand notation. Thus we have:

$$\Delta(f)(g, g') = (f_1 \otimes f_2)(g, g') = f_1(g) f_2(g') = f(gg'). \tag{I.73}$$

It is not difficult to verify the following properties of the co-structures:

$$\begin{aligned}
(id \otimes \Delta)\Delta &= (\Delta \otimes id)\Delta && \text{(I.74)}\\
(id \otimes \epsilon)\Delta(a) &= (\epsilon \otimes id)\Delta(a) = a && \text{(I.75)}\\
m(S \otimes id)\Delta(a) &= m(id \otimes S)\Delta(a) = m(id \otimes S) = \epsilon(a)I && \text{(I.76)}
\end{aligned}$$

and

$$\begin{aligned}
\Delta(ab) &= \Delta(a)\Delta(b), \quad \Delta(I) = I \otimes I, && \text{(I.77)}\\
\epsilon(ab) &= \epsilon(a)\epsilon(b), \quad \epsilon(I) = 1, && \text{(I.78)}\\
S(ab) &= S(b)S(a), \quad S(I) = I && \text{(I.79)}
\end{aligned}$$

where $a, b \in A = Fun(G)$ and $m$ is the multiplication map $m(a \otimes b) \equiv ab$. The product in $\Delta(a)\Delta(b)$ is the product in $A \otimes A : (a \otimes b)(c \otimes d) = ab \otimes cd$.

**Proof:**

(i)

$$\begin{aligned}
(id \otimes \Delta)\Delta(f)(g, g', g'') &= f(g) \otimes \Delta(f)(g', g'')\\
&= f(g) \otimes f(g'g'')\\
&= f(gg'g'')\\
&= f(gg') \otimes f(g'')\\
&= (\Delta \otimes id)\Delta(f)(g, g', g'') && \text{(I.80)}
\end{aligned}$$

$f_1 \otimes f_{21} \otimes f_{22} = f_{11} \otimes f_{12} \otimes f_2$

(ii)

$$\begin{aligned}
(id \otimes \epsilon)\Delta(f)(g) &= f_1(g)\epsilon(f_2) \\
&= f_1(g)f_2(1_G) \\
&= f(g1_G) = f(g) \\
&= f(1_G g) \\
&= \epsilon(f_1)f_2(g) \\
&= (\epsilon \otimes id)\Delta(f)(g) \qquad \text{(I.81)}
\end{aligned}$$

(iii)

$$\begin{aligned}
m(S \otimes id)\Delta(f)(g) &= m(Sf_1 \otimes f_2) \\
&= f_1(g^{-1})f_2(g) \\
&= f(g^{-1}g) \\
&= f(1_G) = \epsilon(f) \\
&= f(gg^{-1}) \\
&= m(id \otimes S)\Delta(f)(g) \qquad \text{(I.82)}
\end{aligned}$$

(iv)

$$\begin{aligned}
\Delta(fh)(g,g') &= (fh)(gg') \\
&= f(gg')h(gg') \\
&= \Delta(f)(g,g')\Delta(h)(g,g') \qquad \text{(I.83)}
\end{aligned}$$

(v)

$$\begin{aligned}
\epsilon(fh) &= (fh)(1_G) \\
&= f(1_G)h(1_G) \\
&= \epsilon(f)\epsilon(h) \qquad \text{(I.84)}
\end{aligned}$$

(vi)

$$\begin{aligned}
(Sfh)(g) &= (fh)(g^{-1}) \\
&= h(g^{-1})f(g^{-1}) \\
&= (Sh)(Sf) \qquad \text{(I.85)}
\end{aligned}$$

### I.2.6 Example Where $G$ is a Group of Matrices

We define $A = Fun(G)$ to be the algerba of polynomials in the matrix elements ${T^a}_b$ of this representation of $G$ and $\det T^{-1}$.

$$\Delta({T^a}_b)(g,g') = {T^a}_b(gg') = {T^a}_b(gg') = {T^a}_c(g){T^c}_b(g'), \tag{I.86}$$

Therefore

$$\Delta({T^a}_b) = {T^a}_c \otimes {T^c}_b. \tag{I.87}$$

using (I.70) and (I.71), one finds:

$$\begin{aligned} \epsilon({T^a}_b) &= \delta^a_b && \text{(I.88)} \\ S({T^a}_b) &= {(T^{-1})^a}_b. && \text{(I.89)} \end{aligned}$$

Thus the algebra $A = Fun(G)$ of polynomials in the elements ${T^a}_b$ and $\det T^{-1}$ is a Hopf algebra with co-structures defined by (I.87)-(I.89) and (I.77)-(I.79).

**Equivalence of the information contained in the group and the Hopf algebra**

**Noncommutative deformation, $Fun_q(G)$, of $Fun(G)$**

Noncommutative deformation $Fun_q(G)$ of $Fun(G)$. Since $G$ is a group, $Fun(G)$ is a Hopf algerba; the noncommutative deformation of $Fun(G)$ is usually calleda Quantum group. The term stems from the fact that the deformation is obtained by quantizing a Poisson structure of the algebra ([??]).

## I.3 Duality

for all $\psi, \phi \in A'$ and $a, b \in A$

$$\begin{aligned} < \psi\phi, a > = < \psi \otimes \phi, \Delta(a) >, &\qquad < I, a >= \epsilon(a) && \text{(I.90)} \\ < \Delta(\psi), a \otimes b > = < \psi, ab >, &\qquad \epsilon(\psi) =< \psi, I > && \text{(I.91)} \\ < S(\psi), a > \;\; &= \;\; < \psi, S(a) > && \text{(I.92)} \end{aligned}$$

where $<\psi\otimes\phi, a\otimes b>\equiv<\psi, a><\phi, b>$.

**Example:**

generators of the group vector space $e_i$ and the generators of the dual space $e^i$.

$$m(g_1 \otimes g_2) := g_1 \cdot g_2. \tag{I.93}$$

If we consider its action on the generator of the group

$$m(e_i \otimes e_j) = m_{ij}^k e_k. \tag{I.94}$$

$\eta$ sends every number to the same number times the identity element, i.e., $\mathbf{e}\,\eta : \mathbb{R} \to \mathbb{R}(G)$

$$\eta(\lambda) = \lambda\mathbf{e} \tag{I.95}$$

The **antipode** - $S : \mathbb{R}(G) \to \mathbb{R}(G)$, $g \mapsto S(g)g^{-1}$

$$\begin{aligned} < m(g_1 \otimes g_2), f > &= =< g_1 \otimes g_2, \Delta(f) > \\ &= < g_1 \otimes g_2, f_1 \otimes f_2 > \\ &= < g_1 f_{(1)} >< g_2 f_{(2)} > . \end{aligned} \tag{I.96}$$

the coproduct map: $\Delta$

$$(\lambda_1 g_1 + \lambda_2 g_2) \cdot g_3 = \lambda_1 g_1 \cdot g_3 + \lambda_2 g_2 \cdot g_3 \tag{I.97}$$

group algebra $\mathbb{R}(G)$.

As this has the structure of a vector space we may form a dual vector space via some inner product Fun$(G)$ of functions on $G$. This bilinear inner product will take values from $< .,. >: \mathbb{R}(G)\otimes\text{Fun}(G) \to \mathbb{R}$. A natural choice is simply as functions on the group space $< g, f >:= f(g)$.

given the inner product between these two vector spaces and given an operator on one of them we can define its dual action, that is its adjoint, acting on the other.

generators of the group vector space $e_i$ and the generators of the dual space $e^i$.

$$m(g_1 \otimes g_2) := g_1 \cdot g_2. \tag{I.98}$$

Since these identities hold for arbitrary elements $a, b$, we will conclude that $A'$ is a Hopf algerba.

coassociativity of $\Delta$ corresponds to associativity of the multiplication in the dual and similarly associativity to coassociativity in the dual.

**coalgebra**

$$
\begin{aligned}
< (id \otimes \Delta)\Delta(\psi), a \otimes b \otimes c > &= < \psi_1 \otimes \Delta(\psi_2), a \otimes b \otimes c > \\
&= < \psi_1 \otimes \psi_2, a \otimes bc > \\
&= < \psi_1 \psi_2, a(bc) > = < \psi_1 \psi_2, (ab)c > \\
&= < \psi_1 \otimes \psi_2, ab \otimes c > \\
&= < \Delta(\psi_1) \otimes \psi_2, a \otimes b \otimes c > \\
&= < (\Delta \otimes id)\Delta(\psi), a \otimes b \otimes c >
\end{aligned} \tag{I.99}
$$

$$(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$$

**algebra**

$$
\begin{aligned}
< \psi(\phi\varphi), a > &= < \psi \otimes \phi\varphi, \Delta(a) > \\
&= < \psi \otimes \phi\varphi, a_1 \otimes a_2 > \\
&= < \psi \otimes \phi \otimes \varphi, a_1 \otimes \Delta(a_2) > \\
&= < \psi \otimes \phi \otimes \varphi, \Delta(a_1) \otimes a_2 > \\
&= < \psi\phi \otimes \varphi, a_1 \otimes a_2 > \\
&= < (\psi\phi)\varphi, a >
\end{aligned} \tag{I.100}
$$

$$\psi(\phi\varphi) = (\psi\phi)\varphi$$

Check:

**(i)**

$$
\begin{aligned}
< \Delta(\psi\phi), a \otimes b > &= < \psi\phi, ab > \\
&= < \psi \otimes \phi, \Delta(ab) > \\
&= < \psi \otimes \phi, \Delta(a)\Delta(b) > \\
&= < \psi_1 \otimes \phi_1 \otimes \psi_2 \otimes \phi_2, \Delta(a) \otimes \Delta(b) > \\
&= < \Delta(\psi)\Delta(\phi), a \otimes b >
\end{aligned} \tag{I.101}
$$

as required. .... Implying

$$\Delta(\psi\phi) = \Delta(\psi)\Delta(\phi)$$

**(ii)**

$$\begin{aligned} < (S\psi_1)\psi_2, a > &= < S\psi_1 \otimes \psi_2, a_1 \otimes a_2 > \\ &= < \psi_1 \otimes \psi_2, Sa_1 \otimes a_2 > \\ &= < \psi, (Sa_1)a_2 > \\ &= < \psi, I > \epsilon(a) \\ &= \epsilon(\psi)\epsilon(a) \\ &= < I\epsilon(\psi), a > \end{aligned} \tag{I.102}$$

Implying

$$(S\psi_1)\psi_2 = I\epsilon(\psi)$$

**(iii)**

$$\begin{aligned} < \Delta(I), a \otimes b > &= < I, ab > \\ &= < I, a >< I, b > \\ &= < I \otimes I, a \otimes b > \end{aligned} \tag{I.103}$$

$$\Delta(I) = I \otimes I.$$

**(iv)**

$$\begin{aligned} \epsilon(\psi\phi) &= < \psi\phi, I > \\ &= < \psi \otimes \phi, I \otimes I > \\ &= < \psi, I >< \phi, I > \\ &= \epsilon(\psi)\epsilon(\phi) \end{aligned} \tag{I.104}$$

□

A subspace $I \subseteq C$ is a **coideal** if $\Delta \subseteq I \otimes C + C \otimes I$ and if $\epsilon(I) = 0$.

if $I$ is a coideal then, the quotient $C/I$ is a coalgebra with comultiplication induced from $\Delta$.

### I.3.1 Hopf algebra from a Lie group: Universal envelopping algebra

#### First account

The universal enveloping algebra $U(g)$ of a Lie algebra $g$, i.e. the algebra, with unit $I$, of polynomials in the generators $\chi_i$ modulo the commutation relations

$$[\chi_i, \chi_j] = C_{ij}^{\ k} \chi_k. \tag{I.105}$$

We define the co-structures on the generators as:

$$\begin{aligned}
\Delta(\chi_i) &= \chi_i \otimes I + I \otimes \chi_i, \quad \Delta(I) = I \otimes I && \text{(I.106)}\\
\epsilon(\chi_i) &= 0, \qquad\qquad\qquad\quad \epsilon(I) = 1 && \text{(I.107)}\\
S(\chi_i) &= -\chi_i, \qquad\qquad\qquad S(I) = I && \text{(I.108)}
\end{aligned}$$

and extend them to all $U(g)$ by requiring $\Delta$ and $\epsilon$ to be linear and multiplicative, $\Delta(\chi\chi') = \Delta(\chi)\Delta(\chi')$, $\epsilon(\chi\chi') = \epsilon(\chi)\epsilon(\chi')$, while $S$ is linear and antimultiplicative. We have to check that the maps $\Delta, \epsilon, S$ are well defined. Since $[\chi, \chi']$ is linear in the generators we have

$$\Delta[\chi, \chi'] = [\chi, \chi'] \otimes I + I \otimes [\chi, \chi'],$$

at the same time, using that $\Delta$ is multiplicative we have

$$\begin{aligned}
\Delta[\chi, \chi'] &= \Delta(\chi)\Delta(\chi') - \Delta(\chi')\Delta(\chi)\\
&= \chi\chi' \otimes I + \chi \otimes \chi' + \chi' \otimes \chi + I \otimes \chi\chi' - (\chi \leftrightarrow \chi')\\
&= [\chi, \chi'] \otimes I + I \otimes [\chi, \chi']
\end{aligned} \tag{I.109}$$

#### Second account

The coproduct, counit, and antipode are

$$\Delta a = a \otimes 1 + 1 \otimes a, \qquad \epsilon a = 0, \qquad Sa = -a \tag{I.110}$$

$\Delta$ algebra map i.e., $\Delta(ab) = \Delta(a)\Delta(b)$, $S$ antialgebra map i.e., $S(ab) = S(b)S(a)$.

$[a, b]$ is a member of the algebra, for $\Delta$ to be consitant we must have $\Delta[a, b] = [a, b] \otimes 1 + 1 \otimes [a, b]$. We can easily check this:

$$\begin{aligned}\Delta[a,b] &= \Delta(ab)-\Delta(ba)\\ &= (a\otimes 1+1\otimes a)(b\otimes 1+1\otimes b)-a\leftrightarrow b\\ &= (ab\otimes 1+1\otimes ba+a\otimes b+b\otimes a)-a\leftrightarrow b\\ &= [a,b]\otimes 1+1\otimes[b,a].\end{aligned} \tag{I.111}$$

If $S$ is to be a consistent we must have $S([a,b])=-[a,b]$, we check this:

$$S([a,b])=S(ab)-S(ba)=(-b)(-a)-(-a)(-b)=-[a,b] \tag{I.112}$$

An element of a general Hopf algebra which has this linear form

$$\Delta a=a\otimes 1+1\otimes a \tag{I.113}$$

is called **primative**.

The Hopf algerbra $U(g)$ is noncommutative, $\xi\xi'\neq\xi'\xi$ but is cocommutative, i.e. for all $\xi\in U(g)$, $\xi_1\otimes\xi_2=\xi_2\otimes\xi_1$. This follows from the modulo commutation property (I.105):

$$\begin{aligned}\xi_1\otimes\xi_2 &= (\xi_1+[\xi_2,\xi_1])\otimes(\xi_2+[\xi_1,\xi_2])\\ &= \xi_2\otimes\xi_1.\end{aligned} \tag{I.114}$$

**Tensor product universal enveloping algebra**

tensor Hopf algebra

$$\mathcal{T}(V)=k\oplus V\oplus(V\otimes V)\oplus\ldots \tag{I.115}$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the quotient of $\mathcal{T}(\mathfrak{g})$ modulo the ideal generated by $\xi\otimes\eta-\eta\otimes\xi-[\xi,\eta]$.

