Chapter 2

Algebraic Quantum Gravity, Reduced Phase Quantization and the Master constraint Path Integral

2.1 Introduction

In LQG the Hamiltoian or Master constraint operator necessarily changes the number of degrees of freedom on which the semiclassical state, that it acts on, depends. The fluctuations of the degrees of freedom added by the operator are therefore not suppresed by the semiclassical state.

Semiclassical states are graph dependent, no normalizable semiclassical states depending on all graphs.

Try to use only one fundamental graph.

Algebraic quantum gravity is a new approach to canonical quantum gravity suggested by loop quantum gravity. But in ontrast to loop quantum gravity, the quantum kinematics of algebraic quantum gravity is determined by an abstract $^*-$algebra generated by a countable set of elementary operators labeled by a single algebraic graph with countably infinite number of edges, while in loop quantum gravity the elementary operators are labelled by a collection of embedded graphs with a finite number of edges.

The missing information about the topology and differential structure of the spacetime manifold as well as about the background metric to be approximated is supplied by coherent states.

Definition An oriented algebraic graph is an abstract graph specified by its adjacency matrix α, which is an $N \times N$ matrix. One of its entries α_{IJ} stand for the number of edges that start at vertex I and end at vertex J. The valence of the vertex I is given by

$$v_I = \sum_J (\alpha_{IJ} + \alpha_{JI}).$$

$\square$

In the qunatization procedure, one fixes a specific cubic algebric graph with a countably infinite number of edges and with the valence of each vertex $v_I = 2 \times \dim(\Sigma)$.

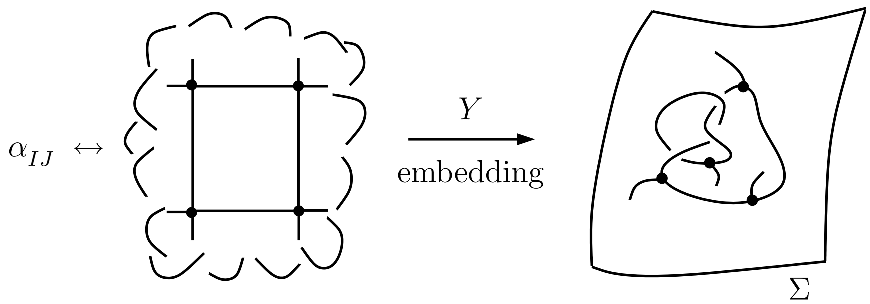

Figure 2.1: The quantum theory is independent of knottedness of a graph, topology and differentiability of a manifold. This information is attained when the abstract graph is embedded into a differential manifold when we do semi-classical analysis.

Given such data, the corresponding coherent state defines a sector in the ITP which can be identified with a usual QFT on the given manifold and background. Thus, AQG contains QFT on all curved spacetimes at once, providing contact with the familiar low energy physics and possibly has something to say about topology change. Topology is semiclassical concept, so AQG may incorporate topology change.

2.1.1 Differences Between AQG and LQG

Topology:

In LQG it must be provided.

In AQG it is absent.

Differenetiable structure:

In LQG it must be provided.

In AQG it is absent.

Size of Hilbert spaces:

For LQG the set Γ of all finite embedded graphs is uncountable.

For AQG the ITP of a countable number of Hilberts spaces of which at least countably infinite many are at least two dimensional is not separable.

The two Hilbert spaces are not directly elated to each other.

In LQG one needs all graphs because the algebra of elementary operators contains the holonomies along all possible paths and those are obtained from a fixed given path through the natural action of the diffeomorphism group.

In AQG the action of the infinitesimal diffeomorphisms preserves the algebraic graph and so there is no need to take all algebraic graphs into account.

AQG possibly can deal with topology change in the sense that it incorporates the semi-classical limits for all topologies while the corresponding states belong to the same Hilbert space.

2.2 Algebraic Quantum Gravity

Given the algebraic graph α, we define a quantum $^*-$algebra by associating with each edge e an element $A(e)$ of a compact, connected, semisimple Lie group G and an element $E_j(e)$ take value in its Lie algebra g. These are subject to the commutation relations

$$[\hat{A}(e), \hat{A}(e')] = 0,$$
$$[\hat{E}_j(e), \hat{A}(e')] = i\hbar \mathcal{Q}^2 \delta_{e,e'} \frac{\tau_j}{2} \hat{A}(e),$$
$$[\hat{E}_j(e), \hat{E}_k(e')] = -i\hbar \mathcal{Q}^2 \delta_{e,e'} f_{jkl} \hat{E}_l(e),$$

and $^*-$relations

$$\hat{A}(e)^\dagger = [\hat{A}(e)^{-1}]^T, \qquad \hat{E}_j(e)^\dagger = \hat{E}_j(e),$$

where $\mathcal{Q}$ stands for the coupling constant, τ_j are the generators in the Lie algebra and f_{jkl} the structure constants.

When we test the semiclassical limit we should specify an embedding map Y which maps the algebraic graph to an embedded one. With this specific embedding, we will see the correspondence between the classical algeba of elementary observables and the quantum $^*-$algebra.

$$A_0(e) := A_0(Y(e)) := \mathcal{P} \exp(\int_{Y(e)} A_0) \tag{2.1}$$

$$E_0(e) := \int_{S_e} \epsilon_{abc} dx^a \wedge dx^b A_0(\rho_e)\ (E_0)^c(x)\ A_0(\rho_e(x))^{-1} \tag{2.2}$$

which we will refer to as holonomies and electric fluxes respectively.

