Fluid Dynamics

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Chapter 1

Fluid Mechanics

1.1 Introduction

1.1.1 Definition of a Fluid

The defining property of a fluidis that it cannot withstand, shearing forces, however small, without sustained motion. Since both gases and liquids have this property, both are fluids and subject to a unified treatment as far as their macroscopic motion is concerned.

The rate at which the fluid deforms continuously depends not only on the magnitude of the applied force but also on a property of the fluid called its viscosity - its resistance to deformation and flow.

Solids also deform when sheared, but a position of equilibrium is soon reached in which elestic forces induced by the deformation of the solid exactly counterbalance the applied shear force, and further deformation ceases.

1.1.2 Continuum

There are a large number of atoms or molecules in any given volume. For example the number of gas molecules in $(1\mu cm)^3$ is 2.687×10^7. Therefore gases and liquids can be considered as continuum, for example water appears continuous, wind appears continuous (even though we cant see it).

1.1.3 Physical Properties

What are the physical properties of a fluid? Density, viscocity and temperature and pressure, velocity and acceration.

$$\begin{aligned} \text{Density} \quad & \varrho = \varrho(\vec{x}, t) \\ \text{Voscosity} \quad & \mu = \mu(\vec{x}, t) \\ \text{Temperature} \quad & T = T(\vec{x}, t) \\ \text{Voscosity} \quad & p = p(\vec{x}, t) \\ \text{Temperature} \quad & \vec{u} = \vec{u}(\vec{x}, t). \end{aligned} \tag{1.1}$$

1.2 Kinematics

Kinematics involves the description of a fluids motion without considering forces.

1.2.1 Velocity

Velocity is the rate of change of position with time, i.e.

$$\vec{u} = \frac{d\vec{x}}{dt}. \tag{1.2}$$

There are two possible descriptions:

Lagrangian

(a) In the Lagrangian description we follow a particle which starts at position $\vec{x} = \vec{x}_0$ at time $t = t_0$ and we follow it and observe its velocity $\vec{q}(\vec{x}_0; t, t_0)$. In this case $\vec{x}$ and t are dependent variables.

Newton's laws and conservation of mass and energy, apply directly to each particle. However, fluid flow is a continuum phenomenon and it is not possibly to to track each "particle". Let's consider a second way of describing fluid motion.

Eulerian

(b) In the Eulerian description we define the velocity of the fluid particle which is situated at $\vec{x}$ at time t to be $\vec{u}(\vec{x}, t)$.

$$\vec{u}(\vec{x}, t) = (U_1(x, y, z, t), U_1(x, y, z, t), U_1(x, y, z, t)) \tag{1.3}$$

with $\vec{x}$ and t as independent variables.

Here one is not concerned about the location or velocity of any particular particle, but rather about the velocity of whatever particle happens to be at a particular location of interest at a particular time.

1.2.2 Acceleration of Fluid Particle

(a) Lagrangian description

$$\vec{a} = (\vec{x}_0; t, t_0) = \frac{\partial \vec{q}}{\partial t}. \tag{1.4}$$

(b) Eulerian.

The velocity field $\vec{u}$ gives the velocity of the fluid at someplace in the fluid. The acceleration is not simply given by $\partial\vec{u}/\partial t$ as this represents the rate at which the velocity $\vec{u}(x, y, z)$ changes at a fixed point in space. What we need is the rate at which the velocity changes for a particular piece of fluid. After time dt the piece of fluid will have moved to a new position. The particle at position $(\vec{x}, t)$ moves to $(\vec{x} + d\vec{x}, t + dt)$ where $d\vec{x} = \vec{u}(\vec{x}, t)dt$

$$\vec{a}(\vec{x}, t) = \frac{\vec{u}(\vec{x} + d\vec{x}, t + dt) - \vec{u}(\vec{x}, t)}{dt} \tag{1.5}$$

With use of the Taylor expansion we have

$$\vec{u}(\vec{x} + d\vec{x}, t + dt) = \vec{u}(\vec{x}, t) + \frac{\partial \vec{u}}{\partial t}dt + \frac{\partial \vec{u}}{\partial x_i}dx_i \tag{1.6}$$

so that

$$\vec{a}(\vec{x}, t) = \frac{\partial \vec{u}}{\partial t} + \frac{dx_i}{dt}\frac{\partial \vec{u}}{\partial x_i} \tag{1.7}$$

which can be written as

$$\vec{a}(\vec{x}, t) = \underbrace{\frac{\partial \vec{u}}{\partial t}}_{local\ change} + \underbrace{\vec{u} \cdot \nabla \vec{u}}_{advective}. \tag{1.8}$$

We define the substantial derivative by

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla. \tag{1.9}$$

1.3 General Deformations

Consider an infintesimal volume element of the fluid at time t_0. After an infintesimal time later δt this volume element will have suffered a deformation that can be considered the combination of a translation, linear expansion, rotation and shear (see fig 1.1).

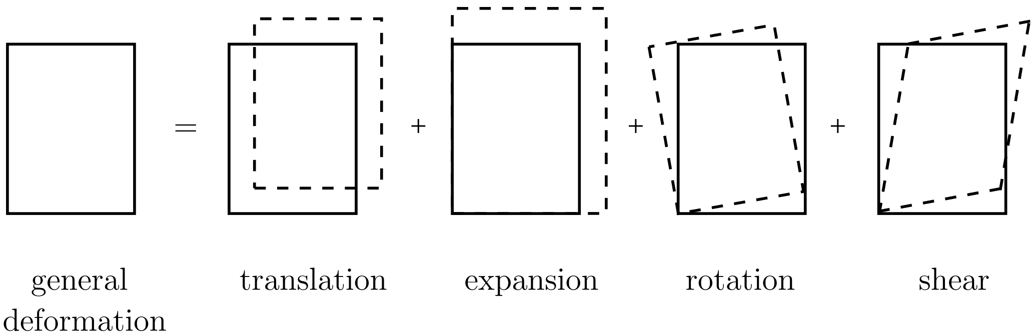

Figure 1.1: Rotation.

A translation is when the position of a fluid element is changed without changing its size or shape, see fig 2. After a time δt the fluid element will be shifted in the $x-$direction an amount $u_x \delta t$ and shifted in the $y-$direction an amount $u_y \delta t$ - see fig 1.2.

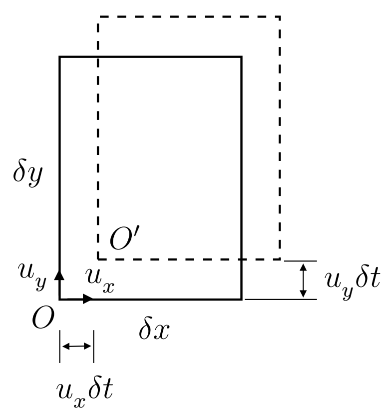

Figure 1.2: Pure translation of fluid element.

Expansion

Let us consider the case where we have linear expansion in the $x-$direction. At time t_0 the $x-$component of the velocity at position $(x + \delta x, y)$ where δx is small is given by the first order Taylor expansion

$$u(x+\delta x, y, z, t_0) = u(x, y, z, t_0) + \frac{\partial u}{\partial x}\delta x.$$

The situation we consider is shown in fig 1.3.

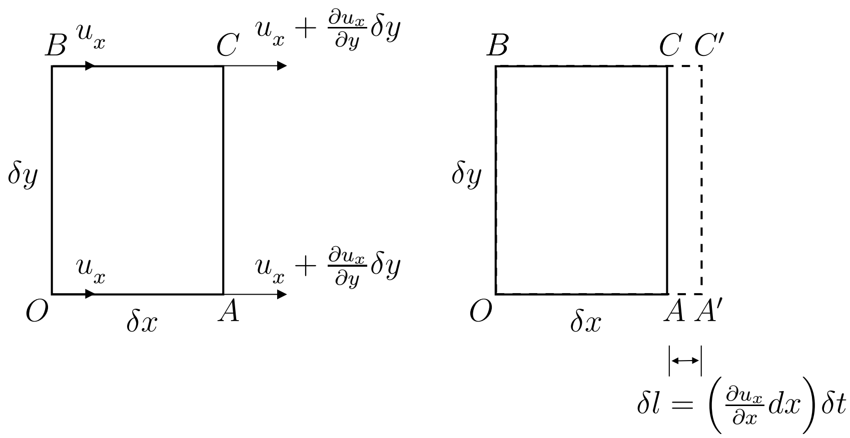

Figure 1.3: An expansion of fluid element in the $y-$direction.

The change in volume after time δt will then be

$$\begin{aligned}
\delta V &= (\delta x + \delta l)\delta y \delta z - \delta x \delta y \delta z \\
&= \delta l \delta y \delta z \\
&= [u(x+\delta x, y, z)\delta t - u(x, y, z)\delta t]\delta y \delta z \\
&= \left(\frac{\partial u_x}{\partial x}\delta x\right)\delta t(\delta y \delta z) \qquad (1.10)
\end{aligned}$$

and the rate of change in volume per unit volume is then

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{\partial u_x}{\partial x}.$$

If velocity gradients $\partial u_y/\partial y$ and $\partial u_z/\partial z$ are also present, it can easily be shown that

$$\frac{1}{\delta V}\frac{d\delta V}{dt} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \nabla \cdot \vec{u}.$$

Rotation and shear

We now consider the effects of rotation and shear. For simplicity we consider motion in the $x - y$ plane, however these considerations are easily extended to the more general case. The situation is the one shown in fig 1.4. We have that the change in angle of the line OA is

$$\delta\alpha = \left(\frac{\partial u_y}{\partial x}\delta x\right)\delta t$$

and the change in angle of the line OB is

$$\delta\beta = \left(\frac{\partial u_x}{\partial y}\delta y\right)\delta t.$$

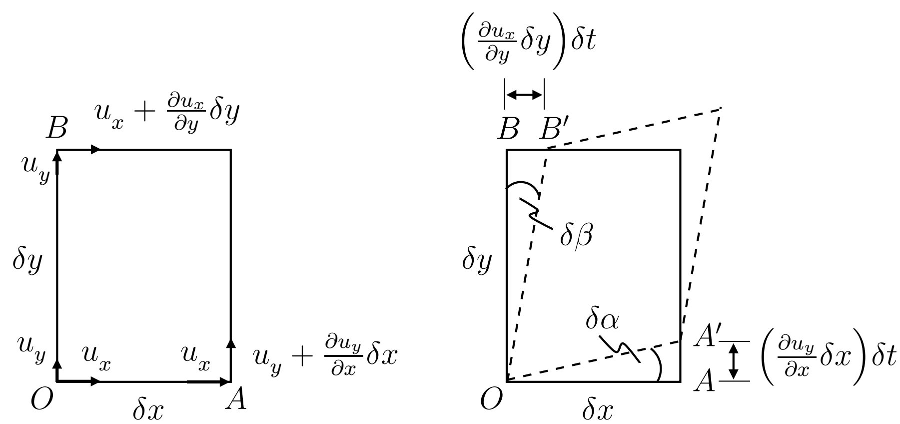

Figure 1.4: Rotation and shear.

Note that if $\partial u_y/\partial x$ is positive, ω_{OA} will be counterclockwise. If $\partial u_x/\partial y$ is positive then ω_{OB} will be clockwise.

Rotation

We consider pure rotations first where after time δt the element has rotated by a small angle θ in which case

$$\begin{aligned} \vartheta &= \delta\alpha = \frac{\frac{\partial u_y}{\partial x}\delta x\delta t}{\delta x} = \frac{\partial u_y}{\partial x}\delta t \\ &= -\delta\beta = -\frac{\frac{\partial u_x}{\partial y}\delta x\delta t}{\delta x} = -\frac{\partial u_x}{\partial y}\delta t. \end{aligned} \quad (1.11)$$

Rotations are obviously volume preserving.

Shear

Shear type strains are where the initial fluid element is changed into a parallelogram.

$$\begin{aligned} \vartheta &= \delta\alpha = \frac{\partial u_y}{\partial x}\delta t \\ &= \delta\beta = \frac{\partial u_x}{\partial y}\delta t. \end{aligned} \tag{1.12}$$

The area of the parallelogram is $|OA||OB|\sin(\pi - 2\delta\alpha)$ or $|OA||OB|\cos(2\delta\alpha)$. Taylor expanding we get

$$|OA||OB|\left(1 - \frac{1}{2!}(2\delta\alpha)^2 + \ldots\right)$$

so to first order the area is unchanged.

Extracting rotation and shear

When we have general 'angular' deformations we extract the part that corresponds to rotation by taking the average of the angular velocities with counterclockwise rotation considered positive,

$$\begin{aligned} \omega &= \frac{1}{2}\left(\frac{\delta\alpha}{\delta t} - \frac{\delta\beta}{\delta t}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \end{aligned} \tag{1.13}$$

We extract the shear part from

$$\begin{aligned} \sigma &= \frac{1}{2}\left(\frac{\delta\alpha}{\delta t} + \frac{\delta\beta}{\delta t}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right) \end{aligned} \tag{1.14}$$

For pure rotation (1.14) is zero, but for pure shear we have $\delta\alpha = \delta\beta$ and (1.13) is zero.

1.4 Rate of Strain Tensor

$$u_i(\vec{x}+\vec{r}) = u_i(\vec{x}) + r_j \frac{\partial u_i}{\partial x_j} + \mathcal{O}(r^2) \tag{1.15}$$

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.16}$$

$$\begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \tag{1.17}$$

1.5 Rate of Rotation Tensor

Define

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = -\omega_{ji}. \tag{1.18}$$

This has three independent components. Behave like a vector. Only happens in three dimensions.

We define the vorticity as

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} \tag{1.19}$$

which is written in vector notation as $\nabla \times \vec{\omega}$.

The rate of rotation tensor can be written in terms of $\vec{\omega}$

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{1.20}$$

We can now see clearly the meaning of the term $\omega_{ij}r_j$ in the Taylor series expansion for the displacements

$$\omega_{ij}r_j = \frac{1}{2}\epsilon_{ijk}\omega_j r_k \tag{1.21}$$

which in vector notation is

$$\frac{1}{2}\vec{\omega} \times d\vec{r}. \tag{1.22}$$

Say $\vec{\omega} = \omega\vec{k}$ and $\vec{r} = r\vec{i}$

$$\frac{1}{2}\vec{\omega} \times d\vec{r} = \frac{1}{2}\omega r\vec{j}.$$

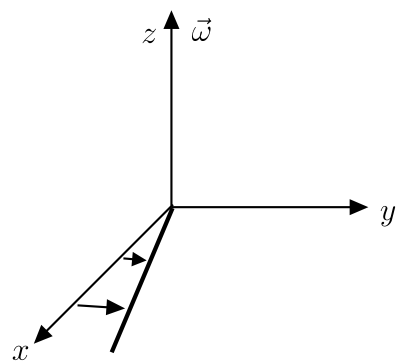

Figure 1.5: Rotation.

$$u_i(\vec{x}+\vec{r}) = u_i(\vec{x}) + e_{ij}r_j + \frac{1}{2}\epsilon_{ijk}\omega_j r_k + \mathcal{O}(r^2) \tag{1.23}$$

1.5.1 Decomposition of $\partial u_i/\partial x_j$

The tensor $\partial u_i/\partial x_j$ can be written

$$\begin{aligned}
\frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \\
&= e_{ij} + \omega_{ij} \\
&= \frac{1}{3}e_{kk}\delta_{ij} + \left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right) + \frac{1}{2}\epsilon_{ijk}\omega_k \\
&= \frac{1}{3}\theta\delta_{ij} + \gamma_{ij} + \frac{1}{2}\epsilon_{ijk}\omega_k
\end{aligned} \tag{1.24}$$

where we have defined the expansion parameter θ as e_{kk}. From (1.24)

$$\sum_i \frac{\partial u_i}{\partial x_i} = \theta$$

where we have used $\sum_i \gamma_{ii} = 0$. The term

$$\frac{1}{3}\theta\delta_{ij}$$

in (1.24) represents the isotropic expansion part, that is it is the volume change 'averaged' in each direct equally. Also from (1.24) we have

$$\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \epsilon_{ijk}\omega_k$$

If $i \neq j$ then

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}.$$

represents pure shear. The non-isotropic part of the expansion is

$$\gamma_{ii} = e_{ii} - \frac{1}{3}\theta \qquad \text{(no summation implied)}$$

as e_{ii} (no summation implied) is the volume change in the $ith-$direction and is minus the ith part of isotropic expansion.

