Appendix J

Category Theory and Topos

J.1 A Very Gentle Introduction to Category Theory

John Baez website http://math.ucr.edu/home/baez/README.html

In especially simple and clear way.

sets
functions
diagrams of sets and functions
categories
functors
natural transformations
diagrams of categories, functors and natural transformations 2-categories
2-functors
2-natural transformations
modifications

understand an object by placing it in a category and studying its relation with other objects of the same category by their morphisms, or related categories through functors.

For brevity we will simplify the situation considerably, leaving out significant details.

pass from a an element-based approach to an arrow-based one approach to an arrow based one in the category of sets.

Examples of categories

Category	obects	morphism
Set	sets	function
Top	topology	homeomorphism
Grp	group	homomorphism
$\mathcal{P}$	elements	$p \leq q$
Dgrph	vertices	paths

Example Every directed graph can be regarded as a category if we take the vertices to be the objects and the arrows to be the paths in the graph.

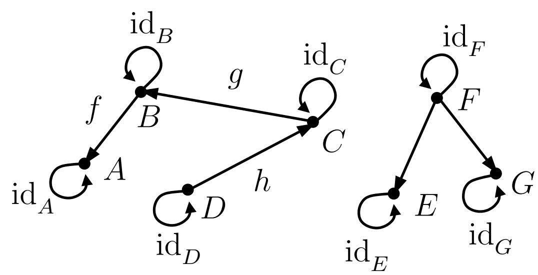


Figure J.1: Every directed graph can be regarded as a category if we take the vertices to be the objects and the arrows to be the paths in the graph.

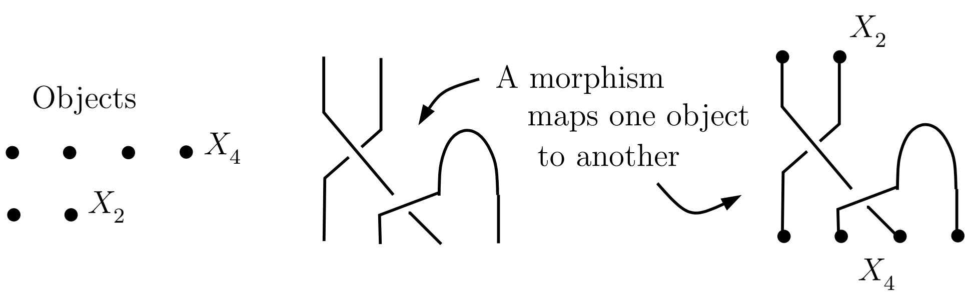


Figure J.2: is in Hom$(4, 2)$

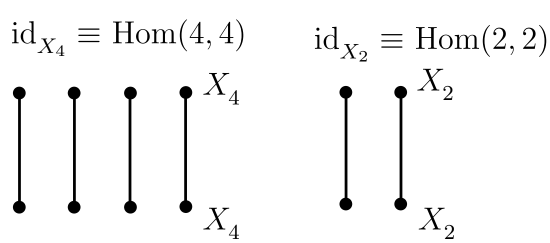


Figure J.3: Different identities X_2 id$_{X_2}$.

Hom(4, 2) =

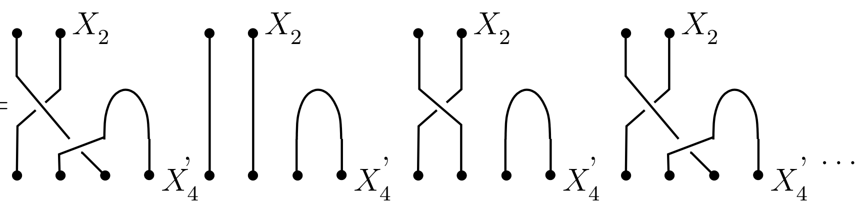


Figure J.4: Different identities X_2 id_{X_2}. is in Hom(4, 2)

Things to note about these:

1. There are objects and morphisms between them.

2. Composition may not be defined for all pairs of objects.

3. If there is an arrow from X to Y, there may not be an arrow from Y to X.

4. When composition is defined between object X and object Y, and composition between object Y and object Z, composition between X and Y is defined.

5. Objects have different identities. We will denote the identity of the object X as id_X.

Definition A category $\mathcal{C}$ consists of "objects", $\mathfrak{ob}C$, and "morhisms" (or "arrows"), f, between them, such that

(i) If $f : C_1 \to C_2$ and $g : C_2 \to C_3$ are morhisms , then there exists a morphism $g \circ f : C_1 \to C_3$.

The morphisms are often denoted $\mathrm{Hom}(C_1, C_2)$ for f. The composition of morphisms is then expressed as

$$\mathrm{Hom}(C_1, C_2) \times \mathrm{Hom}(C_2, C_3) \to \mathrm{Hom}(C_1, C_3) \tag{J.1}$$

(ii) It is assumed that the identity map for C, $\mathrm{id} : C \to C$, is a morphism for every object C of $\mathcal{C}$.

The axioms on the morphisms and composition are very weak, with no conditions imposed on the objects other than they belong to a certain class. This so means that many diverse mathematical structures can be seen as a category.

Example A **monoid** is a set X with an operation called multiplication xy for $x, y \in X$, which is associative and has unit $e \in X$, satisfying $ex = xe = x$ for all. A monoid can be regarded as a category with one object (i.e. X), and an arrow for every $x \in X$.

As a group G is a special case of a monoid, any group can be thought of as a category with the single object G and the elements of the group $g \in G$ as the arrows.

Example **Set** is the category which has the class of all sets as objects, and functions between sets as arrows.

Example **Grp** is the category which has the class of all groups as objects, and functions between between groups as arrows.

Example A partially ordered set is . This can be regarded as a category in which the objects are the elements of $\mathcal{P}$; and if $p, q \in \mathcal{P}$, an arrow from p to q is defined to exist if $p \leq q$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

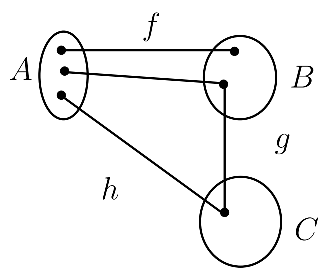


Figure J.5: x

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

commutes if and only if $g(f(x)) = h(x)$ for all $x \in X$. We write

$$g \circ f = h. \tag{J.2}$$

Definition The 'opposite' of a category $\mathcal{C}$, denoted $\mathcal{C}^{op}$, is obtained by reversing the arrows of $\mathcal{C}$, while keeping the same objects. The arrow $f : A \longrightarrow B$ in $\mathcal{C}$ becomes the arrow $f^{op} : B \longrightarrow A$ in $\mathcal{C}^{op}$. Composition is ordered the other way around. Insted of

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{J.3}$$

we have

$$C \xrightarrow{g^{op}} B \xrightarrow{f^{op}} A \tag{J.4}$$

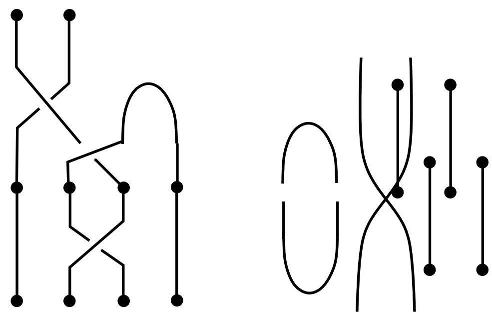


Figure J.6: tangleCat4 is in Hom(4, 2)

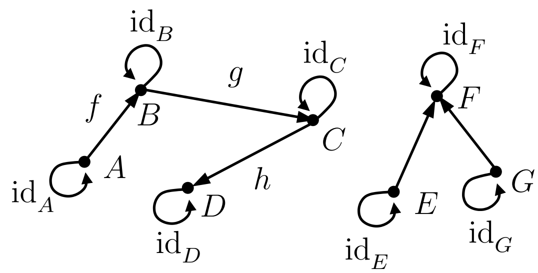


Figure J.7: The 'opposite' category of the graph in Fig(J.1).

equivalences a group with a different way of renaming the elements does not really give you back a different group.

Example A category where the objects are discrete events and the arrows are causal relations between events. $p \to q$ means p preceeds q or $p \le q$. Arrows are closed under composition because $p \to q$ and $q \to r$ implies $p \to r$. Arrows are associative $(p \to q) \to r = p \to (q \to r)$.

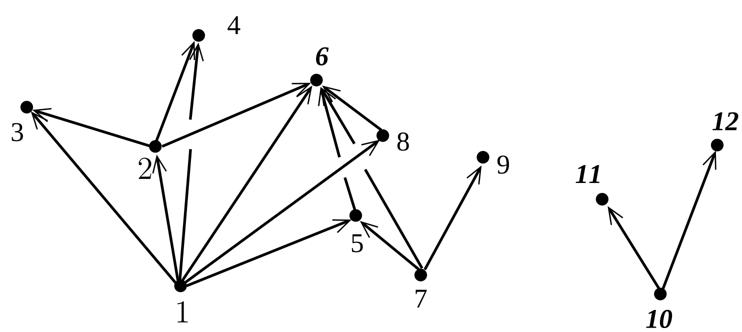


Figure J.8: causalsetex1. Taken from [366].

Definition Functors F are the structure preserving assignments between categories. In other words, a functor is a *homomorphism* between categories.

1. to each object A in $\mathcal{C}$, there is an object in $\mathcal{D}$ $F(A)$;
2. to each object arrow $f : B \to A$ in $\mathcal{C}$
3. $F(1_A) = id_{F(A)}$

$$F(f \circ g) = F(f) \circ F(g). \tag{J.5}$$

We have just encountered in the last section an important example of a functor, that of the fundamental group.

Example

$$Past : \longrightarrow \mathbf{Set} \tag{J.6}$$

Past as a functor it outputs the set $Past(p)$ for the object p for each p, (see fig.(J.1)). For each causal relation $p \to q$ and outputs the function that takes you from the set $Past(p)$ to the set $Past(q)$.

that assigns to each $p \in \mathcal{C}$ its past

$$Past(p) = \{r \in \mathcal{C} : r < p\}. \tag{J.7}$$

and to each causal relation, when it appears in the causal set, a function

$$Past_{pq} : Past(p) \longrightarrow Past(q), \tag{J.8}$$

which includes the identity map $Past_{pp} : Past(p) \longrightarrow Past(p)$.

Theorem J.1.1 *If $f : X \to Y$ is a homeomorphism of X onto Y, the induced homomorphism $f_* : \pi(X, p) \to \pi(Y, fp)$ is an isomorphism onto for any basepoint p in X.*

If pathwise connected topological spaces X and Y are homeomorphic, their fundamental groups are isomorphic. This can be expressed as saying their is a functor from the category of topological spaces to the category of groups. The functor F map a topological space X to the fundamental group $\pi(X, p)$, i.e., $F(X) = \pi(X, p)$ and maps homeomorphisms into isomorphisms, i.e., $F(f) = f_*$.

One can even define maps between functors, and so on and on. We then come into the wonderful world of *n-categories*, which is, John Baez mathematics underlying GR and quantum mechanics. Finds application in many things including also homotopy theory...