As one can show, if the classical theory is equipped with the following Poisson brackets

$$\begin{aligned} \{(A_0)^j_a(x), (A_0)^k_b(y)\} &= 0 \\ \{(E_0)^j_a(x), (E_0)^k_b(y)\} &= 0 \\ \{(E_0)^j_a(x), (A_0)^k_b(y)\} &= Q^2 \delta^a_b \delta^k_j \delta(x,y) \end{aligned} \tag{2.3}$$

then the quantities () satisfy

2.2.1 AQG Hilbert Space

A natural representation of $\mathfrak{U}$ of AQG is the von Neumann's ITP

$$\mathcal{H}_{AQG} = \otimes_{e \in E(\alpha)} L_2(\mathbf{G}, d\mu_H). \tag{2.4}$$

AQG space of states is the closure of the linear span of elementary states denoted $\otimes_f := \otimes_e f_e$:

$$\otimes_f = \otimes_e f_e, \qquad f_e = f_e(h_e) \in L_2(\mathbf{G}, d\mu_H). \tag{2.5}$$

Two elements $\otimes_f$ and $\otimes_{f'}$ in $\mathcal{H}^{\otimes}$ are said to be strongly equivalent if

$$\sum_e | < f_e, f'_e >_{\mathcal{H}_e} -1|$$

converges. We denote by $[f]$ the strong equivalence class containing $\otimes_f$. It turns out that two elements in $\mathcal{H}^{\otimes}$ are orthogonal if they lie in different strongly equivalence clases. Hence the infinite tensor Hilbert space $\mathcal{H}^{\otimes}$ can be decomposed as a direct sum of the Hilbert subspaces (sectors) $\mathcal{H}^{\otimes}_{[f]}$ which are the closure of strongly equivalence clases $[f]$. Furthermore, even though the Hilbert space $\mathcal{H}^{\otimes}$ is non-separable each sector $\mathcal{H}^{\otimes}_{[f]}$ is separble and has a natural Fock space structure. The basic elements of the quantum algebra are represented on $\mathcal{H}^{\otimes}$ in the obvious way

$$
\begin{aligned}
\hat{A}(e)\otimes_f &:= [A(e)f_e]\otimes[\otimes_{e'\neq e}f_{e'}], \\
\hat{E}_j(e)\otimes_f &:= [i\hbar\mathcal{Q}^2 X^e_f f_e]\otimes[\otimes_{e'\neq e}f_{e'}].
\end{aligned}
\tag{2.6}
$$

For each embedding Y for every sector isomrphic with $\mathcal{H}_{Y(\alpha)}\subset\mathcal{H}_{LQG}$.

2.2.2 Quantum Dynamics

diffeomorphism transformations are not meaningful in the algebraic formulation as the algebraic graph is not embedded in a manifold. We can implement operator corresponding to diffeomorphism generators which have meaning when the algebraic graph is embedded in a manifold.

We employ the (extended) master constraint

$$
\hat{M} := \sum_{v\in V(\alpha)}[\hat{G}_j(v)^\dagger\hat{G}_j(v)+\hat{D}_j(v)^\dagger\hat{D}_j(v)+\hat{H}_j(v)^\dagger\hat{H}_j(v)]
$$

to implememnt algebraic versions of diffeomorphism.

Let $L(v,e_1,e_2)$ denote the set of minimal loops starting at v along e_1 and ending at v along e_2^{-1}. Recall a loop β is said to be minimal provided that there is no other loop within α satisfying the same restrictions with fewer edges transversed.

where $V := \sum_{v\in V(\alpha)} V_v$.

$$
\begin{aligned}
Q_v^{(r)} &:= \frac{1}{E(v)}\sum_{e_1\cap e_2\cap e_3=v}\epsilon_v(e_1,e_2,e_3)\mathrm{Tr}((\hat{A}(e_1)[\hat{A}(e_1)^{-1},V_v^{(r)}]) \\
&\times\ (\hat{A}(e_2)[\hat{A}(e_2)^{-1},V_v^{(r)}])\ (\hat{A}(e_3)[\hat{A}(e_3)^{-1},V_v^{(r)}]))
\end{aligned}
\tag{2.7}
$$

Gauss constraint

$$
\hat{G}_j(v) := \hat{\mathcal{Q}}_v^{1/2}\sum_{e\ at\ v}\hat{E}_j(e)
$$

Spatial diffeomorphism constraint

$$
\begin{aligned}
\hat{D}_j(v) &:= \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \\
&\times \sum_{\beta \in L(v,e_1,e_2)} \mathrm{Tr}(\tau_j [\hat{A}(\beta) - \hat{A}(\beta)^{-1}] \, \hat{A}(e_3) \, [\hat{A}(e_3)^{-1}, \sqrt{\hat{V}_v}])
\end{aligned} \tag{2.8}
$$

Euclidean Hamiltonian constraint (up to overall factor)

$$
\begin{aligned}
\hat{H}_E^{(r)}(v) &:= \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(v, e_1, e_2)|} \\
&\times \sum_{\beta \in L(v,e_1,e_2)} \mathrm{Tr}([\hat{A}(\beta) - \hat{A}(\beta)^{-1}] \, \hat{A}(e_3) \, [\hat{A}(e_3)^{-1}, \hat{V}_v^{(r)}])
\end{aligned} \tag{2.9}
$$

Lorentzian Hamiltonian constraint (up to overall factor)

$$
\begin{aligned}
\hat{T}(v) &:= \frac{1}{E(v)} \sum_{e_1 \cap e_2 \cap e_3 = v} \epsilon_v(e_1, e_2, e_3) \\
&\times \mathrm{Tr}((\hat{A}(e_1)[\hat{A}(e_1)^{-1}, [\hat{H}_E^{(1)}, \hat{V}]]) \, (\hat{A}(e_2)[\hat{A}(e_2)^{-1}, [\hat{H}_E^{(1)}, \hat{V}]]) \\
&\times (\hat{A}(e_3)[\hat{A}(e_3)^{-1}, \sqrt{\hat{V}_v}]))
\end{aligned} \tag{2.10}
$$

$$
\hat{H}(v) = \hat{H}_E^{(1/2)}(v) + \hat{T}(v); \tag{2.11}
$$

When we test the semiclassical limit of these operators we should specify an embedding map Y which maps an algebraic graph to an embedded one. With this specific embedding, we can see the correspondence between the classical algeba of elementary observables and the quantum $^*-$algebra.

2.3 Semi-Classical Analysis

We want to show that AQG is a canonical quantization of classical General Relativity including matter. now, the classical theory is formulated on manifolds diffeomorphic to $\mathbb{R} \times \sigma$ where σ is a three manifold of arbitrary toploogy.

