Appendix K

Loop Quantum Gravity Quantization of String Theory

K.1 Introduction

String theory starts with the idea that fundamental particles are not point-like but are excitations of a one dimensional string. To remove the tachyon, one adds fermions and requires that the string be supersymmetric.

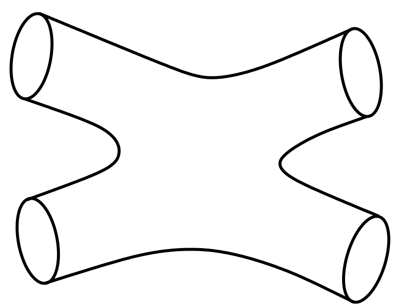

Figure K.1: stringfig1.

In defining a perturbative expansion, they must make a choice for the background metric of the spacetime will be. They don't really want the spacetime metric to appear at all in the formulation of the theory. They want it to be something that emerges as a prediction of the theory.

Argue that it is natural to first address these difficulties in the simplest possible scenario of general relativity. What is learnt will be useful in the future to tackle more elaborate theories of string theory.

surfaces swept out by a string in spacetime (called world sheets). In analogy with minimization of the proper time of a particle solutions to the equations of motion are surface swept out that are extremal with respect to the Minkowski metric.

Since the process of first quantization a string is equivalent to second quantization of a point particle theory on teh worldsheet that possesses gauge symmetries, it is a quantum field theory.

K.1.1 Unification via Strings

One of the wonderful things about string theory is the natural way it gives rise to a unification of all the particles and forces.

there should be a massless particle of spin 2 arising from the vibration of the strings. It was appropriate to identify this massless field with gravity.

"The vibrations of a string include states that correpsond to all the known kinds of matter and forces. The graviton, the particle that carries the gravitational force, comes out of the vibrations of loops (i.e., closed strings). The phton, carrier of the electromagnetic force, also emerges from the vibrations of a string. The more complicated gauge fields, in terms of our understanding of the strong and weak nuclear forces, also comes out automatically; that is, string theory predicts generally that there are gauge fields similar to these, although it does not predict the particular mix of forces we see in nature."

The equations of motion do not determine the

parametrization is a symmetry transformation that does not change the physical state of the system. We will call this parameter space the *abstract world-sheet*. Puting it into the same class of theories as General Relativity - the *background independent* techniques of loop quantum gravity.

of the closed bosonic string in flat Minkowskian spacetime.

K.2 Unresolved Problems in Conventional String Theory

Among thye fields that live on the worldsheet are the imbedding coordinates, X^a, where $X^a(\sigma)$ are coordinates in the background spacetime and σ are the two dimensional coordinates on the worldsheet. The original form of the string action seemd difficult to quantize. Progress was made when Polyakov werote the worldsheet theory in terms of the metric on the worldsheet $h_{\alpha\beta}$, so as to form the action,

$$\int d^2\sigma \sqrt{h} h^{\alpha\beta}(\partial_\alpha X^a)(\partial_\beta X^b) g_{ab}(X(\sigma))$$

where g_{ab} is the metric of the background metric. The two dimensional coordinates of the worldsheet an be fixed so that

$$h_{\alpha\beta} = \eta_{\alpha\beta} e^{\phi},$$

where η is the metric of flat, two dimensional spacetime. This leaves unfixed the third component of the metric, represented by ϕ, which we see here is the conformal factor.

Polyakov noticed that the quantization of the X^a fields on the worldsheet will in general give rise to dependence on ϕ in spite of the fact that the calssical actionis conformally invariant. However, it was noticed that this anomaly is absent if the dimension of the (Minkowski) background spacetime is 26 diemnsions!

Supersymmetry gives rise to additional necesary conditions on the background spacetime that it must satisfy.

K.2.1 String Theory in Non-Statinary Spacetimes

For there to be a consistent quantum mechanical description of strings moving and interacting in that geometry it turns out that a necessary condition for the string theory to be consistent is that, to a certain approximation, the geometry is a solution to the equations of a higher dimensional version of general relativity. So in a sense the the equations of general relativity come about from conditions for a string to move consistently.

More precisely, a necessary condition for a perturbative theory to be consisitent is that the two dimensional world sheet quantum field theory that defines the theory be conformally invariant. This maens that the conformal anomaly on the two dimensional worldsheet vanishes. To leading order in l_{string} this condition is equivalent to the Einstein equations of the background manifold.

However, this is the situation only in the original twenty-six dimensional bosonic string theory. Worldsheet supersymmetry $\rightarrow$ space-time supersymmetry. Space-time supersymmetry requires background spacetime geometry admits a timelike Killing vector field, that is, that the background spacetime geometry is not evolving with time!

In the absence of a construction of string theory on general time-dependent spacetime or compelling arguent for its existence, it can not be asserted that all of general relativity can be derived from string theory.

However, the LQG string does not require supersymmetry, and hence could be viable in time dependent background spacetime geometries.

Supersymmetry requires fermions and bosons to come in pairs consisting of one of each, with the same mass. This is not observed in nature.

K.2.2 Problems Arising From Extra Dimensions

Problems?!! Yes.

Roger Penrose argues that most of the compactified spaces that extra dimensions curl up into will quickly colapse to singularities. To show this, he apllies to the spacetime backgrounds of these string theories the Hawking-Penrose theorems in cosmological solutions.

K.2.3 Positive Cosmological Constant and the Landscape (More is Less)

If we want the theory to give a positive cosmological constant, so as to agree with observation, there is at present evidence for around 10^{500} such theories.

An attempt to construct a unique theory of nature leads instead to 10^{500} theories.

Gary Horowitz and colleagues have found a possible instability that afflicts not just the landscape of new theories but all solutions that involve the six-dimensional Calabi-Yau spaces.

K.3 The Approach to Quantization

Quantisation is the problem of deriving the mathematical framework of a quantum mechanical system from the mathematical framework of the corresponding classical mechanical system. A method of quantisation must contain a map $\mathcal{Q}$ from the set of classical observables to the set of quantum observables with the following properties:

The steps in quantization.

$$\mathcal{Q} : f \to \mathcal{Q}(f) \tag{K.1}$$

of operators $\mathcal{Q}(f)$ on some Hilbert space to classical observerables f.

1) Lineararity,

$$\mathcal{Q}(\alpha f + g) = \alpha \mathcal{Q}(f) + \mathcal{Q}(g), \quad \text{for all } \alpha \in \mathbb{C}, f, g \in C^{\infty}(\mathcal{M}), \tag{K.2}$$

2) The constant function 1 is mapped to the identity operator or matrix $\mathbb{I}$,

$$\mathcal{Q}(1) = \mathbb{I}. \tag{K.3}$$

3)

$$\mathcal{Q}(f)^* = \mathcal{Q}(f). \tag{K.4}$$

In the early days of quantum theory, Dirac proposed that the quantum analogue of the Poisson bracket of two classical observables should be the commutator of the corresponding quantum observables.

4)

$$[\mathcal{Q}(f), \mathcal{Q}(g)] = -i\hbar \mathcal{Q}(\{f, g\}). \tag{K.5}$$

5) If $\{f_1, f_2, \ldots, f_k\}$ is a complete set of observerables, $\{\mathcal{Q}(f_1 0, \mathcal{Q}(f_2), \ldots, \mathcal{Q}(f_k)\}$ is a complete set of operators.

However, there is a problem. If we demand some particular complete set of observerables to satisfy (K.5), then some alternative complete set of observerables don't neccessarly satisfy (K.5), there may be extra higher order terms in $\hbar^2$ resulting in a kind of deformation quantization for this other complete set of observerables. So it is not ruled out that different choices of complete sets will lead to inequivalent quantum theories (i.e., to inequivalent predictions for the results of experiments). There isn't an axiomatic scheme for quantization that will tell us how to find a complete set of observerables, or which ones to choose. Extraneous considerations are required to construct the quantum theory, such as symmetries of the classical theory which may make one complete set more 'natural' than another.

We will use an algebraic approach to quantizing the bosonic string, the approach known as algebraic quantum field theory to be precise. This is a formulation of QFT on an axiomatic basis: that is, starting from what seem to be physically necessary and mathematically precise principles which any QFT would have to satisfy, and then finding QFTs which actually satisfy them.

An algebraic quantum theory is specified by giving the algebraic structure of the observables but not the Hilbert space on which they act. GNS construction is used to obtain a Hilbert space with set of operators which realizes a representation of the observable operator algebra.

For the LQG string again the Poincare group is represented unitarily but weakly discontinuously. However, we can approximate the generators in terms of teh corresponding Weyl operators using some tiny but finite parameter ϵ. Since there are a finite number of operators in the corresponding C^*-algebra, an appeal to Fell's theorem and using the continuity of the Weyl operators in the Fock representation guarentees that we find a state in the folium of the LQG string with respect to which the expectation values of the approximate Poincare generators coincide with their vacuum (or higher excited state) expectation values in the Fock representation to arbitrary precision δ.

K.4 String Theory

K.5 Nambu-Goto String

The functions $X^\mu(\sigma, \tau)$ specify points in Minkowskian spacetime of the string worldsheet. They map from a two dimensional parameter space (the abstract worldsheet), to coordinates in Minkowski spacetime.

$$ds^2 = h_{ab} d\xi^a d\xi^b \tag{K.6}$$

$$ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} d\xi^a d\xi^b \tag{K.7}$$

hence

$$h_{ab} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^a} \frac{\partial X^\nu}{\partial \xi^b} \tag{K.8}$$

The action is

$$S_{NG} = -\frac{1}{2\pi\alpha'} \int_\Sigma d^2\xi (-h)^{1/2} \tag{K.9}$$

where $h = \det h_{ab}$ and α is related to the string tension as $T = \frac{1}{2\pi\alpha'}$.

$$\begin{aligned} ||\, d\xi^1 \times\; d\xi^2|| &= ||\, d\xi^1||\; ||\, d\xi^2|| \sin\theta \\ &= \sqrt{||\, d\xi^1||^2 ||\, d\xi^2||^2 (1 - \cos^2\theta)} \\ &= \sqrt{||\, d\xi^1||^2 ||\, d\xi^2||^2 - (||\, d\xi^1 \cdot\; d\xi^2||)^2} \\ &= \sqrt{|q_{11} q_{22} - (q_{12})^2|}\, d\sigma_1\, d\sigma_2 \\ &= \sqrt{\det q}\, d^2\sigma \end{aligned} \tag{K.10}$$

$$A(S) = \int d^2\sigma \sqrt{-\det h} \tag{K.11}$$

This is the so-called Nambu-Goto action. by using the matrix h_{ab}:

$$
(h_{ab}) := \begin{pmatrix} (\dot{X})^2 & \dot{X} \cdot X' \\ \dot{X} \cdot X' & (X')^2 \end{pmatrix}. \tag{K.12}
$$

$$
\begin{aligned}
S_{NG} &= -T \int_\Sigma (\partial_\tau X^\mu \partial_\sigma X_\mu - \partial_\sigma X^\mu \partial_\tau X_\mu)^{1/2} \; d\sigma \, d\tau, \\
&= -T \int_\Sigma (\partial_\tau X^\mu \partial_\sigma X^\nu \eta_{\mu\nu})^{1/2} \; d\sigma \, d\tau.
\end{aligned} \tag{K.13}
$$