1.6 Conservation of Mass

1.6.1 Continuity Equation

Consider an arbitrary volume V bounded by a surface S so the mass in V is

$$\int_V \varrho dV$$

and conservation of mass means the rate of change of mass equals the mass flux across the boundary

$$\frac{\partial}{\partial t}\int_V \varrho dV = -\int_S \varrho\vec{u}\cdot\vec{n}dS \tag{1.25}$$

The minus sign is there as mass flux leaing the volume would reduce the mass. Using the Divergence Theorem

$$\int_V \left[\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \vec{u}) \right] dV = 0 \tag{1.26}$$

since the volume is arbitrary the integrand must vanish

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \vec{u}) = 0 \tag{1.27}$$

This is the differential form of the continuity equation.

1.6.2 Direct Derivation

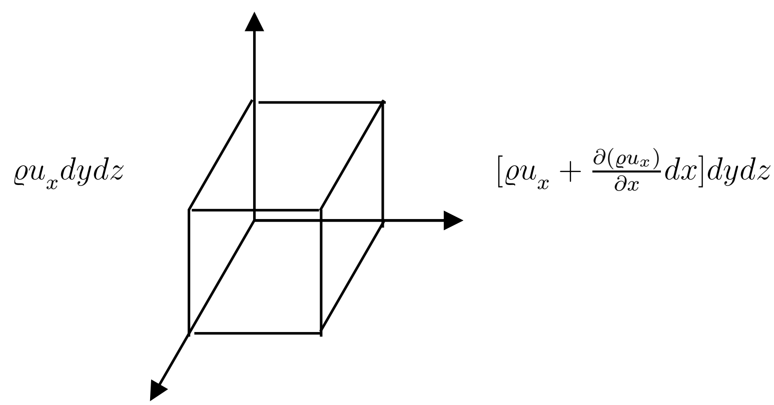

Figure 1.6: Uniform flow over a flat plate $n = 1$.

Consider the mass flux through the x faces

$$\begin{aligned} x_{flux} &= \left[\varrho u_x + \frac{\partial}{\partial x}(\varrho u_x) dx \right] dydz - \varrho u_x dydz \\ &= \frac{\partial}{\partial x}(\varrho u_x) dxdydz. \end{aligned}$$

similarly for the y and z faces

$$\begin{aligned} y_{flux} &= \frac{\partial}{\partial y}(\varrho u_y) dxdydz \\ z_{flux} &= \frac{\partial}{\partial z}(\varrho u_z) dxdydz \end{aligned} \tag{1.28}$$

The total net mass outflux must balance the rate of decrease of mass within the cubical element which is

$$-\frac{\partial \varrho}{\partial t} dxdydz$$

Combining the above equations, the balance of mass outflux with mass decrease is then described by the equation

$$\left[\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho u_x) + \frac{\partial}{\partial y}(\varrho u_y) + \frac{\partial}{\partial z}(\varrho u_z)\right] dxdydz = 0.$$

Or

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x}(\varrho u_x) + \frac{\partial}{\partial y}(\varrho u_y) + \frac{\partial}{\partial z}(\varrho u_z) = 0.$$

which is just (1.27).

1.6.3 Simplifications

Written out the continuity equation is

$$\frac{\partial \varrho}{\partial t} + \varrho \nabla \cdot \vec{u} + \vec{u} \cdot \nabla \varrho = 0 \tag{1.29}$$

For constant ϱ the continuity equation linear.

This can be written in terms of the substantial derivative as

$$\frac{D\varrho}{Dt} + \varrho \nabla \cdot \vec{u} = 0$$

A simplification would be steady flow in which case the continuity equation becomes

$$\nabla \cdot (\varrho \vec{u}) = 0.$$

If the density is constant $\varrho = \varrho_0$, that is, the fluid is incompressible the continuity equation becomes

$$\nabla \cdot \vec{u} = 0 \tag{1.30}$$

So $\vec{u}$ divergence free and hence solenoidal, that is, there is a vector $\vec{A}$ such that

$$\vec{u} = \nabla \times \vec{A}.$$

We exclusively deal with incompressible flows in these notes. Note that this equation can be expressed in terms of the rate of strain tensor as $e_{kk} = 0$.

1.7 Pathlines and Streamlines

1.7.1 Pathlines

The locus of an actual particle $\vec{X}(t)$ is called a pathline. Pathlines are associated with the Lagrangian description. The corresponding acceleration of the particle is then

$$\frac{d\vec{X}}{dt} = \underbrace{\vec{q}}_{Lagrangian} = \underbrace{\vec{u}(\vec{x}, t)}_{Eulerian} . \tag{1.31}$$

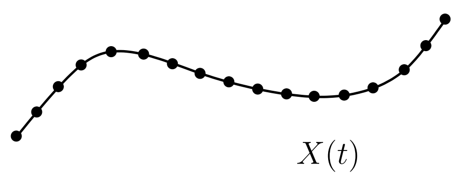

Figure 1.7: A pthline.

Pathlines are associated with the Lagrangian description.

1.7.2 Streamlines

We can draw lines which are always tangent to the fluid velocity at an instant in time, these are called streamlines. With steady flow (this is when the velocity of fluid doesn't change at any point with time) the streamlines dont change with time and in this case the particle's path coincides with the streamline. If we have unsteady flow the streamlines vary from instant to instant and in general the particle's path does not coindide with a streamline as the particle can end up on another streamline. That is with time unsteady flow pathlines and streamlines are not necessarily coincident.

Streamlines are defined by the three simulataneous equations written as

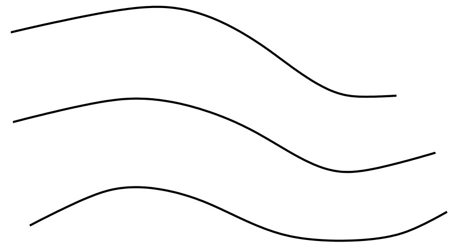

Figure 1.8: Streamlines.

$$\frac{d\vec{x}}{ds} = \lambda \vec{u}(\vec{x}, t) \tag{1.32}$$

or

$$\frac{dx}{ds}\vec{x} + \frac{dy}{ds}\vec{y} + \frac{dz}{ds}\vec{z} = \lambda(u_x\vec{x} + u_y\vec{y} + u_z\vec{z}) \tag{1.33}$$

where s is a parameter along the streamline. The parameter s should not be confused with time, the above equations are integrated while keeping time fixed. The resulting curves give us the streamlines at an instant in time.

The streamline passing through a closed curve which does not lie on a surface generated by streamlines form a tubular surface. The fluid contained in such surface is called a stream tube.

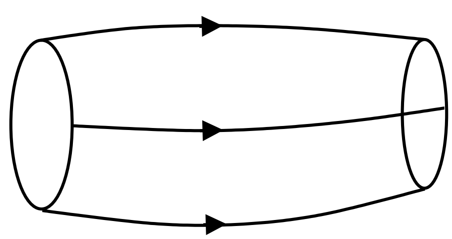

Figure 1.9: Stream tube.

1.8 Streamfunction and the Velocity Potential

For incompressible flow

$$\nabla \cdot \vec{u} = 0 \tag{1.34}$$

Here we breifly introduce the method using the so-called streamfucntion. We will go into this in more detail in future chapters. The streamfunction method is introduced for 2D motion. It was extended to axisymmetric flow in thre dimensions - then called Stoke's streamfucntion.

For steady flows pathlines and streamlines coincide.

1.8.1 Streamfunction in 2D

Imcompressibility condition

The imcompressibility condition

$$\nabla \cdot \vec{u} = 0. \tag{1.35}$$

In two dimensions ($\varrho = Const.$) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.36}$$

If we choose

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{1.37}$$

then (1.35) is automatically satisfied:

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) = 0.$$

Streamfunction giving Streamlines

We now prove that $\psi = Const.$ is a streamline in steady flow. First note tat in general for two infinitesimally close points $a = (x, y)$ and $b = (x + dx, y + dy)$ we have from calculus that

$$\begin{aligned} \psi(x + dx, y + dy) - \psi(x, y) &= \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \\ &= \nabla \psi \cdot d\vec{r}. \end{aligned} \tag{1.38}$$

where $d\vec{r}$ is the vector frompoint a to point b. If a and b are points on a curve defined by $\psi = C$ where C is a constant, then $d\vec{r}$ is tangent to the curve $\psi = C$ at a and

$$0 = \psi(x + dx, y + dy) - \psi(x, y) = \nabla\psi \cdot d\vec{r}.$$

Implying that the vector $\nabla\psi$ is normal to the curve $\psi = C$. If we can show that everywhere $\vec{u}\cdot\nabla\psi = 0$, using the formula for $\vec{u}$ in terms of ψ given by (1.37), then we will have proved the result. This easily follows

$$\vec{u} \cdot \nabla\psi = \frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial x} + \left(-\frac{\partial\psi}{\partial x}\right)\frac{\partial\psi}{\partial y} = 0. \tag{1.39}$$

Flux Through a curve

We have already defined the flux through a surface by summing over $\vec{u} \cdot \vec{n}dS$ where dS is the area of an infintesimal area element with unit normal vector $\vec{n}$. The two dimensional analogue of the volume flux as the sum of terms

$$\vec{u} \cdot \vec{n}dl$$

along a given curve and where dl is the length of an infintesimal line element.

We first consider the simple case of the volume flux Φ through the curve $x = Const.$ in the $x-$direction from say a to b,

$$\Phi = \int_a^b u dy = \int_a^b \frac{\partial\psi}{\partial y} dy = \psi|_b - \psi|_a. \tag{1.40}$$

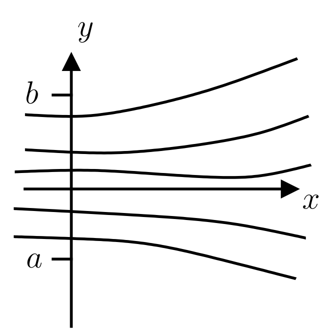

Figure 1.10: Stress.

Now consider any curve Γ from point P to point Q. The volume flux through a curve is the integral of the dot product of the flow velocity vector (u, v) and the normal to the curve. If (dx, dy) is the infinitesimal displacement from one point of the curve to another, the unit normal $\vec{n}$ to the curve there multiplied by dl is $(dy, -dx)$, i.e.

$$\vec{n}dl = dy\vec{e}_x - dx\vec{e}_y. \tag{1.41}$$

This is easily seen from

$$(dx\vec{e}_x + dy\vec{e}_y) \cdot (dy\vec{e}_x - dx\vec{e}_y) = 0$$

and

$$\begin{aligned}(dy\vec{e}_x - dx\vec{e}_y) \cdot (dy\vec{e}_x - dx\vec{e}_y) &= dx^2 + dy^2 = dl^2 \\ &= (\vec{n}dl) \cdot (\vec{n}dl).\end{aligned}$$

The vector (1.41) points 'outward' as can be understood from considering fig (1.8).

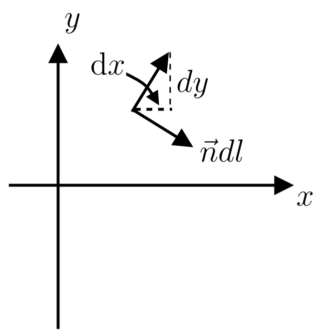

Figure 1.11: Say two points on a curve have tangent vector $dx\vec{e}_x + dy\vec{e}_y$ then $dy\vec{e}_x - dx\vec{e}_y$ points 'outward' - this can be seen here following from the fact that dx and dy are positive.

The volume flux through the curve Γ with end points P and Q is

$$\begin{aligned}\Phi &= \int_P^Q \vec{u} \cdot \vec{n}dl \\ &= \int_P^Q (u\vec{e}_x + v\vec{e}_y) \cdot (dy\vec{e}_x - dx\vec{e}_y) \\ &= \int_P^Q (udy - vdx) \\ &= \int_P^Q \left(\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy\right) \\ &= \int_P^Q \nabla\psi \cdot d\vec{r}\end{aligned} \tag{1.42}$$

where we have used (1.37). By the theorem proved in section B.4 we then have

$$\Phi = \psi|_P - \psi|_Q. \tag{1.43}$$

Note that the flux only depends on the values ψ take at the end points, and so the flux is independent of which curve we choose between the end points.

1.8.2 Stokes Streamfunction - 3D

Since it is axisymmetric it can be written in terms of two coordinates. There are two different choices of coordinates systems. This will be much expanded on in future chapters.

1.8.3 The Velocity Potential

The velocity potential arises in flow where the vorticity is zero.

Thus $\vec{\omega} = \nabla \times \vec{u} = 0$, that is, irrotational implies that we can write $\vec{u}$ as

$$\vec{u} = \nabla\phi \tag{1.44}$$

where ϕ is the velocity potential. And is the fluid is incompressible

$$\nabla^2\phi = 0 \tag{1.45}$$

that is it satisfies the Laplace equation.

1.9 Vortex Lines and Circulation

1.9.1 Vortex Lines

Define vorticity $\vec{\omega} = (\omega_1, \omega_2, \omega_3)$

A line to which vorticity vectors are tangent at all its points is called a vortex line. The differential equations for a vortex line are,

$$\frac{d\vec{x}}{ds} = \lambda\vec{\omega}. \tag{1.46}$$

Voretx lines have a direction of $\vec{\omega}$ and have a density in any region proportional to the magnitude of $\vec{\omega}$. As $\vec{\omega} = \nabla \times \vec{u}$, we have that $\nabla \cdot \vec{\omega} = 0$. So vortex lines are like lines of a magnetic force - they will form closed loops.

1.9.2 Circulation

The circulation Γ along any line is defined by the line integral

$$\Gamma = \int_A^B \vec{u} \cdot d\vec{x} = \int_A^B u_i dx_i. \tag{1.47}$$

There is a simple relationship between the circulation around a closed curve and the vorticity of the fluid over any surface bounded by that curve.

Consider the following by Stokes teorem

$$\begin{aligned} \int_C \vec{u} \cdot d\vec{x} &= \int_S \nabla \times \vec{u} \cdot d\vec{S} \\ &= \int_S \vec{\omega} \cdot \vec{n} dS = \int_S \omega_n dS \end{aligned} \tag{1.48}$$

normal component of vorticity through that surface.

Chapter 2

Fluid Dynamics - Euler Equations

2.1 Body Forces and Surface Forces

2.1.1 Body Forces (i.e. External Forces)

If F_i denotes the component of body forces per unit mass acting in the x_i-direction, the $i-th$ component of the resulting body force acting on V is

$$\int_V \varrho F_i dV. \tag{2.1}$$

2.1.2 Surface Forces

Let F_i denote body forces per unit mass in the $i-th$ direction acting upon a surface whose normal is in the $j-th$ direction and denoted by σ_{ij}

$$F_i = \sigma_{ij} dS_j \tag{2.2}$$

2.2 Ideal (or Perfect) Fluid

An ideal (or perfect) fluid is one which can exert no shearing stress across any surface

$$\sigma_{ij} = -p\delta_{ij} \tag{2.3}$$

that is the stress tensor for a perfect fluid is diagonal

$$\sigma = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \tag{2.4}$$

$$\int_S \sigma_{ij} dS_j = -\int_S p dS_i \tag{2.5}$$

this is done ignoring viscous effects, that is $\mu = 0$ and the fluid is inviscid.