In AQG, as apposed to LQG, semiclassical tools developed for background independent quantum field theories already available can be aplied to the operators encoding the dynamics, and not just to the kinematic operators.

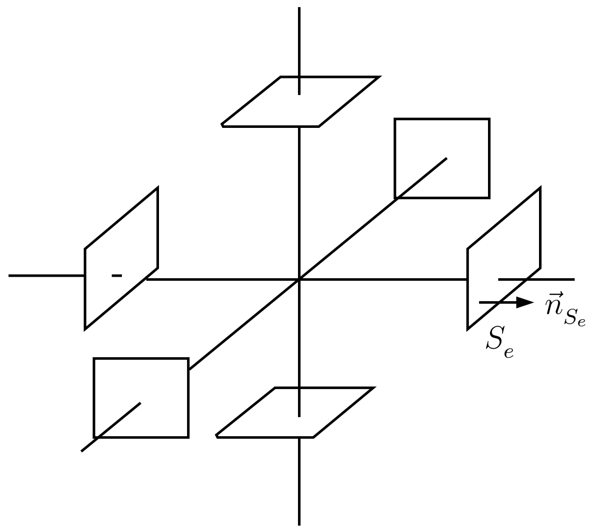

Figure 2.2: Cubic algebraic graph.

Choose Σ, embedding $Y : \alpha \to \Sigma$, surfaces S_e. We choose embedding Y such that γ is dual to a certain triangulation γ^*.

Choose classical field configuration $(A_0(x), E_0(x))$.

The precision with which the semiclassical limit is reached depends on the choice of embedding (its "fineness" with respect to the background metric to be approximated) which is a feature of the state.

Compute

$$g_e(A_0, E_0) := \exp(i\tau_j E_0^j(S_e))\ A_0(Y(e)) \in G^{\mathbb{C}} \tag{2.12}$$

$$\psi_{e;A_0,E_0} := \psi^{t_e}_{g_e;(A_0,E_0)} \tag{2.13}$$

Hence the coherent state on the whole graph is represented by an infinite tensor product state:

$$\psi_{A_0,E_0} := \otimes_e \psi_e, \qquad \psi_e(h_e) := \sum_\pi \dim(\pi) e^{-t_e\lambda_\pi} \chi(g_e h_e^{-1}) \tag{2.14}$$

It is important to keep in mind that (2.13) is a state in the abstrct graph Hilbert space, we just use all the data σ, Y, A_0, E_0, etc in order to construct specific elements of the abstract ITP Hilbert space.

Minimum uncertainty states for $\mathfrak{U}_{AQG}$

$$< \psi_{A_0,E_0}, \hat{A(e)}\psi_{A_0,E_0} > = A_0(e), \qquad < \psi_{A_0,E_0}, \hat{E_j(S)}\psi_{A_0,E_0} > = E_{j0}(S) \tag{2.15}$$

$$< \Delta\hat{A}(e) >< \Delta\hat{E_j}(S) > = \frac{1}{2}| < [\hat{A(e)}, \hat{E_j(S)}] > | \tag{2.16}$$

With the semiclassical state we just constructed, the expectation value of the above (extended) master constraint operator can be calculated and its semiclassical limit tested.

In the non-Abelian case, except in special cases, the volume operator cannot be diagonalized analytically, which is prohibited so far as any explicit calculations involving the quantum dynamics. The substitution of the correct non-Abelian gauge group $SU(2)$ of the canonical formulation of General Relativity by the incorrect Abelian gauge group $U(1)^3$. This is crucial in order that semiclassical calculations can be carried out. The calculation in [222] shows that the result of the exact non-Abelian calculation matches precisely the results of the Abelian approximation.

The result of the calculation is

$$< \psi^t_{\alpha,A_0,E_0}, \hat{\mathbf{M}}\psi^t_{\alpha,A_0,E_0} > = \lim_{t\to\infty} \hat{\mathbf{M}}^{cubic}(A_0, E_0) = \lim_{\epsilon\to\infty} \hat{\mathbf{M}}(A_0, E_0). \tag{2.17}$$

Consequently, it has been shown that Algebraic Qunatum Gravity is a theory of quantum gravity which has the same infinitesimal generators as General Relativity. Thus the problem of whether the semiclassical sector includes General Relativity, that is still unsolved within the framework of Loop Quantum Gravity, is significantly improved in the context of Algebraic Quantum Gravity. Additionaly, it has been shown that the next-to-leading order term of the expectation value which can be interpreted as fluctuations of $\hat{\mathbf{M}}$ are finite.

2.4 Reduced Phase Space Quantization

2.4.1 Brown-Kuchar Framework

It is based on the introduction of a dust field, whose world-line identifies a preferred time-like direction. This direction plays the role of time. The constraints are modified by terms due to the matter field

2.4.2 Relational Framework

Deparametrized Theories

For deparametrized theories it is possible to find canonical coordinates consisting of two sets of canonical pairs (P^I, T_I) and (q^a, p_a) respectively (where the Poisson brackets between elements of the first and second set vanish) such that the constraints C_I can be rewritten in the equivalent form

$$C_I = P_I + h_I(q^a, p_a) \tag{2.18}$$

that is, they no longer depend on the variables T^I. This is a special case and most gauge systems cannot be written in this form. Even with dust General Relativity is a priori not of this form, however, one can reduce it to this form with additional manipulation.

2.4.3 Motivation for Abstract (Algebraic) Graphs

1. $\mathcal{H}$ too large (not separable) due to graph uncountability

2. vast overcounting of the number of degrees of freedom

In free scalar field theory on Minkowski space the quantum configuration space consists of Schwarz distributions rather than smooth functions. The label set of the fields consists of test functions of rapid decrease which are dense in the Hilbert space of square integrable functions on $\mathbb{R}^3$ and there exists a countable orthonormal basis of that Hilbert space.