Symmetries of the Nambo-Goto action

D-dimensional Poincare invariance

$$
X'^\mu(\xi) = \Lambda^\mu_\nu X^\nu(\xi) + a^\mu \tag{K.14}
$$

Two-dimensional diffeomorphism invariance - reparametrization of the worldsheet

$$
\begin{aligned}
\xi'^\mu &= \xi'^\mu(\xi) \\
X'^\mu(\xi') &= X^\mu(\xi).
\end{aligned} \tag{K.15}
$$

$$
\begin{aligned}
\pi_\mu &:= \alpha' \frac{\delta S}{\delta \dot{X}^\mu(y)} \\
&:= \alpha' \left(\frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} - \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \ddot{X}^\mu} \right) \right) \\
&= \alpha' \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} \quad \text{as there areno second derivatives of } X \text{ in the action.}
\end{aligned} \tag{K.16}
$$

$$
\begin{aligned}
\pi^\tau_\mu &:= \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X') X'_\mu - (X')^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X') - (\dot{X})^2 (X')^2}} \\
\pi^\sigma_\mu &:= \frac{\partial \mathcal{L}}{\partial X'^\mu} = -\frac{T_0}{c} \frac{(\dot{X} \cdot X') \dot{X}_\mu - (\dot{X})^2 \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X') - (\dot{X})^2 (X')^2}}
\end{aligned} \tag{K.17}
$$

$$
\{X^\mu(t,x), X^\nu(t,x')\} = \{\pi_\mu(t,x), \pi_\nu(t,x')\} = 0, \{\pi_\mu(t,x), X^\nu(t,x')\} = \alpha' \delta^\nu_\mu \delta(x,x'). \tag{K.18}
$$

$$
\begin{aligned}
D_a &:= \pi_\mu X^\mu_{,a} = 0, \\
C &:= \frac{1}{2}(\eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q)) = 0
\end{aligned}
\tag{K.19}
$$

where

$$
q_{ab}(y) = [(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})^{1/2}](y) \tag{K.20}
$$

K.6 The Polyakov String

The Nambo-Goto action is not very convenient for quantizing the world sheet theory. It is replaced by another action, which is classically equivalent, the Polyakov action. To do this one needs to introduce another metric $g_{ab}(\sigma,\tau)$ which is taken to be independent of the induced metric h_{ab}. The action for the world sheet

$$
S = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi(-g)^{1/2} g^{ab}(\sigma,\tau)\partial_a X^\mu \partial_b X^\mu \eta_{\mu\nu} \tag{K.21}
$$

with $g = \det(g_{ab})$.

The new metric $g_{ab}(\sigma,\tau)$ is to be thought of as a metric introduced onto the abstract worldsheet. It is easy to see that if the Polyakov action is to invariant under reparmetization, the metric $g_{cd}(\xi)$ must transform as

$$
g'_{ab}(\xi') = \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi). \tag{K.22}
$$

Polyakov action

$\delta S/\delta g_{ab}$. We will need to use the identity

$$
\delta g = -g g_{ab}\delta g^{ab} \quad \Rightarrow \quad \delta(-g)^{1/2} = \frac{1}{2}(-g)^{1/2} g_{ab}\delta g^{ab}. \tag{K.23}
$$

$$
\begin{aligned}
\delta S &= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi \left\{\delta(-g)^{1/2} g^{ab} + (-g)^{1/2}\delta g^{ab}\right\}\partial_a X^\mu \partial_b X^\mu \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi (-g)^{1/2}\delta g^{ab}\left\{-\frac{1}{2} g_{ab} g^{cd}\partial_c X^\mu \partial_d X^\mu + \partial_a X^\mu \partial_b X^\mu\right\} \\
&= -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\xi (-g)^{1/2}\delta g^{ab}\left(-\frac{1}{2} g_{ab} g^{cd} h_{cd} + h_{ab}\right)
\end{aligned}
\tag{K.24}
$$

equations of motion which fixes a relation between the two metrics h_{ab} and g_{ab}

$$h_{ab} = \frac{1}{2} g_{ab} g^{cd} h_{cd}. \tag{K.25}$$

Taking the determinant of both sides

$$(-h)^{1/2} = \frac{1}{2} (-g)^{1/2} g^{cd} h_{cd} \tag{K.26}$$

$$-\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\xi (-h) 1/2 \equiv S_{NG}[X(\xi)]. \tag{K.27}$$

So on-shell the Nambu-Goto and Polyakov theories are equaivalent. the more directly geometric Nambu-Goto string.

In the Polyakov string Weyl invariance introduced artificially by hand then factored out later.

Symmetries of the Polyakov action

$\rightarrow$

introduce τ and σ to describe the motion of the string

The parameterization is arbitrary - a different parameterization of the same path is physically equivalent, and all physical quantities must be independent of this choice. That is, for any two monotonic function $(\tau'(\tau), \sigma'(\sigma))$ and (τ, σ) are the same, where

$$X^a(\tau'(\tau), \sigma'(\sigma)) = X^a(\tau, \sigma) \tag{K.28}$$

$\leftarrow$

D-dimensional Poincare invariance

$$\begin{aligned} X'^{\mu}(\xi) &= \Lambda^{\mu}_{\nu} X^{\nu}(\xi) + a^{\mu} \\ g'_{ab}(\xi) &= g_{ab}(\xi) \end{aligned} \tag{K.29}$$

Two-dimensional reparametrization of the worldsheet

$$
\begin{aligned}
\xi'^{\mu} &= \xi'^{\mu}(\xi) \\
X'^{\mu}(\xi') &= X^{\mu}(\xi) \\
{g'}_{ab}(\xi') &= \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi).
\end{aligned}
\tag{K.30}
$$

Two-dimensional Weyl invariance

$$
\begin{aligned}
X'^{\mu}(\xi) &= X^{\mu}(\xi) \\
{g'}_{ab}(\xi) &= \Omega(\xi) g_{ab}(\xi)
\end{aligned}
\tag{K.31}
$$

quantize a bunch of decoupled simple harmonic oscillators - string oscillation modes.

Conformal invariance is at he heart of the finiteness of string theory- apparently.

K.6.1 Conformal Invariance

Their formulas look so alike that the two are easily mixed up. This often leads to the statements in the literature like:

“conformal transformations are coordinate transformations which affect a Weyl transformation”

even though, as one of the first things one learns in a course on GR, coordinate transformations leave lengths invariant! - we are taught that the metric $g_{\mu\nu}$ transforms in such a way as to leave $d\tau^2$ unchanged, where $d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$. The transformation which affects a Weyl transformation is an *active diffeomorphism* not a coordinate transformation. Consider an active diffeomorphism which sends the point x to the point x' s.t. $x^\mu = f^\mu(x'^\nu)$, this has the effect on the metric

$$
g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial f^\rho}{\partial x'^\mu}\frac{\partial f^\sigma}{\partial x'^\nu} g_{\rho\sigma}(f(x')).
\tag{K.32}
$$

We require the LHS be proportional to $g_{\mu\nu}$.

subsectionReparametrization symmetry to Conformal symmetry

Stretching represents continouis world-sheet coordinate transformations.

The group of general coordinate transformations has as a subgroup conformal transformations (which can be obtained as a residual gauge invariance upon covariant gauge fixing))

K.6.2 Polymer Reresentation

A wide sread expectation belief (not just with LQG people) that geometry will become discrete. as the quantum atom resolved the problem of .. space takes on a natural discretness - and so the momentum oerator is *not* going to be a well-defined operator anymore. It has been reffered to a "pathelogical" but this is physically desirable atribute.

K.7 Worldsheet Background Independence of the String

Background Independence of the Nambu-Goto String

Recall, that the active transformation changes the metric. There is no world-sheet metric so under the active transformation we do generate a new solution.

Same argument apllies to the Polyakov string - world-sheet metric $g_{ab}(\tau, \sigma)$ - if the theory is to be deterministic it should be background independent.

Symmetries of the Nambo-Goto string

not just invariance under coordinate transformations (passive diffeomorphisms)

$$\begin{aligned} \xi'^{\mu} &= \xi'^{\mu}(\xi) \\ X'^{\mu}(\xi') &= X^{\mu}(\xi) \end{aligned} \tag{K.33}$$

but also invariance under active diffeomorphisms

$$\begin{aligned} \xi'^{\mu} &= \xi'^{\mu}(\xi) \\ X^{\mu}(\xi) &\rightarrow X^{\mu}(\xi') \end{aligned} \tag{K.34}$$

Background Independence of the Polyakov String

The metric $h_{ab}(x)$ the string worldsheet picks up from the Minkowski metric η_{ab} of the target space.

shift of emphesis to distances on the abstract world-sheet.

Symmetris of the Polyakov string

not just invariance under coordinate transformations (passive diffeomorphisms). Two-dimensional reparametrization of the worldsheet

$$\begin{aligned} \xi'^{\mu} &= \xi'^{\mu}(\xi) \\ X'^{\mu}(\xi') &= X^{\mu}(\xi) \\ g'_{ab}(\xi') &= \frac{\partial \xi^c}{\partial \xi'^a}\frac{\partial \xi^d}{\partial \xi'^b} g_{cd}(\xi). \end{aligned} \tag{K.35}$$

but also invariance under active diffeomorphisms

$$\begin{aligned} \xi'^{\mu} &= \xi'^{\mu}(\xi) \\ g_{ab}(\xi) &\rightarrow g_{ab}(\xi') \\ X^{\mu}(\xi) &\rightarrow X^{\mu}(\xi') \end{aligned} \tag{K.36}$$

Transformation under a *passive* diffeomorphism:

$$\xi'^a = \xi^a + \epsilon v^a \partial(\xi) \tag{K.37}$$

$$\frac{\partial \xi^c}{\partial \xi'^a} = \delta^c_a + \epsilon \partial_a v^c(\xi) \tag{K.38}$$

$$g'_{ab}(\xi') = (\delta^c_a + \epsilon \partial_a v^c)(\delta^d_a + \epsilon \partial_a v^d) g_{ab}(\xi) = g_{ab}(\xi) + \epsilon(\partial_a v^c + \partial_a v^d) g_{ab}(\xi) \tag{K.39}$$

$$(\delta g)^P_{ab} = g'_{ab}(\xi') - g_{ab}(\xi) = \epsilon(\partial_a v^c + \partial_a v^d) g_{ab}(\xi) \tag{K.40}$$

Transformation under an *active* diffeomorphism:

$$(\delta g)^A_{ab} = \epsilon(v^c \partial_b g_{ac} + v^c \partial_a g_{bc}) \tag{K.41}$$

The Lie derivative generates active diffeomorphisms.

$$\begin{aligned} \{D(u), D(v)\} &= \alpha' D(\mathcal{L}_u v) = \alpha' D(uv') \\ \{D(u), C(v)\} &= \alpha' C(\mathcal{L}_u v) = \alpha' C(uv') \\ \{C(u), C(v)\} &= \alpha' \int dx (u'v - uv') D \end{aligned} \tag{K.42}$$

$C(u) = \int dx u(x) C(x)$ and $D(v) = \int dx v(x) D(x)$

Implications

The theory is finite because of the background independence of the Nambo-Goto string for the same reason as LQG is finite - “short and large distances are gauge equivalent”.