## I.3.2 Quasi-Triangular Hopf Algebras

the map $\tau$

$$\begin{aligned}\tau:\quad & A\otimes A\to A\otimes A\\ & a\otimes b\mapsto b\otimes b.\end{aligned} \tag{I.116}$$

$$\tau(a \otimes b) = b \otimes a. \tag{I.117}$$

Let $\Delta^{op} = \tau \circ \Delta^{op}$ so that, if $a \in A$ and

$$\Delta(a) = \sum_a a_{(1)} \otimes a_{(2)}, \quad \text{then} \quad \Delta^{op}(a) = \sum_a a_{(2)} \otimes a_{(1)}. \tag{I.118}$$

Then $(A, m, \Delta^{op}, \epsilon, \imath, S^{-1})$ forms a Hopf algebra (verify).

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1} \tag{I.119}$$

$$\begin{aligned}(\mathbf{1} \otimes \Delta)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12} = \sum_i A_i A_j \otimes B_j \otimes B_i \\ (\Delta \otimes \mathbf{1})\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23} = \sum_i A_i \otimes A_j \otimes B_i B_j\end{aligned} \tag{I.120}$$

$$(S \otimes \mathbf{1}) = (\mathbf{1} \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1} \tag{I.121}$$

$$\begin{aligned}\mathcal{R}_{12} &= \sum_i A_i \otimes B_i \otimes \mathbf{1} \\ \mathcal{R}_{13} &= \sum_i A_i \otimes \mathbf{1} \otimes B_i \\ \mathcal{R}_{23} &= \sum_i \mathbf{1} \otimes A_i \otimes B_i\end{aligned} \tag{I.122}$$

### I.3.3 Drinfeld's Quantum Double

As this has the structure of a vector space we may form a dual vector space via some inner product $A$ of functions on $A$. This bilinear inner product will take values from $< .,. >: A^* \otimes A \to k$. A natural choice is simply as functions on the vector space $\alpha(a)$. Let $A^* = Hom_k(A, k)$ to be the dual of $A$.

$$D(A) = A \otimes A^{*coop} \tag{I.123}$$

elements $a \otimes \alpha$

$$< \alpha, a >:= \alpha(a) \tag{I.124}$$

$$< \alpha_1 \otimes \alpha_2, a_1 \otimes a_2 >=< \alpha_1, a_1 >< \alpha_2, a_2 > . \tag{I.125}$$

defining multiplication and comultiplication

$$< m^{op}(\alpha_1 \otimes \alpha_2), a >=< \alpha_1 \otimes \alpha_2, \Delta(a) > \quad \text{and} \quad < \Delta^{op}(\alpha), a_1 \otimes a_2 >=< \alpha, m(a_1 \otimes a_2) > \tag{I.126}$$

in components

$$\begin{aligned} e_i e_j = m_{ij}^k e_k, \qquad & \Delta(e_k) = \mu_k^{ij} e_i \otimes e_j \\ e^i e^j = \mu_k^{ij} e^k, \qquad & \Delta(e^k) = m_{ij}^k e^i \otimes e^j \end{aligned} \tag{I.127}$$

lower and upper indexed algebras are dual as bialgebras, the comultiplication of one being the multiplication of the other.

$\mathcal{R}-$matrix

$$\mathcal{R} = \sum_i b_i \otimes b^i, \tag{I.128}$$

comultiplication in $D(A)$,

$$\Delta(\alpha a) = \Delta(\alpha)\Delta(a) = \sum_{a,\alpha} a_{(1)}\alpha_{(1)} \otimes a_{(2)}\alpha_{(2)} \tag{I.129}$$

**Theorem I.3.1** *Let there be a finite dimensional Hopf algebra over $k$ and let $A'^{coop}$ be the Hopf algerba $A^* = Hom_k(A, k)$ with opposite comultiplication. Then there exists a unique quasi-triangular Hopf algebra $(D(A), \mathcal{R})$ given by*

*The $k-$linear map*

$$A \otimes A^{\star coop} \to D(A)$$

$$a \otimes \alpha \mapsto a\alpha$$

*is bijective.*

*$D(A)$ contains $A$ and $A^{\star coop}$ as Hopf subalgebras.*

*The element $\mathcal{R} \in D(A) \otimes D(A)$ is given by*

$$\mathcal{R} = \sum_i b_i \otimes b^i,$$

*where $\{b_i\}$ is a basis of $A$ and $\{b^i\}$ is dual basis in $A'^{coop}$.*

**Example with a finite group**

Let $G$ be a finite group and we assume that $A$ is the group algebra of $G$ i.e. $A = \mathbb{C}[G]$. As a result $A^* = F(G)$, the algebra of functions on $G$, and the quantum double

$$D(\mathbb{C}[G]) = \mathbb{C}[G] \otimes F(G)^{op}$$

is usually denoted $D(G)$. A basis of $D(G)$ is $(x \otimes \delta_g)_{x,g\in G}$ and the Hopf algerba and coalgebra structures are respectively given by:

$$(x \otimes \delta_g) \cdot (y \otimes \delta_h) = xy \otimes \delta_h(y^{-1}gy)\delta_h \tag{I.130}$$

$$\Delta(x \otimes \delta_g) = \sum_{g_1,g_2|g_2g_2=H} (x \otimes g_2) \otimes (x \otimes g_1). \tag{I.131}$$

The action of the antipode on the previous basis is givem by

$$S(x \otimes \delta_g) = x^{-1} \otimes \delta_{x^{-1}g^{-1}x}$$

and the counit is defined by

$$\epsilon(x \otimes \delta g) = \delta(g).$$

The $R$ matrix is given by

$$R = \sum_{x,g} (x \otimes 1) \otimes (1 \otimes \delta_g).$$

Its elements are denoted $(u, \vec{a}) \in SU(2) \times \mathbb{R}^3$

$$(u_1, \vec{a}_1) \cdot (u_2, \vec{a}_2) = (u_1 u_2, u_1 \vec{a}_2 + \vec{a}_1), \tag{I.132}$$

The notation $u\vec{a}$ holds for the action of the vectorial representation of $u$ on the vector $\vec{a}$.

$$ISU(2) = SU(2) \otimes_S \mathbb{R}^3 \tag{I.133}$$

**The Drinfeld double** $DSU(2)$

**1st defifintion:**

The Drinfeld double of a Hopf algebra $A$, for the case of the compact group $SU(2)$, i.e. where $A = \mathbb{C}[SU(2)]$.The Drinfeld double $DSU(2) = \mathbb{C}[SU(2)] \otimes F(SU(2))^{op}$ is a Hopf algebra whose definition given below. In particular, it is a quasi-triangular and admits the group algerba $\mathbb{C}[SU(2)]$ and the algerba of functions $F(SU(2))^{op}$ (with opposite co-product) as sub-Hopf algera.

Hopf algebra structure of $\mathbb{C}[SU(2)]$ is defined by the group law and the following co-algerba relations:

$$\Delta(x) = x \otimes x, \quad S(x) = x^{-1}, \quad \epsilon(x) = \delta(x), \quad \text{for all } x \in SU(2). \tag{I.134}$$

where $\delta$ is the delta distribution on the group $SU(2)$ with respect to the $SU(2)$ Haar measure.

The algebra of functions $F(SU(2))$ is a commutaive and its coalgerba structure is given by:

$$\Delta(f)(x, y) = f(xy), \quad S(f)(x) = f(x^{-1}), \quad \epsilon(f) = f(1),$$

where $f$ is a function and $x, y \in G$.

**2nd defintion:**

The quantum double $DSU(2)$ is defined as a coalgerba as the tensor product of $SU(2)$ with $Fun(SU(2))$ where $Fun(SU(2))$ is a suitable set of functions on $SU(2)$. Its elements are usually denoted $(f, u) \in Fun(SU(2)) \times SU(2)$ in terms of which the algerba is given by

$$(f_1, u_1) \cdot (f_2, u_2) = (f_1 f_2 \circ ad_{u_1}, u_1 u_2), \tag{I.135}$$

and co-algebra by

$$\Delta(f,u) = \sum_{(f)} (f_{(1)}, u) \otimes (f_{(2)}, u). \tag{I.136}$$

where $f, g \in F(SU(2))$, $x, y, a, b \in SU(2)$. The action of the antipode reads

$$S(x \otimes f)(a) = x^{-1} \otimes f(x^{-1}a^{-1}x) \tag{I.137}$$

for any $a \in G$ and the co-unit is simply given by

$$\epsilon(x \otimes f) = f(1).$$

## I.4 Introduction to Quantum Groups

An effective field theory space-time coordinates themselves are noncommutative.

**q deformation of 2-plane.**

$$xy = yx \tag{I.138}$$

$$\begin{aligned} [x,y] &= 0, \quad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right] = 0, \quad \left[\frac{\partial}{\partial x}, y\right] = 0, \quad \left[\frac{\partial}{\partial y}, x\right] = 0 \\ &\left[\frac{\partial}{\partial x}, x\right] = 1, \quad \left[\frac{\partial}{\partial y}, y\right] = 1. \end{aligned} \tag{I.139}$$

$$\begin{aligned} XY &= qYX, \quad \frac{\partial}{\partial X}\frac{\partial}{\partial Y} = q^{-1}\frac{\partial}{\partial X}\frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial X}Y &= qY\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial Y}X = qX\frac{\partial}{\partial Y}, \\ \frac{\partial}{\partial X}X &- q^2X\frac{\partial}{\partial X} = 1 + (q^2-1)Y\frac{\partial}{\partial Y} \end{aligned} \tag{I.140}$$

$$\begin{aligned} AB = qBA, \quad & CD = qDC, \quad AC = qCA, \quad BD = qDB, \\ BC = CB, \quad & AD - DA = (q-q^{-1})BC, \end{aligned} \tag{I.141}$$

and

$$AD - qBC = \det_q T = 1. \tag{I.142}$$

$$\begin{aligned} [n]_q &= \frac{1-q^n}{1-q^{-1}} \\ [n]_q! &= [n]_q[n-1]_q[n-2]_q \dots [2]_q[1]_q, \quad n = 1,2,\dots, \quad [0]_q! := 1 \end{aligned} \tag{I.143}$$

$$[[n]]_q! = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{I.144}$$

$$\begin{aligned} \Delta(A) &= A \otimes A + B \otimes C, \quad \Delta(B) = A \otimes B + B \otimes D, \\ \Delta(C) &= C \otimes A + D \otimes C, \quad \Delta(D) = C \otimes B + D \otimes D. \end{aligned} \tag{I.145}$$

### I.4.1 Q-Deformation of any Classical Lie Algebra

deform any classical Lie algebra. The initial Hopf algebra is the universal enveloping algebra of the Borel subalgebra of a Lie algebra.

$U(sl(2))$, the universal enveloing algebra of $sl(2)$

The classical Lie algebra is generated by $X_+$, $X_-$ and $H$, with commutators

$$[X_+, X_-] = H, \quad [H, X_\pm] = \pm 2X_\pm \tag{I.146}$$

$$\Delta(J) = J \otimes \mathbf{1} + \mathbf{1} \otimes J, \quad J = X_\mp, H. \tag{I.147}$$

The antipode

$$\begin{aligned} \gamma(X_\pm) = -X_\pm, \quad \gamma(H) = -H, \quad \gamma(\mathbf{1}) = \mathbf{1}, \\ \epsilon(X_\pm) = \epsilon(H) = 0, \quad \epsilon(\mathbf{1}) = 1. \end{aligned} \tag{I.148}$$

$$\begin{aligned} e_a e_b = m^c_{ab}\, e_c, \quad \Delta(e^a) = m^a_{cb}\, e^b \otimes e^c \\ e^a e^b = \mu^{ab}_c\, e^c, \quad \Delta(e_a) = \mu^{bc}_a\, e_b \otimes e_c. \end{aligned} \tag{I.149}$$

the antipode $\gamma$

$$\gamma(e_a) = \gamma_a^b e_b, \quad e_a \text{ basis of } A, \tag{I.150}$$

$$\gamma(e^a) = (\gamma^{-1})_b^a e^b, \quad e^b \text{ basis of } A^*. \tag{I.151}$$

$$[X_+, X_-] = [H]_q = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad [H, X_\pm] = \pm 2x_\pm \tag{I.152}$$

Example. Let $G$ be a compact group Lie group. Denote the set of algebraic functions (These are functions that are obtained as matrix elements of finite dimensional representations) on $G$ by $\mathcal{C}(G)$. Then $\mathcal{C}(G)$ forms a commutative Hopf algebra as follows:

$$\begin{array}{ccc} & \mathbf{1}(g) = \mathbf{1}, & (f \cdot g)(g) = f(g)h(g), \\ \epsilon\, f = f(e), & \Delta f(g, g') = f(gg'), & (Sf)(g) = f(g^{-1}) \end{array}$$

where $f, h \in \mathcal{C}(G)$ and $g, g' \in G$ and $e$ denotes the unit element in $G$. For the coproduct observe the implicit isomorphism $\mathcal{C}(G) \otimes \mathcal{C}(G) \cong \mathcal{C}(G \times G)$ which holds due to the Peter-Weyl theorem:

$$G \times G = \oplus$$

given $f, h \in \mathcal{C}(G)$ there exist a unique $F \in \mathcal{C}(G \times G)$

$$f(g) \times h(g') =$$

### I.4.2 Quantum Poincare Lie Algebra

quite general method to deform the Hopf algebra $U(g)$, the universal enveloping algebra of a given Lie algerba $g$. It is based on the twist procedure.

the usual Poincare Lie algerba $iso(3,1)$:

$$\begin{aligned} [P_\mu, P_\nu] &= 0, \\ [P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu), && \text{(I.153)} \\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}), && \text{(I.154)} \end{aligned}$$

### I.4.3 Summary of Quantum Groups

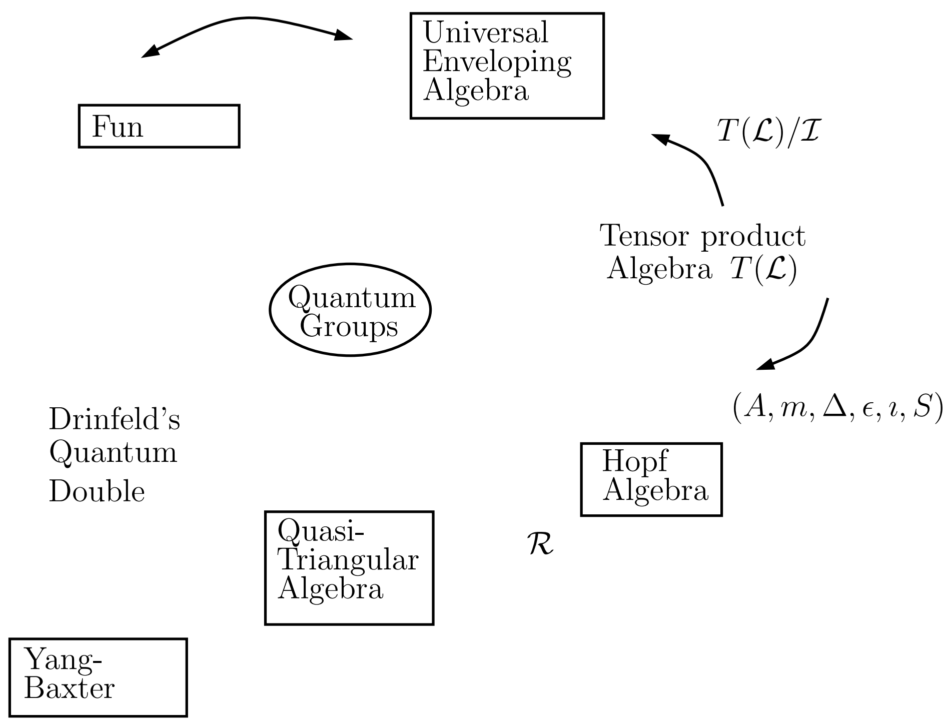



Figure I.4: $\mathcal{I}$ being the ideal generated by . (i) Deformations of commutative Hopf algerbas, of the kind $A = Fun(G)$, are related to deformations of cocommuttative Hopf algebras of the kind $U(g)$ where $g$ is the Lie algebra of $G$. (ii) A non-cocommutative quasi-triangular Hopf algebra is called a quantum group. (iii)

## I.5 WZW Conformal Field Theory

**This section is in very early stage of development. Not really of priority.**

Using: Introduction to conformal field theory by A.N. Schellekens and Quantum Groups in Two-dimensional Physics by C. Gomez et al.