2.3 Conservation of Momentum

We can go through the same process for momentum as with did for mass. The total momentum in the volume V is:

$$\Pi_i = \int_V \varrho u_i dV \tag{2.6}$$

where i runs over the three components of the momentum. We allow the proof to include compressible fluids however will only look at incompressible solution. It's rate of change is just

$$\frac{\partial \Pi_i}{\partial t} = \int_V \frac{\partial(\varrho u_i)}{\partial t} dV \tag{2.7}$$

In the absense of any forces a momentum change within the volume V can occur by momentum flowing across the boundary,

$$\frac{\partial \Pi_i}{\partial t} = -\int_S (\varrho u_i) \vec{u} \cdot \vec{n} dS \tag{2.8}$$

Using the Divergence theorem the surface integral can be written as a volume integral

$$\int_V \left[\frac{\partial(\varrho u_i)}{\partial t} + \nabla \cdot (\varrho u_i \vec{u}) \right] dV = 0 \tag{2.9}$$

As V is arbitrary this implies an equation for the local conservation of mommentum,

$$\frac{\partial(\varrho u_i)}{\partial t} + \nabla \cdot (\varrho u_i \vec{u}) = 0. \tag{2.10}$$

Note the analogy with the mass conservation equation (1.27).

Now we consider the effects of forces. From Newtons $2nd$ law:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{u}).$$

It must be beared in mind that the we write d/dt rather than $\partial/\partial t$, and that the laws of motion apply to particles, not to the control volume. We have “surface forces on S” plus “body forces” in V but assume viscous forces are negligible so surface forces are purely due to pressure. Then we can write for an ideal fluid

$$\frac{\partial}{\partial t}\int_V \varrho u_i dV + \int_S u_i \varrho u_k dS_k = -\int_S p dS_i + \int_V \varrho F_i dV \tag{2.11}$$

where the second term is the outlet momentum flux and the RHS is the sum of surface and body forces. Converting again the flow contributions to a volume integral and then deducing local equations one obtains

$$\frac{\partial(\varrho u_i)}{\partial t} + \nabla_j(\varrho u_i u_j) = -\nabla_i p + \varrho F_i. \tag{2.12}$$

This $d = 3$ expression can readily be generalised to give an expression that is valid in $d + 1$ spacetime.

Alternatively one often proceeds by writing the LHS of (2.12) as

$$\begin{aligned}
\frac{\partial(\varrho u_i)}{\partial t} + \nabla\cdot(\varrho u_i \vec{u}) &= \varrho\frac{\partial u_i}{\partial t} + u_i\frac{\partial \varrho}{\partial t} + [u_i \nabla\cdot(\varrho\vec{u}) + \varrho(\vec{u}\cdot\nabla)u_i] \\
&= \varrho\left[\frac{\partial u_i}{\partial t} + (\vec{u}\cdot\nabla)u_i\right] + u_i\left[\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\vec{u})\right].
\end{aligned}$$

Using the continuity equation, (1.27), this reduces to

$$\varrho\left[\frac{\partial u_i}{\partial t} + (\vec{u}\cdot\nabla)u_i\right] \equiv \varrho\frac{Du_i}{Dt}. \tag{2.13}$$

Using this in (2.12) it can be written

$$\varrho\frac{Du_i}{Dt} + \frac{\partial p}{\partial x_i} - \varrho F_i = 0 \tag{2.14}$$

These are known as the Euler equations of motion.

So in summary the Euler equations can be applied to compressible as well as incompressible flow. In these notes we restrict ourselves to incompressible flow. For an incompressible, ideal fluid the equations of motion are

$$\varrho\frac{D\vec{u}}{Dt} = -\nabla p + \varrho\vec{F} \tag{2.15}$$

$$\nabla\cdot\vec{u} = 0. \tag{2.16}$$

2.4 Conservative Body Forces

$$\vec{F} = -\nabla\Omega \qquad \Omega = scalar \tag{2.17}$$

Thus we can redefine the pressure

$$p' = p + \varrho\Omega$$

so the Euler equation becomes

$$\varrho\frac{D\vec{u}}{Dt} = -\nabla p'. \tag{2.18}$$

2.5 Potential Flow or Irrotational Flow

For vanishing vorticity, that is irrotational flow, $\vec{\omega} = \nabla\times\vec{u} = 0$ implies

$$\vec{u} = \nabla\phi \tag{2.19}$$

and the continuity equation gives us

$$\nabla^2\phi = 0. \tag{2.20}$$

For potential flow the velocity field can thus be found by solving Laplace equation for the velocity potential. Since the Laplace equation is linear we can superimpose solutions. In particular we can take simple flows and superimpose them to form more complicated flows of interest such as the flow around solid objects. In chapter 3 we find simple exact solutions of the Laplace equation in three dimensions, simple flows, and construct more complex flows. We do the same for plane flows in chapter 6.

In this chapter we have not included viscosity. An important aspect of viscous fluids is the non-slip condition at the surface of a solid object. This states that the velocity of the fluid at a solid boundary is zero relative to the boundary. As we shall see potential flows often dont satisfy this non-slip condition - however all liquids have a non-zero viscosity. Can potential flows be of physical interest?

As it turns out for small viscocity (more accurately large Reynolds number, see chapter 8) the viscous effects are concentrated in a thin boundary layer around the surface. Outside the bounary layer the fluid acts as an invisid fluid and so potential flow theory provides both the velocity at the outer edge of the boundary layer and the pressure there (we explain how to find the pressure in the next section). The pressure is not significantly effected by the thin boundary layer, so the pressure calculated from the invisid flow gives the pressure on the surace, and then for example this can be integrated over the body's surface to calculate a lift.

2.6 Irrotational Flow - Bernoulli Pressure

Having constructed potential flows, the Euler equation can then be used to find the pressure. Substitute $\vec{u} = \nabla\phi$ into the Euler equation (2.18),

$$\varrho \frac{D\vec{u}}{Dt} = \varrho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p'. \tag{2.21}$$

Using subscript notation rather than vectors

$$\varrho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p'}{\partial x_i} \tag{2.22}$$

but $u_i = \frac{\partial\phi}{\partial x_i}$

$$\varrho\left(\frac{\partial}{\partial t}\frac{\partial\phi}{\partial x_i} + \frac{\partial\phi}{\partial x_j}\frac{\partial^2\phi}{\partial x_i\partial x_j}\right) = -\frac{\partial p'}{\partial x_i} \tag{2.23}$$

or

$$\varrho\left(\frac{\partial}{\partial x_i}\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left(\frac{\partial\phi}{\partial x_j}\right)^2\right\}\right) = -\frac{\partial p'}{\partial x_i} \tag{2.24}$$

or

$$\frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial t}+\frac{1}{2}\left(\frac{\partial \phi}{\partial x_j}\right)^2+\frac{p'}{\varrho}\right)=0 \tag{2.25}$$

and this implies that

$$\frac{\partial \phi}{\partial t}+\frac{1}{2}\left(\frac{\partial \phi}{\partial x_j}\right)^2+\frac{p'}{\varrho}=Const.=p_\infty \tag{2.26}$$

Therefore

$$p'=p_0-\underbrace{\varrho\frac{\partial \phi}{\partial t}}_{(1)}-\underbrace{\frac{1}{2}\varrho|\vec{u}|^2}_{(2)}\,. \tag{2.27}$$

This is the Bernoulli pressure, where (1) is the rate of change of potential with time and (2) is the dependence quadratically on velocity.

We will prove later that potential flow is not influenced by viscous forces and vorticity.

Chapter 3

Sources and Sinks

As mentioned in the previous chapter, for potential flow the velocity field can be found by solving Laplace equation for the velocity potential. Since the Laplace equation is linear we can superimpose solutions. We can take simple flows and superimpose them to form more complicated flows of interest such as the flow around solid objects. The momentum conservation equation, that is the Euler's equations of motion, can then be used to calculate the pressure.

Exact solutions are easily found and described in spherical or cylindrical coordinates, not just in this chapter but also chapters to come. In the next section we introduce the mathematics for spherical and cylindrical coordinates with more details and derivations given in the appendices D and E.

3.1 Spherical and Cylindrical Coordinates

3.1.1 Spherical Coordinates

Their relation to Cartesian coordinates is

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{3.1}$$

A general position vector is then

$$\vec{r} = r\sin\theta\cos\varphi\vec{e}_x + r\sin\theta\sin\varphi\vec{e}_y + r\cos\theta\vec{e}_z \tag{3.2}$$

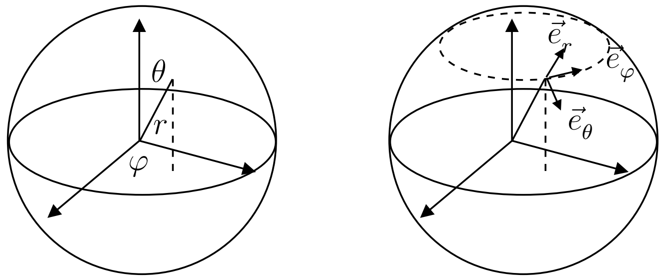

Figure 3.1:

When we work with vectors in spherical polar coordinates, we use a new local vector basis different from $\vec{i}$, $\vec{j}$ and $\vec{k}$ of Cartesian coordinates. Instead we specify vectors as components in the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ of unit vectors. These basis vectors may be visulalised as follows. To see the direction of $\vec{e}_r$, keep θ and φ and increase r. To see the direction of $\vec{e}_\theta$, keep r and φ fixed and increase θ. To see the direction of $\vec{e}_\varphi$, keep r and θ fixed and increase φ. If $\vec{r}$ corresponds to the position vector then mathematically these unit vectors are written

$$\vec{e}_r = \frac{1}{\left|\frac{\partial \vec{r}}{\partial r}\right|}\frac{\partial \vec{r}}{\partial r}, \qquad \vec{e}_\theta = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}\frac{\partial \vec{r}}{\partial \theta}, \qquad \vec{e}_\varphi = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|}\frac{\partial \vec{r}}{\partial \varphi} \tag{3.3}$$

Any arbitrary vector $\vec{v}$ can be written

$$\vec{v} = v_r \vec{e}_r + v_\theta \vec{e}_\theta + v_\varphi \vec{e}_\varphi$$

When working in non-Cartesian coordinates, common notation is

$$h_i = \left|\frac{\partial \vec{r}}{\partial q_i}\right|$$

Here $q_1 = r$, $q_2 = \theta$ and $q_3 = \varphi$. We determine the h_i with the use of (3.2):

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial r} &= \sin\theta\cos\varphi \vec{e}_x + \sin\theta\sin\varphi \vec{e}_y + \cos\theta \vec{e}_z \\
\frac{\partial \vec{r}}{\partial \theta} &= r\cos\theta\cos\varphi \vec{e}_x + r\cos\theta\sin\varphi \vec{e}_y - r\sin\theta \vec{e}_z \\
\frac{\partial \vec{r}}{\partial \varphi} &= -r\sin\theta\sin\varphi \vec{e}_x + r\sin\theta\cos\varphi \vec{e}_y
\end{aligned} \tag{3.4}$$

From these we can check that the basis vectors are indeed orthogonal to each other. We calculate

$$\begin{aligned} h_r^2 \vec{e}_r \cdot \vec{e}_r &= \sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta = 1 \\ h_\theta^2 \vec{e}_\theta \cdot \vec{e}_\theta &= r^2\cos^2\theta\cos^2\varphi + r^2\cos^2\theta\sin^2\varphi + r^2\sin^2\theta = r^2 \\ h_\varphi^2 \vec{e}_\varphi \cdot \vec{e}_\varphi &= r^2\sin^2\theta\sin^2\varphi + r^2\sin^2\theta\cos^2\varphi = r^2\sin^2\theta. \end{aligned} \tag{3.5}$$

to obtain

$$h_r = 1, \qquad h_\theta = r, \qquad h_\varphi = r\sin\theta \tag{3.6}$$

We now look at what $\nabla\phi$ is in spherical polar coordinates. Write

$$\begin{aligned} (\nabla\phi)_i &= (\nabla\phi)\cdot\vec{e}_i(\vec{r}) \\ &= \left(\frac{\partial\phi}{\partial x}\vec{e}_x + \frac{\partial\phi}{\partial y}\vec{e}_y + \frac{\partial\phi}{\partial z}\vec{e}_z\right)\cdot\vec{e}_i(\vec{r}) \end{aligned} \tag{3.7}$$

We have

$$\begin{aligned} \vec{e}_x\cdot\vec{e}_i &= \vec{e}_x\cdot\frac{1}{h_i}\frac{\partial\vec{r}}{\partial q_i} \\ &= \vec{e}_x\cdot\frac{1}{h_i}\frac{\partial}{\partial q_i}\left(x\vec{e}_x + y\vec{e}_y + z\vec{e}_z\right) \\ &= \frac{1}{h_i}\frac{\partial x}{\partial q_i}. \end{aligned}$$

Substutuiting this and analogous results for $\vec{e}_y$ and $\vec{e}_z$ into (3.7) gives

$$\begin{aligned} (\nabla\phi)_i &= \frac{1}{h_i}\frac{\partial\phi}{\partial x}\frac{\partial x}{\partial q_i} + \frac{1}{h_i}\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial q_i} + \frac{1}{h_i}\frac{\partial\phi}{\partial z}\frac{\partial z}{\partial q_i} \\ &= \frac{1}{h_i}\frac{\partial\phi}{\partial q_i} \end{aligned}$$

Therefore we have

$$\nabla\phi = \sum_{i=1}^{3} \vec{e}_i(\vec{r}) \frac{1}{h_i} \frac{\partial \phi}{\partial q_i}. \tag{3.8}$$

In the case of spherical polar coordinates,

$$\nabla\phi = \vec{e}_r \frac{\partial \phi}{\partial r} + \vec{e}_\theta \frac{1}{r} \frac{\partial \phi}{\partial \theta} + \vec{e}_\varphi \frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \varphi}. \tag{3.9}$$

If $\vec{u}$ is the velocity and if $\vec{u} = \nabla\phi$ then we have,

$$u_r = \frac{\partial \phi}{\partial r}, \qquad u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta}, \qquad u_\varphi = \frac{1}{r \sin\theta} \frac{\partial \phi}{\partial \varphi}$$

3.1.2 Cylindrical Coordinates

Their relation to Cartesian coordinates is

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned} \tag{3.10}$$

A general position vector is then

$$\vec{r} = \rho\cos\varphi\vec{e}_x + \rho\sin\varphi\vec{e}_y + z\vec{e}_z \tag{3.11}$$

These basis vectors may be visulalised as follows. To see the direction of $\vec{e}_\rho$, keep φ and z and increase ρ. To see the direction of $\vec{e}_\varphi$, keep ρ and z fixed and increase φ. To see the direction of $\vec{e}_z$, keep ρ and φ fixed and increase z. If $\vec{r}$ corresponds to the position vector then mathematically these unit vectors are written

$$\vec{e}_\rho = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \rho}\right|} \frac{\partial \vec{r}}{\partial \rho}, \qquad \vec{e}_\varphi = \frac{1}{\left|\frac{\partial \vec{r}}{\partial \varphi}\right|} \frac{\partial \vec{r}}{\partial \varphi}, \qquad \vec{e}_z = \frac{1}{\left|\frac{\partial \vec{r}}{\partial z}\right|} \frac{\partial \vec{r}}{\partial z} \tag{3.12}$$

Any arbitrary vector $\vec{v}$ can be written

$$\vec{v} = v_\rho\vec{e}_\rho + v_\varphi\vec{e}_\varphi + v_z\vec{e}_z$$

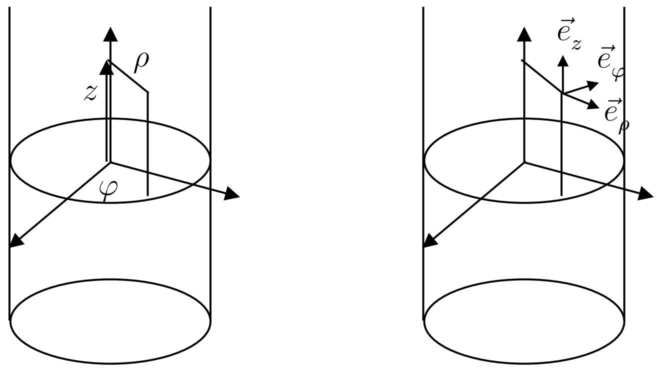

Figure 3.2: Stress.

$$\begin{aligned}\frac{\partial \vec{r}}{\partial \rho} &= \cos\varphi \vec{e}_x + \sin\varphi \vec{e}_y \\ \frac{\partial \vec{r}}{\partial \varphi} &= -\rho\sin\varphi \vec{e}_x + \rho\cos\varphi \vec{e}_y \\ \frac{\partial \vec{r}}{\partial z} &= \vec{e}_z \end{aligned} \tag{3.13}$$

From these we can check that the basis vectors are indeed orthogonal to each other. We calculate

$$\begin{aligned} h_\rho^2 \vec{e}_\rho \cdot \vec{e}_\rho &= \cos^2\varphi + \sin^2\varphi = 1 \\ h_\varphi^2 \vec{e}_\varphi \cdot \vec{e}_\varphi &= \rho^2\sin^2\varphi + \rho^2\cos^2\varphi = \rho^2 \\ h_z^2 \vec{e}_z \cdot \vec{e}_z &= 1. \end{aligned} \tag{3.14}$$

to obtain

$$h_\rho = 1, \qquad h_\varphi = \rho, \qquad h_z = 1 \tag{3.15}$$

As before we have the formula

$$\nabla\phi = \sum_{i=1}^{3} \vec{e}_i(\vec{r}) \frac{1}{h_i} \frac{\partial \phi}{\partial q_i}. \tag{3.16}$$

In the case of cylindrical polar coordinates,

$$\nabla\phi = \vec{e}_\rho \frac{\partial \phi}{\partial \rho} + \vec{e}_\varphi \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi} + \vec{e}_z \frac{\partial \phi}{\partial z}. \tag{3.17}$$

If $\vec{u}$ is the velocity and if $\vec{u} = \nabla\phi$ then we have,

$$u_\rho = \frac{\partial \phi}{\partial \rho}, \qquad u_\varphi = \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi}, \qquad u_z = \frac{\partial \phi}{\partial z}$$

3.2 Uniform Flow

It is easy to see that

$$\phi = Ux \tag{3.18}$$

represents uniform flow with constant velocity U in the $x-$direction.