Physical observables Poisson commute with all the constraints.

$\mathcal{P}_-^R$ are not physical because they evaluate the strings momentum and tension energy at a given value of σ; as in GR the value that a field takes at a point can have no physical meaning.

differential geometry is coordinate independent and so active diffeomorphisms are the only kind of diffeomorphisms.

K.8 Standard Quantization of the Polyakov String

Light-cone gauge fixing. Light-cone coordinates

$$\begin{aligned} X^\pm &= \frac{1}{\sqrt{2}}(X^0 \pm X^1) \\ X^i & \quad I = 1, 2, \ldots, D-1. \end{aligned} \tag{K.43}$$

$$A^\mu B_\mu = -A^+ B^- - A^- B^+ + A^i B^i \tag{K.44}$$

so

$$A_- = -A^+, \quad A_+ = -A^-, \quad A_i = A^i \tag{K.45}$$

K.8.1 Oscillator Expansion

$$X^i(\sigma, t) = X_L^i(\sigma + t) + X_R^i(\sigma - t) \tag{K.46}$$

$$\begin{aligned} X_L^i(\sigma + t) &= \frac{x^i}{2} + \frac{p_i}{2p^+}(t + \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z} - \{0\}} \frac{\alpha_n^i}{n} e^{-2\pi i n(\sigma + t)/\ell} \\ X_R^i(\sigma - t) &= \frac{x^i}{2} + \frac{p_i}{2p^+}(t - \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbf{Z} - \{0\}} \frac{\tilde{\alpha}_n^i}{n} e^{2\pi i n(\sigma - t)/\ell} \end{aligned} \tag{K.47}$$

$$[x^-, p^+] = -i, \;\; [x^i, p_j] = i\delta_{ij}, \quad [\alpha^i_m, \alpha^j_n] = [\tilde{\alpha}^i_m, \tilde{\alpha}^j_n] = m\delta_{ij}\delta_{m,-n} \quad [\alpha^i_m, \tilde{\alpha}^j_n] = 0 \tag{K.48}$$

K.8.2 Gupta-Bleuer Covariant Quantization

The price of maintaining manifest Lorentz covariance is the quantization of non-dynamical degrees of freedom and the introduction of ghosts.

Since the metric is not positive or negative definite, ghost states are introduced which can destroy unitarity.

Do we require unitary evolution with respect to the unphysical parameter τ - probabilities of observable quantities dont depend on it. What physical reason do we have for evolution with respect to an unphysical parameter??

K.9 Dirac Observables of the Nambu-Goto String

The constraints of the theory are represented as quantum operators in the kinematic representation and physical states are those states that are anihilated by the set of constraints. Evolution is generated by constraints, thus implying that Dirac observables are constants of motion. Hence finding enough Dirac observables is equivalent to finding a complete set of constants of motion for the theory. In the case of the Einstein's equations this close to immpossible, however, in the case of the Nambu-Goto string a complete set of constants of motion has been found by Pohlmeyer.

K.10 Invariant charges of the Nambu-Goto String

Lax Pairs Method for Constructing Invariants of Integrable Systems

$$\frac{dL}{dt} \equiv \dot{L} = [M, L]. \tag{K.49}$$

$$L(t) = g(t)L(0)g(t)^{-1} \tag{K.50}$$

where the invertible matrix is determined by the equation

$$M = \frac{dg}{dt}g^{-1}. \tag{K.51}$$

They allow the construction ofconserved quantities.

$$I(L(t)) = I(g(t)L(0)g(t)^{-1}) = I(L(0)) \tag{K.52}$$

such functions are functions of the eigenvalues of L.

$$\mathrm{Tr}(L^n) \tag{K.53}$$

integrability of the system in the sence of Lioville:

(1) the number of independent conserved quantities equals the number of degrees of freedom, and that

(2) these conserved quantities are in involution - are in involution means F_i

$$\{F_i, F_j\} = 0 \tag{K.54}$$

K.10.1 Invariant charges of the Nambu-Goto String

$$\begin{aligned} D &:= 2\pi_\mu X^{\mu\prime}, \\ C &:= \eta^{\mu\nu}\pi_\mu\pi_\nu + \eta_{\mu\nu}X^{\mu\prime}X^{\nu\prime} \equiv \eta^{\mu\nu}\pi_\mu\pi_\nu + \det(q) \end{aligned} \tag{K.55}$$

first step is to introduce a new set of constraints called the Virasoro constraints

$$V_\pm(u) := \pm\frac{1}{2\alpha'}\int_{S^1} dx\; u(C \pm D) \equiv \pm\frac{1}{2\alpha'}\int_{S^1} dx\; \xi\eta_{\mu\nu}Y^\mu_\pm Y^\nu_\pm \tag{K.56}$$

where u is a smearing function and

$$Y^\mu_\pm := \eta^{\mu\nu} \pm X^{\mu\prime}. \tag{K.57}$$

The constraint algebra simplifies to

$$\{V_\pm(u), V_\mp(v)\} = 0, \quad \{V_\pm(u), V_\pm(v)\} = V_\pm(s(u,v)) \tag{K.58}$$

where $(s(u,v)) = u'v - uv'$.

$$\{V_\pm(u), Y_\mp(f)\} = 0, \quad \{V_\pm(u), Y_\pm(f)\} = Y_\pm(uf'). \tag{K.59}$$

$$
\begin{aligned}
p_\mu &= \frac{1}{\alpha'}\int_{S^1} dx\ \pi_\mu \\
J^{\mu\nu} &= \frac{1}{\alpha'}\int_{S^1} dx[X^\nu \eta^{\mu\rho}\pi_\rho - X^\mu \eta^{\nu\rho}].
\end{aligned}
\tag{K.60}
$$

It is straightforward to check that p_μ, $J^{\mu\nu}$ have vanishing Poisson brackets with the $V_\pm(u)$ and are thus strong Dirac observables.

$$
\{p_\mu, Y^\nu_\pm(x)\} = 0, \quad \{J^{\mu\nu}, Y^\rho_\pm(x)\} = (\eta^{\mu\rho}Y^\nu_\pm - \eta^{\nu\rho}Y^\mu_\pm)(x) \tag{K.61}
$$

K.10.2 Algebra of Invariants

The integrability condition for such a Lax pair M, A is the zero curvature condition

$$
F_{\alpha\beta} = \partial_{[\alpha}A_{\beta]} + [A_\alpha, A_\beta] = 0 \tag{K.62}
$$

$$
\mathcal{H}(u,v) = C(u) + D(v), \tag{K.63}
$$

introduce new variables to use instead of the canonical ones - momenta of the left or right movers

$$
Y^\mu_\pm = \pi^\mu \pm X^{\mu\prime}, \tag{K.64}
$$

$$
\dot{Y}_\pm = \{\mathcal{H}(u,v), Y^\mu_\pm\} = [(v \pm u)Y^\mu_\pm]'. \tag{K.65}
$$

More explicitly this equation reads

$$
\frac{d}{dt}\int_{S^1} dx f_\mu(x) Y^\mu_\pm(t,x) = \int_{S^1} dx f_\mu(x)\frac{\partial}{\partial x}\left(Y^\mu_\pm(t,x)(u(x) \pm v(x))\right) \tag{K.66}
$$

“time” evolution.

The invariants that we construct will be functions of $Y^\mu_\pm$, and so we will have to construct

Let τ_I be a basis of $GL(n,\mathbf{R})$ and T^I_μ some complex valued matries.

W define the following connection

$$A_x^{\pm} = Y_{\pm}^{\mu} T_{\mu}^{I} \tau_I \equiv Y_{\pm}^{\mu} T_{\mu}, \quad A_t^{\pm} = (v \pm u) A_x^{\pm}. \tag{K.67}$$

The zero curvature condition then mimics the equations of motion

$$F_{tx}^{\pm} = \partial_t A_x^{\pm} - \partial_x A_t^{\pm} + [A_t^{\pm}, A_x^{\pm}] = \partial_t A_x^{\pm} - \partial_x A_t^{\pm} = \partial_t A_x^{\pm} - \{\mathcal{H}(u,v), A_x^{\pm}\} = 0. \tag{K.68}$$

Need suitable matrix to form a Lax pair.

$$h_c(A^{\pm}) = \mathcal{P} \exp\left(\int_c ds \frac{dc^{\alpha}}{ds} A_{\alpha}^{\pm}\right). \tag{K.69}$$

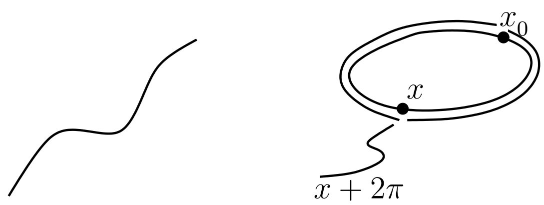

Figure K.2: The.

$$\partial_{\alpha} h_{t,x}(A^{\pm}) := (h_{e_{t,x}}(A^{\pm}))^{-1} h_{e'_{t,x}}(A^{\pm}) \tag{K.70}$$

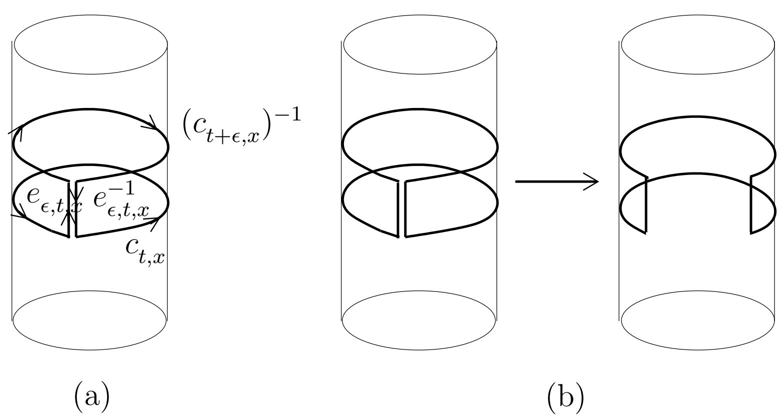

Figure K.3: (a) $c_{t,x} \circ e_{\epsilon,t,x} \circ c_{t+\epsilon,x}^{-1} \circ e_{\epsilon,t,x}^{-1}$. (b) The curve is easily seen to be contractable and hence, because the connection $A^{\pm}$ is flat, its holonomy is equal to one.