Takes up the subject of knot and three-manifold invariants from Chern-Simons theory using the duality with WZW conformal field theory on a boundary, $\Sigma$, of $\mathcal{M}$.

primary fields are fields that transform as

$$\phi = \left[\frac{df}{dz}\right]^h \left[\frac{d\overline{f}}{d\overline{z}}\right]^{\overline{h}} \phi(z, \overline{z}) \tag{I.155}$$

The infinitesimal form of this transformation is

$$\delta\Phi(w,\overline{w}) = h\partial_w\epsilon(w)\Phi(w,\overline{w}) + \epsilon(w)\partial_w\Phi(w,\overline{w}) \tag{I.156}$$

where $f(z) = z + \epsilon(z)$.

Now we consider the quantum version of this transformation.

$$\delta\phi(w,\overline{w}) = [Q_\epsilon, \phi(w,\overline{w})]. \tag{I.157}$$

we use different contours in the two terms

$$[Q_\epsilon, \phi(w,\overline{w})] = \frac{1}{2\pi i}\oint_{|z|>|w|} dz\ \epsilon(z)T(z)\phi(w,\overline{w}) - \oint_{|z|<|w|} dz\ \epsilon(z)\phi(w,\overline{w})T(z) \tag{I.158}$$

$$[Q_\epsilon, \phi(w,\overline{w})] = \frac{1}{2\pi i}\left[\oint_{|z|>|w|} - \oint_{|z|<|w|}\right] dz\ \epsilon(z)R\big(T(z)\phi(w,\overline{w})\big) \tag{I.159}$$

$$[Q_\epsilon, \phi(w,\overline{w})] = \frac{1}{2\pi i}\oint_{C_w} dz\ \epsilon(z)R\big(T(z)\phi(w,\overline{w})\big) \tag{I.160}$$

where $C_w$ is a closed integration contour encircling the point $w$.

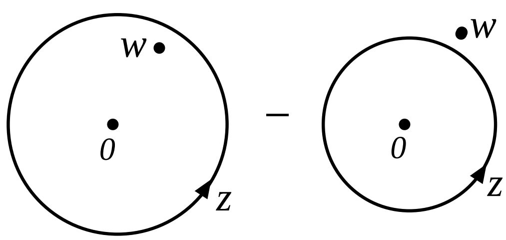



Figure I.5: contourInt. .

Recalling residue theorem:

$$\oint_C f(z)dz = 2\pi i\sum_{i=1}^{n} \mathrm{R}es[f(z_i)] \tag{I.161}$$

if $f(z)$ has an $m$th-order pole at $z_0$ it can be picked out with

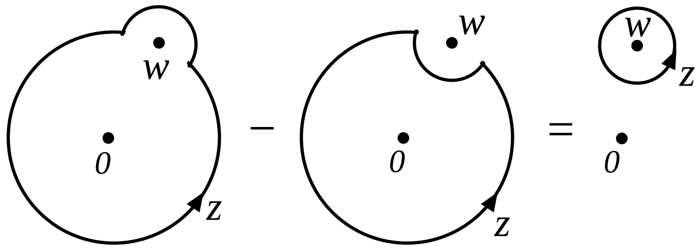



Figure I.6: contourInt2. .

$$\text{R}es[f(z_0)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \tag{I.162}$$

$$R[T(z)\phi(w,\overline{w})] = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_w \phi(w,\overline{w}) + \dots \tag{I.163}$$

the other terms being regular so have no poles - these don't contribute to the integral.

### I.5.1 Representations of the Virasoro Algebra

**Descendants**

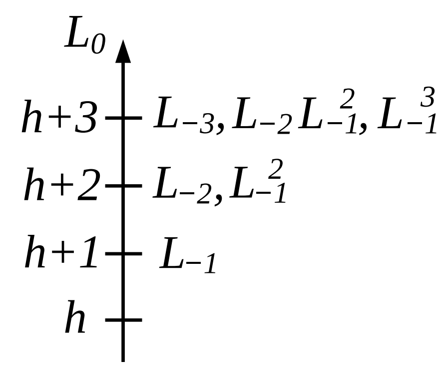



Figure I.7: decsenfig.

Irreducible representations of the Virasoro algebra is called the highest weight representations because they are correspond to a highest weight state $|h>$

$$L_n|h>= 0 \quad n > 0, \qquad L_0|h>= h|h> \tag{I.164}$$

The representation with highest weight $|h>$ consists of the vector space generated by $|h>$ and all descendant fields obtained from $L_{-n}$ $n>0$ on the highest weight.

associated with each irreducible representation $(h,\overline{h})$, we can find a local quantum field $\phi(z,\overline{z})$ such that

$$|h,\overline{h}>=\phi(0,0)|0> \tag{I.165}$$

$$\begin{aligned}[L_n,\phi(w,\overline{w})] &= \frac{1}{2\pi i}\oint z^{n+1}T(z)\phi(w,\overline{w}) \\ &= h(n+1)w^n\phi(w,\overline{w})+w^{n+1}\partial_w\phi(w,\overline{w}),\end{aligned} \tag{I.166}$$

which vanishes if $w=0$ and $n>0$.

### I.5.2 Correlation Functions

**Two-point functions**

$$G(z_1,z_2)=<\phi(z_1)\phi(z_2)>$$

This satisfies the differential equation

$$[\epsilon(z_1)\partial_1+h_1\partial\epsilon(z_1)+\epsilon(z_2)\partial_2+h_2\partial\epsilon(z_1)]G(z_1,z_2)=0,$$

for $\epsilon=1$ this gives

$$(\partial_1+\partial_2)G(z_1,z_2)=0$$

which implies $G(z_1,z_2)=G(z_1-z_2)$. For $\epsilon=z$ and using $\partial_2G(z_1,z_2)=-\partial_1G(z_1,z_2)$

$$[(z_1-z_2)\partial_1+h_1+h_2]G(z_1,z_2)=0.$$

this has the solution

$$G(z_1,z_2)=(z_1-z_2)^{-h_1-h_2}.$$

for the $\epsilon=z^2$ case

$$[z_1^2\partial_1 - z_2^2\partial_2 + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) = 0$$

we find

$$(h_1 - h_2)(z_1 - z_2)G(z_1, z_2) = 0,$$

(exercise) so that $h_1$ must be equal to $h_2$, or else the propagator vanishes. So finally we have

$$G(z_1, z_2) = \begin{cases} C(z_1 - z_2)^{-2h}, & \text{when } h_1 = h_2 = h \\ 0, & \text{otherwise} \end{cases} . \tag{I.167}$$

**Three-point functions**

for $\epsilon = 1$ this gives

$$(\partial_1 + \partial_2 + \partial_3)G(z_1, z_2, z_3) = 0$$

and implies that $G$ is a fucntion of $z_{12} = z_1 - z_2$, $z_{23} = z_2 - z_3$ and $z_{31} = z_3 - z_1$

$$G = G(z_{12} = z_1 - z_2, z_2 - z_3, z_3 - z_1)$$

There are really two independent variables since $z_{31} = z_{23} - z_{12}$. We retain the redundent variable to maintain symmetry. For $\epsilon = z$,

$$z_{12}\frac{\partial G}{\partial z_{12}} + z_{23}\frac{\partial G}{\partial z_{23}} + z_{31}\frac{\partial G}{\partial z_{31}} = -(h_1 + h_2 + h_3)G.$$

We can solve this using separation of variables. Set $G$ equal to

$$G = A(z_{12})B(z_{23})C(z_{13})$$

from which

$$\frac{z_{12}}{A}\frac{\partial A}{\partial z_{12}} = a, \quad \frac{z_{23}}{B}\frac{\partial B}{\partial z_{23}} = b, \quad \frac{z_{31}}{C}\frac{\partial C}{\partial z_{31}} = c,$$

where

$$a+b+c=-(h_1+h_2+h_3)$$

we have then

$$G_{ijk}(z_{12},z_{23},z_{13})=C_{ijk}z_{12}^a z_{23}^b z_{31}^c$$

$\epsilon=z^2$ gives

$$(a+c+2h_1)z_1+(a+b+2h_2)z_2+(b+c+2h_3)z_3=0.$$

As $z_1, z_2$ and $z_3$ are independent we may put $z_1=1, z_2=z_3=0$, $z_1=z_3=0, z_2=1$ and $z_1=z_2=0, z_3=1$ in turn, and obtain

$$a+c+2h_1=a+b+2h_2=b+c+2h_3=0$$

$$a=h_3-h_1-h_2, \quad b=h_1-h_2-h_3, \quad c=h_3-h_1-h_2.$$

$$G_{ijk}^{(3)}(z_{12},z_{23},z_{13})=C_{ijk}z_{12}^{h_3-h_1-h_2}z_{23}^{h_1-h_2-h_3}z_{31}^{h_2-h_3-h_1}.$$

### I.5.3 Conformal Ward Identities

$$<0|\oint\frac{dz}{2\pi i}\epsilon(z)T(z)\phi(w_1)\dots\phi(w_n)|0> \tag{I.168}$$

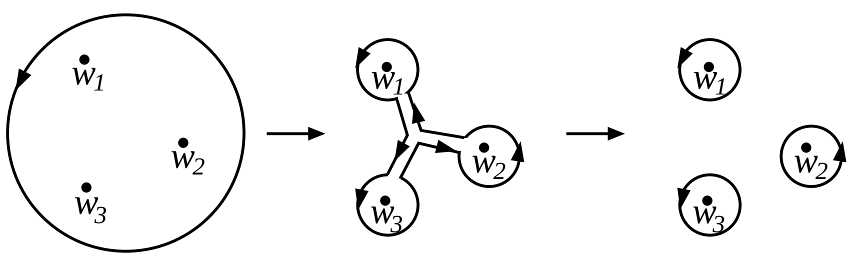



Figure I.8: ConWardfig.

$$\begin{aligned}&\sum_i <0|\phi(w_1)\dots\oint\frac{dz}{2\pi i}\epsilon(z)T(z)\phi(w_i)\dots\phi(w_n)|0>\\ =\ &\sum_i <0|\phi(w_1)\dots\delta_\epsilon\phi(w_i)\dots\phi(w_n)|0>\end{aligned} \tag{I.169}$$

we get

$$
\begin{aligned}
&< 0|T(z)\phi(w_1)\dots\phi(w_n)|0> \\
= \quad & \sum_i \left[\frac{h_i}{(z-w_i)^2} + \frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right] < 0|\phi(w_1)\dots\phi(w_n)|0>
\end{aligned}
\tag{I.170}
$$

### I.5.4 Correlators of Descendants

$$
< 0|\phi_1^{(-k)}(w_1)\phi_2(w_2)\dots\phi_n(w_n)|0> \tag{I.171}
$$

where $\phi_1^{(-k)}$ is the k-$th$ descendant of $\phi$.

$$
\mathcal{L}_{-k}^i = -\frac{(1-k)h_i}{(w_i-w_1)^k} + \frac{1}{(w_i-w_1)^{k-1}}\frac{\partial}{\partial w_i} \tag{I.172}
$$

we put

$$
\mathcal{L}_{-k} := \sum_{i=2}^{n} \mathcal{L}_{-k}^i
$$

we can write the correlation function as

$$
< 0|\phi_1^{(-k)}(w_1)\phi_2(w_2)\dots\phi_n(w_n)|0> = \mathcal{L}_{-k} < 0|\phi_1(w_1)\phi_2(w_2)\dots\phi_n(w_n)|0> \tag{I.173}
$$

### I.5.5 Null State Decoupling

We wish to know if $\|\alpha L_{-2}|h> + \beta L_{-1}L_{-1}|h>\|$ is non positive. To this end we consider the Hermitian matrix

$$
K_2 = \begin{pmatrix} < h|\ L_{-2}^\dagger L_{-2}\ |h> & < h|\ L_{-2}^\dagger L_{-1}L_{-1}\ |h> \\ < h|\ (L_{-1}L_{-1})^\dagger L_{-2}\ |h> & < h|\ (L_{-1}L_{-1})^\dagger L_{-1}L_{-1}\ |h> \end{pmatrix}, \tag{I.174}
$$

noting that

$$
vK_2v^T = \|\alpha L_{-2}|h> + \beta L_{-1}L_{-1}|h>\|
$$

where $\vec{v} = (\alpha, \beta)$. If this matrix were to have negative or zero determinant then there must exist an eigenvector $\vec{v} = (\alpha, \beta)$ with zero or negative eigenvalue. If this were the case the quantity $\|\alpha L_{-2}|h> + \beta L_{-1} L_{-1}|h> \|$ would not be posive.

There are analogous matrices $K_n$ for the n-*th* level. We cannot tell how many positive, zero and negative eigenvalues there are as if $\det K > 0$ there could be an even number of negative eigenvalues.

We are interested in the case of null vectors, the eigenvectors of zero norm.

### I.5.6 Conformal Blocks

A two-dimensional chiral conformal field theory is a quantum field theory defined over cylindrical space time $\mathbb{R} \times S^1_R$ of radius $R$, with coordinates $(\tau, )$. The quantum mechanical degrees of freedom are chiral, which means that the dynamical modes of such a theory purely left-moving right-moving.

Given two representations, $\lambda$ and $\mu$, one can define their *fusion*, namely a *tensor product representation*, $\lambda \otimes \mu$, which is again a unitary representation of $\mathcal{A}$. A chiral algebra is called *rational* if and only if the number of inequivalent, irreducible unitary representations is finite. For a rational chiral algebra, the tensor product of two representations can be decomposed into a direct sum of irreduciable unitary representations.

Let $N^{\lambda_3}_{\lambda_1, \lambda_2}$ denote the multiplicity of $\lambda_3$ as a subrepresentation in the tensor product $\lambda_1 \otimes \lambda_2$. The multiplicities$N^{\lambda_3}_{\lambda_1, \lambda_2}$ are the structure constants of the ring and are called *fusion rules*; for a rational chiral algebra, they are finite non-negative integers.