The streamfunction for uniform flow is

$$\psi = Uy. \tag{3.19}$$

3.3 Point Sources and Sinks

The simplest solution is a point source or sink.

Let us consider Laplace's equation in spherical polar coordinates (r, θ, φ), that is

$$\nabla^2\phi = 0 \tag{3.20}$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \phi}{\partial \varphi^2} = 0 \tag{3.21}$$

Consider the problem with radial symmetry, that is ϕ is a function of r only, that is $\phi = \phi(r)$. Then Laplace's equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) = 0 \tag{3.22}$$

so that

$$\phi = -\frac{A}{r} + B \qquad r \neq 0 \tag{3.23}$$

Without loss of generality we choose $B = 0$ (since we always take the derivative of ϕ in obtaining the velocities), so this solution satisfies Laplaqce's equation except at the origin

$$\phi = -\frac{m}{4\pi r} \tag{3.24}$$

the 4π is a normalising factor.

Note $r = \sqrt{x^2 + y^2 + z^2}$ in Cartesian coordinates and $r = \sqrt{\rho^2 + z^2}$ in cylindrical coordinates.

The potential (3.24) only has radial velocities

$$u_r = \frac{\partial \phi}{\partial r} = \frac{m}{4\pi r^2} \tag{3.25}$$

and this is outflow. We see that the minus sign in (3.24) corresponds to outward flow from the origin, i.e. a source of material. So (3.24) is the velocity potential due to a simple source of strength m at $r = 0$. If we wanted a sink would make the replacement $m \to -m$ and we would have inflow.

The volume flux Φ through a spherical surface at $r = a$ is radial velocity times the area of the sphere,

$$\Phi = \frac{m}{4\pi a^2} \times 4\pi a^2 = m.$$

3.3.1 In Cylindrical Coordinates

In cylindrical coordinates

$$\phi = -\frac{m}{4\pi\sqrt{x^2 + \rho^2}}$$

$$\begin{aligned} u_x &= \frac{\partial \phi}{\partial x} = \frac{mx}{4\pi(x^2 + \rho^2)^{3/2}} = \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} \\ u_\rho &= \frac{\partial \phi}{\partial \varrho} = \frac{m\rho}{4\pi(x^2 + \rho^2)^{3/2}} = -\frac{1}{\varrho}\frac{\partial \psi}{\partial x} \end{aligned} \tag{3.26}$$

so that $\psi(x, \rho)$ is the streamfunction, that is

$$\psi(x, \rho) = -\frac{mx}{4\pi\sqrt{x^2 + \rho^2}}. \tag{3.27}$$

If the source is located at (x_0, y_0, z_0)

$$\phi = -\frac{m}{4\pi\sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}} \tag{3.28}$$

3.4 Doublets

If a source of strength m is situated at $(d, 0, 0)$ and a sink of the same strength at $(-d, 0, 0)$, the potential is a superposition of the two, that is

$$\phi = \frac{m}{4\pi}\left(\frac{1}{\sqrt{(x+d)^2 + y^2 + z^2}} - \frac{1}{\sqrt{(x-d)^2 + y^2 + z^2}}\right) \tag{3.29}$$

If d is infintesimal then we have to first order in d in a Taylor expansion

$$\frac{1}{\sqrt{(x+d)^2 + y^2 + z^2}} = \frac{1}{r} + d\frac{\partial}{\partial x}\frac{1}{r}.$$

Thus to first order in d we have for ϕ

$$\phi = \frac{m}{4\pi}\frac{\partial}{\partial x}\left(\frac{1}{r}2d\right) = -\frac{2md}{4\pi}\frac{x}{r^3}$$

$\lim_{d \to 0} 2md = \mu$

$$\phi = \frac{-\mu x}{4\pi r^3} \tag{3.30}$$

This satisfies Laplaces's equation and is called a doublet. The axis of a doublet is the lines drawn from the sink to the source, in the case the x axis. Likewise with y and z directions

$$\begin{aligned} \phi &= -\frac{\mu_2 y}{4\pi r^3} \\ \phi &= -\frac{\mu_3 z}{4\pi r^3} \end{aligned} \tag{3.31}$$

In general

$$\phi = -\frac{\vec{\mu} \cdot \vec{r}}{4\pi r^3} \tag{3.32}$$

where (μ_1, μ_2, μ_3) and doublet acting in the $\vec{x}$ direction.

3.5 Axi-Symmetric Flow

Spherical Polar coordinates (r, θ, φ)

$$u_r = \frac{\partial \phi}{\partial r} = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}$$

$$u_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$$

We have axi-symmetric flow if $\psi = \psi(r, \theta)$

The formula for the velocities in terms of the streamfunctions will be given in chapter 10.

Cylindrical Polar coordinates (ρ, φ, z)

The velocity potential is

$$u_\rho = \frac{\partial \phi}{\partial \rho} = -\frac{1}{\rho} \frac{\partial \psi}{\partial z}$$

and the streamfunction is

$$u_z = \frac{\partial \phi}{\partial z} = \frac{1}{\rho} \frac{\partial \psi}{\partial \rho}$$

Again the formula for the velocities in terms of the streamfunctions will be given in chapter 10.

3.5.1 Uniform Flow

In spherical Polar coordinates

$$\phi = Ur\cos\theta$$

$$\psi = \frac{1}{2}Ur^2\sin^2\theta$$

as

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta \qquad \text{and} \qquad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta$$

and

$$u_r = \frac{\partial\phi}{\partial r} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = U\cos\theta \qquad \text{and} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -U\sin\theta.$$

In cylindrical coordinates the velocity potential is

$$\phi = Ux$$

The streamfunction is

$$\psi = \frac{1}{2}U\rho^2$$

as

$$u_\rho = \frac{\partial\phi}{\partial\rho} = 0 \qquad \text{and} \qquad u_x = \frac{\partial\phi}{\partial x} = U$$

and

$$u_\rho = -\frac{1}{\rho}\frac{\partial\psi}{\partial x} = 0 \qquad \text{and} \qquad u_x = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = U$$

3.5.2 Source

The velocity potential is

$$\phi = -\frac{m}{4\pi r}$$

The streamfunction is

$$\psi = -\frac{m}{4\pi}\cos\theta$$

and the velocity potential in cylindrical polar coordinates is

$$\phi = -\frac{m}{4\pi\sqrt{z^2+\rho^2}}$$

which is the same function as given above but expressed in cylindrical polar coordinates.

The velocities in spherical coordinates are via the streamfunction

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{m}{4\pi r^2} \qquad \text{and} \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = 0.$$

The velocities in cylinidrical coordinates via the velocity potential are

$$u_\rho = \frac{\partial\phi}{\partial\rho} = -\frac{\partial}{\partial\rho}\frac{m}{4\pi\sqrt{z^2+\rho^2}} = \frac{m\rho}{4\pi\sqrt{z^2+\rho^2}}$$

$$u_z = \frac{\partial\phi}{\partial z} = \frac{mz}{4\pi\sqrt{z^2+\rho^2}}$$

3.5.3 Doublet

The velocity potential in spherical polar coordinates is

$$\phi = -\frac{\mu\cos\theta}{4\pi r^2}$$

The streamfunction in spherical polar coordinates is

$$\psi = \frac{\mu \sin^2\theta}{4\pi r}$$

The velocity potential in spherical polar coordinates via the vector potential is

$$u_r = -\frac{\partial}{\partial r}\frac{\mu\cos\theta}{4\pi r^2} = \frac{\mu\cos\theta}{2\pi r^3} \quad \text{and} \quad u_\theta = -\frac{1}{r}\frac{\partial}{\partial\theta}\frac{\mu\cos\theta}{4\pi r^2} = \frac{\mu\sin\theta}{4\pi r^3}$$

The velocity potential in spherical polar coordinates via the streamfunction is

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\frac{\mu\sin^2\theta}{4\pi r} = \frac{\mu\cos\theta}{2\pi r^3} \quad \text{and} \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\frac{\mu\sin^2\theta}{4\pi r} = \frac{\mu\sin\theta}{4\pi r^3}$$

The velocity potential in cylindrical polar coordinates is

$$\phi = -\frac{\mu z}{4\pi\rho^{3/2}}$$

The velocities in cylinidrical coordinates via the velocity potential are

$$u_\rho = \frac{\partial\phi}{\partial\rho} = -\frac{\partial}{\partial\rho}\frac{\mu z}{4\pi\rho^{3/2}} = \frac{3}{2}\frac{\mu z}{4\pi\rho^{5/2}}$$

$$u_z = \frac{\partial\phi}{\partial z} = -\frac{\mu}{4\pi\rho^{3/2}}$$

3.6 The Half-Body

We can use singular solutions to "build up" the fow around different shape bodies. First example is the so-called "half-body".

Consider what happens when a source is placed in free-stream.

$$\psi(r,\theta) = \underbrace{\frac{1}{2}Ur^2\sin^2\theta}_{uniform} \underbrace{-\frac{m}{4\pi}\cos\theta}_{source} \tag{3.33}$$

We have a dividing streamline for $\psi(r,\theta) = m/4\pi$.

The equation of the "half-body" is given by

$$\frac{m}{4\pi} = \frac{1}{2}Ur^2\sin^2\theta - \frac{m}{4\pi}\cos\theta$$

that is

$$\frac{m}{4\pi}(1+\cos\theta) = \frac{1}{2}Ur^2\sin^2\theta$$

In terms of half angles

$$\frac{m}{4\pi}2\cos^2\frac{\theta}{2} = \frac{1}{2}Ur^2\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2$$

which gives

$$r = \left(\frac{m}{4\pi}\right)^{1/2} cosec\frac{\theta}{2}$$

3.7 Rankine Bodies

A closed body is generated by superimposing a uniform stream plus sources and sinks such that the outflow due to the sources is exactly balanced by the inflow due to the sinks, with the line joining the sources and sinks along the stream direction.

A simply example is a source at $x=-a$ and sink at $x=a$ plus free-stream.

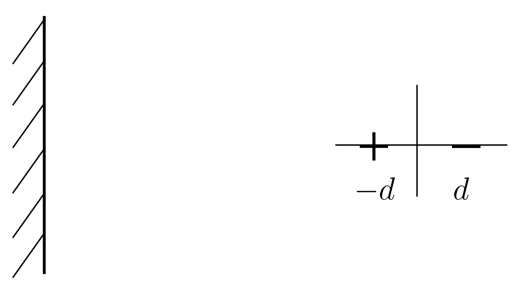

Figure 3.3: .

Potential (spherical polars)

$$\phi = Ur\cos\theta - \frac{m}{4\pi r_1} + \frac{m}{4\pi r_2} \tag{3.34}$$

and streamfunction

$$\psi = \frac{1}{2}Ur^2 \sin^2\theta - \frac{m\cos\theta_1}{4\pi} + \frac{m\cos\theta_2}{4\pi} \tag{3.35}$$

3.8 Flow Around a Shpere

Important problem - many particles are spherical or nearly spherical. Ball games involve spherical objects. Spherical inhomogeneties are used to change flow characteristics for example to "trip" boundary layers or improve mixing in jet engines.

Consider a frame of reference such that the sphere is at rest and the fluid flow at ∞ is U in the $x-$direction.

Figure 3.4: .

Consider the Rankine body - push the soure and sink together, so that we have a dipole (source-doublet). The velocity potential ϕ and streamfunction are

$$\begin{aligned} \phi &= Ur\cos\theta - \frac{\mu\cos\theta}{r^2} \\ \psi &= \frac{1}{2}Ur^2\sin^2\theta + \frac{\mu\sin^2\theta}{r} \end{aligned} \tag{3.36}$$

The first terms correspond to the uniform stream and the second to the dipole.

The radial velocity is

$$v_r = \frac{\partial\phi}{\partial r} = U\cos\theta + \frac{2\mu}{r^3}\cos\theta. \tag{3.37}$$

We require that this velocity is zero on the sphere (a physical consideration) so that the boundary condition is $u_r = 0$ on $r = a$ so that $\mu = -\frac{1}{2}Ua^3$. Then

$$\begin{aligned} \phi &= U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \\ \psi &= \frac{1}{2}U\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta \end{aligned} \tag{3.38}$$

and the velocities are

$$\begin{aligned} u_r &= U\left(1 - \frac{a^3}{r^3}\right)\cos\theta \\ u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta \end{aligned} \tag{3.39}$$

Notice that

(i) the disturbance to the freestream is proportional to $1/r^3$ and

(ii) $u_\theta(r = a) = -\frac{3}{2}U\sin\theta$ and this is the slip velocity.

(iii) Stagnation points

Stagnation points at $r = a$, $\theta = 0$ and $\theta = \pi$, $u_r = u_\theta = 0$

(iv) Pressure distribution on the sphere.

As the flow is steady $\frac{\partial\phi}{\partial t} = 0$ so that the Bernoulli pressure is given by

$$p = p_0 - \frac{1}{2}\varrho|\vec{u}|^2$$

so up stream of the sphere $p = p_\infty$ and $u_x = U$

$$p_\infty = p_0 - \frac{1}{2}\varrho U^2 \quad \text{implying} \quad p_0 = p_\infty + \frac{1}{2}\varrho U^2$$

therefore

$$p = p_\infty + \frac{1}{2}\varrho U^2 - \frac{1}{2}\varrho|\vec{u}|^2.$$

On $r = a$, $u_r = 0$ and $u_\theta - \frac{3}{2}U\sin\theta$ therfore

$$\frac{p - p_\infty}{\varrho} = \frac{1}{2}U^2 \left[1 - \frac{9}{4}\sin^2\theta\right] \tag{3.40}$$

(v) Cavitation (boiling at ordinary temperature)

If the minimum pressure, p_{min}, is less than the saturated vapour pressure p_c then liquid will caviate. The minimum pressure for a sphere occurs at $\theta = \frac{\pi}{2}$

$$p_{min} = p_\infty - \frac{5}{8}\varrho U^2 \tag{3.41}$$

We have $p_{min} \leq p_c$ if

$$p_\infty - \frac{5}{8}\varrho U^2 \leq p_c$$

or

$$U \geq \sqrt{\frac{8}{5}} \left(\frac{p_\infty - p_c}{\varrho}\right)^{1/2}. \tag{3.42}$$

Rearanged this condition becomes

$$\frac{5}{4} \geq \frac{p_\infty - p_c}{\frac{1}{2}\varrho U^2}$$

In hydraulic engineering the cavitation number is defined by

$$\kappa = \frac{p_\infty - p_c}{\frac{1}{2}\varrho U^2}$$

for a sphere one has cavitation at $\kappa = \frac{5}{4}$, experimentally $\kappa < \frac{1}{2}$ for torpedo shaped bodies.