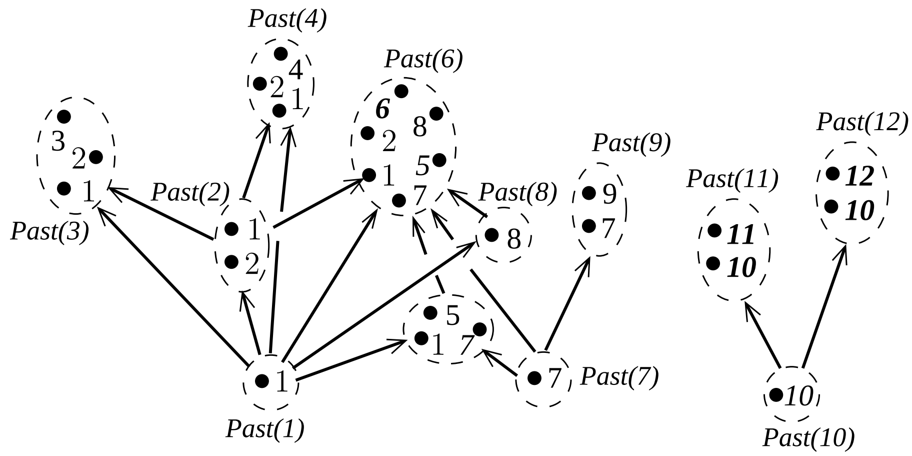


Figure J.9: causalsetex2. Taken from [366].

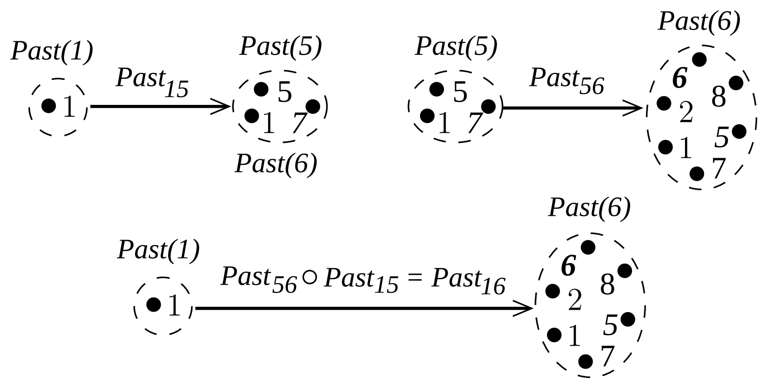


Figure J.10: causalsetex2A. Composition. Associativity.

Definition Natural Transformations Suppose we have two functors F and G from category $\mathcal{C}$ to the category $\mathcal{D}$. Then a *natural transformation* N from F to G assigns to each object X in $\mathcal{C}$ a morphism $N(X) :\rightarrow G(X)$ such that this diagram commutes:

Example "abelianzation", which maps a group to the abelian group. $ab = ba + [a, b]$ so that $ab \sim ba$ so that under induced group multiplication $[a][b] = [ab] = [ba] = [b][a]$. The category $\mathcal{C}$ has only one object, which is H, and the morphisms f are the isomorphisms of the group H. The functors F and G map to the category $\mathcal{D}$, which are themselves groups. The *natural transformation, N, is the "abelianzation"* of $F(H)$ and $G(H)$.

We say a category is small if the collection of objects is a set. The category of all small categories is denoted **Cat**.

A notion of isomorphism only makes sense if the object has some internal structure, for example an isomorphism between groups G and H is when every element of the group G

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, bend left=50, "F(\mathcal{C})"] \arrow[r, bend right=50, "G(\mathcal{C})"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D}
\end{tikzcd}
$$

Figure J.11: NaturalTr0. An arrow η that maps the functor F to the functor G.

$$
\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "n(X)"'] & F(Y) \arrow[d, "n(Y)"] \\
G(X) \arrow[r, "G(f)"] & G(Y)
\end{tikzcd}
$$

Figure J.12: Natural transformation.

is mapped to one element of H. When the objects of a category have no internal structure there is no such thing as an isomorphism. In particular, there no notion of isomorphism between morhpisms!

Example The set of functors $\mathrm{Hom}[\mathcal{P}, G]$. G is a finite connected and compact Lie group.

$\mathrm{Hom}[\mathcal{P}, G]$ is the set of all functors from the groupoid $\mathcal{P}$ to the group G.

Universal Arrows

$u : T \to T'$ satisfying $u \circ h = h'$. Similarly, by the universal property of h' there is a map $u' : T' \to T$ satisfying $u' \circ h' = h$.

We have $u' \circ u : T \to T$ with $(u' \circ u) \circ h = \mathrm{id}_T \circ h$. By uniqueness if arrow, we have that $u' \circ u = \mathrm{id}_T$. Similarly, $u \circ u' = \mathrm{id}_{T'}$. Hence, $T \cong T'$.

Definition Isomorphism (iso) Two objects A and B in a category are said to be *isomorphic* if there exists arrows $f : A \to B$ and $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$.

terminal object is when for every object there is one map to

if there is more than one object and a iso between them then both are terminal isomorphic - so the same as far as the categorical perspective.

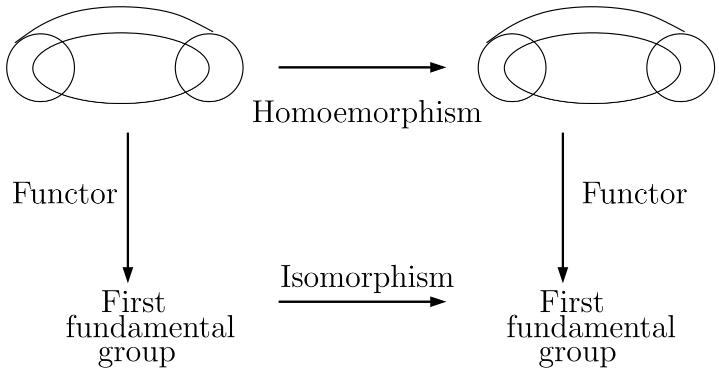
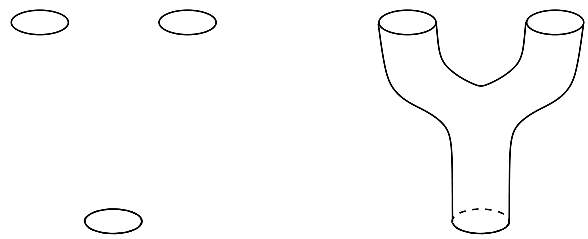


Figure J.13: Natural transformation.



(a) (b)

Figure J.14: Circles are the objects, cobordism is morphism from botoom circle to top circles. 1-d manifolds (circles) represent "space" and cobordism "spacetime".

T and Q with f between T and Q $f : T \to Q$. $u : A \to T$ and $f^{-1} : Q \to T$. Then $f \circ u : A \to Q$ for all A. $f \circ u = v$, $v : A \to Q$ for all A and so Q is also a terminal object.

There is only one morphism from Q to Q and that is the identity id_Q. So T is iso to Q.

But pick one of them and stick with it.

Products

Given two objects A and B, their **product** (if it exists) consists of an object

Definition Product diagram consists of an object P together with to arrows $p_1 : P \to A$ and $p_2 : P \to B$ which has the following properties. Given any object T with arrows $h : T \to A$ and $k : T \to B$, there exists a unique arrow $u : T \to P$ which makes the diagram fig J.1 (b) commute.

Theorem J.1.2 *If P, p_1, p_2 is a product diagram for A and B, and there is an iso $f : Q \xrightarrow{\sim} P$, then Q, q_1, q_2 is alos a product diagram for A and B.*

$$
\begin{tikzcd}
X \times Y \arrow[r] \arrow[dr, "h"'] & T \arrow[d, "u"] \\
& T'
\end{tikzcd}
$$

Figure J.15: universal1.

$$
\begin{tikzcd}
& T \arrow[d, "u"] \\
X \times Y \arrow[ur, "h"] \arrow[r, "h'"] \arrow[dr, "h"'] & T' \arrow[d, "u'"] \\
& T
\end{tikzcd}
$$

Figure J.16: universal2.

from fig. J.1(c) composes $f \circ v : T \xrightarrow{\sim} P$ so that by uniqueness $f \circ v = u$. As u is unique and f was given, $v = f^{-1} \circ u$ is the unique arrow from T to Q composing with $p_1 \circ f$ to give h, and with $p_2 \circ f$ to give k.

J.1.1 Monoidal Categories

Topological qunatum field theory TQFT.

A monoidal category has tensor products of objects and morphisms satisfying the usual axioms, and an object **1** playing the role of identity for the tensor product.

J.2 Topos

If we wish to make the context dependence the central feature of the mathematical formulation the maths that does this is topos!

speculated application to social, political, economic worlds [13].

pass from a an element-based approach to an arrow-based one approach to an arrow based one in the category of sets.

Non-Boolean logic does not assume every statement can be judged true or false, there are some statements upon which one cannot decide. An example of a physical situation where the underlying logic would be non-Boolean is in cosmology where observers can only make judgements upon statements that have to do with their backwards light cone.

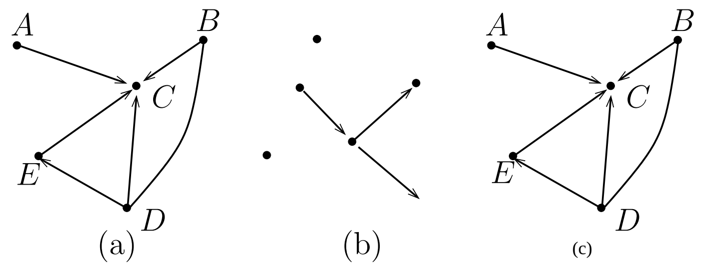


Figure J.17: In especially simple and clear way. (a) Terminal object **Dgrph**. (b) Initial.

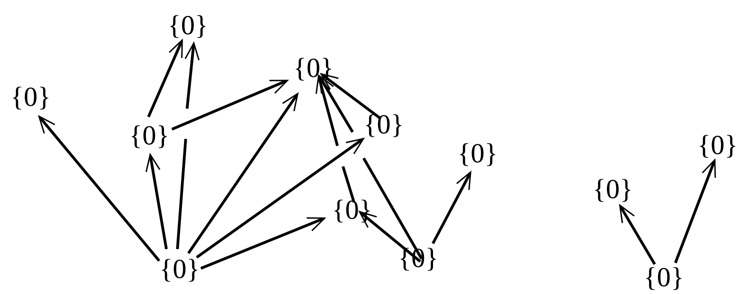


Figure J.18: causalsetex3. Taken from [366].

technique from categorial maths (presheaves) to which we give the physical interpretation of context-dependence.

$$\begin{array}{rcl} \text{fixed assumptions} & \rightarrow & \text{physical statements} \\ \text{to} & & \\ \text{Context category} & \rightarrow & \text{category of physical statements} \end{array}$$

we need physical statements/quantum theory that are observer-dependent. In relational interpretations of quantum theory, context category = all possible systems/enviroments.

Functors between two categories preserve the structure of the first category. In other words, a functor is a homomorphism between categories.