Given number $n$ of irreducible representations, $\lambda_1, \ldots, \lambda_n$, we define the linear space of *conformal blocks* as the space of invariant tensors, i.e., of invariant linear functionals, on the representation space of the tensor product representation $\lambda_1 \otimes \cdots \otimes \lambda_n$.

#### four-point functions

$$F_{ijkl}(z, \overline{z}) = \sum_m C_{ijm} C_{mkl} \mathcal{F}^m_{ijkl}(z) \overline{\mathcal{F}}^m_{ijkl}(\overline{z})$$

#### Decoupling

Conformal blocks form a basis to the space of solutions to the decoupling equations.

## I.6 Classical and Quantum Logic

This difference is the following. In the classical propositional calculus the logical connectives between propositionals obey the distributive law,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{I.175}$$

while the prositional calculus for quantum mechanics $\wedge$ and $\vee$ do not obey this law but a weaker version of it. This is the modular law,

$$\text{if } a \leq c \text{ then } \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{I.176}$$

**Classical logic (absolute):**

'the value of $A$ is $a$'

**Classical logic (contextual):**

'at a certain 'stage' the value of $A$ is $a$'

**Quantum logic:**

'the state of the system is an eigenvector of the operator $\hat{\mathcal{O}}$ with eigenvalue $\lambda$'

The non commutativity of the observable operators results in a non Boolean logic for the propositions.

This is mathematically described by the orthomodular lattice structure of projectors on Hilbert space.

**Quantum logic (relational or contextual):**

'the state of the system $S$ relative to another system, $O$, is an eigenvector of the operator $\hat{\mathcal{O}}$ with eigenvalue $\lambda_O$'

there is no meaning to the state of an isolated system.

quantum mechanical aspect and a distributive causal aspect.

## I.7 Quantum Logic

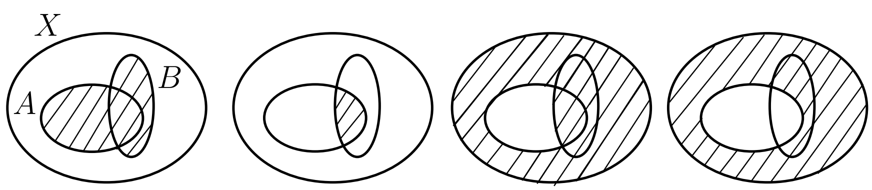
A positive answer to $S$ implies a negative one to $\neg S$.

where the relation of inclusion and the operation of intersection between linear subspaces are the set inclusion and intersection, but where the set complement is replaced by the operation of taking the orthogonal subspace, and the union of two sets by the sum of subspaces.

propositional logic can express relationships that would be far too complicated to express in ordinary language.

| $x$ | $y$ | $x \vee y$ | $x \wedge y$ | $x \Rightarrow y$ | $\neg x$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 0 |
| 1 | 0 | 1 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 |

Example: We define a partial relation between subsets of $X$: if $A \subseteq B$ then $A \leq B$. every two subsets $A, B \in X$ have their union $A \cap B$ is the least upper bound and their intersection $A \cup B$ as the greatest lower bound. Consider any other subset $C$ which is smaller than $A$ and $B$, i.e., $C \subset A$ and $C \subset B$ (i.e., $C \leq A$ and $C \leq B$).




(a) (b) (c) (d)

Figure I.9: If $A \subseteq B$ then $A \leq B$. partially ordered set of subsets of $X$ is a lattice. (a) Their union $A \cap B$ is the l.u.b. (b) $A \cup B$ is the g.l.b.

**Definition** a ***lattice*** is a partially ordered set where the meet and join of any two elements always exists.

**Definition** a lattice is called ***complete*** if and only if the meet and join exist for arbitrary families of elements.

**Lemma I.7.1** *Complete lattice always contains elements 0 and 1.*

**Proof.** Element 0 can be defined as the infimum of all elements of $\mathcal{L}$ and element 1 can be defined as their supremum. Both exist because of the completeness of the lattice.

□

**Definition** Element of the lattice is ***atomic*** if $0 \leq x$ such that there is no $y$ that satisfies $0 \leq y \leq x$.

a join $y = \bigvee x$

The operations $\wedge$ and $\vee$ have the following properties:

$$\begin{array}{rll} \text{(a)}\ \ x \wedge x = x & \text{(b)}\ \ x \vee x = x & \text{(I.177)} \\ \text{(a)}\ \ x \wedge y = y \wedge x & \text{(b)}\ \ x \vee y = y \vee x & \text{(I.178)} \\ \text{(a)}\ \ x \wedge (y \wedge z) = (x \wedge y) \wedge z & \text{(b)}\ \ x \vee (y \vee z) = (x \vee y) \vee z & \text{(I.179)} \\ \text{(a)}\ \ (x \wedge y) \vee x = x & \text{(b)}\ \ (x \vee y) \wedge x = x. & \text{(I.180)} \end{array}$$

$$x \leq y \;\Leftrightarrow\; x \wedge y = x. \tag{I.181}$$

$x \wedge y = x$ and $x \vee y = y$ are equivalent. $x \wedge y = x$, then

$$\begin{aligned} x \vee y &= (x \wedge y) \vee y \\ &= (y \wedge x) \vee y \\ &= y, \end{aligned} \tag{I.182}$$

and similarly $x \vee y = y$ implies $x \wedge y = x$.

In this way we see that characterizing lattices where there were relations are now operations, brings them closer to abstract algebra.

The (I.179 (a)) and (I.178 (a)) imply $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ can be given unambiguous meaning, for example

$$\begin{aligned} x_1 \wedge x_2 \wedge x_3 \wedge x_4 &= (x_1 \wedge x_2) \wedge (x_3 \wedge x_4) \\ &= ((x_1 \wedge x_2) \wedge x_3) \wedge x_4 \\ &= (x_1 \wedge (x_2 \wedge x_3)) \wedge x_4 \\ &= ((x_1 \wedge x_3) \wedge x_2) \wedge x_4 \\ &= (x_1 \wedge x_3) \wedge (x_2 \wedge x_4) \\ &= \ldots \end{aligned} \tag{I.183}$$

Similarly for $x_1 \vee x_2 \vee \cdots \vee x_n$.

A lattice is distributive if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \tag{I.184}$$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{I.185}$$

A *Boolean algebra* is a complemented distributive lattice.

| lattice operation | propositional calculus |
|---|---|
| order relation $\leq$ | implication $\rightarrow$ |
| "meet" $\wedge$ | disjunction "and" $\wedge$ |
| "join" $\vee$ | disjunction "or" $\vee$ |
| "complement" $'$ | negation "not" $\neg$ |

To a given proposition $\mathcal{P}$, we assign a subset $A$ with the interpretation that an element belongs to $A$ if and only if it satisfies the proposition $\mathcal{P}$.

## Orthomodular lattice

$$(a \vee (a^{\perp} \wedge (a \vee b))) = (a \vee b) \tag{I.186}$$

One can weaken the distributaty condition by requiring that (I.190) only hold if $x \leq z$. A lattice is called modular if for all $y$

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \tag{I.187}$$

In an orthocomplemented lattice hold de Morgan laws

$$1^{\perp} = 0, \quad 0^{\perp} \quad (x \vee y)^{\perp} = x^{\perp} \wedge y^{\perp} \quad (x \wedge y)^{\perp} = x^{\perp} \vee y^{\perp}. \tag{I.188}$$

**Definition** Orthocomplemented lattice $\mathcal{L}$ is called **orthomodular** if condition (I.187) holds for $y = x^{\perp}$, that is

$$x \leq z \Rightarrow x \vee (x^{\perp} \wedge z) = z. \tag{I.189}$$

## Logic of Projectors on a Hilbert Space

A projection operator is characterized by *"if you measure again, you will get the same answer"*.

Closed subspaces of a Hilbert space, ordered by inclusion, form a complete lattice, (in a same way to how subsets of a set forms a lattice by inclusion). The greatest lower bound of the subspaces is their intersection, while their least upper bound is the closed span of their union.

projections, we may impose upon the set $L(H)$ the structure of a complete orthocomplemented lattice, defining $P \leq Q$, where $ran(P) \subset ran(Q)$ and $P' = 1 - P$ (so that $ran(P') = ran(P)^{\perp}$). It is straightforward that $P \leq Q$ just in case $PQ = QP = P$. More generally, if $PQ = QP$, then $PQ = PQ$, the meet of $P$ and $Q$ in $L(H)$; also in this case their join is given by $PQ = P + Q - PQ$.

from [367]:

This is still a distributional algebra. Namely, for propositions $P$, $Q$ and $R$, if $P \vee Q$ denotes "$P$ or $Q$", and $P \wedge Q$ means "$P$ and $Q$", then $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$.

the observabls algebra is modified even at the classical level Quantum mechanics is linear, and as a result of the superposition principle, quantum mechanical propositions are not distributive. If $P, Q$ and $R$ are projection operators, $P \vee (Q \wedge R)$ is not equal to $(P \vee Q) \wedge (P \vee R)$.

$\rightarrow$ The logical propositions that represent physical questions in quantum mechanics are of the form the state of the system is an eigenvector of the operator $\hat{\mathcal{O}}$ with eigenvalue $\lambda$ , for an observable $\mathcal{O}$. The non-commutativity of the observable operators gives place to a non-Boolean logic for the propositions, in contrast to the case of the propositions about the phase space of a system in classical physics.This is mathematically described by the orthomodular lattice structure of projectors in Hilbert space, which is often called quantum logic [357]. $\leftarrow$

$- - - - - - - - - - -$

Physical quantities or observables of a quantum entity are described as self-adjoint operators on a complex Hilbert space, where the outcomes of the observable are the numbers of the spectrum of the self-adjoint operator.

The most simple of all observables, an observable with only two possible outcomes, labelled 'yes' and 'no', is described by a self-adjoint operator with only two elements in its spectrum, and this is an orthogonal projection operator.

yes-no-experiments' can be taken as basic in quantum mechanics because a general experiment with more than two outcomes can always be subdivided in different yes-no-experiments, considering each time again a subset and its complement of the outcome set. This means that if we develop a framework for the general description of yes-no-experiments we can reconstruct all the experiments by considering them as compositions of their underlying yes-no-experiments. Much in the same way as a self-adjoint operator is completely defined by means of its spectral resolution, which is the set of orthogonal projections corresponding to the yes-no-experiments that are sub experiments of the observable described by this self-adjoint operator.

In Rovelli's axiomatic approach he postulates that 'yes-no-experiments' are the basic objects.

$- - - - - - - - - - -$

**Gleason's Theorem**

**Quantum Logic**

modern quantum mechanics is based on following assumption

*The partially ordered set of all questions in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of separable, infinite dimensional Hilbert space.*

**Quantum Logic**

An atom of a lattice $\mathcal{L}$ is an element $a$ for which $0 \leq x \leq a$ implies that $x = 0$ or $x = a$. A lattice with 0 is called atomic if for every $x \neq 0$ in $\mathcal{L}$ there is an atom $a \neq 0$ such that $a \leq x$.

Recall that a lattice is called distributive if

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \tag{I.190}$$

One can weaken the distributaty condition by requiring that (I.190) only hold if $x \leq z$. A lattice is called modular if for all $y$

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{I.191}$$

Boolean algebras - algebras for classical logic

## I.7.1 Relational Models

Quantum mechanics has survived all tests and there is no reason to believe that there is any flaw in it.

It does not try to append a reasonable interpretation to the quantum mechanics formalism, nor to postulate as yet unknown new physics, rather it attempts to uncover the physical meaning of quantum mechanics.

nostalgia for the absolute

act of bringing about information

$$\underbrace{\alpha_1 \phi_{q_1} + \alpha_2 \phi_{q_2}}_{t_1} \Longrightarrow \underbrace{\alpha_2 \phi_{q_2}}_{t_2} \tag{I.192}$$

$$\psi_{init} \otimes (\alpha_1 \phi_{q_1} + \alpha_2 \phi_{q_2}) \tag{I.193}$$

if $O'$ performs no measurement,

$$\alpha_1\psi_{q_1} \otimes \phi_{q_1} + \alpha_2\psi_{q_2} \otimes \phi_{q_2}. \tag{I.194}$$

A notation will not appear: the state of the system. The absence of this notion is a prime feature of the relational interpretation here. In place of the notion of state, which refers solely to the system, the notion of the information that a system has about another system is introduced.

With the conceptual foundations of the approach laid out. We describe the program of information-theoretic derivation of quantum theory.

## I.8 Information-Theoretical Derivation of Quantum Theory via Quantum Logic

### I.8.1 Introduction

Rovelli:

> "Quantum mechanics will cease to look puzzling only when we will be able to derive the formalism of the theory from a set of simple physical assertions ("postulates", "principles") about the world. Therefore, we should not try to append a reasonable interpretation to the quantum mechanical formalism, but rather to derive the formalism from a set of experimentally motivated postulates."

From a system of information-theoretic axioms, we use quantum logical techniques for deriving the of quantum theory (Hilbert space structure). Inspired by the ideas of Rovelli's Relational Quantum Mechanics, we then propose a system of information-theoretic axioms and use quantum logical techniques for deriving lattice orthomodularity. Lattice orthomodularity follows from the postulate of finite amount of relevant information, while quantumness is due to the possibility to always obtain new information. To complete the information-theoretic reconstruction of the formalism, additional axioms are proposed from which follow the state space structure and unitary time evolution. Subsequently, Hilbert space, Born rule and unitary dynamics are all formally obtained.

In conclusion, it is shown how a twofold role of the observer as physical system and informational agent leads to a description of measurement by POVM, a necessary ingredient of quantum computation.

**Axioms**

**Postulate 1** (Limited information). There is a maximum amount of relevant information that can be extracted from a system.

**Postulate 2** (Unlimited information). It is always possible to acquire new information about a system.

The motivation for the second postulate is fully experimental. We know that all quantum systems (and all systems are quantum systems) have the property that even if we know their quantum state $|\psi>$ exactly, we can still learn "learn" more about them by performing a measurement of a quantity $O$ such that $|\psi>$ is not an eigenstate of $O$. This is an *experimental* result about the world, coded in quantum mechanics. Postulate 2 expresses this result.

Since the amount of information that $O$ can have about $S$ is limited by postulate 1, when new information is acquired, part of the old relevant-information becomes irrelevant.

The physical content of the general formulism of quantum mechanics is (almost) nothing but a sequence of consequences of two physical facts expressed in postulates 1,2.

**What do we need to reconstruct?**

1) Obtain Hilbert space and prove quantumness.

2) Obtain Born rule with the state space.

3) Obtain unitary dynamics.

**Quantum logical reconstruction of the Hilbert space**

1. Definition of the lattice of yes-no questions.
2. Definition of orthogonal complement.
3. Definition of relevance and proof of orthomodularity.
4. Introduction of the space structure.
5. Lemmas about properties of the space.
6. Definition of the numeric field.
7. Construction of the Hilbert space.

A few notes.

Number 3 - Necessary (but not sufficient) condition to obtain a Hilbert space.

Number 6 - Fields are the "number systems" in maths.