On the other hand

$$\partial_{\alpha} h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), A_{\alpha}^{\pm}(t,x)] \tag{K.71}$$

$$2\partial_{[\alpha}\partial_{\beta]}h_{t,x}(A^{\pm}) = [h_{t,x}(A^{\pm}), F^{\pm}_{\alpha\beta}] \tag{K.72}$$

The zero curvature condition is the integrability condition for the equations

$$T^n_{t,x} := \mathrm{Tr}\left([h_{t,x}(A^{\pm})]^n\right) \tag{K.73}$$

K.10.3 Pohlmeyer Charges

$$Z_{\pm}^{\mu_1\mu_2\cdots\mu_N} = [R^{\mu_1\mu_2\cdots\mu_N}(x) + R^{\mu_2\mu_N\cdots\mu_1}(x) + \cdots + \ldots R^{\mu_N\mu_1\cdots\mu_{N-1}}(x)] \tag{K.74}$$

where

$$R^{\mu_1\mu_2\cdots\mu_N}(x) = \int_x^{x+2\pi} dx_1 \int_{x_1}^{x+2\pi} dx_2 \ldots \int_{x_{N-1}}^{x+2\pi} dx_N Y_{\pm}^{\mu_1}(x_1)\ldots Y_{\pm}^{\mu_1}(x_N). \tag{K.75}$$

It can be shown that, together with $J^{\mu\nu}$, the invariants $Z_{\pm}$ provide a *complete* system of invariants for the string in the sense that one can reconstruct $X^{\mu}(t,x)$ up to gauge transformations (parametrzations) and up to translations in the direction of p_{μ}.

K.11 Worldsheet Background Independent Quantization of the Closed Bosonic Nambu-Goto String

K.11.1 Introduction

in order to highlight the differences and similarities in the two different quantization procedures.

K.11.2 Background Independent Quantization of the Nambu-Goto String

Canonical Quantization

For finite dimensional Hilbert spaces the trace of a commutator should be zero

$$< \psi_n|\hat{P}\hat{Q} - QP|\psi_n >= 0 \neq -i\hbar < \psi_n|\mathbf{I}|\psi_n > \tag{K.76}$$

$$
\begin{aligned}
[Q^n, P] &= Q^n P - P Q^n \\
&= Q^{n-1}(QP - PQ) + Q^{n-1} P Q - P Q^n \\
&= iQ^{n-1} + Q^{n-2}(QP - PQ)Q + Q^{n-2} P Q^2 - P Q^n \\
&= \quad \ldots \\
&= i(n-1)Q^{n-1} + PQ^n - PQ^n \\
&= inQ^{n-1}
\end{aligned}
\tag{K.77}
$$

from this we find that

$$
\begin{aligned}
\|inQ^{n-1}\| = n\|Q^{n-1}\| &= \|[Q^n, P]\| \\
&= \|Q^n P - P Q^n\| \\
&\leq 2\|P\|\|Q^n\| \\
&\leq 2\|P\|\|Q\|\|Q^{n-1}\|
\end{aligned}
\tag{K.78}
$$

which implies that $n \leq 2\|P\|\|Q\|$ for any positive integer $n \in \mathbb{N}$.

so as not to overburden the student with functional analytic issues.

Weyl-Heisenburg algebra

$$
e^{i\alpha P} e^{i\beta Q} = e^{i\beta Q} e^{i\alpha P} e^{i\alpha\beta} \tag{K.79}
$$

this is called the Weyl relation.by differentiating with respect to α andβ and then setting $\alpha = \beta = 0$ we recover the original Heisenberg relation.

$$
U(t)V(s) = V(s)U(t)e^{its} \tag{K.80}
$$

Any representation of the Weyl rerlation is unitarily equivalent to a direct sum of copies of the Schrödinger representation.

von Neumann “Mathematical Foundations of Quantum Mechanics”

$3 + 1$ dimensions

determinisic then, it must be background independent (by background of course we mean the abstract world-sheet) .

Quantum superpositions will result in quantum fluctuations in the world-sheet area. world-sheet area operators - do they have a discrete spectrum? What are the properties of the area operators in the standard quantization of string theory?

A dynamical geometric theory - dont we expect distances and areas to be quantized? If so, should not the absence of a well defined momentum operator to be physically desirable feature? Why is the momentum operator well defined in the standard string quantization with its Fock space Hilbert space?

differential geometry is coordinate independent and so active diffeomorphisms are the only kind of diffeomorphisms. Thiemann uses the coordinate independent formulism so when he talks of diffeomorphisms he is always reffering to active diffeomorphisms.

K.12 The Closed, Bosonic LQG-String

and one can discuss different representations of this algebra afterwards taking care of different physical circumstances.

As we have sketched in section ??, a powerful tool to arrive at such representations is the GNS construction. Hence we will be looking at representations that arise from a linear functional ω on $\mathcal{A}$ whihc is invariant under all the automorphisms α.

here we collect the notation:

Given any element a of the abstract algebra $\mathfrak{U}$,

$$\pi_\omega(a)$$

is the corresponding operator on a Hilbert space of some specific representation.
$\omega \sim$

$$f_\mu^{I,k}(x) = k_\mu \chi_I(x) \tag{K.81}$$

$$\{Y_\epsilon^k(I), Y_{\epsilon'}^{k'}(I')\} = \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k'_\nu \int_{S^1} dx((\partial_x\chi_{I'})\chi_I - \chi_{I'}\partial_x\chi_I) \tag{K.82}$$

$$(\partial_x\chi_{I'}(x))\chi_I(x) = \{-\delta(x, b(I')) + \delta(x, f(I')\}\chi_I(x) \tag{K.83}$$

where $b(I')$ and $f(I')$ denote begining and final point of I' respectively, and similarly for the second term of (K.82). It is easily found that

$$\{Y_\epsilon^k(I), Y_{\epsilon'}^{k'}(I')\} = \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k'_\nu\{[\chi_I]_{\partial I'} - [\chi_{I'}]_{\partial I}\} =: \alpha \tag{K.84}$$

The kinematic quantum algebra will be generated from the Weyl elements

$$\hat{W}_\epsilon^k(I) := e^{i\hat{Y}_\epsilon^k} \tag{K.85}$$

for which we require canonical commutation relations induced from

$$[\hat{Y}_\epsilon^k(I), \hat{Y}_{\epsilon'}^{k'}(I')] = i\hbar \widehat{\{Y_\epsilon^k, Y_{\epsilon'}^{k'}\}} \tag{K.86}$$

It follows from the Baker-Cambell-Hausdorff formula that

$$W_\epsilon^k(I) W_{\epsilon'}^{k'}(I') = \exp(-\ell_s^2 \alpha(\epsilon, k, I; \epsilon, k' I')/2) \exp(i[Y_\epsilon^k(I) + Y_{\epsilon'}^{k'}(I')]) \tag{K.87}$$

since

$$Y_\pm^k(I) + Y_\pm^{k'}(I') = Y_\pm^{k+k'}(I \cap I') + Y_\pm^k(I - I') + Y_\pm^{k'}(I' - I) \tag{K.88}$$

we see that a general element of U can be written as a finite, complex linear combination of elemnets of the form

$$W_+^{k_1 \ldots k_M}(I_1, \ldots, I_M) W_-^{l_1 \ldots l_M}(J_1, \ldots, J_M) \tag{K.89}$$

where

$$W_+^{k_1 \ldots k_M}(I_1, \ldots, I_M) = \exp(i[\sum_{m=1} M Y_\pm^{k_m}(I_m)]) \tag{K.90}$$

where all the I_m are non-empty and mutually non-overlapping

The general **Weyl Relations** are then given by

$$\begin{aligned} W_+(s_1) W_-(s_1') W_+(s_2) W_-(s_2') &= e^{-\frac{i}{2}\ell_s^2[\alpha(s_1,s_2) - \alpha(s_1', s_2')]} W_+(s_1 + s_2) W_-(s_1' + s_2') \\ \left[W_+(s_1) W_-(s_2)\right]^* &= W_+(-s_1) W_-(-s_2) \end{aligned} \tag{K.91}$$

$$h_I^k(X) := \exp(2ik_\mu \int_I dx X^{\mu\prime}) = W_+^k(I) W_-^{-k}(I) \tag{K.92}$$

$$F_I^k(\pi) := \exp(2ik_\mu \int_I dx 1_\mu) = W_+^k(I) W_-^k(I) \tag{K.93}$$

We can combine the local gauge group generated by the Virasoro conatraints and the global Poincare group which itself is the semi-direct product of the Lorentz group with the translation group.

$$G := \text{Diff}(S^1) \times \text{Diff}(S^1) \times \mathcal{P} \tag{K.94}$$

K.13 Implimentation of Pohlmeyer Charges

Algebraic properties:

$$
\begin{aligned}
\pi_\omega(Z^{k_1\ldots k_n}_{\pm,M}) &= \pi_\omega(R^{k_2\ldots k_n k_1}_{\pm,M}) + \pi_\omega(R^{k_1\ldots k_n}_{\pm,M}) + \cdots + \pi_\omega(R^{k_n\ldots k_1 k_{n-1}}_{\pm,M}) \\
(R^{k_2\ldots k_n k_1}_{\pm,M}) &= \sum_{1\le m_1\le\cdots\le m_n\le M} \frac{1}{n!}\sum_{\pi\in S_n} \pi_\omega(Y^{k_{\pi(1)}}_\pm(I_{m_{\pi(1)}}))\ldots(Y^{k_{\pi(n)}}_\pm(I_{m_{\pi(n)}}))
\end{aligned} \tag{K.95}
$$

$$
\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2 k_3}_{\pm,M}) \tag{K.96}
$$

up to commutators, to the expected expression $\pi_\omega(Z^{k_1 k_2 k_3}_\pm) + \pi_\omega(Z^{k_2 k_3 k_3}_\pm)$, while the there are two remaining terms that converge in the semiclassical limit to path ordered integrals of the form

$$
\frac{1}{M}\int_{x_1\ge x_2} d^2x Y^{k_1}_\pm(x_2)Y^{k_2}_\pm(x_2)Y^{k_3}_\pm(x_3) \tag{K.97}
$$

and thus vanish in the large M linit, see below. For the general reations we get similar correction terms whose number depends only on n and which are therefore suppressed compared to the correct leading term $M\to\infty$ and $\ell_s\to 0$.