J.2.1 Basics of Topos

To a given proposition $\mathcal{P}$, we assign a subset A with the interpretation that an element belongs to A if and only if it satisfies the proposition $\mathcal{P}$. In classical logic two-valued logic admit two distinct truth values, namely truth and untruth.

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{J.9}$$

$$
\begin{tikzcd}
A & P \arrow[l, "p_1"'] \arrow[r, "p_2"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "u"] \arrow[dr, "k"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

(a) (b)

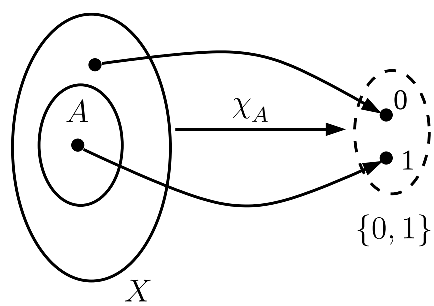
Figure J.19: A product diagram (a) is defined, if u for all T, there an exists a unique arrow u that makes diagram (b) commute.

$$
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "u"] \arrow[dr, "k"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
\text{ and } Q \underset{f^{-1}}{\overset{f}{\rightleftarrows}} P \Longrightarrow
\begin{tikzcd}
 & T \arrow[dl, "h"'] \arrow[d, "f^{-1}\circ u"] \arrow[dr, "k"] & \\
A & Q \arrow[l, "p_1\circ f"] \arrow[r, "p_2\circ f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
 & T \arrow[ddl, "h"'] \arrow[d, "v"] \arrow[ddr, "k"] & \\
 & Q \arrow[d, "f"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

(a) (b) (c)

Figure J.20: Proof of Theorem J.1.2. from (c) we see that $f \circ v = u$ is the unique arrow that makes the diagram commute.

$$A = \chi_A^{-1}(1) \tag{J.10}$$



Note that in nthis setting we will refer to A as the subset of X meaning that there is an inclusion function from A to X.

From a purely technical point of view, it does not matter what truth values and are, so long as they are distinct.

topos: To find out more about the subject to John Baez homepage (http://math.ucr.edu/home/baez/README.html) "fun stuff" and then to the link topos.

1. an initial object (an object like the empty set)
2. a terminal object (an object like a set with one element)

3. binary coproducts (something like the disjoint union of two sets)
4. binary products (something like the Cartesian product of two sets)
5. equalizers (something like the subset of X consisting of all elements x such that $f(x) = g(x)$, where $f, g : X \to Y$)
6. coequalizers (something like the quotient set of X where two elements $f(y)$ and $g(y)$ are identified, where $f, g : Y \to X$)

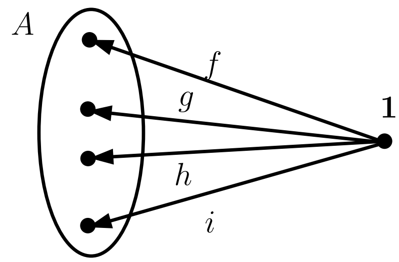
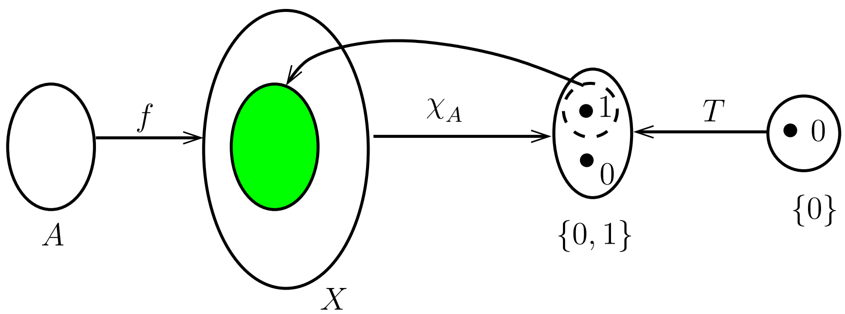


Figure J.21: one-to-one correspondence of elements of A to functions from the terminal set to A.

(3) analogy of $\{0, 1\}$ denoted Ω, intuitively, the elements of Ω are the answers to a natural 'multiple-choice questions' about objects in the objects in the topos, just as "$x \in X$" is a natural for sets.



Category theory notes for ESSLLI

binary products

binary coproducts (something like the disjoint union of two sets) or sums that allow the specification of alternatives

The category of presheaves on $\mathcal{C}$, $\mathbf{Set}^{\mathcal{C}}$, forms a topos.

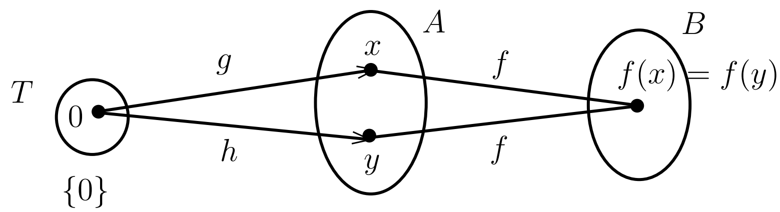
coming from [361]

We will include more details.

Definition An isomorphism (or iso) is a pair of arrows

$$\begin{tikzcd} A \arrow[r, shift right, "g"'] & B \arrow[l, shift right, "f"'] \end{tikzcd}$$

such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. If $f : A \to B$ is an arrow of an ico pair,we say that f is an isomorphism (or f is an iso), and write $f : A \cong B$.



$$g(0) = x \qquad h(0) = y \tag{J.11}$$

f is one-to-one or injective.

Definition The arrow f in the category $\mathcal{C}$ is mono (monic) if and only if for any pair of arrows g, h

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & B \arrow[r, "f"] & C \end{tikzcd}$$

we have that $f \circ g = g \circ f$ implies $g = h$.

Definition The arrow f in the category $\mathcal{C}$ is ***epi*** if and only if for any pair of arrows g, h Z

we have that $g \circ f = f \circ h$ implies $g = h$. We say that it is left cancellable??.

Definition Generalized element: We can think of an arrow to any object B, say $x : A \to B$, as a kind of element of B. x ***the generalized element*** of B

Summary:

$$\begin{array}{rl} x \in_T A,\ y \in_T A & \qquad f(x) = f(y) \quad \Rightarrow x = y \\ x : T \to A,\ y : T \to A & \qquad f \circ x = f \circ y \Rightarrow x = y \end{array} \tag{J.12}$$

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & C \end{tikzcd}$$

Definition (Locally Small). A *category* $\mathcal{C}$ is called locally small if for all objects $X, Y \in \mathcal{C}$, the collection $\mathcal{C}(X, Y)$ is a set.

Definition (Small). A *category* $\mathcal{C}$ is called ***small*** if in addition as locally small, the collection of all objects is a *set*.

??In a mere set elements are either the same or different; there is nothing else to say. The category **Sets** of all sets is not small because there is no set of all sets.??

If the function f's domain is restricted to the shaded region then $f \circ g = f \circ h$ then the fact that would not imply $g = h$, hence f must at least be onto (surjective). In set theory the monic is an inclusive and one-to-one function

Pullbacks

Given any two arrows f and g with the same codomain

$$\begin{tikzcd} A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

Figure J.22: pullback1.

their pullback consists of a pair of arrows p_1, p_2 and an object T such that $f \circ p_1 = p_2 \circ p_2$

$$\begin{tikzcd} D \arrow[r, "p_2"] \arrow[d, "p_1"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

Figure J.23: pullback2.

such that for every pair of arrows $h : T \to A$ and $k : T \to B$ that makes the outer square commutes, i.e. $f \circ h = g \circ k$, there exists a unique arrow $uT :\to D$ $p_1 \circ u = h$ and $p_2 \circ u = k$

$$
\begin{tikzcd}
T \arrow[drr, "k"] \arrow[dr, "u"] \arrow[ddr, "h"'] & & \\
& D \arrow[r, "p_2"'] \arrow[d, "p_1"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}
$$

Figure J.24: pullback3.

$$
\begin{tikzcd}
D \arrow[r, "k"] \arrow[d, "h"'] & E \arrow[r, "m"] \arrow[d, "i"'] & F \arrow[d, "j"] \\
A \arrow[r, "g"'] & B \arrow[r, "f"'] & C
\end{tikzcd}
$$

Figure J.25: pullback10.

Equalizers

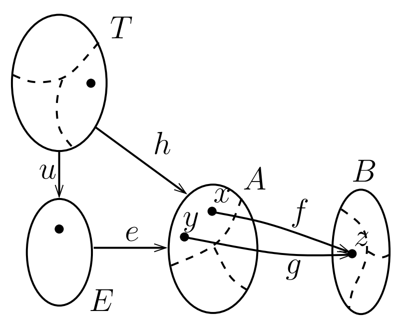


Figure J.26: The equalizer in the category **Set**.

In **Set**, the equalizer of f and g is the set of points x such that $f(x) = g(x)$. In **Grp** it is the kernal of a homomorphism $f : G \to H$ is the equalizer of f and the constant map at the group identity.

Subobject Classifier

a subobject A of X that represents the part of X on which b is true. That b is true is to say that the following diagram commutes

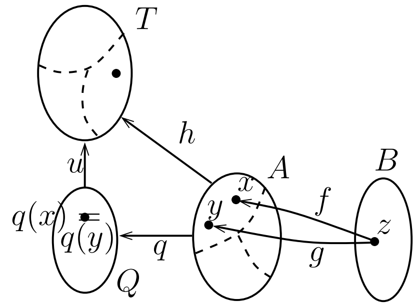


Figure J.27: The coequalizer in the category **Set**.

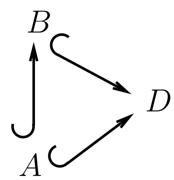


Figure J.28: inclusion. $A \hookrightarrow B$ and $B \hookrightarrow D$ implies $A \hookrightarrow D$. subobject of D

J.2.2 Cartesian Closed Categories

We used the notion of an element using maps from the final object. A cartesian closed category is one in which the arrows can also be seen as being functions. function space, evaluation, composition, etc.

Given sets A and B we have in **Set** the colection of all functions from A to B we denote

$$B^A \tag{J.13}$$

the evaluation map is

$$ev : B^A \times A \to B \tag{J.14}$$

written out in the notation this takes the more familiar form of evaluation:

$$ev(< f, x >) = f(x) \tag{J.15}$$

ev the *evaluation* function.

, a mono b :? →? corresponds to a subset of A and χ_A is the charaeristic morphism ?.

$$
\begin{tikzcd}
A \arrow[r, "\chi_A"] \arrow[d, "!_A"'] & X \arrow[d, "b"] \\
\mathbf{1} \arrow[r, "\mathbf{true}"'] & \{0,1\}
\end{tikzcd}
$$

Figure J.29: subobject of .

$$
\begin{tikzcd}
X \arrow[dr, "x_b"] \arrow[drr, "x"] \arrow[ddr, "!_X"'] & & \\
& A \arrow[r, "\chi_A"] \arrow[d, "!_A"'] & X \arrow[d, "b"] \\
& \mathbf{1} \arrow[r, "\text{true}"'] & \{0,1\}
\end{tikzcd}
$$

Figure J.30: A pullback of b along **true**.