### I.8.2 Reconstruction Attempts

supply $I$-observer with information. The number of questions in $I$, though, can be much larger than $N$, as some of these questions are not independent. In particular, they may be related by implication ($Q_1 \Rightarrow Q_2$), union ($Q_3 = Q_1 \vee Q_2$), and intersection ($Q_3 = Q_1 \wedge Q_2$). One can define an always false question ($Q_0$) and an always true question ($Q_\infty$), negation of a question ($\neg Q$), and a notion of orthogonality as follows: if $Q_1 \Rightarrow \neg Q_2$, then $Q_1$ and $Q_2$ are orthogonal ($Q_1 \perp Q_2$).

#### More Axioms

We will come to Axiom III in the section I.8.10.

Supplementary assumptions:

IV. The logical *or.* $\wedge$ is defined for every pair of questions.

V. The logical *and.* $\vee$ is defined for every pair of questions.

VI. The lattice of questions is complete.

VII. The underlying field of the space of the theory is one of the numeric fields $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{D}$ and the involutory anti-automorphism in this field is continuous.

#### Strings of Answers and Complete Questions

asks the $N$ questions in the family $c$ then the obtained answers form a string

$$s_c = [e_i, \dots, e_N]_c. \tag{I.195}$$

The string $s_c$ can take $2^N = K$ values. We denote them as $s_c^{(1)}$, $s_c^{(2)}, \dots, s_c^{(K)}$ so that

$$\begin{aligned} s_c^{(1)} &= [0, 0, \dots, 0]_c \\ s_c^{(2)} &= [0, 0, \dots, 1]_c \\ &\dots \\ s_c^{(K)} &= [1, 1, \dots, 1]_c \end{aligned} \tag{I.196}$$

$$
\begin{aligned}
Q_c^{(1)} &= [(e_1 = 0) \wedge (e_1 = 0) \wedge \cdots \wedge (e_N = 0)]_c? = \neg Q_1 \wedge \neg Q_2 \wedge \cdots \wedge \neg Q_N \\
Q_c^{(2)} &= [(e_1 = 0) \wedge (e_1 = 0) \wedge \cdots \wedge (e_N = 1)]_c? = \neg Q_1 \wedge \neg Q_2 \wedge \cdots \wedge \ Q_N \\
&\cdots \\
Q_c^{(k)} &= [(e_1 = 1) \wedge (e_1 = 1) \wedge \cdots \wedge (e_N = 1)]_c? = \ Q_1 \wedge \ Q_2 \wedge \cdots \wedge \ Q_N
\end{aligned}
\tag{I.197}
$$

the Questions of kind are referred to as complete questions. Here are some of its properties. It is obvious that $Q_c^{(i)} \wedge Q_c^{(i)} = Q_c^{(i)}$ and $Q_c^{(i)} \vee Q_c^{(i)} = Q_c^{(i)}$. It is also obvious that $Q_c^{(i)} \wedge Q_c^{(j)} = Q_c^{(j)} \wedge Q_c^{(i)}$ and $Q_c^{(i)} \vee Q_c^{(j)} = Q_c^{(j)} \vee Q_c^{(i)}$

$$
\begin{aligned}
\text{“}Q_c^{(i)} \text{ and } Q_\infty\text{”} &= Q_c^{(i)} \qquad & \text{“}Q_c^{(i)} \text{ and } Q_0\text{”} &= Q^0 \\
\text{“}Q_c^{(i)} \text{ or } Q_0\text{”} &= Q_c^{(i)} \qquad & \text{“}Q_c^{(i)} \text{ or } Q_0\text{”} &= Q^\infty
\end{aligned}
\tag{I.198}
$$

$$
\begin{aligned}
Q_c^{(i)} \wedge Q_\infty &= Q_c^{(i)} \qquad & Q_c^{(i)} \wedge Q_0 &= Q_0 \\
Q_c^{(i)} \vee Q_0 &= Q_c^{(i)} \qquad & Q_c^{(i)} \vee Q_0 &= Q_\infty
\end{aligned}
\tag{I.199}
$$

**Lemma I.8.1** *Complete questions $Q_c^{(i)}$ are mutually exclusive, i.e.,*

$$
Q_c^{(i)} \wedge Q_c^{(j)} = Q_0 \qquad \textit{for all } i \neq j. \tag{I.200}
$$

*and for them holds the distributivity law:*

$$
Q_c^{(i)} \vee (Q_c^{(j)} \wedge Q_c^{(k)}) = (Q_c^{(i)} \vee Q_c^{(j)}) \wedge (Q_c^{(i)} \vee Q_c^{(k)}). \tag{I.201}
$$

This implies, from (E.7), also that

$$
Q_c^{(i)} \wedge (Q_c^{(j)} \vee Q_c^{(k)}) = (Q_c^{(i)} \wedge Q_c^{(j)}) \vee (Q_c^{(i)} \wedge Q_c^{(k)}). \tag{I.202}
$$

**Proof:**

Equality to the always false question of the disjunction of any two different complete questions follows immediately from their definition (I.197). For example

$$
\begin{aligned}
& (Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \wedge (Q_1 \wedge \neg Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \\
= \; & (Q_1 \wedge Q_1) \wedge (Q_2 \wedge \neg Q_2) \wedge (\neg Q_3 \wedge \neg Q_3) \wedge (\neg Q_4 \wedge \neg Q_4) \\
= \; & Q_\infty \wedge Q_0 \wedge Q_\infty \wedge Q_\infty \\
= \; & Q_0. \qquad \text{(I.203)}
\end{aligned}
$$

In words, the question "$Q_2$ and $\neg Q_2$" asks something that can never be true, so the question "$Q_c^{(i)}$ and $Q_c^{(j)}$" asks something that can never be true.

**2.** The case $i = j = k$ is trivial.

for $i \neq j = k$

$$\text{"}(Q_c^{(i)} \text{ or } Q_c^{(j)}) \text{ and } (Q_c^{(i)} \text{ or } Q_c^{(j)})\text{"} \equiv \text{"}Q_c^{(i)} \text{ or } Q_c^{(j)}\text{"}.$$

for $i \neq j \neq k$

$$Q_c^{(i)} \vee (Q_c^{(j)} \wedge Q_c^{(k)}) = Q_c^{(i)} \vee Q_0 = Q_c^{(i)} \qquad \text{(I.204)}$$

for $i = j \neq k$

$\square$

**Definition (Material implication)**

**Definition** Questions are **orthogonal** if

$$Q_1 \rightarrow \neg Q_2. \qquad \text{(I.205)}$$

What would be the effect of applying negation to $Q_1 \wedge Q_2$?

$$\neg(Q_1 \wedge Q_2) = \neg\text{"}Q_1 \text{ and } Q_2\text{"} = \text{"}\neg Q_1 \text{ or } \neg Q_2\text{"} = \neg Q_1 \vee \neg Q_2.$$

Say $Q_1 \wedge Q_2 = Q_1$ then

$$\neg Q_1 \wedge \neg Q_2 = \neg Q_2,$$

that is, $\neg Q_2 \leq \neg Q_1$.

**Definition** Orthocomplement $Q^\perp$ is the union of all questions orthogonal to $Q$.

**Lemma I.8.2** *The orthocomplementation as defined in $W(P)$ fulfills the requirements for a lattice orthocomplementation.*

**Proof.** A complemented lattice has a map $x \to x^\perp$, satisfying for all $x, y \in \mathcal{L}$ (i) $x^{\perp\perp} = x$, (ii) $x \leq y \Leftrightarrow y^\perp \leq x^\perp$, (iii) $x \wedge x^\perp = 0$, (iv) $x \vee x^\perp = 1$.

**(i)** The implication $Q_1 \to \neg Q_2$ is equivalent to $Q_2 \to \neg Q_1$. If we take $Q_2$ to be the orthocomplement to $Q_1$, i.e $Q_2 := Q_1^\perp$, this equivalence insures that $(Q_1^\perp)^\perp = Q_2^\perp = Q_1$.

**(iii)** and **(iv)** It is trivial to verify that $Q \wedge Q^\perp = Q_0$ and $Q \vee Q^\perp = Q_\infty$ since $Q^\perp$ is greater or equal to $\neg Q$.

**(ii)** Assume that $Q_1 \leq Q_2$, i.e. $Q_1 \wedge Q_2 = Q_1$. We need to prove that $Q_2^\perp \leq Q_1^\perp$, i.e. $Q_2^\perp \wedge Q_1^\perp = Q_2^\perp$. The left-hand side of this last expression denotes such questions $Q$ that $Q_1 \to \neg Q$ and $Q_2 \to \neg Q$ in all possible measurements.

In their turn, these two conditions holding separately in all measurements imply that it must not be the case that $[(Q_1 \vee Q_2) \wedge \neg Q]$. Note that the equality $Q_1 \wedge Q_2 = Q_1$ implies $\neg Q_1 \vee \neg Q_2 = \neg Q_1$. We get for the negative assumption

$$\begin{aligned} \neg[(Q_1 \vee Q_2) \wedge \neg Q] &= (\neg Q_1 \wedge \neg Q_2) \vee Q \\ &= [(\neg Q_1 \vee \neg Q_2) \wedge \neg Q_2] \vee Q \\ &= \neg Q_2 \vee Q. \end{aligned} \tag{I.206}$$

Recall that this must not be the case. Then negation of the last expression in the line entails that $\neg Q \wedge Q_2$. Since equivalence holds everywhere in (I.206) and we started with $Q_2^\perp \wedge Q_1^\perp$, we conclude that $Q_2^\perp \wedge Q_1^\perp = Q_2^\perp$.

□

**Lemma I.8.3** *By taking all possible unions of sets of complete questions $Q_c^{(i)}$ of the same family $c$*

$$\begin{aligned} &Q_c^{(i)} \\ &Q_c^{(i)} \vee Q_c^{(j)} \\ &Q_c^{(i)} \vee Q_c^{(j)} \vee Q_c^{(k)} \\ &\qquad \cdots \\ &\underbrace{Q_c^{(i)} \vee Q_c^{(j)} \vee \cdots \vee Q_c^{(k)}}_{2^N} \end{aligned} \tag{I.207}$$

*one constructs a Boolean algebra that has $Q_c^{(i)}$ as atoms. Where $Q_0$ and $Q_\infty$serve as the lattice elements 0 and 1.*

**Proof:** A Boolean algebra is a complemented distributive lattice.

Recall a lattice is a partially ordered set where the meet and join of any two elements always exists.

The partial ordering is defined as:

$$x \leq y \Leftrightarrow x \wedge y = x. \tag{I.208}$$

A complemented lattice has a map $x \to x^\perp$, satisfying for all $x, y \in \mathcal{L}$ (i) $x^{\perp\perp} = x$, (ii) $x \leq y \Leftrightarrow y^\perp \leq x^\perp$, (iii) $x \wedge x^\perp = 0$, (iv) $x \vee x^\perp = 1$.

Do all joins

$$(Q_c^{(i_1)} \vee Q_c^{(j_1)} \vee \cdots \vee Q_c^{(k_1)}) \vee (Q_c^{(i_2)} \vee Q_c^{(j_2)} \vee \cdots \vee Q_c^{(k_2)})$$

exist? That is, is (I.207) closed under the joins of any two of its members?

Does the meet of any two members of (I.207) exist?

$$(Q_c^{(i_1)} \vee Q_c^{(j_1)} \vee \cdots \vee Q_c^{(k_1)}) \wedge (Q_c^{(i_2)} \vee Q_c^{(j_2)} \vee \cdots \vee Q_c^{(k_2)})$$

exist?

$$\begin{aligned} Q_c^{(1)} &= \neg Q_1 \wedge \neg Q_2 \\ Q_c^{(2)} &= \neg Q_1 \wedge Q_2 \\ Q_c^{(3)} &= Q_1 \wedge Q_2 \end{aligned} \tag{I.209}$$

We construct elements:

$$v := Q_c^{(1)}, \quad w := Q_c^{(2)}, \quad y := Q_c^{(1)} \vee Q_c^{(2)} \tag{I.210}$$

**1.** First we must show that $v \wedge w$, $v \wedge x$ $v \wedge y$, $w \wedge x$, $w \wedge y$ and $x \wedge y$ exist. And similarly for $\wedge$ replaced by $\vee$.

$$
\begin{aligned}
v \vee w &= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \\
v \vee y &= Q_c^{(1)} \vee (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \\
w \vee y &= Q_c^{(2)} \vee (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \\
y \vee y &= (Q_c^{(1)} \vee Q_c^{(2)}) \vee (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \vee Q_c^{(2)} \\
&= y \qquad \text{(I.211)}
\end{aligned}
$$

$$
\begin{aligned}
v \wedge w &= Q_c^{(1)} \wedge Q_c^{(2)} \\
&= Q_0 \\
v \wedge y &= Q_c^{(1)} \wedge (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(1)} \\
&= v \\
w \wedge y &= Q_c^{(2)} \wedge (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(2)} \\
&= w \\
y \wedge y &= (Q_c^{(1)} \vee Q_c^{(2)}) \wedge (Q_c^{(1)} \vee Q_c^{(2)}) \\
&= Q_c^{(2)} \vee Q_c^{(2)} \\
&= w \qquad \text{(I.212)}
\end{aligned}
$$

**2.** Second must show that there is no element $q$ such that $Q_0 \leq q \leq Q_c^{(i)}$.

We need to show that the only element $q$ in (I.207) such that $q \wedge Q_c^{(1)} = q$ is $Q_0$. By the distributive law (I.201) we have

$$
\begin{aligned}
(Q_c^{(i)} \wedge (Q_c^{(j)} \vee \cdots \vee Q_c^{(k)})) &= (Q_c^{(i)} \wedge Q_c^{(j)}) \vee \cdots \vee (Q_c^{(i)} \wedge Q_c^{(k)}) \\
&= Q_0 \vee \cdots \vee Q_0 \\
&= Q_0 \qquad \text{(I.213)}
\end{aligned}
$$

Hence, the only element such that $Q_0 \leq q \leq Q_c^{(i)}$ is $Q_0$.