Since for spatially diffeomorphism invariant operators (on dust space) such as the Hamiltonian $\hat{\mathbf{H}}$ or any other operationally interesting observable (which does not refer directly to the dust label space) the embedding of a graph is immaterial, we can consider the AQG reformulation as an economic description of the reduced LQG in the sense that diffeomorphism related embeddings would lead to isomorphic sectors superselected by these kind of observables.

2.4.4 Semiclassical Analysis

The semiclassical states depend on a differentiable mainfold χ, an embedding Y of the algebraic graph Γ into χ, a cell complex $Y^*(\Gamma)$ dual to $Y(\Gamma)$ as well as a point (A_0, E_0) in the classical reduced phase space.

2.4.5 Standard Model Hamiltonian on Minkowski Spacetime

It is widely accepted that the framework of QFT on curved spacetimes [11] (touched upon in appendix O) should be an excellent approximation to quantum gravity whenever metric fluctuations are small. In particular, when the background spacetime is Minkowski, the standard model must be reproduced.

Require the construction of a minimum energy eigenstate of $\hat{\mathbf{H}}$ which is simultaneously a minimal uncertainty state Ω for all observables and which is peaked around the flat vacuum (no excitations of observable matter) spacetime. One would study metter excitations of Ω and consider matrix elements of $\hat{\mathbf{H}}$ in such states. Hopefully, the resulting matrix elements of an effective matter Hamiltonian on Minkowski space should be close to the Hamiltonian of the standard model on Minkowski space.

It would be interesting to see whether this background indepndent lattice theory which is manifestly UV finite and non perturbative can explore the non perturbative sector of the standard model such as QCD.

2.4.6 RPSQ Scattering Amplitudes

Physical Hamiltonian defines S-Matrix, scattering theory, Feynman rules.

With a physical Hamiltonian $\hat{\mathbf{H}}$ at our disposal it is possible in principal to perform scattering theory, that is, one can compute matrix elements of the time evolution operator $U(\tau) = \exp(i\tau\hat{\mathbf{H}})$. The analytic evaluation of these matrix elements is too difficult but as in ordinary QFT we may use Fermi's Golden rule and expand, for short time intervals τ, the exponential as

$$U(\tau) = 1_{\mathcal{H}} + i\tau\hat{\mathbf{H}} + \mathcal{O}(\tau^2).$$

There are difficulties in computing the matrix elements of $\hat{\mathbf{H}}$ which invloves the square root of a positive self-adjoint operator. However, since in scattering theory initial and final states are excitations over a ground state which we do not know exactly but presumably can approximate by kinematic coherent states, one can invoke the techniques developed for AGQ to expand the square root of the operator around the square root of its expectation value.

2.5 Algebraic Quantum Gravity and Spin Foams

1. How to make contact with the canonical theory

2. How to rempove the triangulation dependence of the models.

Th extended Master constraint defines a new type of spin foam model which one computes heuristically by

$$\psi_{phys} = \int_{\mathbb{R}} dt \ \exp(it\mathbf{M})\psi \tag{2.19}$$

the physical inner product.

If the expression

$$(\psi, \exp(\pm it\mathbf{M})\psi')$$

is analytic in t (for instance if ψ and ψ' are analytic vectors for M) then it can be considered as the analytic continuation $t \mapsto \mp it$ in t

Notice that $(\psi, \exp(\pm it\mathbf{M})\psi')$ vanishes when ψ, ψ' do not belong to the same sector of the ITP. If we now write

$$\exp(-t\mathbf{M}) = [\exp(-t\mathbf{M}/N)]^N$$

and insert $N-1$ resolutions of unity

$$1_{sector} = \sum_s |s >< s|$$

where $|s>$ denotes a countable orthonormal basis for the given sector then we arrive at a path integral formulation of the physical inner product.

$$\begin{aligned}
(\eta(\psi), \eta(\psi'))_{phys} &:= \int_{\mathbb{R}} dt \ (\psi, \exp(it\mathbf{M})\psi') \\
&= \int_0^\infty dt \, [(\psi, \exp(it\mathbf{M}\psi')) + (\psi, \exp(-it\mathbf{M}\psi'))]
\end{aligned} \tag{2.20}$$

Let us restrict ourselves to the case that the semiclassical theories we want to quantize have compact σ. The appropriate sector of the ITP is then based on the vector $\otimes_{\mathbf{1}} = \otimes_e \mathbf{1}$ where $\mathbf{1}$ is the constant function equal to one. An orthonormal basis for this sector is given by spin ntwork functions defined over all finite subgraphs of the algebraic graph. Then (2.20) defines a concrete spin foam model of General Relativity for which the issue of triangulation dependenceis absent. Details are the subject of future publications by Thiemann et al.

2.6 Master Constraint Path Integral

2.6.1 Introduction

By using the standard technique of skeletonisation and coherent state path-integral, we derive a path-integral

For all the physical models simpler than quantum gravity, the group averaging technique gives the correct physical Hilbert space. We assume this is also the case for the AQG master constraint operator $\hat{\mathbf{M}}$ defined in (), in applying the group averaging technique.

It turns out to be more convenient to use the coherent states in the skeletonisation.

Consider the computation for the group averaging inner product and derive a path-integral formula from the master constraint operator $\hat{\mathbf{M}}$ on the sector $\mathcal{H}_{AL}$.

$$\langle \eta(f) | \eta(f') \rangle := \lim_{\epsilon \to 0} \frac{1}{\ell_P^2} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} \left\langle f \left| \exp[\frac{i}{\ell_P^2} \tau (\hat{\mathbf{M}} - \epsilon)] \right| f' \right\rangle_{Kin} \tag{2.21}$$

2.6.2 Coherent State Path-Integrals

Resolution of identity with coherent states

Single-step amplitude

Evaluation of the overlap function

Computation of the matrix element

The path-integral formula with a classical action in the exponential?