Definition (**Topos**). An elementary topos is a category such that

1. is finitely complete
2. is finitely co-complete,
3. has exponentiation,
4. with a **sub-object classifier**.

Heyting Algebra of $Sub(X)$

Definition (**Topos**). A topos is a **cartesian closed caregory** with a **sub-object classifier**.

The arrow χ_A is called the charateristic morphism of b, this notation comes from the situation in **Set**, where $\Omega = \{0,1\}$.

$m \leq m'$ if there is $f : Y \to Y'$ satisfying $m' \circ f = m$, i.e.,

$$m' \circ f = m \quad \Longleftrightarrow \quad m \leq m'. \tag{J.16}$$

We have a partial order $Sub(X)$ ordered by the smaller-than relation.

$$
\begin{tikzcd}
A \arrow[r, "\chi_A"] \arrow[d, "!_A"'] & X \arrow[d, "b"] \\
\mathbf{1} \arrow[r, "t"'] & \Omega
\end{tikzcd}
$$

Figure J.31: A pullback of b along t.

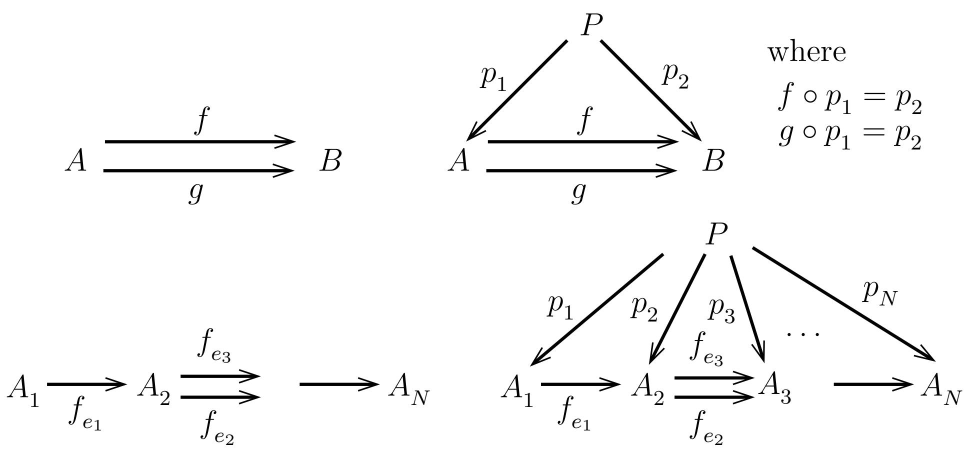


Figure J.32: cones3. (a) $f \circ p_1 = p_2$ and $g \circ p_1 = p_2$. (b) $f_{e_1} \circ p_1 = p_2$, $f_{e_2} \circ p_2 = p_3$, $f_{e_3} \circ p_2 = p_3$, etc

As there always exists f for any subobject m such that

$$\mathrm{id}_X \circ f = m, \tag{J.17}$$

, (i.e. $f = m$), id_X is the greatest element of $Sub(X)$.

$Y \cap Y'$ given by the pullback fig(J.2.2). This seen by noting that for any n, n' such that $n \circ m \leq m''$ and $n' \circ m' \leq m''$, there exists, by defintion of a pullback, a (unique) arrow $u : Y \cap Y' \to X$ such that $m'' \circ u = (m \circ n)$ and $m'' \circ u = (m' \circ n')$, see fig(J.2.2). Hence, $m \circ n \leq m''$ and $m' \circ n' \leq m''$.

A Heyting algebra $\mathbf{H}$ is a latice with greatest and least elements in the which the meet $a \wedge b$ has an implication operator, $\to$, satisfying

$$a \wedge b \leq c \quad a \leq b \to c \tag{J.18}$$

$$c \wedge (a \vee b) = (c \wedge a) \vee (c \wedge b) \tag{J.19}$$

$$\begin{tikzcd}
B^A \times A \arrow[r, "ev"] & B \\
C \times A \arrow[u, "\tilde{g} \times \mathrm{id}_A"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Figure J.33: exponat. in the category **Set**.

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail, "s"'] & 1 \arrow[d, "t"] \\
A \arrow[r] & \Omega
\end{tikzcd}$$

Figure J.34: .

Categorically this corresponds to a ccc with finite coproducts.

Clearly $m \leq m \cup m'$ and $m' \leq m \cup m'$;

$$\begin{aligned}
m \cup m' \circ f &= m \\
m \cup m' \circ f' &= m'
\end{aligned} \tag{J.20}$$

where f is formed by composition of the the top arrow in fig.(J.2.2) with e and f' is formed by composition of the the left arrow with e.

Lemma J.2.1 *In a topos $\mathcal{E}$ each pair of elements of Sub(X) has a least upper bound.*

Suppose there is a subobject $n : Z \rightarrowtail X$ such that $m, m' \leq n$, we wish to show that for any such arrow $m \vee m' \leq n$. By definition of the coproduct there is a unique arrow u from $Y + Y'$ to Z such that the diagram fig.(J.2.2) commutes. This means that $n \circ u$ is a factorization of $[m, m']$.

Since $(m \vee m') \circ e$ is the least such factorization(fig.(J.2.2)), there is an arrow $s : Y \vee Y' \to Z$ such that $n \circ s = m \vee m'$ which shows that $m \vee m' \leq n$.

saying $\wedge$ is the same as saying that it has a right adjoint.

A poset **H** is a Heyting algebra if we have elements

$$
\begin{tikzcd}
P \arrow[r, "f"] \arrow[d, tail, "m"'] & Q \arrow[dl, tail, "n"] \\
X &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[r, "f"] \arrow[d, tail, "m"'] & Q \arrow[dl, tail, "n"] \\
B \arrow[r, tail, "m'"'] & X &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f \circ h"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d, tail, "n"] \\
B \arrow[r, tail, "m'"'] & X
\end{tikzcd}
$$

Figure J.35: Submpinm. smaller than.

$$
\begin{tikzcd}
Z \arrow[dr, "u"] \arrow[drr, "n"] \arrow[ddr, "n'"'] & & \\
& Y \cap Y' \arrow[r, tail] \arrow[d, tail] \arrow[dr, tail, "m''"] & Y \arrow[d, tail, "m"] \\
& Y' \arrow[r, tail, "m'"'] & X
\end{tikzcd}
$$

Figure J.36: Heytlattice. Proof that $Y \cap Y'$ is the greatest lower bound of Y and Y'.

$$T \in \mathbf{H}, \qquad \perp \in \mathbf{H} \tag{J.21}$$

and operations

$$\wedge : \mathbf{H} \times \mathbf{H} \to \mathbf{H}, \qquad \vee : \mathbf{H} \times \mathbf{H} \to \mathbf{H}, \qquad \to : \mathbf{H} \times \mathbf{H} \to \mathbf{H} \tag{J.22}$$

such that $(\mathbf{H}, \wedge, \vee, T, \perp)$ is a lattice and

$$c \wedge a \leq b \quad c \leq a \to b \tag{J.23}$$

for all $a, b, c \in \mathbf{H}$.

$$
\begin{array}{rcll}
\top & = & \mathrm{id}_1 & \\
\perp & = & 0 \rightarrowtail 1 & \text{the uniqe arrow from the intial object.} \\
A \wedge B & = & A \times_{\mathcal{E}} B & \text{the pillback of the A and B.} \\
A \vee B & = & \mathrm{Im}(A + B) & \\
A \to B & = & B^A \rightarrowtail 1 &
\end{array} \tag{J.24}
$$

$$
\begin{tikzcd}
Y + Y' \arrow[dr, "e"] & & Y \arrow[ll, tail] \arrow[dd, tail, "m"] \\
& Y \cup Y' \arrow[dr, tail, "m \cup m'"'] & \\
Y' \arrow[uu, tail] \arrow[rr, tail, "m'"'] & & X
\end{tikzcd}
$$

Figure J.37: Heytlattice2. Proof that $Y \cup Y'$ is the least upper bound of Y and Y'.

$$
\begin{tikzcd}
Y + Y' \arrow[dr, "u"] & & Y \arrow[ll, tail] \arrow[dd, tail, "m"] \\
& Z \arrow[dr, tail, "n"] & \\
Y' \arrow[uu, tail] \arrow[rr, tail, "m'"'] & & X
\end{tikzcd}
$$

Figure J.38: Heytlattice3. Proof that $Y \cup Y'$ is the least upper bound of Y and Y'.

Theorem J.2.2 *For any object X in a topos $\mathcal{E}$, the partially ordered set $Sub_{\mathcal{E}}(X)$ is a Heyting algebra.*

$\mathrm{Sub}_{\mathcal{E}}(X) \cong \mathrm{Sub}_{\mathcal{E}/X}(1)$ together with the fact that $\mathcal{E}/X$ is a topos to conclude that $\mathrm{Sub}_{\mathcal{E}}(X)$ is a Heyting algebra.

Internal Heyting Algebra

subobject lattices $\mathrm{Sub}(X)$ in topos $\mathcal{E}$ induces an internal Heyting algebra on Ω via the isomorphism

$$\mathrm{Sub}_X \cong \mathcal{E}(X, \Omega) \tag{J.25}$$

Presheaves

[361],

The functor category $\mathrm{Set}^{\mathcal{C}^{op}}$ which is also denoted $\hat{\mathcal{C}}$, has presheaves as objects and natural transformations as arrows.

$$
\begin{tikzcd}
Y+Y' & Z \arrow[l, tail] \arrow[d, tail, "m"] \\
Y' \arrow[u, tail] \arrow[ur, "s"] \arrow[r, tail, "m'"'] & X
\end{tikzcd}
$$

Figure J.39: Heytlattice3A. Proof that $Y \cup Y'$ is the least upper bound of Y and Y'.

Yoneda Lemma

A remind of the concepts that will be involved in the statement of the Yoneda lemma:

Let $\mathcal{C}$ be a locally small category (i.e. for every pair of objects A and B the class of arrows $\mathcal{C}(A,B)$ ia a set). For any object A in $\mathcal{C}$ there is a functor $H_A : \mathcal{C}^{op} \to \text{Set}$ which assigns to any object B the set $\mathcal{C}(A,B)$. Any arrow $f : A' \to A$ gives by composition a function $\mathcal{C}(A',A) \to \mathcal{C}(A'',A')$. That is, if $g : A'' \to A'$ then $f \circ g : A'' \to A$ or if $f \in \mathcal{C}(A',A)$ and $g \in \mathcal{C}(A'',A')$ then $f \circ g \in \mathcal{C}(A'',A)$.

$$
\mathrm{Y}_{\mathcal{C}}(B) = \mathrm{Hom}(-,B) : \mathcal{C}^{op} \to \text{Set} \tag{J.26}
$$

For any two categoeires $\mathcal{A}$ and $\mathcal{B}$ there is a category $\mathcal{B}^{\mathcal{A}}$ (also denoted as $[\mathcal{A},\mathcal{B}]$), whose objects are functors from $\mathcal{A}$ to $\mathcal{B}$ and whose morphisms are natural transformations.

X is a functor $\mathcal{C}^{op} \to$**Set**.