**3.** The existence of a complementation. Let us define the map $^\perp$ on (I.207) as

$$(\neg Q_1 \wedge Q_2 \wedge \cdots \wedge \neg Q_{N-1} \wedge Q_N)^\perp := (Q_1 \wedge \neg Q_2 \wedge \cdots \wedge Q_{N-1} \wedge \neg Q_N) \tag{I.214}$$

and the map $^\perp$ on $(Q_c^{(i)} \wedge \cdots \wedge Q_c^{(k)})$ as

$$(Q_c^{(i)} \wedge \cdots \wedge Q_c^{(k)})^\perp := \tag{I.215}$$

(i) Does $x^{\perp\perp} = x$? This is easy to see, for example

$$\begin{aligned} (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4)^{\perp\perp} &= (Q_1 \wedge \neg Q_2 \wedge Q_3 \wedge Q_4)^\perp \\ &= \neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4 \end{aligned} \tag{I.216}$$

So ${Q_c^{(i)}}^{\perp\perp} = Q_c^{(i)}$. Now consider

$$(Q_c^{(i_1)} \wedge Q_c^{(j_1)} \wedge \cdots \wedge Q_c^{(k_1)})^{\perp\perp}$$

(ii) Does $x \leq y$ imply $y^\perp \leq x^\perp$? That is, does $x \wedge y = x$ imply $y^\perp \wedge x^\perp = y^\perp$. Note that only

$$Q_c^{(i)} \wedge Q_c^{(i)} = Q_c^{(i)}, \qquad Q_c^{(i)} \wedge Q_\infty = Q_c^{(i)} \tag{I.217}$$

$$Q_c^{(i)} \wedge Q_c^{(i)} = Q_c^{(i)}, \qquad Q_c^{(i)} \wedge Q_\infty = Q_c^{(i)} \tag{I.218}$$

(iii) Does $x \wedge x^\perp = Q_\infty$?

$$\begin{aligned} & (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \wedge (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4)^\perp \\ = \; & (\neg Q_1 \wedge Q_1) \wedge (Q_2 \wedge \neg Q_2) \wedge (\neg Q_3 \wedge Q_3) \wedge (\neg Q_4 \wedge Q_4) \\ = \; & Q_\infty \wedge \cdots \wedge Q_\infty = Q_\infty. \end{aligned} \tag{I.219}$$

(iv) Does $x \vee x^\perp = Q_0$?

$$
\begin{aligned}
&(\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4) \vee (\neg Q_1 \wedge Q_2 \wedge \neg Q_3 \wedge \neg Q_4)^{\perp} \\
= \;& (\neg Q_1 \vee Q_1) \wedge (Q_2 \vee \neg Q_2) \wedge (\neg Q_3 \wedge Q_3) \wedge (\neg Q_4 \wedge Q_4) \\
= \;& Q_0 \wedge \cdots \wedge Q_0 = Q_0.
\end{aligned}
\tag{I.220}
$$

□

→ Alternatively, one can consider a different family $d$ of $N$ independent yes-no questions and obtain another Boolean algebra with different complete questions as atoms.

It follows, then, from Axiom I that the set of questions $W(P)$ that can be asked to P-observer is algebraically an orthomodular lattice containing subsets that form Boolean algebras.

$$
\begin{array}{ccc}
(Q_c^{(i)} \vee Q_c^{(j)} \vee \cdots \vee Q_c^{(k)}) & \vee & (Q_b^{(i)} \vee Q_b^{(j)} \vee \cdots \vee Q_b^{(k)}) \\
(Q_c^{(i)} \vee Q_c^{(j)} \vee \cdots \vee Q_c^{(k)}) & \wedge & (Q_b^{(i)} \vee Q_b^{(j)} \vee \cdots \vee Q_b^{(k)})
\end{array}
\tag{I.221}
$$

As Rovelli says, "This is precisely the algebraic structure formed by the family of linear subsets of Hilbert space".

This concludes his sketch. The sketch of the Hilbert space construction is not a rigorous derivation due to two key obstacles: First,

**(i) orthomodularity of the lattice was not derived**

and, strictly speaking, from Rovelli s construction one cannot derive it. Second, even if one admits that the lattice is orthomodular,

**(ii) the fact that yes-no questions form an orthomodular lattice and that it contains as subsets Boolean algebras does not yet lead to emergence of the Hilbert space.**

Both these claims were formalized by Gribaum in [376] where all the assumptions needed on the way to rigorous proof made explicit. ←

[20]

*In fact, one may conjecture that this peculiar consistency between the observations of different observers is the missing ingredient for a reconstruction theorem of the Hilbert space formulism of quantum theory. Such a reconstruction theorem is still unavailable. On the basis of reasonable physical assumptions, one is able to derive the structure of an orthomodular lattice containing subsets that form Boolean algebras which "almost," but* *not quite, implies the existence of a Hilbert space and its projector's algebra. Perhaps an appropriate algebraic formulation of the condition of consistency between subsystems could provide the missing hypothesis to complete the reconstruction theorem.*

### I.8.3 Relevance

Information is brought about in the answer to a yes-no question. . Consider $b$ such that it entails the negation of $a : b \Rightarrow \neg a$. If the observer asks $a$ and obtains an answer to $a \ldots$ but then asks a genuine new question $b$, it means that the observer expects either $a$ positive or negative answer to $b$. This, is turn, is only possible if information $a$ is no longer relevant; indeed, otherwise the observer would be bound to always obtain the negative answer to $b$. We say that, by asking $b$, the observer renders the information brought about by asking question $a$ irrelevant.

**Definition Relevance** Question $b$ is called irrelevant with respect to question $a$ if $b \wedge a^{\perp} \neq 0$.

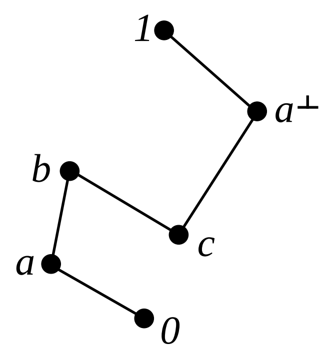



Figure I.10: relevance1.

Trivial in Hilbert spaces: $x \leq y$ are relevant with respect to $y$, all others irrelevant.

Non-trivial if used to derive what in Hilbert space lattices assumed.

### I.8.4 Amount of Information

Two assumptions:

1. If relevance is not lost, the amount of information grows monotonously as new information comes in.

2. The lattice contains all possible information (yes-no questions). Thus, there are sufficiently many questions as to bring about any a priori allowed amount of information.

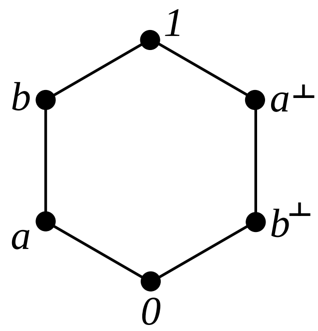



Figure I.11: relevance2.

### I.8.5 Proof of Orthomodularity

By Axiom I there exists a finite upper bound of the amount of relevant information, call it $N$. Select an arbitrary question $a$ and consider a question $\tilde{a}$ such that $\{a, \tilde{a}\}$ bring $N$ bits of information. Then $a^{\perp} \wedge \tilde{a} = 0$.

**Lemma I.8.4** *An orthocomplemented lattice is orthomodular if and only if $a \leq b$ and $a^{\perp} \wedge b = 0$ imply $a = b$.*

Question $b$ is relevant with respect to $a$; and question $\tilde{a}$ is relevant with respect to $b$.

Consider $\{a, b, \tilde{a}\}$. If $b > a$, this sequence preserves relevance and brings about strictly more than $N$ bits of relevant information.

From the contradiction follows $a = b$.

**Proof of Lemma I.8.4**

An orthocomplemented lattice $\mathcal{L}$ is orthomodular if and only if $x \leq z$ and $x^{\perp} \wedge z = 0$ imply $x = z$.

First prove the "only if" case. Say our orthocomplemented lattice $\mathcal{L}$ is orthomodular and that $x \leq z$ and $x^{\perp} \wedge z = 0$. By defintion (I.189), as $x \leq z$, one has

$$x \vee (x^{\perp} \wedge z) = x \vee 0 = z. \tag{I.222}$$

Now we turn to the "if" case. To prove this we assume the lattice is not orthomodular and show this conflicts with $x = z$ for all $x$ and $z$ when $x \leq z$ and $x^{\perp} \wedge z = 0$; as the lattice is either orthomodular or not orthomodular we can conclude that it is orthomodular.

Assume that the lattice is not orthomodular. We need to show is that there exist elements $x$ and $z$ such that

$$x \leq z, \quad x^{\perp} \wedge z = 0, \quad x \neq z. \tag{I.223}$$

Let us use the notation $x < z$ if $x \leq z$ and $x \neq z$. We can then rewrite (I.223) as

$$x < z, \quad x^{\perp} \wedge z = 0. \tag{I.224}$$

By the definition of an orthomodular latice (I.189), there exist elements $y$ and $z$ such that

$$y \leq z, \quad y \vee (y^{\perp} \wedge z) \neq z. \tag{I.225}$$

It can be shown, ([379] Chapter 2, Section 4), that in any lattice we have

$$a \leq b \quad \Rightarrow \quad (c \wedge b) \vee a \leq (c \vee a) \wedge b \quad \text{for all } c. \tag{I.226}$$

Put into this $a = y$, $b = z$ and $c = y^{\perp}$. It follows that

$$(y^{\perp} \wedge z) \vee y \leq (y^{\perp} \wedge y) \wedge z. \tag{I.227}$$

In the right-hand side replace $y^{\perp} \wedge y = 1$, and $1 \wedge z = z$, we then have

$$(y^{\perp} \wedge z) \vee y \leq z. \tag{I.228}$$

From this and (I.225) we obtain

$$(y^{\perp} \wedge z) \vee y < z. \tag{I.229}$$

On the other hand, form de Morgan laws (I.188) one has

$$\begin{aligned} z \wedge (y \vee (y^{\perp} \wedge z)^{\perp}) &= z \wedge (y^{\perp} \wedge (y^{\perp} \wedge z)^{\perp}) \\ &= z \wedge (y^{\perp} \wedge (y \vee z^{\perp})) \\ &= (z \wedge y^{\perp}) \wedge (y \vee z^{\perp}) \\ &= (z \wedge y^{\perp}) \wedge (z \wedge y^{\perp})^{\perp} = 0. \end{aligned} \tag{I.230}$$

Now put $x = y \vee (y^{\perp} \wedge z)$. Equations (I.229) and (I.230) can be rewritten as

$$x < z, \quad x^{\perp} \wedge z = 0. \tag{I.231}$$

□

### I.8.6 Kalmbach's theorem

Infinite-dimensional Hilbert space characterization theorem:

Let $H$ be an infinite-dimensional vector space over real or complex numbers or quaternions. Let $L$ be a complete orthomodular lattice of subspaces of $H$ which satisfies:

(i) Every finite-dimensional subspace of $H$ belongs to $L$.

(ii) For every element $U$ of $L$ and for every finite-dimensional subspace $V$ of $H$, linear sum $U + V$ belongs to $L$.

Then there exists an inner product $f$ on $H$ such that $(H, f)$ is a Hilbert space with L as its lattice of closed subspaces.

### I.8.7 Step 6: Definition of the numeric field

Axiom VII: The underlying numeric field of $V$ is one of the real or complex numbers or quaternions, and the involutory anti-automorphism (conjugation) is continuous.

Substitutes: Solr's theorem assuming existence of an infinite orthonormal sequence of vectors. Also: Zieler, Holland, Landsman

### I.8.8 Step 7: Construction of the Hilbert space

**Theorem I.8.5** *Let $W(P)$ be an ensemble of yes-no questions that can be asked to a physical system and $V$ a vector space over real or complex numbers or quaternions such that a lattice of its subspaces L is isomorphic to $W(P)$. Then there exists an inner product $f$ on $V$ such that $V$ together with $f$ form a Hilbert space.*

### I.8.9 Quantumness

The Hilbert space was soley built using consequences of Axiom I (and supplementary axioms), and Axiom II remained unused through the whole discussion which preceded Theorem E.7.

Quantumness Axiom II: It is always possible to acquire new information about a system.

Criterion: Orthomodular lattice, in order to describe a quantum mechanical system, must be nondistributive.

**Lemma I.8.6** *All Boolean subalgebras of $L(V)$ are proper.*

**Corollary I.8.7** *$W(P)$ is non-Boolean.*

### I.8.10 State space and the Born rule

$$p^{ij} = p(Q_b^{(i)}, Q_c^{(j)}). \tag{I.232}$$

From the way it is defined, the $2^N \times 2^N$ matrix cannot by fully arbitrary.

$$0 \geq p^{ij} \geq 1. \tag{I.233}$$

$$\sum_i p^{ij} \tag{I.234}$$

Rovelli would like to deduce the the existence of the states space and the Born rule from a third axiom [371],

tentative axiom III. Different observers hold information in a consistent way.

No one has completed the programme proposed by Rovelli. Grinbaumm chooses a different approach.

Axiom III Intratheoretic non-contextuality : If information is obtained by an observer, then it is obtained independently of how the measurement was eventually conducted, i.e. independent of the measurement context.

No metainformation : If information $I$ about a system has been brought about, then it happened without bringing in information $J$ about the fact of bringing about information $I$.

Gleason s theorem builds the state space.

[380]:

**Theorem I.8.8 (Gleason).** *Let $f$ be any function from 1-diemensional projections on a Hilbert space of dimension $d > 2$ to the unit interval, such that for each resoltion of the identity in projections $\{P_k\}$, $k = 1 \ldots d$*

$$\sum_{k=1}^{d} P_k = I, \quad \sum_{k=1}^{d} f(P_k) = 1. \tag{I.235}$$

*Then there exists a unique density matrix $\rho$ such that $f(P_k) = Tr(\rho P_k)$.*

### I.8.11 Time and unitary

Assume isomorphism between $W_t(P)$ at different time moments. In other words, time evolution commutes with orthogonal complementation, hence with relevance.

A symmetry, then, is a mapping of vectors in Hilbert space

$$|\psi>\to|\psi'>, \tag{I.236}$$

that preserves the absolute values of inner products

$$|<\varphi|\psi>|=|<\varphi'|\psi'>|, \tag{I.237}$$

for all $|\varphi>$ and $|\psi>$. According to Wigner's theorem, a mapping with this property can always be chosen to be either unitary or anti-unitary. The anti-unitary alternative, while important for discrete symmetries, can be excluded for continuous symmetries. Then the symmetry acts as

$$|\psi>\to|\psi'>=U|\psi>, \tag{I.238}$$

where $U$ is unitary (and linear).

Applying Wigner's theorem, there is a unitary or anti-unitary transformation $U(t_1,t_2)$ : $W_{t_1}(P)\to W_{t_2}(P)$. The unitary transformation is selected by virtue of the condition of continuity in the limit $t_2\to t_1$.

$$Q(t_2)=U(t_2-t_1)Q(t_1)U^{-1}(t_2-t_1) \tag{I.239}$$

The Hamiltonian description follows by conventional arguments (Stone's theorem), these unitary operators form an abelean group and

$U(t_1-t_2)=\exp[-i(t_1-t_2)\mathcal{H}]$.

where $\mathcal{H}$ is a self-adjoint operator on the Hilbert space, the Hamiltonian. By defintion

$$U(t+\delta t,t_0)=(\mathbf{1}-i\delta t\mathcal{H}(t))U(t,t_0). \tag{I.240}$$

Hence

$$\mathcal{H}(t)=i\frac{\partial U(t)}{\partial t}U^{-1}(t). \tag{I.241}$$

The Schrödinger equation follows immediately if we transform from the Heisenberg to the Schrödinger picture.

### I.8.12 POVM description

Twofold role of the observer: Observer is at the same time a physical system (P-observer) and an informational agent (I-observer). Information-based physical theory must give an account of P-observer, while I-observer must remain metatheoretic.

Starting with an orthogonal projector description of measurement and factoring out P-observer, one obtains the general POVM description of measurement.

### I.8.13 List of axioms

Information-theoretic axioms:

I. There is a maximum amount of relevant information that can be extracted from a system.

II. It is always possible to acquire new information about a system.

III. If information $I$ about a system has been brought about, then it happened independently of information $J$ about the fact of bringing about information $I$.