3.9 Line Distributions

We have previously examined cases of a finite number of singularities representing a body (for example oval shaped bodies or spheres).

The next step is to consider an infinite set of singularities each infinitesimally close to its neighbour over a finite line segment.

For example uniform distribution of sources

Streamfunction for a point source at $x = s$ is given in Cartesian coordinates by

$$\psi = -\frac{m(x-s)}{4\pi r} = -\frac{m(x-s)}{4\pi\sqrt{(x-s)^2+y^2+z^2}} \tag{3.43}$$

Let the total strength of the sources be m, take the strength of the source in element δs as $m\delta s/a$, with m constant for a uniform distribution. For a continuous distribution from $s = 0$ to $s = a$ if we let $\delta s \to 0$

$$\psi = -\frac{1}{4\pi a}\int_0^a \frac{m(s)(x-s)}{4\pi\sqrt{(x-s)^2+y^2+z^2}}ds = \frac{m}{4\pi a}[R_2 - R] \tag{3.44}$$

where R_2 is the distance from the point $(a, 0, 0)$ and R the distance to the origin.

Airship forms source + line sink + uniform stream

$$\psi = \frac{1}{2}Ur^2\sin^2\theta + \frac{m}{4\pi a}(R - R_2) - \frac{m}{4\pi}\cos\theta$$

This technique is the basis of slender body theory.

Chapter 4

Viscosity - The Navier-Stokes Equation

4.1 Equations of Motion

$$\frac{d}{dt}\int_V \varrho u_i dV = \int_S \sigma_{ij} n_j dS + \int_V \varrho F_i dV \tag{4.1}$$

Or

$$\int_V \varrho \frac{D\vec{u}}{Dt} dV = \int_S \sigma \cdot \vec{n} dS + \int_V \varrho \vec{f} dV \tag{4.2}$$

by taking the derivative inside the integral, so that it becomes a substantial dreivative. On applying the Divergence theorem,

$$\int_V \nabla \cdot \vec{A} dV = \int_S \vec{A} \cdot \vec{n} dS,$$

and rearranging we obtain

$$\int_V \left(\varrho \frac{D\vec{u}}{Dt} - \varrho \vec{f} - \nabla \cdot \sigma \right) dV = 0 \tag{4.3}$$

Since V is arbitrary the integrand must be indentically zero, that is

$$\varrho \frac{D\vec{u}}{Dt} = \varrho \vec{f} + \nabla \cdot \sigma \tag{4.4}$$

or in subscript notation

$$\varrho \frac{Du_i}{Dt} = \varrho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{4.5}$$

4.2 Newtonian Fluid

4.2.1 Pressure

For a real fluid (that is viscous, heat conducting, ...) in motion, we shall define the (mechanical) pressure by

$$p = -\frac{1}{3} Tr(\sigma_{ij}) = -\frac{1}{3}\sigma_{ii}.$$

In Newtonian fluid there is only a linear dependence on the rate of strain tensor in which case,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} \tag{4.6}$$

If we let $i = j$ and sum (4.6) we obtain

$$\sigma_{ii} = -3p + (2\mu + 3\lambda)e_{ii}$$

However by definition $p = -\frac{1}{3}\sigma_{ii}$, so wemust have the following relation between λ and μ

$$\lambda = -\frac{2}{3}\mu.$$

Therefore the constitutive relation for a Newtonianfluid isgiven by

$$\sigma_{ij} = -p\delta_{ij} + 2\mu\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right)$$

This relation also tells us that the principal axis for σ_{ij} and e_{ij} coincide.

4.2.2 Viscosity

μ is known as the Newtonian viscocity.

4.2.3 Incompressible Newtonian Fluid

$$e_{kk} = \nabla \cdot \vec{u} = 0 \tag{4.7}$$

so

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}. \tag{4.8}$$

This is the constitutive relation on which is nearly all these nots are based.

4.3 Navier-Stokes Equation

Cauchy derived the equation

$$\varrho \frac{Du_i}{Dt} = \varrho f_i + \frac{\partial \sigma_{ij}}{\partial x_j} \tag{4.9}$$

which together with the incompressibility condition for the continuity equation

$$\frac{\partial u_j}{\partial x_j} = 0 \tag{4.10}$$

The stress tensor for a Newtonian fluid is,

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

which on substitution for e_{ij} leads to

$$\sigma_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

This may now be substituted into the Cauchy's equation to yield

$$\varrho \frac{Du_i}{Dt} = \varrho f_i - \frac{\partial p}{\partial x_i} + \mu \left[\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) \right] \tag{4.11}$$

The last term on the RHS of this equation is identically zero because of the continuity equation. The Navier-Stokes equations emerges

$$\varrho \frac{Du_i}{Dt} = \varrho f_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j^2} \tag{4.12}$$

or in vector notation

$$\varrho \frac{D\vec{u}}{Dt} = -\nabla p + \mu \nabla^2 \vec{u} + \varrho \vec{f} \tag{4.13}$$

4.3.1 Conservative Body Force

Often the body force is conservative - that is

$$\vec{f} = -\nabla \Omega \tag{4.14}$$

where Ω is the potential energy per unit mass. For example gravity

$$\vec{f} = -g\vec{k} \tag{4.15}$$

where $\vec{k}$ is the unit vertical upward vector, and hence

$$\Omega = gz. \tag{4.16}$$

Thus we can redefine the pressure as

$$p' = p + \varrho \Omega \tag{4.17}$$

which upon substitution into (4.13) leads to

$$\varrho \frac{D\vec{u}}{Dt} = -\nabla p' + \mu \nabla^2 \vec{u} \tag{4.18}$$

It is convenient to write the equations as

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\varrho} \nabla p' + \nu \nabla^2 \vec{u} \tag{4.19}$$

Writing out the substantial derivative

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\frac{1}{\varrho} \nabla p' + \nu \nabla^2 \vec{u} \tag{4.20}$$

4.4 Alternative Form of the Navier-Stokes Equation and the Vorticity Equation

We first derive a couple of identities using $\epsilon_{ijk}\epsilon_{ki'j'} = (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})$:

$$\begin{aligned}
[\vec{u}\times\vec{\omega}]_i &= \epsilon_{ijk}u_j\omega_k \\
&= \epsilon_{ijk}u_j(\epsilon_{ki'j'}\partial_{i'}u_{j'}) \\
&= \epsilon_{ijk}\epsilon_{ki'j'}u_j\partial_{i'}u_{j'} \\
&= (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})u_j\partial_{i'}u_{j'} \\
&= u_j\partial_i u_j - u_j\partial_j u_i \\
&= \frac{1}{2}\partial_i u^2 - \vec{u}\cdot\nabla u_i \\
&= [\frac{1}{2}\nabla\vec{u}^2 - \vec{u}\cdot\nabla\vec{u}]_i
\end{aligned}$$

and

$$\begin{aligned}
[\nabla\times\vec{\omega}]_i &= \epsilon_{ijk}\partial_j\omega_k \\
&= \epsilon_{ijk}\partial_j(\epsilon_{ki'j'}\partial_{i'}u_{j'}) \\
&= \epsilon_{ijk}\epsilon_{ki'j'}\partial_j\partial_{i'}u_{j'} \\
&= (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})\partial_j\partial_{i'}u_{j'} \\
&= \partial_j\partial_i u_j - \partial_j\partial_j u_i \\
&= [-\nabla^2\vec{u} + \nabla(\nabla\cdot\vec{u})]_i.
\end{aligned}$$

In vector notation these two identities read

$$\vec{u}\cdot\nabla\vec{u} = \frac{1}{2}\nabla|\vec{u}|^2 - \vec{u}\times\vec{\omega} \tag{4.21}$$

and

$$\nabla\times\vec{\omega} = -\nabla^2\vec{u} + \nabla(\nabla\cdot\vec{u}). \tag{4.22}$$

For incompressible flow ($\nabla\cdot\vec{u} = 0$) this second identity becomes

$$\nabla\times\vec{\omega} = -\nabla^2\vec{u}. \tag{4.23}$$

Substitution of (4.21) and (4.23) into (4.20)

$$\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \vec{\omega} = -\nabla \left(\frac{1}{2} |\vec{u}|^2 + \frac{p}{\varrho} \right) - \nu \nabla \times \vec{\omega} \tag{4.24}$$

"This is the 'best' form of the equation to use when considering coordinate systems other than Cartesian coordinates".

We can take the curl of (4.20) using $\nabla \times (\nabla p') = 0$,

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{u} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{u} + \nu \nabla^2 \vec{\omega} \tag{4.25}$$

4.4.1 The Vorticity Equation in Two Dimensions

In two dimensions the first term on the RHS vanishes, that is

$$\vec{\omega} \cdot \nabla \vec{u} = 0. \tag{4.26}$$

Therefore in two dimensions the vorticity satisfies the diffusion equation,

$$\frac{D\vec{\omega}}{Dt} = \nu \nabla^2 \vec{\omega}. \tag{4.27}$$

4.5 Vorticity and Viscocity

It is possible to show that without viscous forces there cannot be a vorticity fieled in the flow which varies with coordinate directions. This is expressed in the following relationship

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \nabla^2 u_i = -\mu \epsilon_{ijk} \frac{\partial \omega_k}{\partial x_j}$$

(recall $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ where $\tau_{ij} = 2\mu e_{ij}$). The equality of the last expression with the central expression can be derived in the following manner

$$
\begin{aligned}
-\mu\epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j} &= -\mu\epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{kmn}\frac{\partial}{\partial x_m}u_n) \\
&= -\mu\epsilon_{ijk}\epsilon_{kmn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}u_n \\
&= -\mu(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\frac{\partial^2 u_n}{\partial x_j \partial x_m} \\
&= -\mu(\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} - \frac{\partial^2 u_i}{\partial x_j \partial x_j}) \\
&= \mu\nabla^2 u_i. \qquad (4.28)
\end{aligned}
$$

where we used $\partial u_j/\partial x_j = 0$.

If invisid flow starts with no vorticity then no vorticity will be produced. To understand this intuitively we note that of the three types of force that can act on a cubic fluid element, the pressure, body forces, and viscous forces, only the viscous shear forces are able to give rotary motion. Hence if the viscous effects are nonexistent, vorticity cannot be introduced.

Chapter 5

Streamfunction in 2-Dimensions

If

$$\nabla \cdot \vec{u} = 0 \tag{5.1}$$

then it follows that there exists a vector field $\vec{A}(\vec{x}, t)$ such that

$$\vec{u} = \nabla \times \vec{A}. \tag{5.2}$$

5.1 Cartesians

As noted in section 1.8.1 the continuity equation in 2D

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

is solved by choosing

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}.$$

The continuity equation is solved if we write $\vec{u} = \nabla \times \vec{A}$ for $\vec{A} = \psi(x, y, t)\vec{z}$, then

$$
\begin{aligned}
\vec{u} = \nabla \times \vec{A} &= \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \psi(x,y,t) \end{vmatrix} \\
&= \frac{\partial \psi}{\partial y}\vec{e}_x - \frac{\partial \psi}{\partial x}\vec{e}_y
\end{aligned} \tag{5.3}
$$

thus

$$
u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{5.4}
$$

5.2 Cylindrical Polar Coordinates

The relation to Cartesian coordinates is

$$
x = r\cos\theta, \qquad y = r\sin\theta.
$$

From section 3.1.2 we see that the formula for the velocity components u_r and u_θ in terms of the velocity potential

$$
u_r = \frac{\partial \phi}{\partial r} \qquad \text{and} \qquad u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}. \tag{5.5}
$$

5.2.1 Relating Velocity to Streamfunction by Solving the Continuity Equation

In cylindrical polar coordinates the continuity equation

$$
\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} = 0 \tag{5.6}
$$

With the choice

$$
u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad \text{and} \qquad u_\theta = -\frac{\partial \psi}{\partial r} \tag{5.7}
$$

the continuity equation is solved.

We obtain these also from the curl in cylindrical polar coordinates

$$\begin{aligned}\vec{u} = \nabla \times \vec{A} &= \frac{1}{r}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & \vec{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ 0 & 0 & \psi(r,\theta,t) \end{vmatrix} \\ &= \frac{1}{r}\frac{\partial \psi}{\partial \theta}\vec{e}_r - \frac{\partial \psi}{\partial r}\vec{e}_\theta \end{aligned} \tag{5.8}$$

We demonstrate that ψ is indeed constant along the streamlines using the formula (5.7) that we have obtained relating the velocity components to this streamfunction. The differential equation for streamlines are

$$\frac{d\vec{x}}{ds} = \lambda \vec{u}.$$

and thus $dr = \lambda u_r ds$ and $rd\theta = \lambda u_\theta ds$. Using this find

$$\begin{aligned} d\psi &= \frac{\partial \psi}{\partial r}dr + \frac{\partial \psi}{\partial \theta}d\theta \\ &= -u_\theta dr + ru_r d\theta \\ &= -u_\theta \lambda u_r ds + u_r \lambda u_\theta ds = 0. \end{aligned} \tag{5.9}$$

5.2.2 Direct Calculation

Here we derive (5.7) directly from (5.4)

Since the streamfunction is defined completely by the geometry of the streamlines, it must be the same in any coordinates system. So if $\psi_C(x,y,t)$ is the streamfunction in Cartesian coordinates, the stream function in polar coordinates will be

$$\psi_P(r,\theta,t) = \psi_C(r\cos\theta, r\sin\theta, t). \tag{5.10}$$

The position vector expressed in Cartesian basis vectors

$$\vec{r} = r\cos\theta\vec{i} + r\sin\theta\vec{j}.$$

Taking the derivative we obtain

$$\frac{\partial \vec{r}}{\partial r} = \cos\theta \vec{i} + \sin\theta \vec{j}.$$

This already has unit length so

$$\vec{e}_r = \cos\theta \vec{i} + \sin\theta \vec{j}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r\sin\theta \vec{i} + r\cos\theta \vec{j}$$

This already has unit length so

$$\vec{e}_\theta = -\sin\theta \vec{i} + \cos\theta \vec{j}$$

$$\begin{aligned} \vec{u} &= u_r\vec{e}_r + u_\theta \vec{e}_\theta \\ &= u_r[\cos\theta \vec{i} + \sin\theta \vec{j}] + u_\theta[-\sin\theta \vec{i} + \cos\theta \vec{j}] \\ &= [u_r\cos\theta - u_\theta \sin\theta]\vec{i} + [u_r\sin\theta + u_\theta\cos\theta]\vec{j} \end{aligned} \tag{5.11}$$

having collected terms in $\vec{i}$ and $\vec{j}$ we see that

$$u = u_r\cos\theta - u_\theta\sin\theta, \qquad v = u_r\sin\theta + u_\theta\cos\theta.$$

Applying the formula for u and v in terms of the stream function

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} = u_r\cos\theta - u_\theta\sin\theta \\ v &= -\frac{\partial \psi}{\partial x} = u_r\sin\theta + u_\theta\cos\theta \end{aligned} \tag{5.12}$$

Solving for u_r and u_θ,

$$\begin{aligned} u_r &= \cos\theta\frac{\partial \psi_C}{\partial y} - \sin\theta\frac{\partial \psi_C}{\partial x} \\ u_\theta &= -\sin\theta\frac{\partial \psi_C}{\partial y} - \cos\theta\frac{\partial \psi_C}{\partial x} \end{aligned} \tag{5.13}$$

Now using the chain rule we can write formula for the derivatives

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} \\ &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} \\ &= -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}. \end{aligned} \tag{5.15}$$

Solving for $\partial/\partial x$ and $\partial/\partial y$ we obtain

$$\begin{aligned} \frac{\partial}{\partial x} &= \cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial y} &= \sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial \theta} \end{aligned} \tag{5.16}$$

Substituting these into (5.13) and using (5.10)

$$\begin{aligned} u_r &= \cos\theta\left[\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right]\psi_P - \sin\theta\left[\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right]\psi_P \\ &= \frac{1}{r}\frac{\partial\psi}{\partial\theta} \end{aligned} \tag{5.17}$$

$$\begin{aligned} u_\theta &= -\sin\theta\left[\sin\theta\frac{\partial}{\partial r} + \cos\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right]\psi_P - \cos\theta\left[\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{1}{r}\frac{\partial}{\partial\theta}\right]\psi_P \\ &= -\frac{\partial\psi}{\partial r} \end{aligned} \tag{5.18}$$

In polar coordinates the formula for the velocity components in terms of the stream function are

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = -\frac{\partial\psi}{\partial r} \tag{5.19}$$

where $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta$.