Definition Let $\mathcal{C}$ be a locally small category. Then

$$
[\mathcal{C}^{op}, \text{Set}](H_A, X) \equiv \text{Set}^{\mathcal{C}^{op}}(H_A, X) \cong X(A) \tag{J.27}
$$

$$
\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, bend left=50, "H_A"{name=U}] \arrow[r, bend right=50, "X"'{name=D}] & \mathbf{Set}
\arrow[from=U, to=D, Rightarrow]
\end{tikzcd}
$$

Figure J.40: Yoneda1. Left hand sidee of J.27.

Definition A representation for a functor is $F : \mathcal{C}^{op} \to$**Set** consists of an object $C \in \mathcal{C}$ and a natural isomorphism

$$\eta_C = (-, C) \cong F. \tag{J.28}$$

If there exists a representation for F then we say the functor F is representable.

Sieves

$$\Omega(p) = \{\text{sieves on } p\} \tag{J.29}$$

Definition (Subfunctor). A functor $A : \mathcal{C}^{op} \to \mathbf{Set}$ is a subfunctor

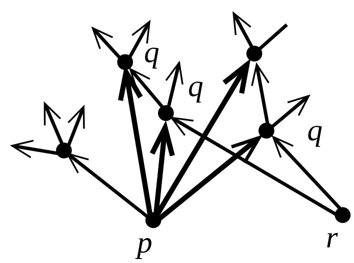


Figure J.41: caustruth1. Taken from [366].

J.2.3 Quantum Causal Histories

The *causal set* $\mathcal{C}$ is a category in which the objects are events $p, q, r \ldots$ and arrows are causal relations between events: $p \leq q$.

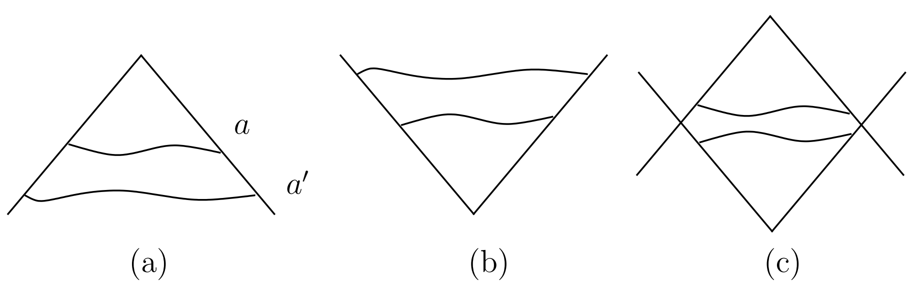


Figure J.42: .

1. The causal past of some event p is the set of all events $r \in \mathcal{C}$ with $r \leq p$. We denote the causal past of p by $P(p)$.

2. The causal future of p is the set of all events $q \in \mathcal{C}$ with $p \leq q$. We denote it by $F(p)$.

3. An *acausal set*, denoted $a, b, c, ...$, is a set of events in $\mathcal{C}$ that are all causally unrelated to each other. Acausal sets plays the role of spacelike hypersurfaces in causal stes.

4. The acausal set a is a complete past of the event p when every event in the causal past $P(p)$ of p is related to some event in a. It is not possible to add an event from $(P(p) - a)$ to a and produce a new acausal set.

5. Similarly, an acausal set b is a complete future of p when every event in the causal future $F(p)$ of p is related to some event in b.

6. A chain is a subset of a poset such that they are connected, this motivates A maximal antichain in the causal set C is an acausal set A such that every event in $(\mathcal{C} - A)$ is causally related to some event in A.

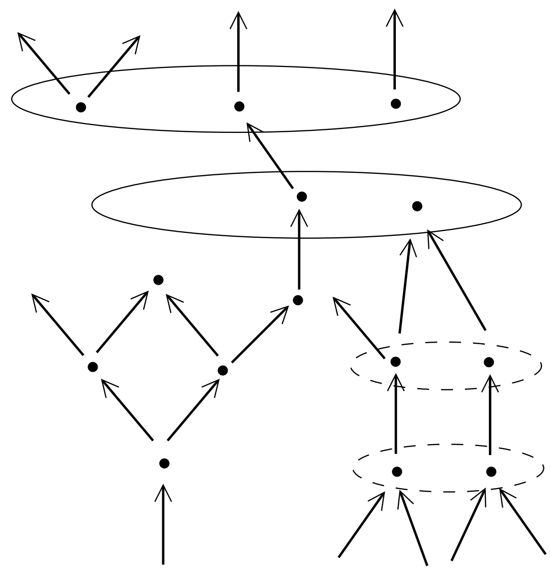


Figure J.43: .

$p = a\mathbf{Set}$:

objects: sets of events $\{r \leq p\}$
arrows: inclusions $\{r \leq p\} \subseteq \{r \leq q\}$ when $p \leq q$.

see definition of a presheaf in the maths glossary.

J.2.4 Consistent Histories Interpretation

Kochen-Specker

J.2.5 Application to Rovelli's Relational Quantum Mechanics

may require the use of more than one Hilbert space, in a way that cannot be 'reduced' to a single Hilbert space.

Initial state of A is $|ON>$. There is an operator $\hat{U}$ in $\mathcal{H}_e \otimes \mathcal{H}_A$.

$$\hat{U} : |\text{spin} > \otimes |\text{ON} > \tag{J.30}$$

Correlated

$$\begin{aligned} (a|\uparrow> +b \quad &\rightarrow \quad a|\uparrow> \otimes |``\uparrow" > +b|\downarrow> \otimes |``\downarrow" > \\ |\uparrow> \otimes |\text{ON} > \quad &\rightarrow \quad |\uparrow> \otimes |``\uparrow" > \\ |\downarrow> \otimes |\text{ON} > \quad &\rightarrow \quad |\downarrow> \otimes |``\downarrow" > \end{aligned} \tag{J.31}$$

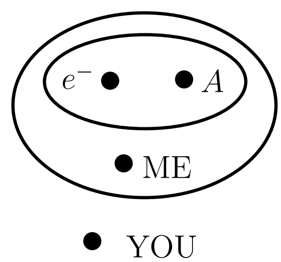


Figure J.44: ME YOU

If I'm going to look consistently say that A the apparatus a new interaction operator of in the Hilbert space of all three of us.

$$\mathcal{H}_e \otimes \mathcal{H}_A \otimes \mathcal{H}_{ME}$$

I should not be passed on a collapsed state by YOU.

When are you going to go from generally a state in the tensor product to a particular one and when does it make sense.

all this makes sense if you tell me which system is supposed to say what about which system. Have all possible

My Hilbert space of all possible systems and subsystems.

a partially ordered set.

Sieves will come from: If you see ME saying e^- is $|\uparrow>$, then you know that A showed $|``\uparrow" >$ and e^- is $|\uparrow>$. i.e., you projected on a particular state implies every **subsystem** of YOU is projected to a particular state.

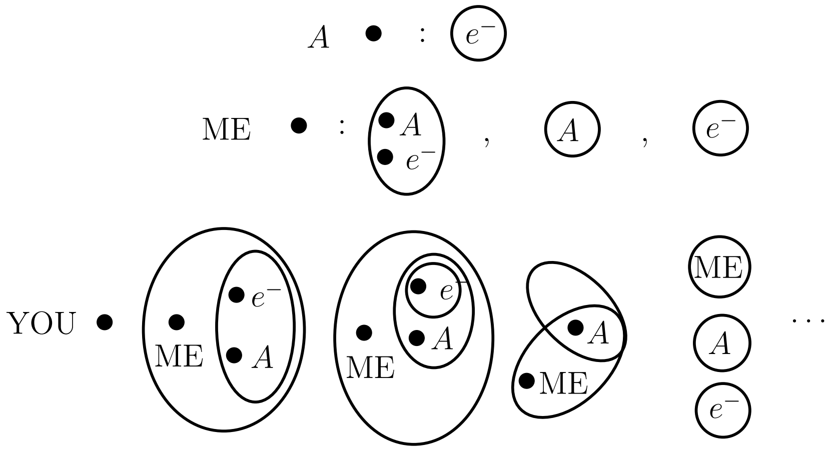


Figure J.45: Partially ordered set. Context for all possible systems and subsystems.

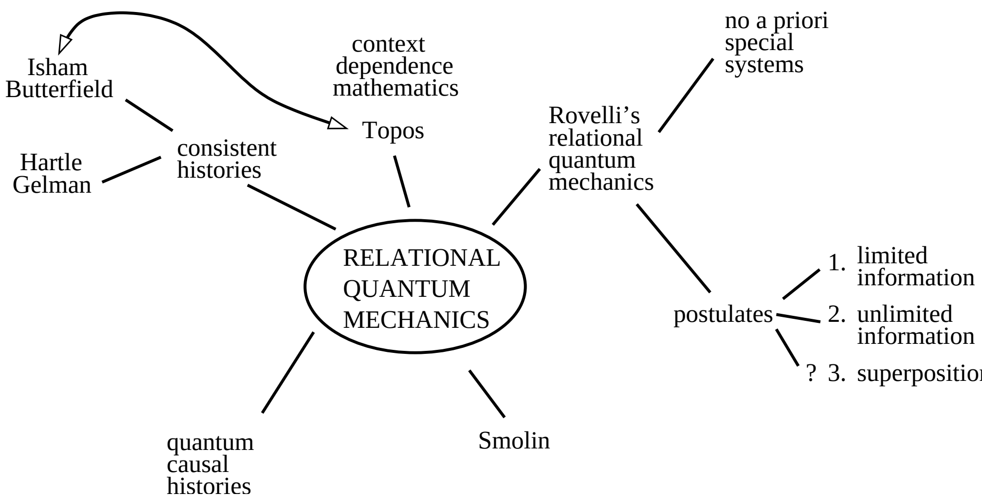


Figure J.46: Relational quantum mechanics.

J.3 Biblioliographical notes

In this chapter I have relied on the following references:

Arun Ram Quantum groups: *A survey of definitions, motivations, and results.*

J.4 Worked Exercises

Proofs

the arrow $f : X \to Y$ in the category $\mathcal{C}$

Given that for any g and h

$$Z \xrightarrow{g,h} X \xrightarrow{f} Y \tag{J.32}$$

we have that $f \circ g = f \circ h$ implies $g = h$.

1. (i) Show that the composition of monos are mono.

(ii) Show that if the composition $f \circ e$ is mono then e is mono.

Answers

1. (i) We have the arrows $e : W \to X$ and $f : X \to Y$ are mono.

we want to show that for any pair g and h

$$Z \xrightarrow{g,h} X \xrightarrow{f \circ e} Y \tag{J.33}$$

we have that $(f \circ e) \circ g = (f \circ e) \circ h$ implies $h = g$.

$$Z \xrightarrow{g,h} W \xrightarrow{e} X \xrightarrow{f} Y \tag{J.34}$$

$$Z \xrightarrow{e \circ g, e \circ h} W \xrightarrow{f} Y \tag{J.35}$$

Since composition is associative the condition $(f \circ e) \circ g = (f \circ e) \circ h$ is the same as $f \circ (e \circ g) = f \circ (e \circ h)$. As f is mono we have: $e \circ g = e \circ h$. Now, as e is mono for any g and h

$$Z \xrightarrow{g,h} W \xrightarrow{e} X \tag{J.36}$$

we have that $e \circ g = e \circ h$ implies $g = h$.