Supplementary assumptions:

IV. The logical *or*. $\wedge$ is defined for every pair of questions. Or more formally:

For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to at least one of the initial question is positive. Thatis, for all $Q_1, Q_2 \in I$ there exists $Q_3$ such that $Q_3 = Q_1 \wedge Q_2$

V. The logical *and*. $\vee$ is defined for every pair of questions. Or more formally:

For any two yes-no questions there exists a yes-no question to which the answer is positive if and only if the answer to both initial questions is positive. That is, all $Q_1, Q_2 \in I$ there exists $Q_3$ such that $Q_3 = Q_1 \vee Q_2$

VI. The lattice of questions is complete.

VII. The underlying field of the space of the theory is one of the numeric fields R, C or D and the involutory anti-automorphism in this field is continuous.

### I.8.14 Open questions

1. Information-theoretic meaning of Axioms IV, V, and VI.
2. Information-theoretic meaning of Axiom VII or of Solr s theorem.
3. Meaning of the probability function $f$ used in Gleason s theorem.
4. Origin of assumptions concerning time evolution.
5. Problem of dimension of the Hilbert space.
6. Superselection rules.

### I.8.15 Example

maintained by the Copenhagen interpretation

is photon is absorbed at time $t$, the cat dies. At time $t'$ the outside observer opens the box.

| photon absorbed | cat alive | observer finds the cat alive |
|---|---|---|
| T | F | F |
| F | T | T |

states $\Phi_1$ $\Phi_2$ interacts with system $S$ with two states $\Psi_1$ $\Psi_2$. If the *up* (*down*) the led goes *on* (*off*).

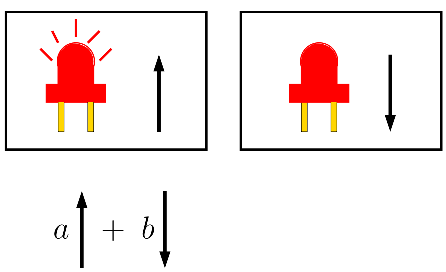


Figure I.12: If the *up* (*down*) the led goes *on* (*off*).

T for true , F for U for undecidable.

| led (S) on | spin (O) up | observer (O') |
|---|---|---|
| T | T | T |
| F | F | F |

T for true with respect to something, F for false with respect to something and U for undecidable with respect to something.

With respect to $S$ before $t'$

| led (S) on | spin (O) up | observer (O') |
| --- | --- | --- |
| T | T | U |
| F | F | U |

## I.9 Heyting Algebra

Heyting algebras are algebraic structures that play a role in theories of intuitionistic logic analogous to that played by Boolean algebras in theories of classical logic.

The axioms for a Heyting algebra are the same as in (I.177), (I.178), (I.179), (I.180), with the following

$$\begin{aligned} (x \Rightarrow x) &= 1, \\ x \wedge (x \Rightarrow y) &= x \wedge y, \\ y \wedge (x \Rightarrow y) &= y, \\ (x \Rightarrow (y \wedge z)) &= (x \Rightarrow y) \wedge (x \Rightarrow z). \end{aligned} \tag{I.242}$$

Every Heyting algebra is distributive.

the observables algebra is modified even at the classical level. Internal observables satisfy a Heyting algebra, which is a weak version of the Boolean algebra of ordinary observables. This is still a distributional algebra. Namely, for propositions $P$, $Q$ and $R$, if $P \vee Q$ denotes "$P$ or $Q$", and $P \wedge Q$ means "$P$ and $Q$", then $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$.

Having both quantum mechanics and internal observables in the same theory requires finding propositions that have a non-distributive quantum mechanical aspect and a distributive causal aspect. The aim of the present paper is to define the histories in which such observables may be encountered.

A boolean algebra is a Heyting algebra in which $x \vee \neg x = 1$.

negation is defined in terms of implication and falsehood as

$$\neg A \equiv A \Rightarrow \perp \tag{I.243}$$

**Topological**

... is a Heyting algebra, where $\mathcal{O}(X)$ is the family of of all open sets in $X$, and partial ordering $\subseteq$ is defined by set inclusion. In $\mathcal{O}(X)$ meet and join are just set-theoretical intersection and union, while the implication operation is given by

$$U \Rightarrow V = \text{ interior of } (X - U) \cup V.$$

taken from B . Residuated Lattices, B1. Algebras for Logics.

### Any Algebra Determines a Logic

For each Heyting algebra $\mathbf{H}$, define $L(\mathbf{H}) = \{\psi(t) : P \models t = 1\}$. Then, $L(\mathbf{H})$ is a logic over intuitionistic logic **Int**.

### Any Logic is Determined by an Algebra

For each logic $\mathbf{L}$ over **Int**, there is an algebra $\mathbf{Q}$ such that $\mathbf{L} = L(\mathbf{Q})$. For instance, take Lindenbaum algebra of $\mathbf{L}$

### Lattice $\longrightarrow$ Algebra $\longrightarrow$ Underlying Logic

## I.10 Biblioliographical notes

In this chapter I have relied on the following references:

Arun Ram Quantum groups: *A survey of definitions, motivations, and results.*

## I.11 Worked Exercies and Details

| Proofs |
|---|

(a) Prove the centre of $\mathbf{g}$ is an ideal.

(b) Show this is precisely the kernal of the adjoint representation of $\mathbf{g}$.

**Answers**

(a) Must show that

$$[[g_1, h_1], [g_2, h_2]] = [[g_2, h_2], [g_1, h_1]] \tag{I.244}$$

for all $g_1, g_2 \in \mathbf{g}$ and for all $h_1, h_2$ in the set of commuting elements. Use $[ab, c] = abc - cab = a[b, c] + [a, c]b$ to write

Proofs

(a) Prove

(b)

**Answers**

(a)

Proofs

ideal generated by $x \otimes x$

$$k \oplus \mathcal{L} \oplus \mathcal{L} \otimes \mathcal{L} \oplus \dots \tag{I.245}$$

Proofs

$$[\Delta]A = \Delta A + \mathbf{1} \otimes x + x \otimes \mathbf{1} = [\Delta A] \quad \text{where } x \in I, \tag{I.246}$$

$$[\mathrm{id}]A = \mathrm{id}A + \mathbf{1} \otimes x + x \otimes \mathbf{1} = [A]. \tag{I.247}$$

$$\begin{aligned} ([\Delta] \otimes [\mathrm{id}]) \circ [\Delta] &= ([\Delta] \otimes [\mathrm{id}]) \circ (\Delta A + \mathbf{1} \otimes x + x \otimes \mathbf{1}) \\ &= \end{aligned} \tag{I.248}$$

$$([\Delta] \otimes [\mathrm{id}]) \circ [\Delta] = ([\mathrm{id}] \otimes [\Delta]) \circ [\Delta] \tag{I.249}$$

Explicate example Hopf algebra.

**The multiplication operation**

$$m \circ (\sigma_a \otimes \sigma_b) = m^c_{ab} \sigma_c \tag{I.250}$$

$$
\begin{aligned}
m \quad \circ \quad & (\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
= \quad & m \circ ( \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} ) \\
= \quad & m^0_{12} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m^1_{12} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + m^2_{12} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + m^3_{12} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{I.251}
$$

$$
\begin{aligned}
m \quad \circ \quad & (\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
= \quad & m^0_{02} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m^1_{02} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + m^2_{02} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + m^3_{02} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{I.252}
$$

$$
\begin{aligned}
m^k_{ij} &= i\epsilon_{ijk}, \ \text{ for } i,j,k = 1,2,3, \ \text{ for } i \neq j. \\
m^0_{aa} &= 1, \qquad \text{for } a = 0,1,2,3 \ , \text{(no summation)} \\
m^b_{0a} &= m^b_{a0} = \delta^b_a, \ \text{ for } a,b = 0,1,2,3.
\end{aligned} \tag{I.253}
$$

$$
m^0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ m^1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ m^2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -i \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \ m^3 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{I.254}
$$

**The coproduct**

$$
\Delta(\sigma_a) = \sigma_a \otimes e + e \otimes \sigma_a = \mu^{bc}_a \sigma_b \otimes \sigma_c \tag{I.255}
$$

$$
\Delta(\sigma_2) = \sigma_2 \otimes e + e \otimes \sigma_2 = \begin{pmatrix} 0 & -i & -i & -i \\ i & 0 & -i & -i \\ i & i & 0 & -i \\ i & i & i & 0 \end{pmatrix} \tag{I.256}
$$

Proofs

Express as matrix equations

$$
\begin{aligned}
m \circ (e_i \otimes e_j) &= m_{ij}^k e_k, \quad \Delta(e_i) = \mu_i^{jk} e_j \otimes e_k \\
\imath(k) = i^j e_j, \quad \epsilon(e_i) &= \epsilon_i, \quad S(e_i) = S_i^j e_j
\end{aligned} \tag{I.257}
$$

**Answers**

**(1)** (I.62)

$$
\begin{aligned}
m \circ (\mathrm{id}_A \otimes m)(e_i \otimes e_j \otimes e_k) = m \circ (e_i \otimes m_{jk}^l e_l) = m_{jk}^l m_{il}^m e_m \\
m \circ (m \otimes \mathrm{id}_A)(e_i \otimes e_j \otimes e_k) = m \circ (m_{ij}^l e_l \otimes e_k) = m_{ij}^l m_{lk}^m e_m
\end{aligned} \tag{I.258}
$$

$$
\Longrightarrow \qquad m_{jk}^l m_{il}^m = m_{ij}^l m_{lk}^m \tag{I.259}
$$

**(2)** (I.63)

$$
(\mathrm{id}_A \otimes \Delta) \circ \Delta(e_i) = \mu_i^{jk} (\mathrm{id}_A \otimes \Delta) \circ (e_j \otimes e_k) = \mu_i^{jk} \mu_k^{lm} (e_j \otimes e_l \otimes e_m) \tag{I.260}
$$

$$
\begin{aligned}
(\Delta \otimes \mathrm{id}_A) \circ \Delta \circ e_i &= \mu_i^{jk} (\Delta \otimes \mathrm{id}_A) \circ (e_j \otimes e_k) = \mu_i^{jk} \mu_j^{lm} (e_l \otimes e_m \otimes e_k) \\
&= \mu_i^{km} \mu_k^{jl} (e_j \otimes e_l \otimes e_m)
\end{aligned} \tag{I.261}
$$

$$
\Longrightarrow \qquad \mu_i^{jk} \mu_k^{lm} = \mu_i^{km} \mu_k^{jl} \tag{I.262}
$$

**(3)** (I.64) **unit condition,**

**(4)** (I.65) **counit condition,** $\epsilon : A \to k$

$$
m \circ (\mathrm{id}_A \otimes \epsilon) \circ \Delta(a) = \mu_i^{jk} m \circ (\mathrm{id}_A \otimes \epsilon) \circ e_j \otimes e_k = \mu_i^{jk} m_{lm}^k e_j \epsilon(e_k) \tag{I.263}
$$

**(5)** (I.66) **$\Delta$ is an algebra homomorphism,** $\epsilon : A \to k$

$$
\Delta \circ m = (m \otimes m) \circ (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_A) \circ (\Delta \otimes \Delta), \tag{I.264}
$$

**(6)** (I.67) **$\epsilon$ is an algebra homomorphism,** $\epsilon : A \to k$, $\epsilon(e_i) \mapsto k_i$

$$
\epsilon \circ m(e_i \otimes e_j) = \epsilon \otimes \epsilon(e_i \otimes e_j), \tag{I.265}
$$

$$
m_{ij}^k k_k = k_i k_j \tag{I.266}
$$

**(7)** (I.68) $S(e_i) = S_i^j e_j$

$$m\circ(\mathrm{id}_A\otimes S)\circ\Delta(e_i) = \mu_i^{jk} m\circ(\mathrm{id}_A\otimes S)\circ(e_j\otimes e_k) = \mu_i^{jk}S_k^l m\circ(e_j\otimes e_l) = (\mu_i^{jk}S_k^l m_{jl}^m)\; e_m \quad \text{(I.267)}$$

$$m(S\otimes \mathrm{id}_A)\circ\Delta(e_i) = \mu_i^{jk}S_j^l m\circ(e_l\otimes e_k) = (\mu_i^{jk}S_j^l m_{lk}^m)\; e_m \quad \text{(I.268)}$$

$$\imath\circ\epsilon(e_i) = \imath(k) = k\mathrm{id}_A \quad \text{(I.269)}$$

$$\mu_i^{jk}S_k^l m_{jl}^m = \mu_i^{jk}S_j^l m_{lk}^m = \quad \text{(I.270)}$$

Proofs

Verify that $(A, m, \Delta^{op}, \epsilon, \imath, S^{-1})$ forms a Hopf algebra,

**Answer**

$(A, m, \Delta^{op}, \epsilon, \imath, S^{-1})$ forms a Hopf algebra if:

$$\begin{array}{ll}
(1) & m\circ(\mathrm{id}_A\otimes\epsilon)\circ\Delta^{op} = m\circ(\epsilon\otimes\mathrm{id}_A)\circ\Delta^{op} = \mathrm{id}_A \\
(2) & (\mathrm{id}_A\otimes\Delta^{op})\circ\Delta^{op} = (\Delta^{op}\otimes\mathrm{id}_A)\circ\Delta^{op}, \\
(3) & \Delta^{op}\circ m = (m\otimes m)\circ(\mathrm{id}_A\otimes\tau\otimes\mathrm{id}_A)\circ(\Delta^{op}\otimes\Delta^{op}), \\
(4) & m\circ(\mathrm{id}_A\otimes S^{-1})\circ\Delta^{op} = m(S^{-1}\otimes\mathrm{id}_A)\circ\Delta^{op} = \imath\circ\epsilon.
\end{array} \quad \text{(I.271)}$$

Assume (2) is true,

$$\Delta^{op}(e_i) = \mu_i^{jk}e_k\otimes e_j = \mu_i^{kj}e_j\otimes e_k \quad \text{(I.272)}$$

$$(\mathrm{id}_A\otimes\Delta^{op})\circ\Delta^{op}(e_i) = \mu_i^{kj}(\mathrm{id}_A\otimes\Delta^{op})\circ(e_j\otimes e_k) = \mu_i^{kj}\mu_k^{ml}(e_j\otimes e_l\otimes e_m) \quad \text{(I.273)}$$

$$\begin{aligned}
(\Delta^{op}\otimes\mathrm{id}_A)\circ\Delta^{op}\circ e_i &= \mu_i^{kj}(\Delta^{op}\otimes\mathrm{id}_A)\circ(e_j\otimes e_k) = \mu_i^{kj}\mu_j^{ml}(e_l\otimes e_m\otimes e_k) \\
&= \mu_i^{km}\mu_k^{jl}(e_j\otimes e_l\otimes e_m)
\end{aligned} \quad \text{(I.274)}$$

$$\implies \qquad \mu_i^{kj}\mu_k^{ml} = \mu_i^{mk}\mu_k^{lj} \quad \text{(I.275)}$$

If we swap around $m$ and $j$, we have agreement with (I.262) and so we have verified (2).

Assume (4) is true,

The same as in but we swap the upper two indices of $\mu$ replace $S$ by its inverse, so that we arrive at the condition:

$$\mu_i^{kj}(S^{-1})_k^l m_{jl}^m = \mu_i^{kj}(S^{-1})_j^l m_{lk}^m = \tag{I.276}$$

$$\mu_i^{kj}(S^{-1})_k^l m_{jl}^m = \mu_i^{kj}(S^{-1})_j^l m_{lk}^m = \tag{I.277}$$

bf Prove:

Verify that $(A, m^{op}, \Delta, \epsilon, \imath, S^{-1})$ forms a Hopf algebra,

Proofs

**Proof:**

**1**

$$\begin{aligned} 0 &= [(z_1^2 - z_2^2)\partial_1 + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) \\ &= [(z_1 + z_2)(z_1 - z_2)\partial_1 + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) \\ &= [-(z_1 + z_2)(h_1 + h_2) + 2h_1 z_1 + 2h_2 z_2]G(z_1, z_2) \end{aligned}$$

**2**

Proofs

Prove the equivalence of (I.184) and (I.185).