2.6.3 Formulation of the Master Constraint Path-Integral

Resolution of identity with coherent states

insert in the resolution of identity with coherent states

$$
\begin{aligned}
&\left\langle f \left| [1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon)]^N \right| f' \right\rangle_{Kin} \\
= \; & \int dg_N \ldots dg_1 dg_0 \left\langle \tilde{\psi}^t_{g_N} \left| 1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon) \right| \tilde{\psi}^t_{g_{N-1}} \right\rangle_{Kin} \times \\
& \times \left\langle \tilde{\psi}^t_{g_{N-1}} \left| 1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon) \right| \tilde{\psi}^t_{g_{N-2}} \right\rangle_{Kin} \ldots \left\langle \tilde{\psi}^t_{g_1} \left| 1+\frac{i\tau}{\ell_P^2 N}(\hat{\mathbf{M}}-\epsilon) \right| \tilde{\psi}^t_{g_0} \right\rangle_{Kin} \\
& \times \left\langle f \middle| \tilde{\psi}^t_{g_N} \right\rangle_{Kin} \left\langle \tilde{\psi}^t_{g_0} \middle| f' \right\rangle_{Kin}
\end{aligned} \tag{2.22}
$$

where the measure

$$
dg = \prod_{e\in E(\gamma)} \frac{d^3p(e)dh(e)}{t^3} + \mathcal{O}(t^\infty)
$$

up to an overall constant.

Single-step amplitude

First let us compute the single-step amplitude

$$
< \tilde{\psi}^t_{g_i} | 1+\frac{i\tau}{\ell_p^2 N}(\mathbf{M}-\epsilon) | \tilde{\psi}^t_{g_{i-1}} >_{Kin} .
$$

The overlap function $< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin}$ is sharply peaked at $g_i = g_{i-1}$ in a Gaussian fashion (with width $\sqrt{t}$).

In the semiclassical limit $t \to 0$ we have

$$
\begin{aligned}
& \lim_{t\to 0} < \tilde{\psi}^t_{g_i} | 1+\frac{i\tau}{\ell_p^2 N}(\mathbf{M}-\epsilon) | \tilde{\psi}^t_{g_{i-1}} >_{Kin} \\
= \; & \lim_{t\to 0} \left[1+\frac{i\tau}{\ell_p^2 N} \frac{< \tilde{\psi}^t_{g_i} | \mathbf{M}-\epsilon | \tilde{\psi}^t_{g_{i-1}} >_{Kin}}{< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin}} \right] < \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin} \\
= \; & \lim_{t\to 0} \left[1+\frac{i\tau}{\ell_p^2 N} \frac{< \tilde{\psi}^t_{g_i} | \mathbf{M}-\epsilon | \tilde{\psi}^t_{g_i} >_{Kin}}{< \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_i} >_{Kin}} \right] < \tilde{\psi}^t_{g_i} | \tilde{\psi}^t_{g_{i-1}} >_{Kin}
\end{aligned} \tag{2.23}
$$

As the master constraint has the correct semiclassical limit

$$\begin{aligned} &\lim_{t\to 0} < \tilde{\psi}^t_{g_i}|1+\frac{i\tau}{\ell_p^2 N}(\mathbf{M}-\epsilon)|\tilde{\psi}^t_{g_{i-1}} >_{Kin} \\ = \; &\lim_{t\to 0}\left\{1+\frac{i\tau}{\ell_p^2 N}(\mathbf{M}[g_i]-\epsilon)\right\} < \tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}} >_{Kin} \end{aligned}$$

$$< \tilde{\psi}^t_{g_i}|\mathbf{M}|\tilde{\psi}^t_{g_i} >$$

Evaluation of the overlap function

Let us evaluate the overlap function

$$< \tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}} >_{Kin} = \prod_{e\in E(\gamma)} \frac{< \psi^t_{g_i(e)}|\psi^t_{g_{i-1}(e)} >}{\|\psi^t_{g_i(e)}\|\,\|\psi^t_{g_{i-1}(e)}\|}$$

where

$$\psi^t_{g_i(e)} = \sum_{2j_e=0}^{\infty}(2j_e+1)e^{-tj_e(j_e+1)/2}\chi_{j_e}\big(g(e)h^{-1}(e)\big) \tag{2.24}$$

is the complexifier coherent state on the edge e. If we set $n = 2j+1$

$$< \psi^t_{g_i}|\psi^t_{g_{i-1}} >_{Kin} = \frac{e^{t/4}}{2\sinh(z_{i,i-1})}\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i,i-1}} \tag{2.25}$$

Therefore

$$< \tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}} >_{Kin} \;=\; \prod_{e\in E(\gamma)} \frac{\sqrt{|\sinh(z_i)\sinh(z_{i-1})|}}{\sinh(z_{i,i-1})} \frac{\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i,i-1}}}{\sqrt{\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_i}}\sqrt{\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i-1}}}} \tag{2.26}$$

We now use the Poisson resumation formula

$$\sum_{n\in\mathbb{Z}} f(ns) = \frac{1}{s}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} dxe^{2\pi inx/s}f(x) \tag{2.27}$$

with $s = \sqrt{t_e}$. This gives

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} ne^{-tn^2}e^{nz_{i,i-1}} &= \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}}\sqrt{t}ne^{tn^2}e^{\sqrt{t}nz_{i,i-1}/\sqrt{t}} \\
&= \frac{1}{t}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} dxe^{2\pi inx/\sqrt{t}}xe^{-x^2}e^{xz_{i,i-1}/\sqrt{t}} \\
&= \frac{1}{t}\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} dxxe^{-x^2}e^{(2\pi in+z_{i,i-1})x/\sqrt{t}} \\
&= C\frac{1}{t^{3/2}}\sum_{n\in\mathbb{Z}}(2\pi in+z_{i,i-1})^2e^{(2\pi in+z_{i,i-1})^2/4t} \qquad (2.28)
\end{aligned}$$

only the term with $n = 0$

$$\begin{aligned}
<\tilde{\psi}^t_{g_i}|\tilde{\psi}^t_{g_{i-1}}>_{Kin} &\approx \prod_{e\in E(\gamma)}\frac{\sqrt{|\sinh(z_i)\sinh(z_{i-1})|}}{\sinh(z_{i,i-1})}\frac{z_{i,i-1}e^{z^2_{i,i-1}/t}}{\sqrt{|z_ie^{z_i^2/t}|}\sqrt{|z_{i-1}e^{z^2_{i-1}/t}|}} \\
&= \prod_{e\in E(\gamma)}\frac{z_{i,i-1}\sqrt{|\sinh(z_i)\sinh(z_{i-1})|}}{\sqrt{|z_iz_{i-1}|}\sinh(z_{i,i-1})}e^{[z^2_{i,i-1}-z_i^2-z^2_{i-1}]/t} \qquad (2.29)
\end{aligned}$$