(ii) We want to show that $e \circ g = e \circ h$ implies $g = h$. Suppose $e \circ g = e \circ h$, applying f to both sides gives $f \circ (e \circ g) = f \circ (e \circ h)$ or $(f \circ e) \circ g = (f \circ e) \circ h$ and since the composition $f \circ e$ is mono $g = h$, hence e is mono too.

2. (i)

$$W \xrightarrow{e} X \xrightarrow{f} Y \xrightarrow{g,h} Z \tag{J.37}$$

The condition $g \circ (f \circ e) = h \circ (f \circ e)$ is the same as $(g \circ f) \circ e = (h \circ f) \circ e$. Now, since e is epi we have $g \circ f = h \circ f$. As f is epi $g = h$, and hence we have shown the composition $f \circ e$ is epi too.

implies $g = h$.

Products.

Consider any arrows $a : T \to E$ and $b : T \to E$ such that $e \circ a = e \circ b$. We wish to show this implies $a = b$.

$$
\begin{tikzcd}
T \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
T \arrow[d, "a"'] \arrow[dr, "h = a \circ e"] & & \\
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
$$

(a) (b)

Figure J.47: Steps in proof of theorem .

we could have done the same for b in plase of a, and as $b \circ e = a \circ e$, the unique arrow that makes this diagram the same equalizer is b. Hence $a = b$.

Products.

1 Show there is a

2 Show there is a

Answer

$$
\begin{tikzcd}
 & Q \arrow[dl, "p_1"'] \arrow[d, "u"] \arrow[dr, "p_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A & Q \arrow[l, "p_1"'] \arrow[r, "p_2"] & B \\
 & P \arrow[ul, "p_1"] \arrow[u, "v"'] \arrow[ur, "p_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & P \arrow[dl, "p_1"'] \arrow[d, "u \circ v"] \arrow[dr, "p_2"] & \\
A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & A
\end{tikzcd}
\Rightarrow u \circ v = \mathrm{id}_A
$$

(a) (b) (c)

Figure J.48: Steps in proof of theorem **??**.

$$
\begin{tikzcd}
 & A \times B \arrow[dl, "h \circ p_1"'] \arrow[d, "h \times k"] \arrow[dr, "k \circ p_2"] & \\
A & A' \times B' \arrow[l, "p_1"] \arrow[r, "p_2"'] & B
\end{tikzcd}
$$

Figure J.49: Implies $h \times k$ is unique E.7.

Twist arrows are natural

Prove:

$$tw \circ f \times g = g \times f \circ tw \tag{J.38}$$

$$\begin{tikzcd}
A \times B \arrow[r, "\sim"] \arrow[d, "f \times g"'] & B \times A \arrow[d, "f \times g"] \\
A' \times B' \arrow[r, "\sim"] & B' \times A'
\end{tikzcd}$$

Figure J.50: .

Answer

$$\begin{tikzcd}
A \times B \arrow[r, "f \times g"] \arrow[dr, "p_2"'] & A' \times B' \arrow[r, "\sim"] \arrow[dr, "p'_2"'] & B' \times A' \arrow[d, "q'_1"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Figure J.51: prod9.

(a) (b)

$$\begin{tikzcd}
A \times B \arrow[r, "\sim"] \arrow[dr, "p_2"'] & B \times A \arrow[r, "g \times f"] \arrow[d, "q_1"] & B' \times A' \arrow[d, "q'_1"] \\
& B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Figure J.52: prod10.

$$q'_1 \circ tw \circ f \times g = q'_1 \circ g \times f \circ tw \tag{J.39}$$

Quotient categories.

$p_1 \circ e$ $p_2 \circ e$

$$
\begin{tikzcd}
A \times B \arrow[r, "g \circ p_2"] \arrow[d, "f \times g"'] & B' \\
A' \times B' \arrow[r, "\sim"'] & B' \times A' \arrow[u, "q_1'"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \times B \arrow[r, "\sim"] \arrow[d, "g \circ p_2"'] & B \times A \arrow[d, "g \times f"] \\
B' & B' \times A' \arrow[l, "q_1'"]
\end{tikzcd}
$$

Figure J.53: prod11.

$$
\begin{tikzcd}
E \arrow[r, "p_1 \circ e"] \arrow[d, "p_2 \circ e"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"'] & C
\end{tikzcd}
$$

Figure J.54: .

$$p_1 \circ e \tag{J.40}$$

$p_1 \circ e$ $p_2 \circ e$

$$p_1 \circ e \tag{J.41}$$

The morphism e equalizes f and g because we have

$$
\begin{aligned}
f \circ e &= p_1 \circ < f, g > \circ e = (p_1 \circ \Delta_B) \circ f' = \mathrm{id}_B \circ f' = f' && \text{(J.42)} \\
g \circ e &= p_2 \circ < f, g > \circ e = (p_2 \circ \Delta_B) \circ f' = \mathrm{id}_B \circ f' = f' && \text{(J.43)}
\end{aligned}
$$

Quotient categories.

For

i) $f, g : X \to Y$ and $h : Y \to Z$, if $f \sim_{X,Y} g$ then $h \circ f \sim_{X,Y} h \circ g$

ii) $f : X \to Y$ and $g, h : Y \to Z$, if $g \sim_{Y,Z} g$ then $g \circ f \sim_{X,Z} h \circ f$.

2 Show there is a functor $\mathcal{C} \to \mathcal{C}/\sim$, which takes each arrow of $\mathcal{C}$ to its equivalence class.

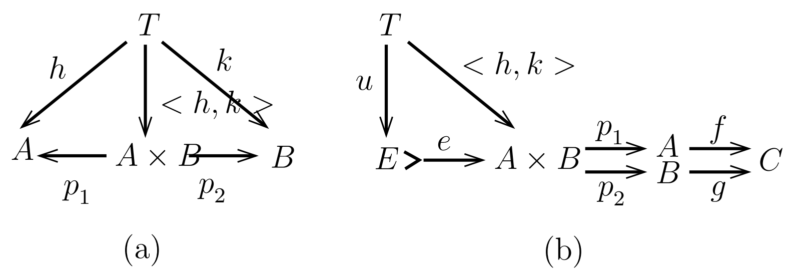


Figure J.55: this one.

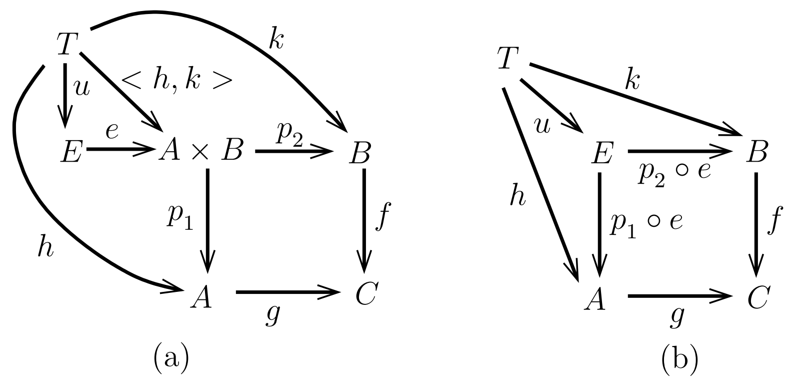


Figure J.56: this one.

Answer

We need to show that composition is representation independent, i.e., that

for $[f]_{X,Y} : X \to Y$, $[g]_{Y,Z} : Y \to Z$ and $[h]_{Y,Z} : Y \to Z$ then: $[h]_{Y,Z} \circ [f]_{X,Y} = [h]_{Y,Z} \circ [g]_{X,Y}$

and for $[g]_{X,Y} : X \to Y$, $[h]_{Y,Z} : Y \to Z$ and $[f]_{Y,Z} : Y \to Z$ then: $[g]_{Y,Z} \circ [f]_{X,Y} = [h]_{Y,Z} \circ [f]_{X,Y}$

condition 1. above is the same as $[f]_{X,Y} = [g]_{X,Y}$, if $[h \circ f]_{X,Y} = [h \circ g]_{X,Y}$

$$[h]_{Y,Z} \circ [f]_{X,Y} = [h \circ f]_{X,Z} = [h \circ g]_{X,Z} = [h]_{Y,Z} \circ [g]_{X,Y} \tag{J.44}$$

$$[g]_{Y,Z} \circ [f]_{X,Y} = [h \circ f]_{X,Z} = [g \circ f]_{X,Z} = [g]_{Y,Z} \circ [f]_{X,Y} \tag{J.45}$$

2 $F : \mathcal{C} \to \mathcal{C}/\sim$. Does $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

Proofs

From the diagram fig(E.7), it is easy to see that

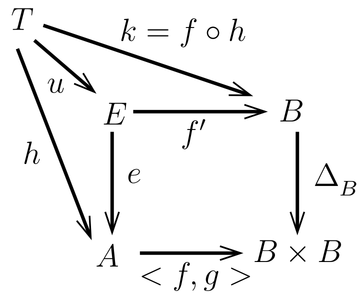


Figure J.57: this one.

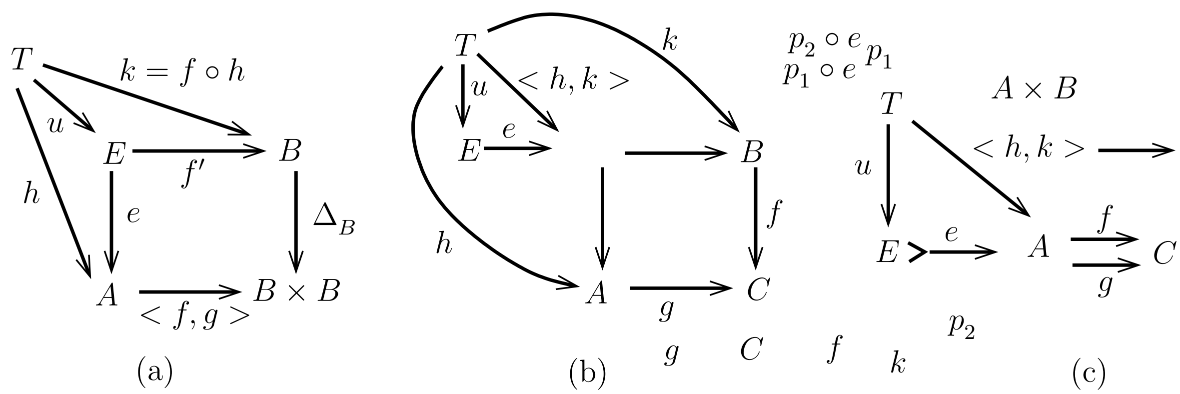


Figure J.58: this one.

$$g \circ p_2 \circ h = g \circ p_2 \circ k, \tag{J.46}$$

p_1 can be composed with k or h p_2 can be composed with k or h to form arrows from T to A and from T to B respectively.

$$p_1 \circ k = p_1 \circ h, \qquad p_2 \circ k = p_2 \circ h \tag{J.47}$$

Therefore the two diagrams in figs J.4(a) and (b) coincide except possibly for the arrows h and k. However, by definition of a pullback there exists a unique arrow $u : T \to A \times_C B$ such that diagrams shown in figs(a) and (b) should commute. Hence $h = k$.

co-products

Proofs

Obviously the conditions $f \circ p_1 = p_3$ and $g \circ p_2 = p_3$ are equivalent to the condition that the square in diagram fig(J.4) commutes. Similar observation for the cone T q_1, q_2. Whence, the condition that C, p_1, p_2 be a limit is equalent to the condition that C, p_1, p_2 be a pullback.