**proof (1)**

$x \wedge y = x$ and $x \vee y = y$ are equivalent. $x \wedge y = x$, then

$$\begin{aligned} x \vee y &= (x \wedge y) \vee y \\ &= (y \wedge x) \vee y \\ &= y, \end{aligned} \tag{I.278}$$

and similarly $x \vee y = y$ implies $x \wedge y$. Since

$x \wedge x = x$ , we have $x \leq x$

We have proved that $\leq$ is a partially order relation. We now show that $x \wedge y$ is the greatest lower bound of $x$ and $y$. Since

$(x \wedge y) = x$ and $(x \wedge y) \vee y = (x \vee y) \wedge y = y$

Now we show that if $z \leq x$ and $z \leq y$ then $z \leq x \wedge y$ as it should to be.

$z \wedge x = z$ and $z \wedge y = z$, then

$$\begin{aligned} z \wedge (x \wedge y) &= (z \wedge x) \wedge y \\ &= z \wedge y \\ &= y, \end{aligned} \tag{I.279}$$

so $z \leq x \wedge y$

**proof ()**

First note that by (I.184) $w \wedge (x \vee z) = (w \wedge x) \vee (w \wedge z)$. We write $w = (x \vee z)$

$$\begin{aligned} (x \vee y) \wedge (x \vee z) &= [(x \vee y) \wedge x] \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \vee y) \wedge z] \\ &= x \vee [(x \wedge z) \vee (y \wedge z)] \qquad \text{by (I.184) again} \\ &= [x \vee (x \wedge z)] \vee (y \wedge z) \\ &= x \vee (y \wedge z), \end{aligned} \tag{I.280}$$

Proofs

an analogous definition can be made in $\mathbf{K} - \mathbf{v}.\mathbf{s}.^{op}$, but then defines in $\mathbf{K} - \mathbf{v}.\mathbf{s}.$ a dual to the definition of an algebra.

Proofs

$$\bigvee A := a_1 \vee a_2 \vee a_3 \vee \cdots \vee a_N \tag{I.281}$$

say $a_2 \leq a_1$, $a_1 \leq a_3$

$$\begin{aligned} \bigvee A &:= (a_1 \vee a_2) \vee a_3 \vee a_4 \\ &= (a_1 \vee a_3) \vee a_4 \\ &= a_3 \vee a_4 \end{aligned} \tag{I.282}$$

complete homomorphism if for every $A \subseteq B$

$f(\bigvee A) = \bigvee\{f(a)|a \in A\}$ and $f(\bigwedge A) = \bigwedge\{f(a)|a \in A\}$

Exercise. Show $1 = \bigwedge \emptyset$ and $0 = \bigvee \emptyset$. Conclude that a complete homomorphism preserves 1,0 and complements.

$\bigwedge \emptyset = \emptyset = \emptyset \wedge 1 = 1$

$\bigvee \emptyset = 0 \bigvee \emptyset = 0$

Exercise. Show that $\bigwedge A = \neg \bigvee \{\neg a | a \in A\}$ and conclude that if a function preserves all $\bigvee$'s, 1 and complements, it is a complete homomorphism.

Wigner's theorem

Wigner's unitary-antiunitary theorem. Let $H$ be a complex Hilbert space and $\alpha : H \to H$ be a surjective (onto) map with the property that

$$| < \alpha x, \alpha y > | = | < x, y > | \quad (x, y \in H).$$

Then $\alpha$ is of the form

$$\alpha x = \varphi(x) U x \quad (x, y \in H)$$

where $U : H \to H$ is either unitary or antiunitary operator (that is, $U$ is either an inner product preserving bijection (onto and one-to-one) or a bijection conjugate linear map with the property that $< Ux, Uy >=< x, y >$ for all $x, y \in H$) and $\varphi : H \to \mathbb{C}$ is a so-called phase-function which means that its values are modulus one.

Details: Matched pairs and Bicrossproducts.

S. Majid, *Algebraic Approach to Quantum Gravity.*

hepth0604130 S. Majid, *Algebraic Approach to Quantum Gravity II: Noncommutative Spacetime*, [hep-th/0604130].

hepth0604132 S. Majid, *Algebraic Approach to Quantum Gravity III: Noncommutative Riemannian Geometry*, [hep-th/0604132].

Double cross product $X \simeq G \bowtie M$

$$h \triangleleft e = h, \qquad (h \triangleleft g_1) \triangleleft g_2 = h \triangleleft (g_1 g_2) \tag{I.283}$$
$$e \triangleright g = g, \qquad h_1 \triangleright (h_2 \triangleright g) = (h_1 h_2) \triangleright g \tag{I.284}$$
$$e \triangleleft g = e, \qquad (h_1 h_2) \triangleleft g = (h_1 \triangleleft (h_2 \triangleright g))(h_2 \triangleleft g) \tag{I.285}$$
$$h \triangleright e = e, \qquad h \triangleright (g_1 g_2) = (h \triangleright g_1)((h \triangleleft g_1) \triangleright g_2) \tag{I.286}$$

where the first two equations say that we have an action, .

Having such a pair, one can introduce the following group structure on $G \times H$:

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\}, \\ \{g, h\}^{-1} &= \{h^{-1} \triangleright g^{-1}, h^{-1} \triangleleft g^{-1}\} \end{aligned} \tag{I.287}$$

**Subgroups**

For $g_1 = g_2 = e$

$$\begin{aligned} \{e, h_1\} \circ \{e, h_2\} &= \{e(h_1 \triangleright e), (h_1 \triangleleft e)h_2\} \\ &= \{e, h_1 h_2\} \end{aligned} \tag{I.288}$$

For $h_1 = h_2 = e$

$$\begin{aligned} \{g_1, e\} \circ \{g_2, e\} &= \{g_1(e \triangleright g_2), (e \triangleleft g_2)e\} \\ &= \{g_1 g_2, e\} \end{aligned} \tag{I.289}$$

**special cases**

Semigroups are a special case of a matched pair: Say $h \triangleleft g = h$ for all $h \in H$ and $g \in G$

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\} \\ &= \{g_1(h_1 \triangleright g_2), h_1 h_2\} \end{aligned} \tag{I.290}$$

if $h_1 \triangleright g_2 = h_1 g_2 h_1^{-1}$ then it is a semigroup. Now, say $h \triangleright g = g$ for all $h \in H$ and $g \in G$

$$\begin{aligned} \{g_1, h_1\} \circ \{g_2, h_2\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)h_2\} \\ &= \{g_1 g_2, (h_1 \triangleleft g_2)h_2\} \end{aligned} \tag{I.291}$$

if $h_1 \triangleleft g_2 = g_2 h_1 g_2^{-1}$ then it is a semigroup.

**Check** Verify inverse:

$$
\begin{aligned}
\{g,h\}\circ\{g,h\}^{-1} &= \{g,h\}\circ\{h^{-1}\triangleright g^{-1}, h^{-1}\triangleleft g^{-1}\} \\
&= \{g(h\triangleright(h^{-1}\triangleright g^{-1})), (h\triangleleft(h^{-1}\triangleright g^{-1})(h^{-1}\triangleleft g^{-1})\} \\
&= \{g((hh^{-1})\triangleright g^{-1}), (hh^{-1})\triangleleft g^{-1}\} \\
&= \{g(e\triangleright g^{-1}), e\triangleleft g^{-1}\} \\
&= \{gg^{-1}, e\} \\
&= \{e,e\}
\end{aligned} \tag{I.292}
$$

$$
\begin{aligned}
\{g,h\}^{-1}\circ\{g,h\} &= \{h^{-1}\triangleright g^{-1}, h^{-1}\triangleleft g^{-1}\}\circ\{g,h\} \\
&= \{(h^{-1}\triangleright g^{-1})((h^{-1}\triangleleft g^{-1})\triangleright g), ((h^{-1}\triangleleft g^{-1})\triangleleft g)h\} \\
&= \{h^{-1}\triangleright(g^{-1}g), h^{-1}\triangleleft(g^{-1}g)h\} \\
&= \{h^{-1}\triangleright e, (h^{-1}\triangleleft e)h\} \\
&= \{e, hh^{-1}\} \\
&= \{e,e\}
\end{aligned} \tag{I.293}
$$

Verify associativity:

$$
\begin{aligned}
(\{g_1,h_1\}\circ\{g_2,h_2\})\circ\{g_3,h_3\} &= \{g_1(h_1\triangleright g_2), (h_1\triangleleft g_2)h_2\}\circ\{g_3,h_3\} \\
&= \{(g_1(h_1\triangleright g_2))\ (\ ((h_1\triangleleft g_2)h_2)\triangleright g_3), (((h_1\triangleleft g_2)h_2)\triangleleft g_3)h_3\}
\end{aligned} \tag{I.294}
$$

$$
\begin{aligned}
\{g_1,h_1\}\circ(\{g_2,h_2\}\circ\{g_3,h_3\}) &= \{g_1,h_1\}\circ\{g_2(h_2\triangleright g_3), (h_2\triangleleft g_3)h_3\} \\
&= \{g_1(h_1\triangleright(g_2(h_2\triangleright g_3))\ ), (h_1\triangleleft(g_2(h_2\triangleright g_3))\ )((h_2\triangleleft g_3)h_3)\}
\end{aligned} \tag{I.295}
$$

Consider the “$h$” term in the last line of (I.294)

$$
\begin{aligned}
((h_1\triangleleft g_2)h_2)\triangleleft g_3 &= ((h_1\triangleleft g_2)\triangleleft(h_2\triangleright g_3))(h_2\triangleleft g_3) \\
&= (h_1\triangleleft(g_2(h_2\triangleright g_3)))(h_2\triangleleft g_3)
\end{aligned} \tag{I.296}
$$

where we used (I.285) in the first line and (I.283) in the second. Consider the “$g$” term in the last line of (I.294)

$$
\begin{aligned}
h_1\triangleright(g_2(h_2\triangleright g_3)) &= (h_1\triangleright g_2)((h_1\triangleleft g_2)\triangleright(h_2\triangleright g_3)) \\
&= (h_1\triangleright g_2)(\ ((h_1\triangleleft g_2)h_2)\triangleright g_3))
\end{aligned} \tag{I.297}
$$

where we used (I.286) in the first line and (I.284) in the second. Substituting both these results back into the equations they were taken from proves associativity.

□

**Conversely** For a group $X$, which factorizes into two subgroups $G$ and $H$, we recover a matched pair $G$ and $H$, with left and right actions derived from (I.287).

$$\begin{aligned}\{g,h\} &= \{g,h\}\circ\{e,e\} \\ &= \{g(h\triangleright e),(h\triangleleft e)e\},\end{aligned} \tag{I.298}$$

implying $h\triangleright e=e$ and $h\triangleleft e=h$. Similarly,

$$\begin{aligned}\{g,h\} &= \{e,e\}\circ\{g,h\} \\ &= \{e(e\triangleright g),(e\triangleleft g)h\},\end{aligned} \tag{I.299}$$

implies $e\triangleright g=g$ and $e\triangleleft g=e$.

from associativity of the group we can derive the action and matched pair conditions. Put $g_1=g_2=e$

$$\begin{aligned}(\{e,h_1\}\circ\{e,h_2\})\circ\{g_3,h_3\} &= \{e(h_1\triangleright e),(h_1\triangleleft e)h_2\}\circ\{g_3,h_3\} \\ &= \{(e(h_1\triangleright e))\ (\ ((h_1\triangleleft e)h_2)\triangleright g_3),(((h_1\triangleleft e)h_2)\triangleleft g_3)h_3\} \\ &= \{((h_1h_2)\triangleright g_3),((h_1h_2)\triangleleft g_3)h_3\}\end{aligned} \tag{I.300}$$

$$\begin{aligned}\{e,h_1\}\circ(\{e,h_2\}\circ\{g_3,h_3\}) &= \{e,h_1\}\circ\{e(h_2\triangleright g_3),(h_2\triangleleft g_3)h_3\} \\ &= \{e(h_1\triangleright(e(h_2\triangleright g_3))\ ),(h_1\triangleleft(e(h_2\triangleright g_3))\ )((h_2\triangleleft g_3)h_3)\} \\ &= \{h_1\triangleright(h_2\triangleright g_3))\ ),(h_1\triangleleft(h_2\triangleright g_3)\ )((h_2\triangleleft g_3)h_3)\}\end{aligned} \tag{I.301}$$

implying

$$(h_1h_2)\triangleright g_3=h_1\triangleright(h_2\triangleright g_3)$$

and

$$(h_1h_2)\triangleleft g_3=(h_1\triangleleft(h_2\triangleright g_3))(h_2\triangleleft g_3)$$

$h_2 = h_3 = e$

$$
\begin{aligned}
(\{g_1, h_1\} \circ \{g_2, e\}) \circ \{g_3, e\} &= \{g_1(h_1 \triangleright g_2), (h_1 \triangleleft g_2)e\} \circ \{g_3, e\} \\
&= \{(g_1(h_1 \triangleright g_2))\ (\ ((h_1 \triangleleft g_2)e) \triangleright g_3), (((h_1 \triangleleft g_2)e) \triangleleft g_3)e\} \\
&= \{g_1(h_1 \triangleright g_2)((h_1 \triangleleft g_2) \triangleright g_3), (h_1 \triangleleft g_2) \triangleleft g_3\}
\end{aligned}
\tag{I.302}
$$

$$
\begin{aligned}
\{g_1, h_1\} \circ (\{g_2, e\} \circ \{g_3, e\}) &= \{g_1, h_1\} \circ \{g_2(e \triangleright g_3), (e \triangleleft g_3)e\} \\
&= \{g_1(h_1 \triangleright (g_2(e \triangleright g_3))\ ), (h_1 \triangleleft (g_2(e \triangleright g_3))\ )((e \triangleleft g_3)e)\} \\
&= \{g_1(h_1 \triangleright (g_2 g_3)\ ), h_1 \triangleleft (g_2 g_3)\}
\end{aligned}
\tag{I.303}
$$

implying

$$h_1 \triangleright (g_2 g_3) = (h_1 \triangleright g_2)((h_1 \triangleleft g_2) \triangleright g_3)$$

$$h_1 \triangleleft (g_2 g_3) = (h_1 \triangleleft g_2) \triangleleft g_3$$

$\square$

Consider $hg \in X$ where $h \in H$, $g \in G$. By unique factorization assumption, this element is equal to $(h \triangleright g)(h \triangleleft g)$ for some elements $h \triangleright g \in G$ and $h \triangleleft g \in H$. This defines the maps $\triangleright : H \times G \to G$ and $\triangleleft : H \times G \to H$.

$\square$

**Bicrosproduct**

The backreaction of $H$ on $G$ can be used to make a semidirect product structure and render a Hopf algebra

$\mathbb{C}[H] \blacktriangleright\!\!\triangleleft U(\mathfrak{g})$