Computation of the matrix element

$$\begin{aligned}
&\left\langle f\left|\exp[\frac{i}{\ell_p^2}\tau(\mathbf{M}-\epsilon)]\right|f'\right\rangle_{Kin} \\
= &\int dg_N\cdots dg_0 <\tilde{\psi}^t_{g_N}|\tilde{\psi}^t_{g_{N-1}}>_{Kin}\cdots<\tilde{\psi}^t_{g_1}|\tilde{\psi}^t_{g_0}>_{Kin} \\
&\times\exp\left[i\frac{\tau}{\ell_p^2N}\big(\mathbf{M}[g_N]-\epsilon+tF^t(g_N,g_{N-1})\big)\right]\cdots\exp\left[i\frac{\tau}{\ell_p^2N}\big(\mathbf{M}[g_1]-\epsilon+tF^t(g_1,g_0)\big)\right] \\
&\times\overline{f(g_N)}f'(g_0)
\end{aligned}$$

where $f(g) :=<\tilde{\psi}^t_g|f>_{Kin}$.

The product of overlap functions

$$
\begin{aligned}
&< \tilde{\psi}^t_{g_N}|\tilde{\psi}^t_{g_{N-1}} >_{Kin}< \tilde{\psi}^t_{g_{N-1}}|\tilde{\psi}^t_{g_{N-2}} >_{Kin} \cdots < \tilde{\psi}^t_{g_1}|\tilde{\psi}^t_{g_0} >_{Kin} \\
= & \prod_{e\in E(\gamma)} \frac{z_{N,N-1}\sqrt{|\sinh(p_N)\sinh(p_{N-1})|}}{\sqrt{p_N p_{N-1}}\sinh(z_{N,N-1})} e^{[z^2_{N,N-1}-\frac{1}{2}p^2_N-\frac{1}{2}p^2_{N-1}]/t} \\
& \times \frac{z_{N-1,N-2}\sqrt{|\sinh(p_{N-1})\sinh(p_{N-2})|}}{\sqrt{p_{N-1}p_{N-2}}\sinh(z_{N-1,N-2})} e^{[z^2_{N-1,N-2}-\frac{1}{2}p^2_{N-1}-\frac{1}{2}p^2_{N-2}]/t} \cdots \\
& \cdots \times \prod_{e\in E(\gamma)} \frac{z_{1,0}\sqrt{|\sinh(p_1)\sinh(p_0)|}}{\sqrt{p_1 p_0}\sinh(z_{1,0})} e^{[z^2_{1,0}-\frac{1}{2}p^2_1-\frac{1}{2}p^2_0]/t} \\
= & \prod_{e\in E(\gamma)} \frac{\sinh(p_N)}{p_N} e^{-p^2_N/2t} \frac{\sinh(p_{N-1})}{p_{N-1}} e^{-p^2_{N-1}/t} \cdots \frac{\sinh(p_1)}{p_1} e^{-p^2_1/t} \frac{\sinh(p_0)}{p_0} e^{-p^2_0/2t} \\
& \times \frac{z_{N,N-1}}{\sinh(z_{N,N-1})} e^{z^2_{N,N-1}/t} \cdots \frac{z_{1,0}}{\sinh(z_{1,0})} e^{z^2_{1,0}/t} \sqrt{\frac{p_0 p_N}{\sinh(p_0)\sinh(p_N)}}
\end{aligned}
$$

Combining the results

$$
\begin{aligned}
& \left\langle f \left| \exp[\frac{i}{\ell^2_p}\tau(\mathbf{M}-\epsilon)] \right| f' \right\rangle_{Kin} \\
= & \int \prod_{e\in E(\gamma)} \prod_{i=0}^{N} \frac{dh_i d^3 p_i}{t^3} \frac{\sinh(p_i)}{p_i} e^{-p^2_i/t} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} e^{z^2_{k,k-1}/t} \\
& \times \exp\left[i\frac{\tau}{\ell^2_p k}\big(\mathbf{M}[g_k]-\epsilon+tF^t(g_k,g_{k-1})\big)\right] \overline{f(g_N)} f'(g_0)
\end{aligned} \tag{2.30}
$$

Define Lie algebra variables θ_k such that

$$
h_k = e^{\theta_k\cdot\tau/2},
$$

$z_{k,k-1}$ is found to be

$$
\begin{aligned}
\cosh(z_{k,k-1}) &= \frac{1}{2}tr(h_{k-1}h^\dagger_k e^{-i(p_k+p_{k+1})\cdot\tau/2}) \\
&= \frac{1}{2}tr(e^{i\left[-(p_k+p_{k+1})+i(\theta_k-\theta_{k-1})\right])\cdot\tau/2}) \\
&= \cosh\left(\sqrt{[\frac{-(p_k+p_{k-1})}{2}+\frac{(\theta_k+\theta_{k-1})}{2}]^2}\right)
\end{aligned} \tag{2.31}
$$

Therefore,

$$z_{k,k-1}^2 - \frac{1}{2}p_k^2 - \frac{1}{2}p_{k-1}^2 = -\frac{1}{4}\left[(p_k - p_{k-1})^2 + (\theta_k - \theta_{k-1})^2 + 2i(p_k + p_{k-1})\cdot(\theta_k - \theta_{k-1})\right] \quad (2.32)$$

We insert this result back into (2.30) and obtain

$$\begin{aligned}
&\left\langle f \left| \exp[\frac{i}{\ell_p^2}\tau(\mathbf{M} - \epsilon)] \right| f' \right\rangle_{Kin} \\
= &\int \prod_{e\in E(\gamma)} \prod_{i=0}^{N} \frac{dh_i d^3 p_i}{t^3} \frac{\sinh(p_i)}{p_i} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} \overline{f(g_N)} f'(g_0) \\
&\times \exp\left\{ -i\frac{(p_k + p_{k-1})\cdot(\theta_k - \theta_{k-1})}{2t} - \frac{1}{4t}\left[(p_k - p_{k-1})^2 + (\theta_k - \theta_{k-1})^2\right] + \right. \\
&\left. + i\frac{\tau}{\ell_p^2 k}[\mathbf{M}[g_k] - \epsilon + tF^t(g_k, g_{k-1})] \right\} \qquad (2.33)
\end{aligned}$$

The path-integral formula with a classical action in the exponential?