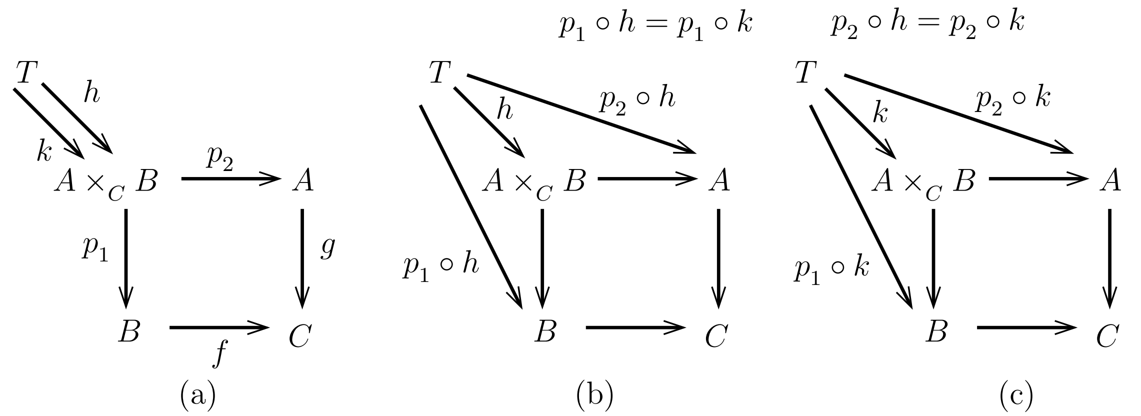


Figure J.59: Since $p_1 \circ k = p_1 \circ h$ and $p_2 \circ k = p_2 \circ h$, $h = k$.

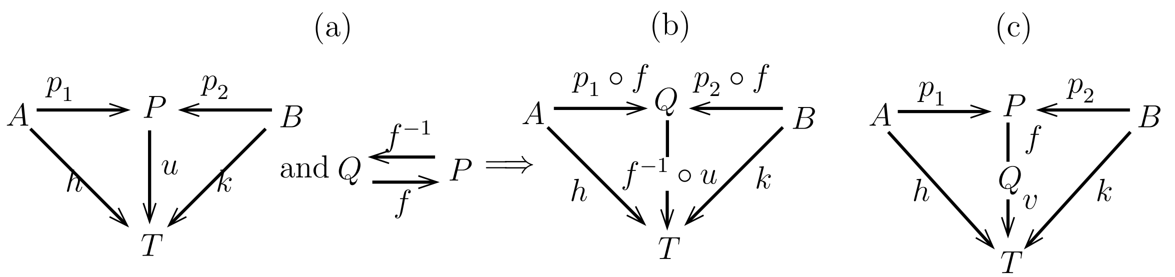


Figure J.60: Proof of Theorem J.1.2. from (c) we see that $f \circ v = u$ is the unique arrow that makes the diagram commute.

Limits as Equalizers of Products

A T-element $x_i \in_T A_i$ for each vertex i for every $f_e : A_j \to A_k$ we have $f_e(x_j) = x_k$.

Then D has a limit if and only if there is an equalizer for r and s.

Consider any cone T, $h_i : T \to A_i$ over D.

Questions on intersection pullbacks

1.

2.(a) Show that for any X, and any $x \in_X A$ that $x \in i \cap j$ if and only if $x \in i$ and $x \in j$.

(b) Explain why this is just the defition of a pullback in this notation.

Proof

1.

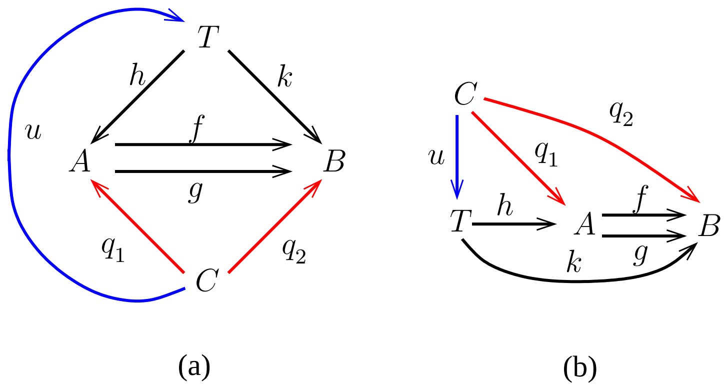


Figure J.61: limits3.

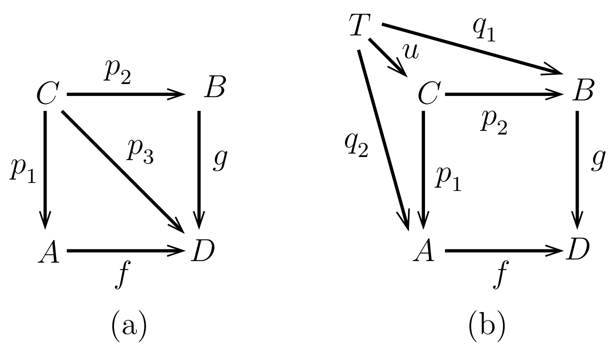


Figure J.62: limits4.

Conversely, suppose the outer square is a pullback $q : T \to E$ and $t : T \to A$ with $i \circ q = f \circ t$.

So that there is a unique $u : T \to D$ with $t = h \circ u$ and $m \circ q = (m \circ k) \circ u$, see fig(J.4(a)).

Thus $m \circ (k \circ u) = m \circ q$ and $i \circ (k \circ u) = i \circ q$. As m and i are the projection arrows of a pullback, $q = k \circ u$. Then $t = h \circ u$ and $q = k \circ u$. Must prove u is unique.

2.

for any object X, and any $x \in_X A$ has $x \in i \cap j$ if and only if $x \in i$ and $x \in j$.

Say there is h such that $x = i \circ h$ and $x = j \circ k$, (i.e. $x \in i$ $x \in j$). Then the outer square commutes, i.e., $i \circ h = j \circ k$. From the definition of a pullback we know that there is a (unique) arrow $u : X \to S \cap T$ such that $x = i \cap j \circ u$.

Conversely, suppose there is an arrow $u : X \to S \cap T$ such that $x = i \cap j \circ u$.

3.

Note $m \circ q = f \circ p$. Since $n \circ (h \circ q) = f \circ p$, see fig (J.4(a)). Now as F is a pullback, there is a unique $u : E \to F$ with $p = r \circ u$ and $h \circ q = s \circ u$, (J.4(b)). Check that the whole diagram commutes.

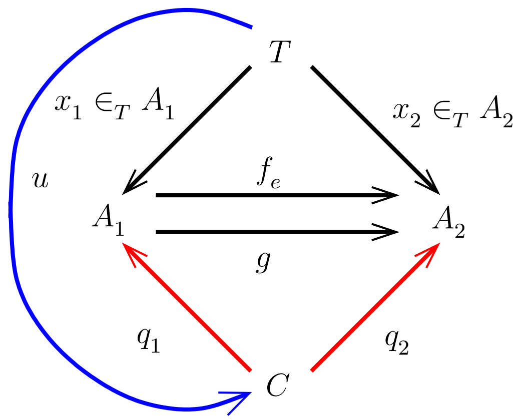


Figure J.63: limits5. The T-element $x_1 \in_T A_1$ for the vertex 1 for $f_e : A_1 \to A_2$ we have $f_e(x_1) = x_2$

Heyting Algebras

Proofs

Show that any Heyting algebra is a *distributive* lattice:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \tag{J.48}$$

for all $a, b, c \in \mathbf{H}$.

Sub(X) forms a Heyting Algebra

Questions

1. Prove that the pullback $Y \cap Y'$ is the greatest lower bound for Y and Y'.

Proof

1.

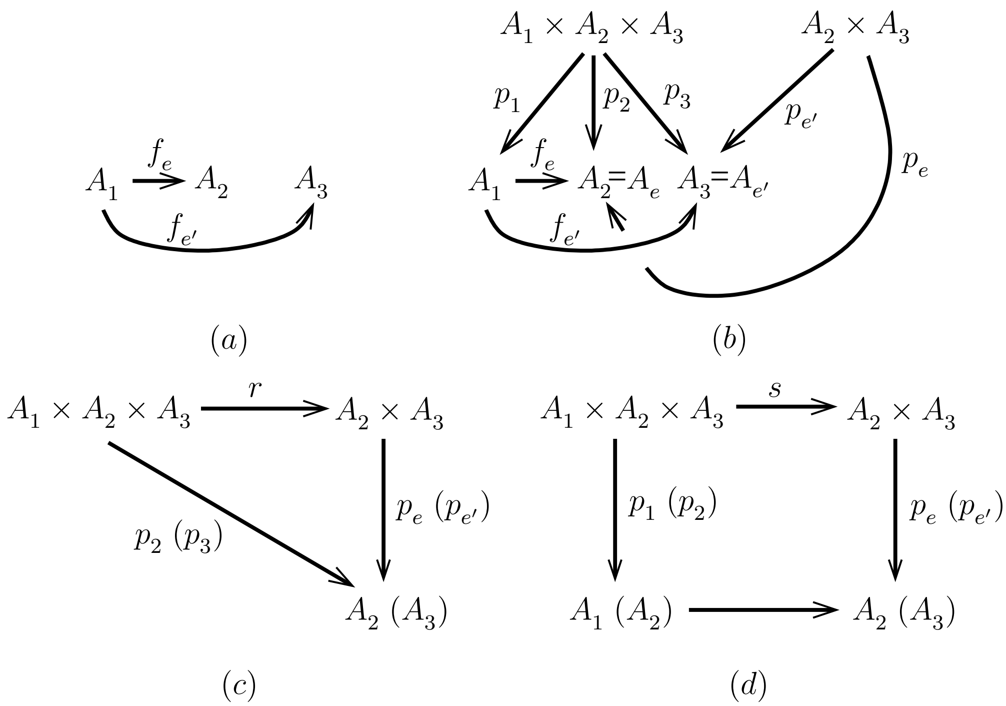


Figure J.64: limEqProd. (a) For diagram D. (b)