$$
Z^{k_1\ldots k_n}_\pm := k_{\mu_1}\ldots k_{\mu_n} Z^{\mu_1\ldots\mu_n}_\pm \tag{K.98}
$$

$$
\frac{W^k_\pm([a,b]) - W^{-k}_\pm([a,b])}{2i[b-a]} = k_\mu Y^\mu_\pm(\frac{a+b}{2}) + \mathcal{O}((b-a)^2) \tag{K.99}
$$

$$
\pi_\omega(Z^{k_1\ldots k_n}_\pm) = \lim_{\mathcal{P}\to S^1} \pi_\omega(Z^{k_1\ldots k_n}_{\pm,\mathcal{P}}) \tag{K.100}
$$

where $\mathcal{P} = \{I_m; m = 1,\ldots,M := |\mathcal{P}|\}$ is any partition of $[2\pi]$ into consecutive intervals

$$
\begin{aligned}
\pi_\omega(Z^{k_1\ldots k_n}_{\pm,\mathcal{P}}) &= \pi_\omega(R^{k_1\ldots k_n}_{\pm,\mathcal{P}}(0)) + \pi_\omega(R^{k_2\ldots k_n k_1}_{\pm,\mathcal{P}}(0)) + \cdots + \pi_\omega(R^{k_n k_1\ldots k_{n-1}}_{\pm,\mathcal{P}}(0)) \\
\pi_\omega(R^{k_2\ldots k_n k_1}_{\pm,\mathcal{P}}(0)) &= \sum_{m_1=1}^{M}\sum_{m_2=m_1}^{M}\cdots\sum_{m_n=m_{n-1}}^{M} \frac{1}{(2i)^n n!}\sum_{\pi\in S_n}\times \\
&\quad [\pi_\omega(W^{k_{\pi(1)}}_\pm(I_{m_{\pi(1)}})) - \pi_\omega(W^{-k_{\pi(1)}}_\pm(I_{m_{\pi(1)}}))]\cdots\times \\
&\quad \cdots\times[\pi_\omega(W^{k_{\pi(1)}}_\pm(I_{m_{\pi(n)}})) - \pi_\omega(W^{-k_{\pi(1)}}_\pm(I_{m_{\pi(n)}}))]
\end{aligned} \tag{K.101}
$$

$$f_\mu^s(x) = \sum_{I\in\gamma(s)} \chi_I(x) k_\mu^I(s) \tag{K.102}$$

such that $\exp(iY_\pm(f^s)) = W_\pm(s)$.

$$W_\pm^{k+k'}((I_m),(I_{m'})) := \exp(Y_\pm^k(I_m) + Y_\pm^{k'}(I_{m'})) \tag{K.103}$$

K.14 Classical Limit of the Quantum Pohlmeyer Charges

$$W_\pm(s,m_0) := \exp(i\sum_{I\in\gamma} k_\mu^I(s)\int_I dx[\pi_0^\mu(x) \pm X_0^{\mu\prime}]) \tag{K.104}$$

Adjointneess Relatons

The hilbert space can be written as an uncoountable direct sum

$$\mathcal{H}_\omega^\pm = \overline{\oplus_\gamma \mathcal{H}_{\omega,\gamma}^\pm} \tag{K.105}$$

Algebraic Properties

$$\pi_\omega(Y_\pm^k(I)) := \frac{1}{2i}[\pi_\omega(W_\pm^k(I)) - \pi_\omega(W_\pm^{-k}(I))] \tag{K.106}$$

K.15 Worked Examples and Details

Details: Diff .

$$\{Y_\pm^\mu, Y_\pm^\nu\} = \{\eta^{\mu\rho}\pi_\rho \pm X^{\mu\prime}, \eta^{\mu\rho}\pi_\rho \pm X^{\mu\prime}\} \tag{K.107}$$

we get four terms however two of which are $\{\pi_\rho,\pi_\sigma\}$ and $\{X^{\mu\prime}, X^{\nu\prime}\}$ which are both zero, so

$$\begin{aligned}\{Y_\pm^\mu, Y_\pm^\nu\} &= \pm[\eta^{\mu\rho}\partial_y\{\pi_\rho(x), X^\nu(y)\} + \eta^{\nu\sigma}\partial_x\{X^\nu(x), \pi_\sigma(y)\}] \\ &= \pm\alpha'\eta^{\mu\nu}[\partial_y\delta(x,y) - \partial_x\delta(x,y)]\end{aligned} \tag{K.108}$$

Similarly, we find

$$\begin{aligned}\{Y^\mu_\pm, Y^\nu_\mp\} &= \{\eta^{\mu\rho}\pi_\rho \pm X^{\mu\prime}, \eta^{\mu\rho}\pi_\rho \mp X^{\mu\prime}\} \\ &= \pm\alpha'\eta^{\mu\nu}\left[\partial_y\delta(x,y) + \partial_x\delta(x,y)\right]\end{aligned} \tag{K.109}$$

First note:

$$\begin{aligned}\eta_{\mu\nu}\eta_{\sigma\rho}\{Y^\mu_\pm(x)Y^\nu_\pm(x), Y^\sigma_\pm(y)Y^\rho_\pm(y)\} &= 2\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)\{Y^\nu_\pm(x), Y^\sigma_\pm(y)Y^\rho_\pm(y)\} \\ &= 4\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\{Y^\nu_\pm(x), Y^\sigma_\pm(y)\}\end{aligned} \tag{K.110}$$

$$\begin{aligned}\{V_\pm(u), V_\pm(v)\} &= \frac{1}{4\alpha'^2}\int_{S^1} dx \int_{S^1} dy u(x)\eta_{\mu\nu}v(y)\eta_{\sigma\rho}\{Y^\mu_\pm(x)Y^\nu_\pm(x), Y^\sigma_\pm(y)Y^\rho_\pm(y)\} \\ &= \frac{1}{\alpha'^2}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\{Y^\nu_\pm(x), Y^\sigma_\pm(y)\} \\ &= \pm\frac{1}{\alpha'}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\eta^{\nu\rho}[\partial_y\delta(x,y) - \partial_x\delta(x,y)] \\ &= \pm\frac{1}{\alpha'}\int_{S^1} dx(u'v - uv')\eta_{\mu\nu}Y^\mu_\pm(x)Y^\nu_\pm(y) \\ &= V_\pm(u'v - uv').\end{aligned} \tag{K.111}$$

$$\begin{aligned}\{V_\pm(u), V_\mp(v)\} &= \frac{1}{\alpha'^2}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\mp(y)\{Y^\nu_\pm(x), Y^\sigma_\mp(y)\} \\ &= \pm\frac{1}{\alpha'}\int_{S^1} dx \int_{S^1} dy u(x)v(y)\eta_{\mu\nu}\eta_{\sigma\rho}Y^\mu_\pm(x)Y^\nu_\pm(y)\eta^{\nu\rho}[\partial_y\delta(x,y) + \partial_x\delta(x,y)] \\ &= \pm\frac{1}{\alpha'}\int_{S^1} dx\partial_x(uvY^\mu_\pm(x)Y^\nu_\pm(x))\eta_{\mu\nu} \\ &= = 0.\end{aligned} \tag{K.112}$$

(K.108)

$$\begin{aligned}\{Y_\pm(f), Y_\pm(g)\} &= \int_{S^1} dx \int_{S^1} dy f_\mu(x)g_\nu(y)\{Y^\mu_\pm(x), Y^\nu_\pm(y)\} \\ &= \pm\alpha' \int_{S^1} dx \int_{S^1} dy f_\mu(x)g_\nu(y)\eta^{\mu\nu}[\partial_y\delta(x,y) - \partial_x\delta(x,y)] \\ &= \pm\alpha'\eta^{\mu\nu}\int_{S^1} dx\left(\partial_x f_\mu(x)g_\nu(x) - f_\mu\partial_x g_v(x)\right)\end{aligned} \tag{K.113}$$

$$\begin{aligned}\{Y_\pm(f),Y_\mp(g)\} &= \pm\alpha'\eta^{\mu\nu}\int_{S^1}dx\,(\partial_x f_\mu(x)g_\nu(x)+f_\mu\partial_x g_v(x))\\ &= \pm\alpha'\eta^{\mu\nu}\int_{S^1}dx\partial_x(f_\mu(x)g_\nu(x))\\ &= \pm\alpha'\eta^{\mu\nu}[f_\mu(x)g_\nu(x)]_0^{2\pi}=0 \qquad \text{(K.114)}\end{aligned}$$

Details: Constraint algebra im terms of Virasoro constraints.

Check:

(a) $\{V_\pm(u),Y_\mp(f)\}=0$ and $\{V_\pm(u),Y_\pm(f)\}=Y_\pm(uf')$

(b) $[\alpha_u^\pm(t)](Y_\mp(x))=Y_\mp(x),\quad [\alpha_u^\pm(t)](Y_\pm(x))=([]Y_\pm)(x)$

Proof:

(a)

$$\begin{aligned}\{V_\pm(u),Y_\mp(f)\} &= \pm\frac{1}{\alpha'}\int_{S^1}dx\int_{S^1}dy\;u(x)f_\sigma(y)\;\eta_{\mu\nu}\left(Y^\nu_\pm(x)\{Y^\mu_\pm(x),Y^\sigma_\mp(y)\}\right)\\ &= \int_{S^1}dx\int_{S^1}dy\;u(x)f_\sigma(y)\;\eta_{\mu\nu}\eta^{\mu\sigma}Y^\nu_\pm(x)(\partial_y\delta(x,y)+\partial_x\delta(x,y))\\ &= -\int_{S^1}dx[u(x)Y^\nu_\pm(x)\partial_x f_\mu(x)+f_\mu(x)\partial_x(u(x)Y^\nu_\pm(x))]\\ &= -\int_{S^1}dx[u(x)Y^\mu_\pm(x)\partial_x f_\mu(x)-u(x)Y^\nu_\pm(x)\partial_x f_\nu(x)]\\ &= 0. \qquad \text{(K.115)}\end{aligned}$$

Noting first that $\{Y^\mu_\pm Y^\nu_\pm,Y^\sigma_\pm\}=Y^\nu_\pm\{Y^\mu_\pm,Y^\sigma_\pm\}+Y^\mu_\pm\{Y^\nu_\pm,Y^\sigma_\pm\}$

$$\begin{aligned}\{V_\pm(u),Y_\pm(f)\} &= \pm\frac{1}{2\alpha'}\int_{S^1}dx\int_{S^1}dy\;u(x)f_\sigma(y)\;\eta_{\mu\nu}2\left(Y^\nu_\pm(x)\{Y^\mu_\pm(x),Y^\sigma_\pm(y)\}\right)\\ &= \int_{S^1}dx\int_{S^1}dy\;u(x)f_\sigma(y)\;\eta_{\mu\nu}(\eta^{\mu\sigma}Y^\nu_\pm(x)\partial_y\delta(x,y)-\partial_x\delta(x,y))\\ &= \int_{S^1}dxf_\mu(x)\;\partial_x(u(x)Y^\nu_\pm(x))-u(x)Y^\nu_\pm(x)\;\partial_x f_\mu(x)\\ &= -2\int_{S^1}dxY_\pm(x)u(x)\partial_x f_\mu(x)\\ &\equiv -2Y_\pm(uf') \qquad \text{(K.116)}\end{aligned}$$

in the first line we used that $\eta_{\mu\nu}Y^\nu_\pm\{Y^\mu_\pm, Y^\sigma_\pm\} = \eta_{\mu\nu}Y^\mu_\pm\{Y^\nu_\pm, Y^\sigma_\pm\}$, and in the second line we used (K.108).