We make the following approximations:

(i) We assume the fluctuation F^t is negligible.

(ii)

With the above approximations, (2.33) simplifies to

$$\begin{aligned}
&\left\langle f \left| \exp[\frac{i}{\ell_p^2}\tau(\mathbf{M} - \epsilon)] \right| f' \right\rangle_{Kin} \\
= &\int \prod_{e\in E(\gamma)} \prod_{i=0}^{N} \frac{dh_i d^3 p_i}{t^3} \left[\prod_{j=0}^{N} \frac{\sinh(p_j)}{p_j} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})} \right] \overline{f(g_N)} f'(g_0) \\
&\times \prod_{k=1}^{N} \exp\left\{ -i\frac{(p_k + p_{k-1})}{2t}\cdot(\theta_k - \theta_{k-1}) + i\frac{\tau}{\ell_p^2 k}[\mathbf{M}[g_k] - \epsilon + tF^t(g_k, g_{k-1})] \right\} \\
&\qquad (2.34)
\end{aligned}$$

which is the analogue of the path-integral of the "Master Action" on the continuum

$$
\begin{aligned}
S_{Master} &= \lim_{N\to\infty}\sum_{k=1}^{N}\frac{\tau}{N}\frac{(p_k+p_{k-1})}{2t}\cdot\frac{(\theta_k-\theta_{k-1})}{\tau/N} - \lim_{N\to\infty}\sum_{k=1}^{N}\frac{\tau}{N}\frac{1}{\ell_p^2}\mathbf{M}[p,\theta] \\
&= \int_{t_i}^{t_f} dt\int_{\Sigma} p_j\partial_t\theta^j - \int_{t_i}^{t_f} dt\mathbf{M}[p,\theta] \qquad (2.35)
\end{aligned}
$$

we insert (2.34) back into () and obtain

$$
\begin{aligned}
& < \eta(f)|\eta(f') > \\
= & \lim_{\epsilon\to 0}\frac{1}{\ell_p^2}\int_{-\infty}^{\infty}\int \prod_{e\in E(\gamma)}\prod_{i=0}^{N}\frac{dh_i d^3p_i}{t^3}\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\overline{f(g_N)}f'(g_0) \\
& \times\prod_{k=1}^{N}\exp\left\{-\frac{i}{\ell_p^2}\left[\sum_{e\in E(\gamma)} a^2\frac{(p_k+p_{k-1})}{2}(\theta_k-\theta_{k-1})^a - \frac{\tau}{N}(\mathbf{M}[g_k]-\epsilon)\right]\right\} \qquad (2.36)
\end{aligned}
$$

2.6.4 LQG Master Constraint Path-Integral in terms of the Original Hamiltonian Constraints

The τ integral involved is

$$
\int_{-\infty}^{\infty}\frac{d\tau}{2\pi}\prod_{k=1}^{N}\exp\left\{\frac{\tau}{N}(\mathbf{M}[g_k]-\epsilon)\right\} = \delta(\sum_{k=1}^{N}\mathbf{M}[g_k]-\epsilon)
$$

So that

$$
\begin{aligned}
& < \eta(f)|\eta(f') > \\
= & \lim_{\epsilon\to 0} N\int \prod_{e\in E(\gamma)}\prod_{i=0}^{N}\frac{dh_i d^3p_i}{t^3}\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\overline{f(g_N)}f'(g_0) \\
& \times\prod_{k=1}^{N}\exp\left[-\frac{i}{\ell_p^2}\sum_{e\in E(\gamma)} a^2\frac{(p_k+p_{k-1})}{2}(\theta_k-\theta_{k-1})^a\right]\delta(\sum_{k=1}^{N}\mathbf{M}[g_k]-\epsilon) \qquad (2.37)
\end{aligned}
$$

The master constraint has the following expression:

$$\mathbf{M}=\sum_{v\in V(\gamma)}\frac{G_{j,v}}{V_v^{1/2}}\frac{G_{j,v}}{V_v^{1/2}}+\frac{D_{j,v}}{V_v^{1/2}}\frac{D_{j,v}}{V_v^{1/2}}+\frac{H_v}{V_v^{1/2}}\frac{H_v}{V_v^{1/2}} \tag{2.38}$$

We need to deal with

$$\delta\big(\sum_{k=1}^{N}\sum_{v\in V(\gamma)}\left\{\frac{G_{j,v}}{V_v^{1/2}}[g_k]\frac{G_{j,v}}{V_v^{1/2}}[g_k]+\frac{D_{j,v}}{V_v^{1/2}}[g_k]\frac{D_{j,v}}{V_v^{1/2}}[g_k]+\frac{H_v}{V_v^{1/2}}[g_k]\frac{H_v}{V_v^{1/2}}[g_k]\right\}-\epsilon\big) \tag{2.39}$$