Chapter 6

Plane Potential Flow

6.1 Potential Flow

a) Potential flow has no vorticity

$$\omega_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_k} = 0$$

b) Potential flow is not influenced by viscous forces. Recall the stress tensor can be written: $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ where τ_{ij} is the viscous stress tensor - see (4.9). The viscous term in the Navier-Stokes equation is

$$\frac{\partial \tau_{ij}}{\partial x_j} = \mu \nabla^2 u_i.$$

From $\nabla \times \vec{\omega} = \nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u} = -\nabla^2 \vec{u}$ (where we have used $\nabla \cdot \vec{u} = 0$),

$$\mu \nabla^2 u_i = -\mu \epsilon_{ijk}\frac{\partial \omega_k}{\partial x_j} = 0.$$

c) Potential flow satisfies Laplace equation,

$$0 = \nabla \cdot \vec{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi.$$

6.2 Complex Representation

We will show in the following that any analytic function represents a two dimensional flow.

Incompressible - $\nabla \cdot \vec{u} = 0$ implies that there exists a vector field $\vec{A} = \psi \vec{k}$ such that $\vec{u} = \nabla \times \psi \vec{k}$ or

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x} \tag{6.1}$$

Irrotational - $\vec{\omega} = \nabla \times \vec{u} = 0$ implies there is ϕ such that $\vec{u} = \nabla \phi$ or

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}. \tag{6.2}$$

Using (6.2) in the continuity equation $\nabla \vec{u} = 0$ gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla^2 \phi = 0.$$

Using that potential flow has no vorticity we find using (6.1)

$$-\omega = -\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = \nabla^2 \psi = 0.$$

Thus both the velocity potetial ϕ and the streamfrunction ψ the Laplace equation.

These imply

$$\frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}, \qquad -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y} \tag{6.3}$$

which are the Cauchy-Riemann equations. These are necessary condition for the complex function $\phi(x, y) + i\psi(x, y)$ to be analytic, as we shall now show. First we must note that

$$f'(z) \text{ exists and is unique } \Longleftrightarrow f(z) \text{ ia an analytic function}$$

requiring that a derivative of a complex function should be the same independent of the direction we take the limit in the complex plane. We have

$$
\begin{aligned}
f'(z) &= \lim_{\Delta z \to 0} \frac{1}{\Delta z}[f(z+\Delta z) - f(z)] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}[\phi(x+\Delta x, y) - \phi(x,y) + i\psi(x+\Delta x, y) - i\psi(x,y)] \\
&= \lim_{i\Delta y \to 0} \frac{1}{\Delta x}[\phi(x, y+\Delta y) - \phi(x,y) + i\psi(x, y+\Delta y) - i\psi(x,y)] \qquad (6.4)
\end{aligned}
$$

Equating the real parts of thes last two give lines give

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

and equating imaginary parts

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}.$$

Thus we have that the Cauchy-Riemann equations as a necessary requirement for a complex function to be analytic.

$$\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}$$

$$\frac{dw}{dz} = u - iv \qquad (6.5)$$

Note that

$$\left|\frac{dw}{dz}\right| = \sqrt{u^2 + v^2}$$

which is the magnitude of the velocity. $\frac{dw}{dz}$ is the complex velocity and is analytic since the derivative of an analytic function is analytic.

6.3 Simple Flows

(b) Uniform flows

(b) Flow near a stagnation point

(c) Flow around a corner

(i) Flows for $0 \leq \theta \leq \pi/n$
(ii) Flows for $0 \leq \theta \leq 2\pi/n$

6.3.1 Uniform Flows

Consider uniform flow in the $x-$direction,

$$u_x = u = U, \quad v = 0$$

which implies that $\phi = Ux+k$, $\psi = Uy+k'$ where k and k' are arbitrary amd do not effect the velocity or Cauchy-Riemann equations as these all involve derivatives, and hence can be dropped without loss of generality. The complex potential is $w(z) = Uz$ and

$$\frac{dw}{dz} = U = U + i0$$

this is where the stream is aligned with the $x-$direction.

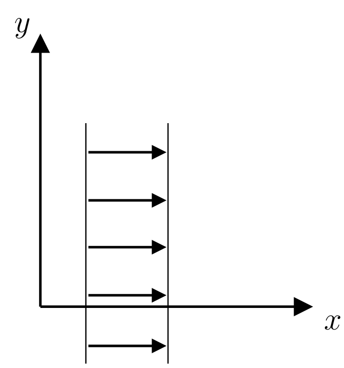

Figure 6.1: Uniform motion in the $x-$direction..

We now calculate the velocity field from the complex potential $w = Ue^{-i\alpha}z$.

$$\frac{dw}{dz} = Ue^{-i\alpha} = U(\cos\alpha - i\sin\alpha)$$

$$\begin{aligned} u &= U\cos\alpha \\ v &= U\sin\alpha. \end{aligned} \tag{6.6}$$

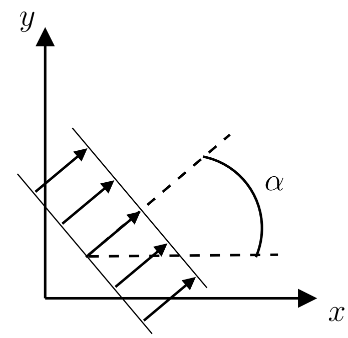

Figure 6.2: Uniform motion at an angle α

Obviously $\sqrt{u^2+v^2} = U$.

It is easy to show that multiplying any complex potential $w(z)$ by $e^{-i\alpha}$ rotates the velocity vector anti-clockwise by an angle of α at each point in space: as

$$\frac{dw(z)}{dz} = u(x,y) - iv(x,y)$$

then

$$\begin{aligned}\frac{d(w(z)e^{-i\alpha})}{dz} &= [u(x,y) - iv(x,y)][\cos\alpha - i\sin\alpha] \\ &= u(x,y)\cos\alpha - v(x,y)\sin\alpha - i[u(x,y)\sin\alpha + v(x,y)\cos\alpha] \\ &= \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}u(x,y) \\ v(x,y)\end{pmatrix}. \qquad (6.7)\end{aligned}$$

Note that we are rotating the velocity vector at each point in space rather than rotating about the origin.

6.3.2 Flow Near a Stagnation point

A stagnation point z_s has both velocity components zero, i.e. $u = v = 0$ at $z = z_s$ thus

$$\left.\frac{dw}{dz}\right|_{z=z_s} = 0. \qquad (6.8)$$

For some potential $w(z)$ (analytic about z_s) consider a Taylor expansion of $w(z)$ about z_s to invetigate flow near a stagnation point.

$$w(z) = w(z_s) + (z - z_s)w'(z_s) + \frac{1}{2}(z - z_s)^2 w''(z_s) + \mathcal{O}(|z - z_s|^3)$$

so that

$$w(z) = w(z_s) + \frac{1}{2}(z - z_s)^2 w''(z_s) + \mathcal{O}(|z - z_s|^3). \tag{6.9}$$

Assuming that $w''(z_s) \neq 0$ it may be written in polar coordinate form $w''(z_s) = Ae^{i\alpha}$ $A > 0$ and α real. The term $w(z_s)$ is a constant and so does not effect the resulting velocity field and so can be neglected. The flow resulting from (6.9) is the same as the flow resulting from $(1/2)z^2 Ae^{i\alpha} + \mathcal{O}(|z|^3)$ but shifted by an amount x_s in the $x-$direction and an amount y_s in the $y-$direction. We are considering the flow near the stagnation point and so the terms represented by $\mathcal{O}(|z - z_s|^3)$ are to be neglected. We know that a factor $e^{i\alpha}$ rotates the velocity vector by an angle of α clock-wise at each point in space.

In the case of monomials in $z - z_s$, we will find that we can obtain the flow by performing a rotation about the point z_s. First put $z - z_s = (x - x_s) + i(y - y_s)$. Consider the $(x'y')$ axis defined by $x' = x - x_s$ and $y' = y - y_s$. Then consider this axis rotated through $-\alpha/2$ and becoming the $X - Y$ axis so that

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\alpha/2) & \sin(\alpha/2) \\ -\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}. \tag{6.10}$$

(see fig 6.3 and note for example that $X = a$ and $Y = 0$ corresponds to $x' = a\cos(\alpha/2)$ and $y' = -a\sin(\alpha/2)$) this is effected via

$$\begin{aligned} (z - z_s) &= (X + iY)e^{-\frac{i\alpha}{2}} \\ &= [X\cos(\alpha/2) + Y\sin(\alpha/2)] + i[-X\sin(\alpha/2) + Y\cos(\alpha/2)] \end{aligned} \tag{6.11}$$

Now $X + iY$ can also by written in polar form $Re^{i\theta}$

Then the complex potenetial in $X - Y$ coordinates is

$$\begin{aligned} w &= C + \frac{1}{2}(X + iY)^2 \\ &= Re(C) + \frac{1}{2}(X^2 - Y^2) + i[Im(C) + AXY]. \end{aligned} \tag{6.12}$$

Using

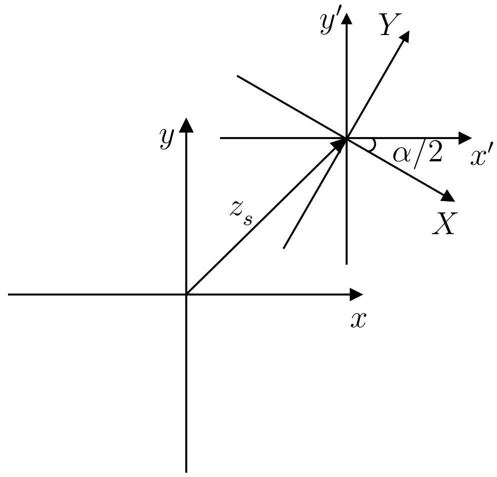

Figure 6.3: Flow at a stagnation point.

$$w = \tilde{\phi}(X,Y) + i\tilde{\psi}(X,Y)$$

$$\tilde{\psi}(X,Y) = AXY + constant \tag{6.13}$$

and

$$u_X = \frac{\partial \tilde{\psi}(X,Y)}{\partial Y} \qquad \text{and} \qquad u_Y = -\frac{\partial \tilde{\psi}(X,Y)}{\partial X}. \tag{6.14}$$

So

$$u_X = AX \qquad \text{and} \qquad u_Y = -AY. \tag{6.15}$$

So the flow in the neighbourhood of z_s in the $X-Y$ frame is the same as the flow generated by the complex potential $\frac{1}{2}z^2$.

6.3.3 Flow Around a Corner

A natural extension of the preceding analysis is to consider the complex potential

$$w(z) = Az^n, \tag{6.16}$$

In polar coordinates the complex potential becomes

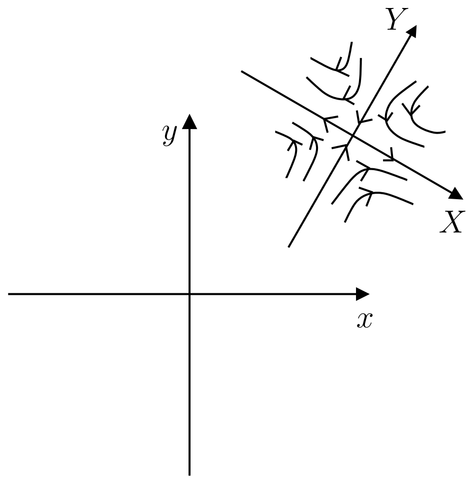

Figure 6.4: Flow at a stagnation point.

$$Ar^n e^{in\theta} = Ar^n \cos(n\theta) + iAr^n \sin(n\theta) \tag{6.17}$$

Thus the velocity potential and the streamfunction can be identified as

$$\begin{aligned} \phi &= Ar^n \cos(n\theta) \\ \psi &= Ar^n \sin(n\theta) \end{aligned} \tag{6.18}$$

The complex velocity is then

$$\frac{dw}{dz} = Anz^{n-1}. \tag{6.19}$$

The streamfunction in spherical polar coordinates is given by

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$u_\theta = -\frac{\partial \psi}{\partial r}$$

$$u_r = Anr^{n-1}\cos(n\theta), \qquad u_\theta = -Anr^{n-1} sin(n\theta) \tag{6.20}$$

Consider the streamlines $\Psi = 0$. This obviously implies that $u_\theta = 0$, which could concur with there being a wall through which there is no flow. In this case $0 \leq \theta \leq \pi/n$.

At $\theta = 0$ and $\theta = \pi/n$ $(n \neq 1)$ we have $u_\theta = 0$, while at $\theta = \pi/2n$ $u_r = 0$ and u_θ has its maximum value. Flow is parallel to the walls at the walls - the velocity component perpendicular to the wall is zero as it should be. At the point $r = 0$ we have $u_r = u_\theta = 0$ so the velocity of the fluid is zero there and we have a stagnation point.

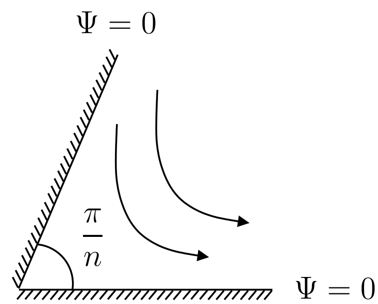

Figure 6.5: Flow at a corner.

These solutions corresponing to integer value n can be extended by having the wall at $2\pi/n$ as shown in fig (5.4) while retaining the same velocity formula (6.20) but now with the variable θ extended to the range $0 \leq \theta \leq 2\pi/n$. Where there was a wall there is now a stagnation point streamline. The streamfunction is zero on the walls and on the stagnation point streamline.

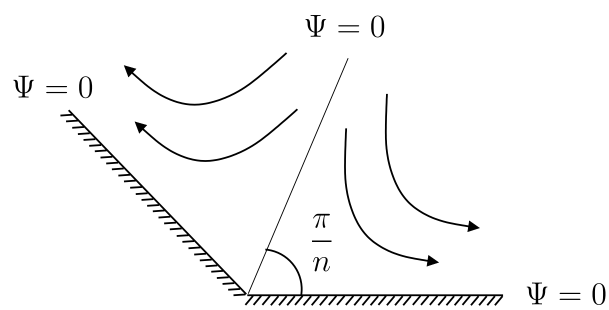

Figure 6.6: Extended solution from flow at a corner with walls at $\theta = 0$ and $\theta = 2\pi/n$.