(b)

From $\{V_\pm(u), Y_\mp(x)\} = 0$ it is obvious that

$$\{V_\pm(u), Y_\mp(x)\}_{(n)} = 0 \quad \text{for all } n \geq 1 \tag{K.117}$$

hence the only non-zero term in the sum (E.7) is the first $n = 0$.

$$\{V_\pm(u), Y^\mu_\pm(x)\}_{(2)} = \{V_\pm(u), \partial_x u(x) Y^\mu_\pm(x)\} = \partial_x(\partial_x u(x))Y^\rho_\pm(x) \tag{K.118}$$

$$\{V_\pm(u), Y_\pm(f)\}_{(n)} = \frac{\partial^n u(x)}{\partial x^n} Y_\pm(x) \tag{K.119}$$

where $u_{(n)}$ is defined inductively as $u^{(n+1)} = \partial_x(u_{(n)})$

$$\{V_\pm(u), Y_\pm(f)\}_{(n)} = \sum_{n=0}^{\infty} \frac{t^n}{n!}\frac{\partial^n u(x)}{\partial x^n} Y^\mu_\pm(x) \tag{K.120}$$

$$[\alpha^+_u(t)](F) = F[(\varphi^u_t)^* Y_+, Y_-], \quad [\alpha^-_u](F) = F[Y_+, \varphi^* Y_-] \tag{K.121}$$

Details: Diff .

Prove:

$$\alpha_a(Y_\pm) \;:=\; \sum_{n=0}^{\infty} \frac{1}{n!}\{a^\mu p_\mu, Y_\pm\}_{(n)} = Y_\pm \tag{K.122}$$

$$\alpha_\Lambda(Y_\pm) \;:=\; \sum_{n=0}^{\infty} \frac{1}{n!}\{\Lambda_{\mu\nu}J^{\mu\nu}, Y_\pm\}_{(n)} = \exp(\Lambda_{\mu\nu}\tau^{\mu\nu}) \cdot Y_\pm \tag{K.123}$$

Proof:

part I

It is obvious from the Poisson brackets that

$$\{\pi_\mu(y), Y^\nu_\pm(x)\} = \pm\alpha' \delta^\nu_\mu \partial_x \delta(x,y), \quad \{X^\mu(y), Y^\nu_\pm(x)\} = \alpha' \eta^{\mu\nu}\delta(x,y). \tag{K.124}$$

$$\begin{aligned}\{p_\mu, Y^\nu_\pm(x)\} &= \frac{1}{\alpha'}\int_{S^1} dy\{\pi_\mu(y), Y^\nu_\pm(x)\}\\ &= \pm\delta^\nu_\mu\,\partial_x\int_{S^1} dy\delta(x,y)\\ &= \pm\delta^\nu_\mu\,\partial_x 1 = 0. \end{aligned} \tag{K.125}$$

$$\begin{aligned}\{J^{\mu\nu}, Y^\rho_\pm(x)\} &= \frac{1}{\alpha'}\int_{S^1} dy\{X^\nu(y)\eta^{\mu\sigma} - X^\mu(y)\eta^{\nu\sigma})\pi_\sigma(y), Y^\rho_\pm(x)\}\\ &= \frac{1}{\alpha'}\int_{S^1} dy(\eta^{\mu\sigma}\{\pi_\sigma(y)X^\nu(y), Y^\rho_\pm(x)\} - \mu\leftrightarrow\nu)\\ &= \frac{1}{\alpha'}\int_{S^1} dy(\eta^{\mu\sigma}(\pi_\sigma(y)\{X^\nu(y), Y^\rho_\pm(x)\} + \{\pi_\sigma(y), Y^\rho_\pm(x)\}X^\nu(y)) - \mu\leftrightarrow\nu)\\ &= \int_{S^1} dy(\eta^{\mu\sigma}(\pi_\sigma(y)\eta^{\nu\rho}\delta(x,y) \pm \delta^\rho_\sigma\partial_x\delta(x,y)X^\nu(y)) - \mu\leftrightarrow\nu)\\ &= \eta^{\mu\rho}(\eta^{\nu\sigma}\pi_\sigma \pm \partial_x X^\nu) - \mu\leftrightarrow\nu\\ &= (\eta^{\mu\rho}Y^\nu_\pm - \eta^{\nu\rho}Y^\mu_\pm)(x). \end{aligned} \tag{K.126}$$

part II

$$\begin{aligned}\{p_\nu, V_\pm(u)\} &= \frac{1}{2\alpha'^2}\int_{S^1} dxu(x)\eta_{\mu\nu}\{p_\nu, Y^\mu_\pm(x)Y^\nu_\pm(x)\}\\ &= \frac{1}{\alpha'}\int_{S^1} dxu(x)\eta_{\mu\nu}\{p_\nu, Y^\mu_\pm(x)\}Y^\nu_\pm(x) = 0. \end{aligned} \tag{K.127}$$

$$\begin{aligned}\{J^{\mu\nu}, V_\pm(u)\} &= \frac{1}{\alpha'}\int_{S^1} dxu(x)Y^\sigma_\pm(x)\{J^{\mu\nu}, Y^\sigma_\pm(x)\}\\ &= \frac{1}{\alpha'}\int_{S^1} dxu(x)Y^\sigma_\pm(x)(\eta^{\mu\rho}Y^\nu_\pm(x) - \eta^{\nu\rho}Y^\mu_\pm(x)\\ &= \frac{1}{\alpha'}\int_{S^1} dxu(x)(Y^\mu_\pm(x)Y^\nu_\pm(x) - Y^\nu_\pm(x)Y^\mu_\pm(x)) \end{aligned} \tag{K.128}$$

$$\begin{aligned}\{\Lambda_{\mu\nu}J^{\mu\nu}, Y_\pm\}_{(2)} &\equiv \{\Lambda_{\mu_2\nu_2}J^{\mu_2\nu_2}, \{\Lambda_{\mu_1\nu_1}J^{\mu_1\nu_1}, Y_\pm\}\}\\ &= \{\Lambda_{\mu_2\nu_2}J^{\mu_2\nu_2}, \eta^{\mu_1\rho_1}Y^{\nu_1}_\pm - \eta^{\nu_1\rho_1}Y^{\mu_1}_\pm\} \end{aligned} \tag{K.129}$$

Details: ‘time” evolution .

Prove:

$$\dot{Y}_{\pm} = \{\mathcal{H}(u,v), Y_{\pm}\} = [(v \pm u)Y^{\mu}_{\pm}]' \tag{K.130}$$

explicitly this equation reads

$$\frac{d}{dt}\int_{S^1} dx f_{\mu}(x) Y^{\mu}_{\pm}(t,x) = \int_{S^1} dx f_{\mu}(x) \frac{\partial}{\partial x}\big(Y^{\mu}_{\pm}(t,x)(u(x) \pm v(x))\big) \tag{K.131}$$

Proof:

We consider $Y_+(f)$ and $Y_-(f)$ separately and using (K.59)

$$\begin{aligned}
\{\mathcal{H}(u,v), Y_+(f)\} &\equiv \{V_+(v+u) + V_-(v-u), Y_+(f)\} \\
&= Y_+((u+v)f') + 0 \\
&= \int_{S^1} dx Y^{\mu}_+(x)(u+v)(x)\partial_x f_{\mu}(x) \\
&= -\int_{S^1} dx f_{\mu}(x)\; \partial_x(Y^{\mu}_+(x)(u+v)(x)) \\
&\equiv -[(v+u)Y^{\mu}_+]'(f),
\end{aligned} \tag{K.132}$$

$$\begin{aligned}
\{\mathcal{H}(u,v), Y_-(f)\} &\equiv \{V_+(v+u) + V_-(v-u), Y_-(f)\} \\
&= 0 + Y_-((u-v)f') \\
&= \int_{S^1} dx Y^{\mu}_-(x)(u-v)(x)\partial_x f_{\mu}(x) \\
&= -\int_{S^1} dx f_{\mu}(x)\; \partial_x(Y^{\mu}_-(x)(u-v)(x)) \\
&\equiv [(v-u)Y^{\mu}_-]'(f)
\end{aligned} \tag{K.133}$$

Details: Diff .