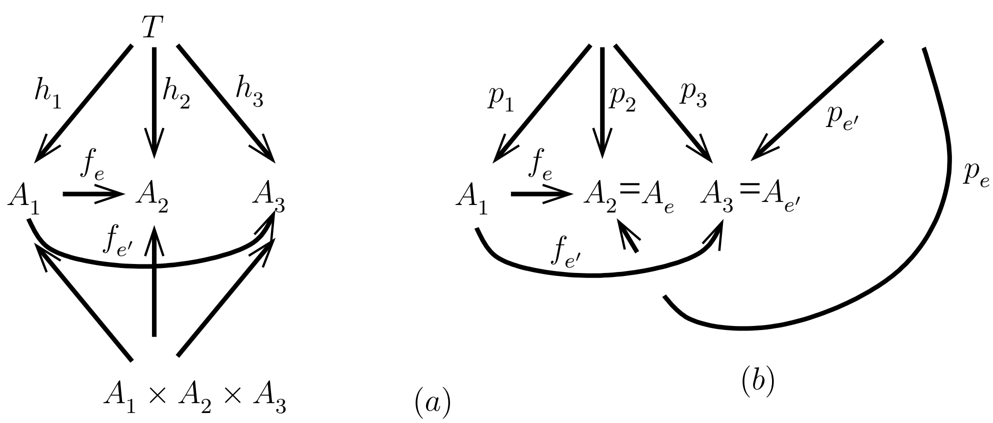


Figure J.65: limEqProd1. (a) For diagram D. (b)

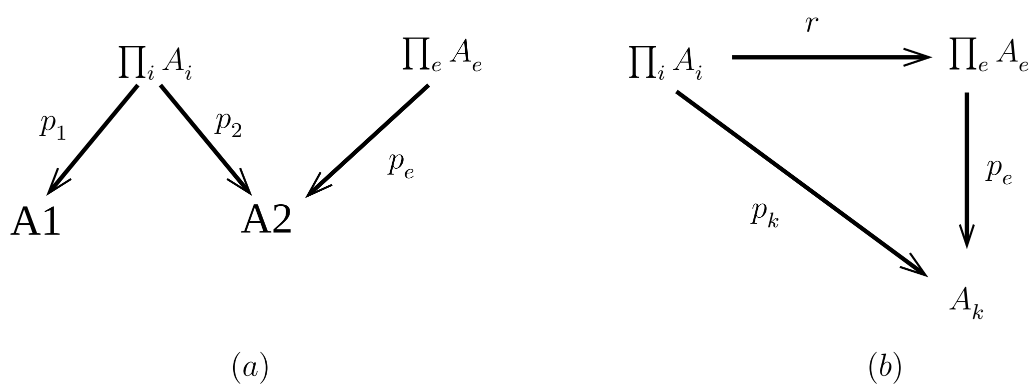


Figure J.66: limEqProd2.

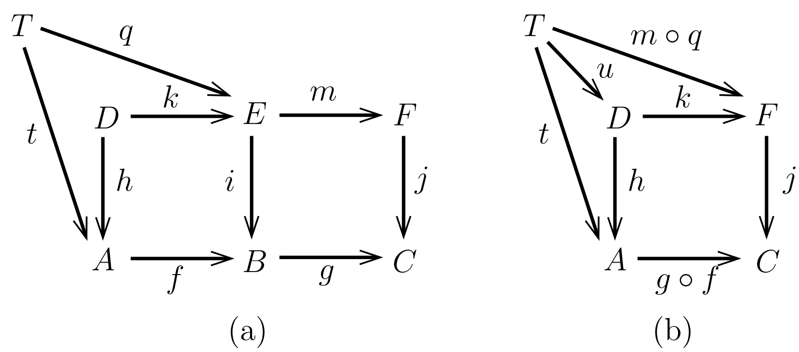


Figure J.67: pullback11.

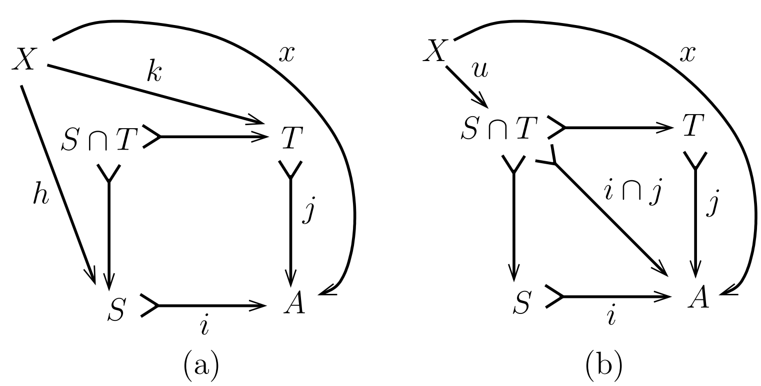


Figure J.68: intersectcomp1.

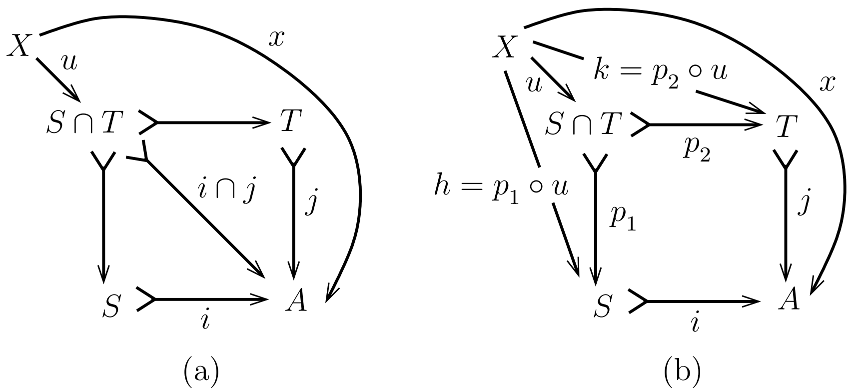


Figure J.69: intersectcomp2.

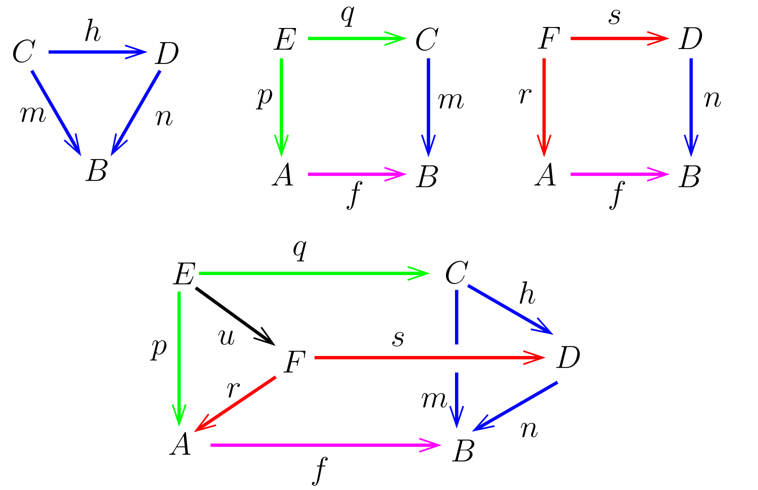


Figure J.70: pullback4.

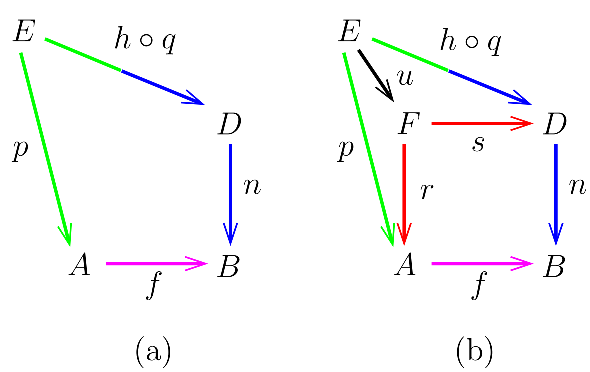


Figure J.71: pullback5.

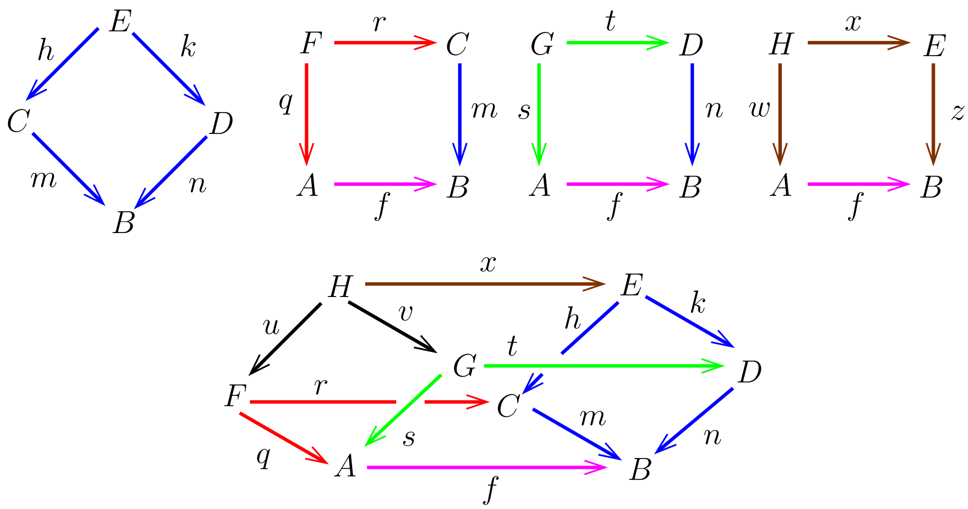


Figure J.72: pullback6.

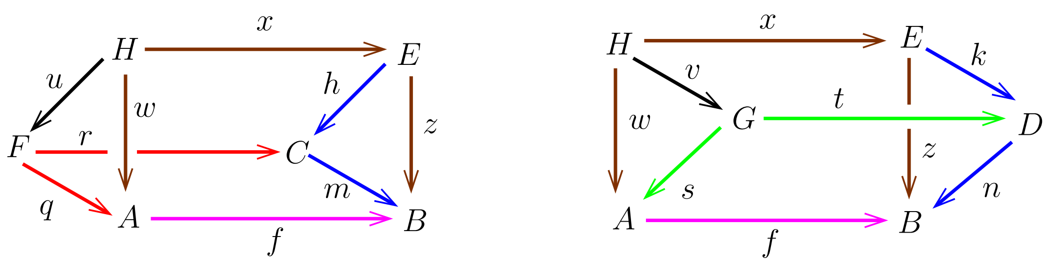


Figure J.73: pullback7.

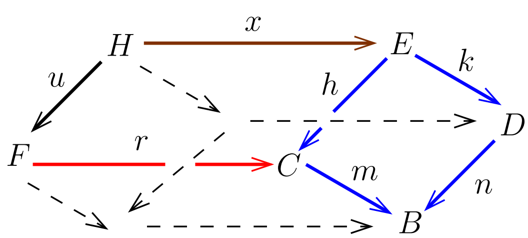


Figure J.74: pullback8.

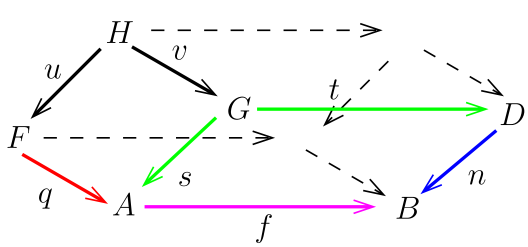


Figure J.75: pullback9.