Fisrt consider the type of integral

$$\begin{aligned}
& \lim_{\epsilon\to 0}\int dx_1 dx_2\cdots dx_N\delta(\sum_{i=1}^{N}x_ix_i-\epsilon)f(\vec{x}) \\
= & \lim_{\epsilon\to 0}\int dx_2\cdots dx_N\frac{1}{2\sqrt{\epsilon-\sum_{i=2}^{N}x_ix_i}}\left[f(x_1=\sqrt{\epsilon-\sum_{i=2}^{N}x_ix_i})+f(x_1=-\sqrt{\epsilon-\sum_{i=2}^{N}x_ix_i})\right] \\
= & \left\{\lim_{\epsilon\to 0}\int dx_2\cdots dx_N\frac{1}{2\sqrt{\epsilon-\sum_{i=2}^{N}x_ix_i}}\right\}f(\vec{x}=0) \\
\equiv & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)f(\vec{x}=0) \\
= & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\int dx_1dx_2\cdots dx_N\prod_{i=1}^{N}\delta(x_i)f(\vec{x})
\end{aligned} \tag{2.40}$$

Consider the type of integral

$$\begin{aligned}
& \lim_{\epsilon\to 0}\int dt_1\cdots dx_M\delta(\sum_{i=1}^{N}x_i(\vec{t})x_i(\vec{t})-\epsilon)f(t_1,\ldots,t_M) \\
= & \lim_{\epsilon\to 0}\int dx_1\ldots dx_N dt_{N+1}\ldots dt_M\frac{1}{\det(\partial x/\partial t)}\delta(\sum_{i=1}^{N}x_ix_i-\epsilon)\tilde{f}(x_1,\cdots,x_N,t_{N+1},\cdots,t_M) \\
= & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\int dx_1\ldots dx_N dt_{N+1}\ldots dt_M\frac{1}{\det(\partial x/\partial t)}\prod_{i=1}^{N}\delta(x_i)\tilde{f}(x_1,\cdots,x_N,t_{N+1},\cdots,t_M) \\
= & \lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\int dt_1\cdots dx_M\prod_{i=1}^{N}\delta(x_i(\vec{t}))f(t_1,\ldots,t_M)
\end{aligned} \tag{2.41}$$

Therefore we obtain the result:

$$\lim_{\epsilon\to 0}\delta(\sum_{i=1}^{N} x_i(\vec{t})x_i(\vec{t})-\epsilon)=\lim_{\epsilon\to 0}\mathcal{N}(\epsilon)\prod_{i=1}^{N}\delta(x_i) \tag{2.42}$$

$$\begin{aligned}
&\prod_{k=1}^{N}\prod_{v\in V(\gamma)}\delta^3\big(\frac{G_{j,v}}{V_v^{1/2}}[g_k]\big)\delta^3\big(\frac{D_{j,v}}{V_v^{1/2}}[g_k]\big)\delta^3\big(\frac{H_v}{V_v^{1/2}}[g_k]\big)\\
=&\prod_{k=1}^{N}\prod_{v\in V(\gamma)} V_v^{7/2}\delta^3\big(G_{j,v}[g_k]\big)\delta^3\big(D_{j,v}[g_k]\big)\delta^3\big(H_v[g_k]\big)\\
=&\prod_{k=1}^{N}\prod_{v\in V(\gamma)} d^3\Lambda_{v,k}d^3N_{v,k}dN_{v,k}\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)}V_v^{7/2}[g_k]\right]\\
&\times\prod_{k=1}^{N}\exp\left\{-\frac{i}{\ell_p^2}\sum_{v\in V(\gamma)}(\Lambda^i_{v,k}G_{i,v}[g_k]+N^i_{v,k}D_{i,v}[g_k]+N_{v,k}H_k g_k])\right\}
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&<\eta(f)|\eta(f')>\\
=&\lim_{\epsilon\to 0}\mathcal{N}(\epsilon)N\int\prod_{e\in E(\gamma)}\left[\prod_{i=0}^{N}\frac{dh_i d^3p_i}{t^3}\right]\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)}d^3\Lambda_{v,k}d^3N_{v,k}dN_{v,k}\right]\\
&\times\left[\prod_{k=1}^{N}\prod_{v\in V(\gamma)}V_v^{7/2}[g_k]\right]\left[\prod_{j=0}^{N}\frac{\sinh(p_j)}{p_j}\prod_{k=1}^{N}\frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right]\sqrt{\frac{p_0p_N}{\sinh(p_0)\sinh(p_N)}}\\
&\times\prod_{k=1}^{N}\exp\left\{-\frac{i}{\ell_p^2}\sum_{e\in E(\gamma)}a^2<p>_k\frac{(\theta_k-\theta_{k-1})^a}{\Delta T_k}\Delta T_k\right.\\
&\qquad\left.+\sum_{v\in V(\gamma)}(\Lambda^i_{v,k}G_{i,v}[g_k]+N^i_{v,k}D_{i,v}[g_k]+N_{v,k}H_k g_k])\right\}\times\overline{f(g_N)}f'(g_0)
\end{aligned} \tag{2.44}$$

Here the graph γ is a finite graph with finite number of edges and vertices.

$$
\begin{aligned}
& \mathcal{D}\mu[g,\Sigma,N^a,N] \\
= \ & \prod_{e\in E(\gamma)} \left[\prod_{i=0}^{N} \frac{dh_i d^3p_i}{t^3}\right] \left[\prod_{k=1}^{N} \prod_{v\in V(\gamma)} d^3\Lambda_{v,k} d^3N_{v,k} dN_{v,k}\right] \left[\prod_{k=1}^{N} \prod_{v\in V(\gamma)} V_v^{7/2}[g_k]\right] \\
& \times \left[\prod_{j=0}^{N} \frac{\sinh(p_j)}{p_j} \prod_{k=1}^{N} \frac{z_{k,k-1}}{\sinh(z_{k,k-1})}\right] \sqrt{\frac{p_0 p_N}{\sinh(p_0)\sinh(p_N)}}
\end{aligned}
\tag{2.45}
$$

We choose a reference vector Ω and define the path-integral representation of the group averaging physical inner product

$$
<\eta(f)|\eta(f')>_\Omega = \lim_{\epsilon\to 0} \frac{\int_{\mathbb{R}} dt < f|e^{it(\mathbf{M}-\epsilon)}|f'>}{\int_{\mathbb{R}} dt < \Omega|e^{it(\mathbf{M}-\epsilon)}|\Omega>}
\tag{2.46}
$$