Prove:

(a) $\partial_x h_{t,x}(A^{\pm}) = [h_{t,x}, A^{\pm}_x(t,x)]$

(b) $\partial_t h_{t,x}(A^{\pm}) = [h_{t,x}, A^{\pm}_t(t,x)]$

(c) Using the functional chain rule:

$$\frac{\delta F(x)}{\delta G(z)} \equiv \int dy \, \frac{\delta F(x)}{\delta J(y)} \, \frac{\delta J(y)}{\delta G(z)} \tag{K.134}$$

to prove $\{\mathcal{H}(u,v), h_{t,x}(A^{\pm})\} = \partial_t h_{t,x}(A^{\pm})$

Proof:

(a)

$$\begin{aligned}
\partial_x h_{t,x} &= \partial_x[(h_{e_{t,x}})^{-1} h_{e'_{t,x}}] \\
&= (h_{e_{t,x}})^{-1}\partial_x h'_{e_{t,x}} + (\partial_x (h_{e_{t,x}})^{-1})h'_{e_{t,x}} \\
&= (h_{e_{t,x}})^{-1}\partial_x h'_{e_{t,x}} - (h_{e_{t,x}})^{-1}\partial_x h_{e'_{t,x}}(h_{e_{t,x}})^{-1} h_{e'_{t,x}} \\
&= (h_{e_{t,x}})^{-1}\partial_x h_{e'_{t,x}} A^{\pm} - A^{\pm}(h_{e_{t,x}})^{-1}\partial_x h_{e'_{t,x}} \\
&= [h_{t,x}, A^{\pm}_x(t,x)]
\end{aligned} \tag{K.135}$$

(b)

$c_{t,x} \circ e_{\epsilon,t,x} \circ c^{-1}_{t+\epsilon,x} \circ e^{-1}_{\epsilon,t,x}$ is contractible. Now, zero curvature we make the holonomy of the this contractible curve equal to one:

$$h_{t,x}(A^{\pm})h_{\epsilon,t,x}(A^{\pm})h^{-1}_{t+\epsilon,x}(A^{\pm})h^{-1}_{\epsilon,t,x}(A^{\pm}) = 1 \tag{K.136}$$

Recall that $h_{t,x}(A^{\pm}) := h_{t+\epsilon,x}$ so that we have from (K.136)

$$h_{t,x}(A^{\pm})h_{\epsilon,t,x}(A^{\pm}) = h_{\epsilon,t,x}(A^{\pm})h_{t+\epsilon,x}(A^{\pm}) \tag{K.137}$$

$$h_{\epsilon,t,x}(A^{\pm}) := \mathcal{P}\exp(\int_{t,x}^{t+\epsilon,x} ds A^{\pm}) = \mathbf{1} + \epsilon A^{\pm} + \mathcal{O}(\epsilon^2) \tag{K.138}$$

Putting all this together we have

$$\begin{aligned}
\partial_t h_{t,x}(A^{\pm}) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\, h_{t+\epsilon,x}(A^{\pm}) - \partial_t h_{t,x}(A^{\pm})\,\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(h_{e_\epsilon,t,x}(A^{\pm})^{-1}[h_{e_\epsilon,t,x}(A^{\pm})^{-1}h_{t+\epsilon,x}(A^{\pm})] - h_{t,x}(A^{\pm})\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(h_{e_\epsilon,t,x}(A^{\pm})^{-1}[h_{t,x}(A^{\pm})^{-1}h_{e_\epsilon,t,x}(A^{\pm})] - h_{t,x}(A^{\pm})\right) \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(1 - \epsilon A^{\pm}\right) h_{t,x}(A^{\pm})\left(1 + \epsilon A^{\pm}\right) \\
&= [h_{t,x}, A^{\pm}_t(t,x)]
\end{aligned} \tag{K.139}$$

(c)

$$\begin{aligned}\{\mathcal{H}(u,v), h_{t,x}(A^{\pm})\} &= \int_{S^1} dy\ \{\mathcal{H}(u,v), Y^{\mu}_{\pm}(t,y)\}\ \frac{\delta h_{t,x}(A^{\pm})}{\delta Y^{\mu}_{\pm}(t,y)} \\ &=: \int_{S^1} dy\ [\partial_t Y^{\mu}_{\pm}(t,y)]\ \frac{\delta h_{t,x}(A^{\pm})}{\delta Y^{\mu}_{\pm}(t,y)} \\ &= \partial_t h_{t,x}(A^{\pm}) \end{aligned} \tag{K.140}$$

Details: Diff .

$$\partial_x h_{t,x} = [h_{t,x}, A^{\pm}_x] \tag{K.141}$$

$$\partial_t h_{t,x} = [h_{t,x}, A^{\pm}_t] \tag{K.142}$$

differentiating (K.141) with respect to t gives

$$\begin{aligned}\partial_t\partial_x h_{t,x} &= [\partial_t h_{t,x}, A^{\pm}_x] + [h_{t,x}, \partial_t A^{\pm}_x] \\ &= [[h_{t,x}, A^{\pm}_t], A^{\pm}_x] + [h_{t,x}, \partial_t A^{\pm}_x]\end{aligned} \tag{K.143}$$

where we have substituted (K.142) in the second line of (K.143). Differentiating (K.142) with respect to x gives, doing

$$\partial_x\partial_t h_{t,x} = [[h_{t,x}, A^{\pm}_x], A^{\pm}_t] + [h_{t,x}, \partial_t A^{\pm}_t] \tag{K.144}$$

subtracting (K.144) from (K.143) gives

$$(\partial_t\partial_x - \partial_x\partial_t)h_{t,x} = [h_{t,x}, \partial_t A^{\pm}_x - \partial_x A^{\pm}_t] + [[h_{t,x}, A^{\pm}_t], A^{\pm}_x] - [[h_{t,x}, A^{\pm}_x], A^{\pm}_x] \tag{K.145}$$

now using the Jacobi identity

$$[[h_{t,x}, A^{\pm}_t], A^{\pm}_x] + [[A^{\pm}_x, h_{t,x}], A^{\pm}_t] + [[A^{\pm}_t, A^{\pm}_x], h_{t,x}] \equiv 0 \tag{K.146}$$

in (K.145) we get our final result

$$
\begin{aligned}
(\partial_t\partial_x - \partial_x\partial_t)h_{t,x} &= [h_{t,x}, \partial_t A_x^{\pm} - \partial_x A_t^{\pm} + [A_t^{\pm}, A_x^{\pm}]] \\
&\equiv [h_{t,x}, F_{\alpha\beta}^{\pm}].
\end{aligned} \tag{K.147}
$$

Details: Pohlmeyer Charges.

Pohlmeyer Charges

$$
\{\mathcal{P}^\mu(\sigma), \mathcal{P}^\nu(\sigma')\} = -\eta_{\mu\nu}\delta'(\sigma - \sigma') \tag{K.148}
$$

The modes of the Virasoro constraints are

$$
L_m := \frac{1}{2}\int d\sigma e^{-im\sigma}\eta_{\mu\nu}\mathcal{P}^\mu(\sigma)\mathcal{P}^\nu(\sigma) \tag{K.149}
$$

$$
\{L_m, \mathcal{P}^\mu(\sigma)\} = \left(e^{-im\sigma}\mathcal{P}^\mu(\sigma)\right)' \tag{K.150}
$$

The **Pohlmeyer invariants** $Z^{\mu_1\cdots\mu_N}$ are defined as

$$
Z^{\mu_1\cdots\mu_N}(\mathcal{P}) := \frac{1}{N}\int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \ldots \int_{\sigma^N}^{\sigma^1+2\pi} d\sigma^N \mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\ldots\mathcal{P}^{\mu_N}(\sigma^N) \tag{K.151}
$$

Poisson commute with all L_m.

First note that if $F(\sigma^1, \sigma^2, \cdots, \sigma^N$ is any function which is p-eriodic with period 2π in each of its N arguments, the cyclically permutated path-ordered integral over F is equal to the integral used in (K.151)

$$
\begin{aligned}
\Bigg[\int_{0<\sigma^1<\sigma^2<\cdots<\sigma^N<2\pi} d^N\sigma &+ \int_{0<\sigma^N<\sigma^1<\cdots<\sigma^{N-1}<2\pi} d^N\sigma \\
&+ \cdots + \int_{0<\sigma^{N-1}<\sigma^N<\cdots<\sigma^{N-2}<2\pi} d^N\sigma\Bigg] F(\sigma^1, \sigma^2, \ldots, \sigma^N) \\
&= \int_0^{2\pi} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^1 \cdots \int_{\sigma^{N-1}}^{\sigma^1+2\pi} d\sigma^N F(\sigma^1, \sigma^2, \ldots, \sigma^N)
\end{aligned} \tag{K.152}
$$

This follows...

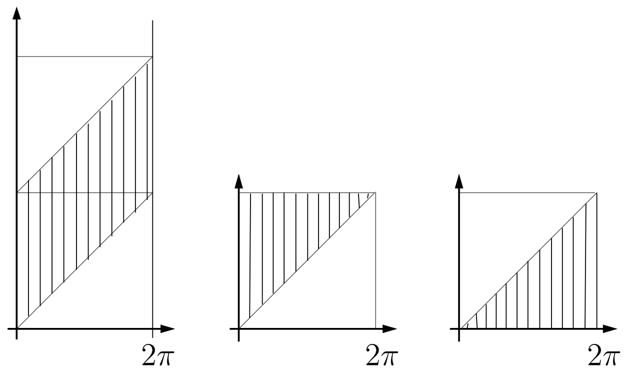

Figure K.4: .

implies cyclic permutation

$$Z^{\mu_1\mu_2\mu_3}(\mathcal{P}) = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 \left(\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\mathcal{P}^{\mu_3}(\sigma^3)+ \right. \\ \left. \mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2)\mathcal{P}^{\mu_2}(\sigma^3) + \mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2)\mathcal{P}^{\mu_1}(\sigma^3)\right). \tag{K.153}$$

It can be used to express the variation as

$$\delta Z^{\mu_1\mu_2\mu_3} = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 \left(\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)\delta\mathcal{P}^{\mu_3}(\sigma^3)+ \right. \\ \left. \mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2)\delta\mathcal{P}^{\mu_2}(\sigma^3) + \mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2)\delta\mathcal{P}^{\mu_1}(\sigma^3)\right). \tag{K.154}$$

because we can cyclically permute the integration variables. But if one now sets $\delta\mathcal{P}(\sigma) = \{L_m, \mathcal{P}^\mu(\sigma)\}$ one gets, using (K.150)

$$\delta Z^{\mu_1\mu_2\mu_3} = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3 \left(\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2)(\xi\mathcal{P}^{\mu_3}(\sigma^3))'+ \right. \\ \left. \mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2)(\xi\mathcal{P}^{\mu_2}(\sigma^3))' + \mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2)(\xi\mathcal{P}^{\mu_1}(\sigma^3))'\right). \tag{K.155}$$

integrating over σ^3, which is straightforward as

$$\int_{\sigma^2}^{\sigma^1+2\pi} d\sigma^3(\xi\mathcal{P}(\sigma^3))' = \left[\xi\mathcal{P}(\sigma^3)\right]_{\sigma^2}^{\sigma^1+2\pi} \tag{K.156}$$

where for brevity we have set $\xi = e^{-im\sigma}$

$$\delta Z^{\mu_1\mu_2\mu_3} = \frac{1}{3}\int_0^{2\imath} d\sigma^1 \int_{\sigma^1}^{\sigma^1+2\pi} d\sigma^2 \qquad \big(\xi\mathcal{P}^{\mu_3}\mathcal{P}^{\mu_1}(\sigma^1)\mathcal{P}^{\mu_2}(\sigma^2) - \mathcal{P}^{\mu_1}(\sigma^1)\xi\mathcal{P}^{\mu_2}\mathcal{P}^{\mu_3}(\sigma^2)$$
$$+ \quad \xi\mathcal{P}^{\mu_2}\mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^2) - \mathcal{P}^{\mu_3}(\sigma^1)\xi\mathcal{P}^{\mu_1}\mathcal{P}^{\mu_2}(\sigma^2)$$
$$+ \quad \xi\mathcal{P}^{\mu_1}\mathcal{P}^{\mu_2}(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^2) - \mathcal{P}^{\mu_2}(\sigma^1)\xi\mathcal{P}^{\mu_1}\mathcal{P}^{\mu_3}(\sigma^2)\big)\,. \tag{K.157}$$

and used the notation $\xi\mathcal{P}^{\mu_3}\mathcal{P}^{\mu_1}(\sigma^1) \equiv \xi(\sigma^1)\mathcal{P}^{\mu_3}(\sigma^1)\mathcal{P}^{\mu_1}(\sigma^1)$, etc.