(i) Flows for $0 \leq \theta \leq \pi/n$

Flow for $n = 2$ and walls at $\theta = 0$ and $\theta = \pi/2$

Here we can consider Cartesian coordinates.

$$w(z) = Az^2 = A(x + iy)^2 = A(x^2 - y^2) + i2Axy$$

The complex velocity can then be obtained from

$$\frac{dw(z)}{dz} = 2Az$$
$$= 2A(x+iy).$$

Implying

$$u = 2Ax$$
$$v = -2Ay$$

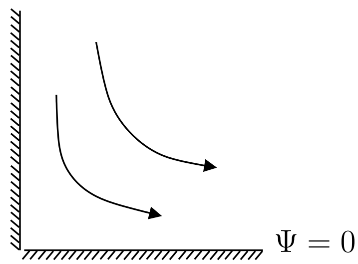

Figure 6.7: Flow around a right-angled corner, $n = 2$.

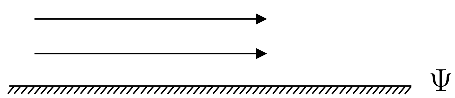

Figure 6.8: Uniform flow over a flat plate $n = 1$.

(ii) Flows for $0 \leq \theta \leq 2\pi/n$

Stagnation point flow when $n = 2$ **and walls at** $\theta = 0$ **and** $\theta = \pi$

Or if we allow

In chapter 7 we will reconsider this flow but for non-zero viscocity where we demand that $u = v = 0$ at the wall but that the flow is still described by $\psi = Axy$ as $y \to \infty$.

Flow over a wedge $1 \leq n \leq 2$

The wedge angle is

$$\vartheta = 2\pi - \frac{2\pi}{n} = 2\pi\frac{n-1}{n}$$

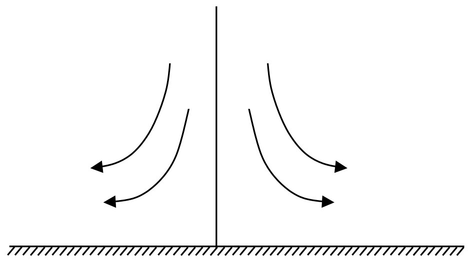

Figure 6.9: Stagnation point flow when $n = 2$.

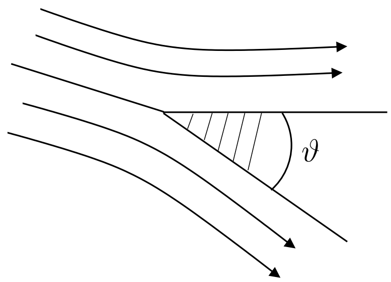

Figure 6.10: Flow over a wedge $1 \leq n \leq 2$.

Uniform flow over and below a plate for $n = 1$ with walls at $\theta = 0$, $\theta = \pi$ and $\theta = 2\pi$

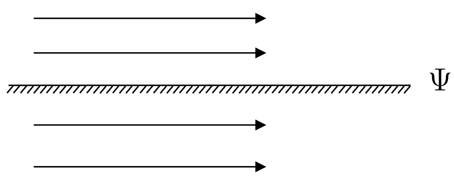

Figure 6.11: Uniform flow over a flat plate $n = 1$.

6.4 Singularities

(a) Source (and Sink)

(b) Vortex

(c) 3 (or Doublet)

6.4.1 Source (and Sink)

The complex potential for a two-dimensional source located at $z = 0$ is

$$w(z) = m \ln z \tag{6.21}$$

with $m > 0$, this gives us

$$\frac{dw}{dz} = \frac{m}{z}.$$

Recall that in polar coordinates $m \ln z = m(\ln r + i\theta)$, so that the streamfunction is given by $\psi = m\theta$

So the streamlines (ψ is a constant) are straight lines radiating from the source

$$\frac{dw}{dz} = \frac{m}{r}e^{-i\theta}, \qquad \left|\frac{dw}{dz}\right| = \frac{m}{r}$$

From $u - iv = \frac{dw}{dz}$ we have

$$u = \frac{m}{r}\cos\theta, \qquad v = \frac{m}{r}\sin\theta \tag{6.22}$$

6.4.2 Vortex

Consider the potential

$$w(z) = \Gamma i \ln z, \tag{6.23}$$

with derivative

$$\frac{dw}{dz} = \frac{i\Gamma}{z}$$

As

$$w(z) = \Gamma i(\ln(re^{i\theta})) = \Gamma(i \ln r - \theta)$$

This implies that the velocity potential is $\phi = -\Gamma\theta$ and $\psi = \Gamma \ln r$

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0$$

and

$$u_\theta = -\frac{\partial\psi}{\partial r} = -\frac{\Gamma}{r}$$

The circulation around a closed curve C given by

$$\oint_C \vec{u}\cdot d\vec{r}$$

let C be a circle of radius a so on this curve $dr = 0$

$$\oint_C \vec{u}\cdot d\vec{r} = \int_0^{2\pi} u_\theta a d\theta = -\frac{\Gamma}{a}a2\pi = -\Gamma 2\pi \tag{6.24}$$

6.4.3 Dipole (or Doublet)

Place a source and sink on the $x-$axis a distance d from the origin. The complex potential is

$$w(z) = m\ln(z+d) - m\ln(z-d)$$

which can be rewritten as

$$w(z) = m\ln\left(\frac{1+\frac{d}{z}}{1-\frac{d}{z}}\right).$$

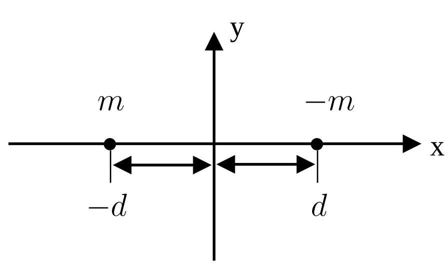

Figure 6.12: Doublet.

We have the Taylor expansions

$$\begin{aligned} \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \\ \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \dots \end{aligned}$$

Let $d \to 0$ and use a Taylor series expansion

$$\begin{aligned} w(z) &= m \ln\left(1 + \frac{d}{z}\right) - m \ln\left(1 - \frac{d}{z}\right) \\ &= m\left(\frac{d}{z} - \frac{1}{2}\frac{d^2}{z^2} + \frac{1}{3}\frac{d^3}{z^3} - \dots\right) - m\left(-\frac{d}{z} - \frac{1}{2}\frac{d^2}{z^2} - \frac{1}{3}\frac{d^3}{z^3} + \dots\right) \\ &= \frac{2md}{z}\left(1 + \frac{1}{3}\frac{d^2}{z^2} + \dots\right) \end{aligned} \tag{6.25}$$

Now we consider the dual limit of $d \to 0$ whilst $m \to \infty$ and call the limit of $2md$ μ so we have

$$w(z) = \frac{\mu}{z} \tag{6.26}$$

the direction of a dipole is source to sink, that is plus to minus. Writting

$$w(z) = \frac{\mu}{z} = \frac{\mu}{x+iy} = \frac{\mu(x-iy)}{x^2+y^2}$$

we can extract the imaginary part which gives us

$$\psi = -\frac{\mu y}{x^2+y^2}$$

Streamlines are given by ψ constant say ψ_0.

$$x^2 + y^2 = -\frac{\mu y}{\psi_0}$$

or

$$x^2 + \left(y + \frac{\mu}{2\psi_0}\right)^2 = \frac{\mu^2}{4\psi_0^2}$$

which are circles with centre at $(0, \frac{\mu}{2\psi_0})$ and radius $\frac{\mu}{2|\psi_0|}$.

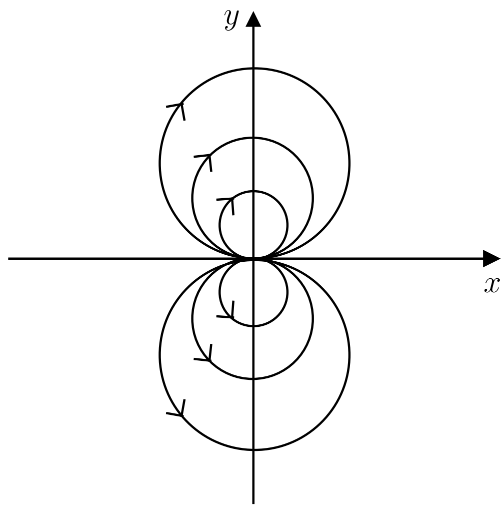

Figure 6.13: Streamlines for a dipole.

More generally the complex potential for a dipole at $z_0 = x_0 + iy_0$ which makes an angle α with the $x-$axis is given by

$$w(z^*) = \frac{\mu e^{i\alpha}}{z^* - z_0^*}. \tag{6.27}$$

First note that the complex potential

$$\frac{\mu}{z^*}$$

invokes the same flow pattern as

$$\frac{\mu}{z}.$$

Then substituting

$$x \mapsto x' = x - x_0 \quad y \mapsto y' = y - y_0$$

shifts the origin to (x_0, y_0) and this would be given by the complex potential

$$\frac{\mu}{z^* - z_0^*}.$$

Then multiplying by $e^{i\alpha}$ gives

$$\begin{aligned}\frac{\mu e^{i\alpha}}{x' - iy'} &= \frac{\mu(x'+iy')(\cos\alpha + i\sin\alpha)}{x'^2+y'^2} \\ &= \mu\frac{(x'\cos\alpha - y'\sin\alpha) + i(x'\sin\alpha + y'\cos\alpha)}{x'^2+y'^2} \end{aligned} \tag{6.28}$$

so if we write $X = x'\cos\alpha - y'\sin\alpha$ and $Y = x'\sin\alpha + y'\sin\alpha$

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}. \tag{6.29}$$

6.5 Flow Around a Circular Cylinder

From the choice of complex potential

$$w(z) = Uz + \frac{\mu}{z} \tag{6.30}$$

we obatin the complex velocity

$$\frac{dw}{dz}(z) = U - \frac{\mu}{z^2} \tag{6.31}$$

which is analytic except at the origin. We have stagnation points at $z = \pm a$ (that is where $x = \pm a$) if and only if $\mu = Ua^2$. Using this value for μ and then writting out the complex potential we obatin

$$w(z) = U\left(z + \frac{a^2}{z}\right) = U\left(x + iy + \frac{a^2}{x+iy}\right) \tag{6.32}$$

and for ψ, the imaginary part of w, we obatin

$$\psi = U\left(y - \frac{a^2 y}{x^2+y^2}\right) = Uy\left(\frac{x^2+y^2-a^2}{x^2+y^2}\right) \tag{6.33}$$

The streamlines are given by ψ being a constant and a dividing streamline is when $\psi = 0$ for $x^2 + y^2 = a^2$ which is the equation of a circle of radius a, if we consider this dividing

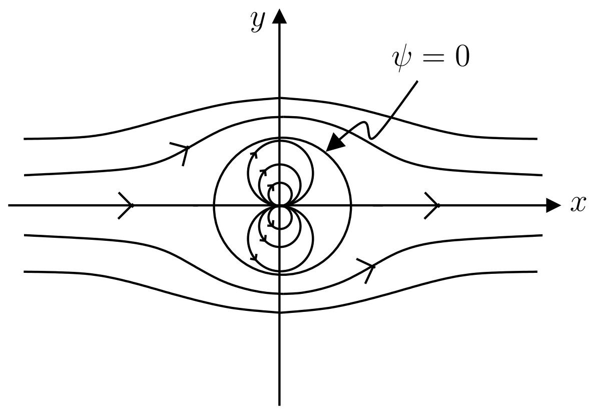

Figure 6.14: Superposition of uniform flow and a dipole results in $\psi = 0$ as a solid object, a cylinder.

streamline to be solid, we then have a cylinder. In terms of polar coordinates the cylinder is given by $z = ae^{i\theta}$ so that

$$\begin{aligned}\frac{dw}{dz} &= U\left(1 - \frac{a^2}{z^2}\right) \\ &= U(1 - e^{-2i\theta}) \\ &= Ue^{-i\theta}(e^{i\theta} - e^{-i\theta}) \\ &= 2iU \sin\theta e^{-i\theta} \end{aligned} \tag{6.34}$$

therefore

$$\left|\frac{dw}{dz}\right|^2 = 4U^2 \sin^2\theta$$

By Bernoulli's equation,

$$\frac{p}{\varrho} + \frac{1}{2}|\vec{u}|^2 = k,$$

at infinity

$$\frac{p_\infty}{\varrho} + \frac{1}{2}U^2 = k$$

and at cylinder

$$\frac{p}{\varrho} + \frac{1}{2}4U^2 \sin^2\theta = k,$$

Equating the last two expressions (both equal to k) we obtain

$$\frac{p}{\rho} = \frac{p_\infty}{\rho} + \frac{1}{2}U^2(1 - 4\sin^2\theta). \tag{6.35}$$

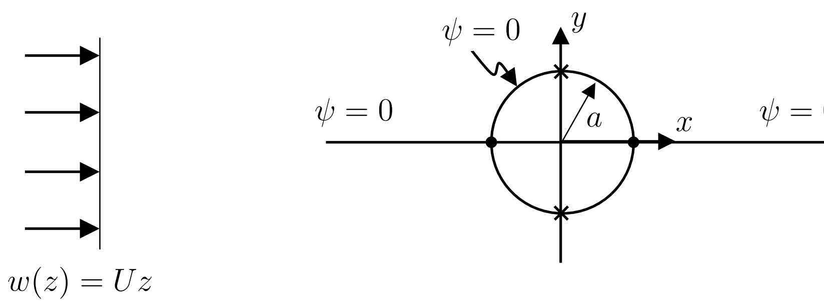

Figure 6.15: Flow around a cylinder. We have a dividing streamline for $\psi = 0$ as a circle of radius a about the origin. Another solution for $\psi = 0$ is the line $y = 0$. The points denoted by black circles denote stagnation points, namely where $dw/dz = 0$. The x's denote points of minimum pressure. Cavitation occurs here first.

Note that the pressure is the same at diametrically opposite points on the cylinder. Therefore the total force on the cylinder $\int\int p\vec{n}dS = 0$ since all the pressures cancel.

$$\vec{n} = cos\theta\vec{i} + \sin\theta\vec{j}$$

Notice that minimum pressure occurs when $\theta = \pm\pi/2$ that is when $\sin^2\theta = 1$. Therefore from (6.35) we have

$$p_{min} = p_\infty - \frac{3}{2}\varrho U^2.$$

We require the pressure p is always greater than zero, thus

$$p_\infty \geq \frac{3}{2}\varrho U^2$$

When p_∞ is equal to $\frac{3}{2}\varrho U^2$ cavitation will occur and the equations will no longer apply.

Note that p_{max} occurs at $\theta = 0$ and at π i.e. at the stagnation points, this is always the case for steady flow.

6.6 Flow Around a Rotating Cylinder

We can obtain flow around a rotating cylinder if we add a vortex to the previous solution.

6.6.1 New Complex Potential

We add a vortex to the previous complex potential (6.30) to obtain

$$w(z) = U\left(z + \frac{a^2}{z}\right) + i\Gamma \ln \frac{z}{a} \tag{6.36}$$

which is equivalent to

$$w(z) = U\left(z + \frac{a^2}{z}\right) + i\Gamma \ln z - \underbrace{i\Gamma \ln a}_{constant}.$$

The derivative is

$$\frac{dw}{dz} = U\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{z}. \tag{6.37}$$

Now

$$\psi = Im(w) = U\left(y - \frac{a^2 y}{x^2 + y^2}\right) + \Gamma \ln r - \Gamma \ln a \tag{6.38}$$

so that the dividing streamline $\psi = 0$ is given by $r = a$, but $y = 0$ is no longer a streamline:

$$\psi(x, y = 0) = \Gamma \ln x - \Gamma \ln a \neq const.$$

and so the flow is no longer symmetric about the $x-$axis.