$$\{L_m, Z^{\mu_1\cdots\mu_N}(\mathcal{P})\} = 0 \tag{K.158}$$

$$W^{\mathcal{P}}[A] = \mathrm{Tr}P\exp\left(\int_0^{2\pi} A\cdot\mathcal{P}(\sigma)\;d\sigma\right) \tag{K.159}$$

$$W^{\mathcal{P}}[A] = \sum_{n=0}^{\infty} Z^{\mu_1\cdots\mu_n}\mathrm{Tr}(A_{\mu_1}\dots A_{\mu_n}) \tag{K.160}$$

Details: Pohlmeyer Charges.

one would like to define

$$\pi_\omega(Z_\pm^{k_1\dots k_n}) := \lim_{|\mathcal{P}|\to S^1}\pi_\omega(Z_{\pm,\mathcal{P}}^{k_1\dots k_n}) \tag{K.161}$$

show that we have

(1) $\pi_\omega(Z_\pm^{k_1\dots k_n}) = 0$ in the weak operator topology on $\mathcal{H}_\omega$ while

(2) $\pi_\omega(Z_\pm^{k_1\dots k_n}) = \infty$ in the strong operator topology on $\mathcal{H}_\omega$, the divergence being of order of $M^{n/2}$.

Details: .

$$\alpha(\epsilon,k,I;\epsilon',k',I') := \epsilon\alpha'\delta_{\epsilon,\epsilon'}\eta^{\mu\nu}k_\mu k_{\mu'}\alpha(I_m,I_{m'}) \tag{K.162}$$

where

$$\alpha(I_m, I_{m'}) = (\chi_{I_{m'}})|_{\partial I_m} - (\chi_{I_m})|_{\partial I_{m'}} = \begin{cases} 0 & m = m' \text{ or } |m-m'| > 1 \\ -1 & m = m'+1 \\ 1 & m' = m+1 \end{cases} \tag{K.163}$$

$$\pi_\omega(W_\pm^{k+k'}((I_m),(I_{m'}))) := \pi_\omega\left(\exp(Y_\pm^k(I_m) + Y_\pm^{k'}(I_{m'}))\right) \tag{K.164}$$

$$\begin{aligned} [\pi_\omega(W_\pm^k(I_m)), \pi_\omega(W_\pm^{k'}(I'))] &= \pi_\omega(W_\pm^k(I))\pi_\omega(W_\pm^{k'}(I_{m'})) - \pi_\omega(W_\pm^{k'}(I_{m'}))\pi_\omega(W_\pm^k(I)) \\ &= (e^{-\ell_s^2\alpha(k,I;k',I')/2} - e^{-\ell_s^2\alpha(k',I';k,I)/2})\ \pi_\omega\left(\exp(Y_\pm^k(I_m) + Y_\pm^{k'}(I_{m'}))\right) \\ &= -2i\sin(-\ell_s^2\alpha(k,I;k',I')/2)\ \pi_\omega(W_\pm^{k+k'}((I_m),(I_{m'}))) \end{aligned} \tag{K.165}$$

$$\begin{aligned} [\pi_\omega(Y_\pi^k(I_m)), \pi_\omega(Y_\pi^l(I_{m'}))] &= -\frac{1}{4}[\pi_\omega(W_\pm^k(I_m)) - \pi_\omega(W_\pm^{-k}(I_{m'})), \pi_\omega(W_\pm^l(I_{m'})) - \pi_\omega(W_\pm^{-l}(I_{m'}))] \\ &= -\frac{1}{4}\Big\{[\pi_\omega(W_\pm^k(I_m)), \pi_\omega(W_\pm^l(I_{m'}))] - [\pi_\omega(W_\pm^k(I_m))\pi_\omega(W_\pm^{-l}(I_{m'}))] \\ &- [\pi_\omega(W_\pm^{-k}(I_{m'})), \pi_\omega(W_\pm^l(I_{m'}))] + [\pi_\omega(W_\pm^{-k}(I_{m'})), \pi_\omega(W_\pm^{-l}(I_{m'}))]\Big\} \end{aligned} \tag{K.166}$$

$$[\pi_\omega(Y_\pi^k(I_m)), \pi_\omega(Y_\pi^l(I'_m))] = -\frac{i}{2}\sin(\mp\ell_s^2[k\cdot l]\alpha(I_m, I_{m'})/2)\times \tag{K.167}$$

Details: Pohlmeyer Charges.

Abbreviating

$$\begin{aligned} F_\pm^{k_1k_2k_3} &:= \pi_\omega(Y_\pm^{k_1}(I_{m_1}))\times \\ &\times [\pi_\omega(Y_\pm^{k_2}(I_{m_2}))\pi_\omega(Y_\pm^{k_3}(I_{m_3})) + \pi_\omega(Y_\pm^{k_3}(I_{m_3}))\pi_\omega(Y_\pm^{k_2}(I_{m_2})) \\ &+ \pi_\omega(Y_\pm^{k_3}(I_{m_2}))\pi_\omega(Y_\pm^{k_2}(I_{m_3})) + \pi_\omega(Y_\pm^{k_2}(I_{m_3}))\pi_\omega(Y_\pm^{k_3}(I_{m_2}))]. \end{aligned} \tag{K.168}$$

$$\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2k_3}_{\pm,M}) = \frac{1}{2}\sum_{m_1=1}^{M}\sum_{1\le m_2\le m_3\le M} F^{k_1k_2k_3}_{\pm}$$
$$= \frac{1}{2}\left(\sum_{1\le m_1\le m_2\le m_3\le M} + \sum_{1\le m_2< m_1\le m_3\le M} + \sum_{1\le m_2\le m_3< m_1\le M}\right) F^{k_1k_2k_3}_{\pm} \tag{K.169}$$

the case $m_1 = m_2 = m_3$ is taken care of in the first summation. We can change the condition $< m_1$ in the summations of the last two terms to include equality, i.e. $\le m_1$, by subtracting off two terms

$$\pi_\omega(Z^{k_1}_{\pm,M})\pi_\omega(Z^{k_2k_3}_{\pm,M}) = \frac{1}{3!}\left(\sum_{1\le m_1\le m_2\le m_3\le M} + \sum_{1\le m_2\le m_1\le m_3\le M} + \sum_{1\le m_2\le m_3\le m_1\le M}\right) 3F^{k_1k_2k_3}_{\pm}$$
$$- \frac{1}{2}\left(\sum_{1\le m_2\le m_3\le M} + \sum_{1\le m_2\le m_3\le M}\right) F^{k_1k_2k_3}_{\pm}. \tag{K.170}$$

Details: Pohlmeyer Charges.

$$\sum_{0\le m_1\le m_2\le M} \ldots = \sum_{m_1,m_2=0}^{M} \Theta(m_2-m_1)\ \ldots \tag{K.171}$$

$$[\pi_\omega \tag{K.172}$$

Details: Classical Limit of the Pohlmeyer Charges.

$$\psi^{\pm}_{\gamma,L,m_0} := \sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s')}\overline{W_\pm(s')}\pi_\omega(W_\pm(s'+s_0))\Omega_\omega \tag{K.173}$$

$$\pi_\omega(W_\pm(s))\pi_\omega(W_\pm(s'+s_0)) = e^{\mp i\ell_s^2\alpha(s,s_0+s')/2}\pi_\omega(W_\pm(s+s'+s_0)) \tag{K.174}$$

$$\begin{aligned}\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0} &= \sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s')}\overline{W_\pm(s',m_0)}e^{\mp i\ell_s^2\alpha(s,s_0+s')/2}\pi_\omega(W_\pm(s+s'+s_0))\Omega_\omega \\ &= \sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'-s)}\overline{W_\pm(s'-s,m_0)}e^{\mp i\ell_s^2\alpha(s,s_0+s'-s)/2}\pi_\omega(W_\pm(s'+s_0))\Omega_\omega\end{aligned} \tag{K.175}$$

where we have used the translation invariance of the lattice to change s' to $s'-s$. From the bilinearity and antisymmetry of the function

$$\alpha(s,s') := [k^I(s)\cdot k^{I'}(s')]\alpha(I,I') \tag{K.176}$$

we have that

$$\begin{aligned}\alpha(s,s+s'+s_0) &= \alpha(s,s_0)+\alpha(s,s')+\alpha(s,-s) \\ &= \alpha(s,s_0)+\alpha(s,s')+0,\end{aligned} \tag{K.177}$$

and hence

$$\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0} = W_\pm(s,m_0)e^{\mp i\ell^2\alpha(s,s_0)/2}\sum_{s'\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'-s)}\overline{W_\pm(s',m_0)}e^{\mp i\ell^2\alpha(s,s')/2}\pi_\omega(W_\pm(s'+s_0))\Omega_\omega \tag{K.178}$$

$$\frac{<\psi^\pm_{\gamma,L,m_0}|\pi_\omega(W_\pm(s))\psi^\pm_{\gamma,L,m_0}>}{\|\psi^\pm_{\gamma,L,m_0}\|^2} \tag{K.179}$$

$$\begin{aligned}\|\psi^\pm_{\gamma,L,m_0}\|^2 &= \sum_{s'\in S_{\gamma,L}}\sum_{s''\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'+s'')}W_\pm(s'')\overline{W_\pm(s')} <\pi_\omega(W_\pm(s''+s_0))\Omega_\omega|\pi_\omega(W_\pm(s'+s_0))\Omega_\omega> \\ &= \sum_{s'\in S_{\gamma,L}}\sum_{s''\in S_{\gamma,L}} e^{-\frac{t}{2}\lambda(s'+s'')}\delta_{s',s''} \\ &= \sum_{s'\in S_{\gamma,L}} e^{-t\lambda(s')}\end{aligned} \tag{K.180}$$