6.6.2 Velocity Components

The radial and angular velocity components can be obtained from the formula

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = -\frac{\partial \psi}{\partial r}.$$

The streamfunction (6.38) written in terms of r and θ is

$$\psi(r,\theta) = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \Gamma\ln r - \Gamma\ln a$$

from which we obtain velocity components

$$\begin{aligned} u_r &= U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \\ u_\theta &= -U\left(1 - \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{r} \end{aligned} \tag{6.39}$$

6.6.3 Stagnation Points

Finding the stagnation points via

$$\frac{dw}{dz} = 0$$

gives the quadratic equation

$$z^2 - a^2 + \frac{i\Gamma}{U}z = 0$$

for z, with solution

$$z_s = -\frac{i\Gamma}{2U} \pm \sqrt{-\frac{\Gamma^2}{(2U)^2} + a^2}. \tag{6.40}$$

Obviously if $\Gamma = 0$ (non-rotating cylinder) then the stagnation points would be at $z_S = \pm a$ as was found before. There are three cases for $\Gamma > 0$ to consider here:

(i) $0 < \dfrac{\Gamma^2}{(2U)^2} < a^2$ $\left(0 < \dfrac{\Gamma}{2U} < a\right)$. Here the square-root in (6.40) is real and the mod-square of z is

$$|z_s|^2 = \left(-\frac{\Gamma^2}{(2U)^2} + a^2\right) + \frac{\Gamma^2}{(2U)^2} = a^2$$

and hence the stagnation points lie on the cylinder's surface. As $0 < \sin\beta < 1$ for $0 < \beta < \pi/2$, there is a β in this range such that $\Gamma = 2aU\sin\beta$. We must solve

$$a \sin\beta = \frac{\Gamma}{2U}$$

resulting in

$$\begin{aligned} z_s &= -ia\sin\beta \pm \sqrt{a^2 - a^2\sin^2\beta} \\ &= -ia\sin\beta \pm a\cos\beta. \end{aligned} \tag{6.41}$$

The situation is symmetric about the $y-$axis.

(ii) $\frac{\Gamma}{2U} = a$. It is obvious from (6.40) that there is a single stagnation point at $z = -ia$.

(iii) $\frac{\Gamma}{2U} > a$. Here the square-root in (6.40) becomes imaginary and we get one stagnation point inside $r = a$,

$$z_s = -\frac{i\Gamma}{2U} + i\sqrt{\frac{\Gamma^2}{(2U)^2} - a^2}.$$

and one outside,

$$z_s = -\frac{i\Gamma}{2U} - i\sqrt{\frac{\Gamma^2}{(2U)^2} - a^2}$$

(both) lying on the negative imaginary axis.

6.6.4 Force on a Cylinder

At the cylinder, $z = ae^{i\theta}$

$$\left.\frac{dw}{dz}\right|_{z=ae^{i\theta}} = U\left(1 - e^{-2i\theta}\right) + \frac{i\Gamma}{a}e^{-i\theta}$$

which can be rewritten

$$\frac{dw}{dz} = ie^{-i\theta}\left(2U\sin\theta + \frac{\Gamma}{a}\right)$$

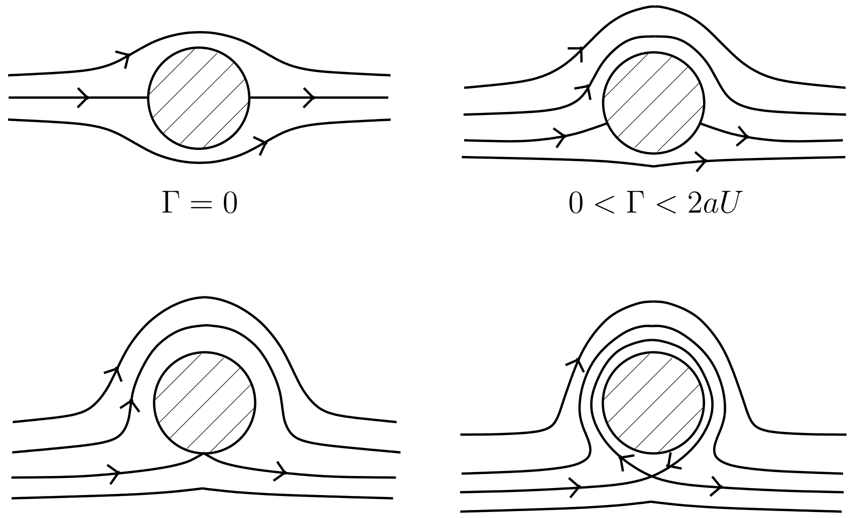

Figure 6.16: Flow around a clockwise rotating cylinder for different values of Γ.

and so

$$\left|\frac{dw}{dz}\right|^2 = \vec{u}^2 = \left(2U\sin\theta + \frac{\Gamma}{a}\right)^2$$

This splits as

$$\vec{u}^2 = 4U^2\sin^2\theta + \frac{4\Gamma U}{a}\sin\theta + \frac{\Gamma^2}{a^2}$$

The first parts are the previous velocity and the other is the addad circulation, from Bernoulli's equation $\frac{p}{\varrho} + \frac{1}{2}U^2 = k$

By Bernoulli's equation, $\dfrac{p}{\varrho} + \frac{1}{2}|\vec{u}|^2 = k$ first at infinity

$$\frac{p_\infty}{\varrho} + \frac{1}{2}U^2 = k$$

and then at cylinder

$$\frac{p}{\varrho} + \frac{1}{2}\left(4U^2\sin^2\theta + \frac{4\Gamma U}{a}\sin\theta + \frac{\Gamma^2}{a^2}\right) = k,$$

Equating the last two expressions (both are equal to k) we obtain

$$\frac{p}{\varrho} = \frac{p_\infty}{\varrho} + \frac{1}{2}U^2 - \frac{1}{2}\left(4U^2 \sin^2\theta + \frac{4\Gamma U}{a}\sin\theta + \frac{\Gamma^2}{a^2}\right) \tag{6.42}$$

which gives the pressure. Now

$$\vec{F} = \int_S -p\vec{n}dS$$

for a cylinder

$$\vec{F} = -\int_0^{2\pi} p(\theta)\vec{e}_r a d\theta \tag{6.43}$$

where $\vec{F}$ is the force per unit length, and

$$\vec{e}_r = \cos\theta\vec{e}_x + \sin\theta\vec{e}_y.$$

As $\int_0^{2\pi}\cos\theta d\theta = \int_0^{2\pi}\sin\theta d\theta = \int_0^{2\pi}\sin^3 d\theta = \int_0^{2\pi}\sin^2\theta\cos\theta d\theta = 0$ because of symmetry, the only term that contributes to $\vec{F}$ is

$$-\frac{1}{2}\varrho\frac{4\Gamma U}{a} sin\theta\vec{e}_r$$

So that

$$\vec{F} = 2\varrho\Gamma U\int_0^{2\pi}\sin\theta(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y)d\theta \tag{6.44}$$

The integral $\int_0^{2\pi}\sin\theta\cos\theta d\theta = 0$ by symmetry. We are left with the integral $\int_0^{2\pi}\sin^2\theta d\theta$,

$$\int_0^{2\pi}\sin^2\theta d\theta = \frac{1}{2}\int_0^{2\pi}(1-\cos 2\theta) = \frac{1}{2}\int_0^{2\pi}d\theta = \pi.$$

Therefore $\vec{F}$ is the force per unit length is

$$\vec{F} = 2\varrho\Gamma U\pi\vec{e}_y. \tag{6.45}$$

Notice that the force is upward.

We are familiar with the lift generated by a curving ball (backspin and top spin). The flow associated with a rotating cylinder two dimensional and is more easy to understand.

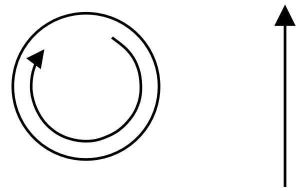

Figure 6.17: Flow around a rotating cylinder results in an upward force on the cylinder.

6.7 Other Methods for Obtaining Potentials

(a) Method of images

(b) Continuous Distributions

(c) Conformal transformations

6.7.1 Method of Images

$$w(z) = m\ln(z-a) + m\ln(z+a) \tag{6.46}$$

The image of a dipole at $z = a$ is a dipole is a dipole at $z = -a$ with axis reflected in the $y-$axis.

$$\begin{aligned} w &= -\frac{\mu e^{i\alpha}}{z-a} - \frac{\mu e^{i(\pi-\alpha)}}{z+a} \\ &= -\frac{\mu e^{i\alpha}}{z-a} + \frac{\mu e^{-i\alpha}}{z+a} \end{aligned} \tag{6.47}$$

For $z = iy$ the complex potential w becomes

$$w = \frac{\mu e^{i\alpha}}{a-iy} + \frac{\mu e^{-i\alpha}}{a+iy}$$

which is obviously real (being the sum of a complex number and its complex conjugate) and hence corresponds to $\psi = 0$.

6.7.2 Continuous Distributions

So far we have only discussed discrete distributions of singularities. Let pregress to an infinite set of singularities each infinitesimally close to its neighbour, over a finite line segment.

Take strength of source in elements δs as $m(s)\delta s/a$ (total strength is m)

$$w(z) = \int_0^a \frac{m(s)}{a} \ln(z-s)ds \tag{6.48}$$

and

$$\frac{dw}{dz} = u - iv = \int_0^a \frac{m(s)}{a} \frac{1}{z-s} ds \tag{6.49}$$

we can consider the case we have already seen, which is m is a constant. The intragl is easy to perform

$$\int_0^a \frac{1}{z-s} ds = [\ln(z-s)]_0^a = \ln(z-a) - \ln z,$$

so we end up with

$$\frac{dw}{dz} = \frac{m}{a} \ln\left(\frac{z-a}{z}\right) \tag{6.50}$$

If we write $z = re^{i\theta}$ and $z - a = r_1 e^{i\theta_1}$,

$$\frac{dw}{dz} = \frac{m}{a} \ln\left(\frac{r_1 e^{i\theta_1}}{re^{i\theta}}\right) = \frac{m}{a} \ln(\frac{r_1}{r}) + i\frac{m}{a}(\theta_1 - \theta)$$

Therefore $u = \frac{m}{a} \ln(\frac{r_1}{r})$ and $v = \frac{m}{a}(\theta_1 - \theta)$.

The technique is used in slendre body theory.

6.7.3 Conformal Transformations

Consider the angle preserving transformation $\xi = \xi(z)$. We can use this technique to find the complex potential for a large variety of geometries.

Chapter 7

Boundary Conditions

The Navier-Stokes equation

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\varrho}\nabla p + \nu\nabla^2\vec{u}$$

is a second order differential equation in space. This implies that each of the 3 velocity components must have two boundary conditions for the problem to be well-posed.

7.0.4 Flow Past a Stationary Finite Solid Body

There are two sets of boundary conditions here, one on hte body and the other at ∞. At ∞ the flowtends to the free stream of constant velocity U_∞ int the $x-$direction, that is

$$\vec{U} \to (U_\infty, 0, 0)$$

On the body surface S we have

1. kinematic condition, that is no flow through the surface

$$(\vec{u}\cdot\vec{n})_S = 0.$$

2. Non-Slip condition

In this case the velocities in tangental and binormal direction (the binormal is the cross product of the tangential and normal vector and so is orthogonal to the normal and given tangential vector. It also tangential to the surface but orthognal to the given tangential vector) are zero (that is the fluid 'sticks' to the surface - hence the non-slip condition).

Thus the velocity on the surface can be set identically equal to zero because the velocity in all three orthogonal directions (i.e. normal tangential and binormal) are zero, that is

$$\vec{u} = 0 \quad \text{on} \quad S.$$

Note however if the surface were moving with a velocity $\vec{U}$ and the fluid at infinity was at rest we would have

$$\begin{aligned} \vec{u} &\rightarrow \quad \text{at} \quad \infty \\ \vec{u} &= \vec{U} \quad \text{on} \quad S. \end{aligned} \tag{7.1}$$

3. Free-surface conditions.

(a)

Let $F(\vec{x}, t) = 0$ be the equation of any surface moving with the fluid (for example ocean surface), then a prticle of fluid remains on the surface. This tells us that the total (substantial) derivative of F must be indentically zero, that is

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \vec{u} \cdot \nabla F = 0.$$

For example with two-dimensional water waves we might define the free surface height by $y = \zeta(x, t)$

FIG HERE

The kinematic condition for the free surface

$$F = \zeta(x, t) - y = 0$$

becomes

$$\frac{\partial \zeta}{\partial t} + u \frac{\partial \zeta}{\partial x} - v = 0.$$

(b)

Chapter 8

Similarity, Scaling, Reynold's Number

Let us now scale the Navier-Stokes equation with respect to the following characteristic scales of the flow field

$$\begin{aligned} U &= \text{Velocity scale} \\ L &= \text{Characteristic length} \\ \omega &= \text{Characteristic frequency.} \end{aligned} \tag{8.1}$$

The Navier-Stokes equation yet again is

$$\underbrace{\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}}_{Inertial} = - \underbrace{\frac{1}{\varrho}}_{pressure}\nabla p \; + \underbrace{\nu\nabla^2\vec{u}}_{viscous} \tag{8.2}$$

($\nu = \mu/\varrho$). The order (size of) the respective terms is as follows

1.$\dfrac{\partial \vec{u}}{\partial t} = \mathcal{O}(\omega U)$

2. $\vec{u}\cdot\nabla\vec{u} = \mathcal{O}\left(\frac{U^2}{L}\right)$

3. $\nu\nabla^2\vec{u} = \mathcal{O}\left(\frac{\nu U}{L^2}\right)$.

1.

The ratio of (1) to (3) and (2) to (3) as follows

$$\mathcal{O}\left(\frac{\omega U}{\nu U/L^2}\right) = \mathcal{O}\left(\frac{\omega L^2}{\nu}\right)$$

2.

$$\mathcal{O}\left(\frac{U^2/L}{\nu U/L^2}\right) = \mathcal{O}\left(\frac{UL}{\nu}\right)$$

The quantity $\frac{\omega L^2}{\nu}$ is often referred to as the oscillatory Reynolds number (which is related to the Strouhal number) while,

$$R = \frac{UL}{\nu} = \varrho\frac{UL}{\mu} \tag{8.3}$$

is the famous Reynolds number which represents the ratio of inertial forces to viscous forces. If $R >> 1$ then inertial forces dominate the flow field. Conversely, if $R << 1$ then viscous forces dominate the flow field.

8.1 Dynamic Similarity

We can make the Navier-Stokes equation dimensionless if the units of time, distance and velocity are scaled in accordance with

$$\begin{aligned} t &= \frac{L}{U}t' \\ \vec{x} &= L\vec{x}' \\ \vec{u} &= U\vec{u}' \end{aligned} \tag{8.4}$$

That is, distance is measured in multiples of L, time in multiples of U/T and velocity in multiples of U. We also have

$$p = \frac{F}{A} = (\varrho L^3)\frac{U^2}{L}F'\frac{1}{L^2}\frac{1}{A'} = \varrho U^2 p'.$$

With these new variables, the derivatives get changed from $\partial/\partial x$ to $(1/L)\partial/\partial x'$, and so on. We have for the terms of the Navier-Stokes equation

$$
\begin{aligned}
\varrho\frac{\partial \vec{u}}{\partial t} &\rightarrow \frac{\varrho U^2}{L}\frac{\partial \vec{u}}{\partial t} \\
\varrho\vec{u}\cdot\nabla\vec{u} &\rightarrow \frac{\varrho U^2}{L}\vec{u}'\cdot\nabla'\vec{u}' \\
\nabla p &\rightarrow \frac{\varrho U^2}{L}\nabla' p' \\
\mu\nabla^2\vec{u} &= \frac{\mu U}{L^2}\nabla'^2\vec{u}'
\end{aligned} \tag{8.5}
$$

Substituting these in gives

$$
\frac{\partial \vec{u}'}{\partial t'} + \vec{u}'\cdot\nabla'\vec{u}' = -\frac{1}{\varrho}\nabla' p' + \frac{1}{R}\nabla'^2\vec{u}'. \tag{8.6}
$$

Whatever the scale, flows with the same R "look" the same - in terms of appropriate scaled x', y', z' and t'.

The basis of tests on scaled down models in a wind or water tunnel based on the concept of dynamical similarity that is experiments should be conducted at the same Reynolds number.

We are able to predict the values quantities to be expected on a prototype from measurements on a model.

For example: for a ship the Reynolds number R is the same in a water channel as an ocean. As ν is the same because the fluid is the same, then if L decreases, U must increase to keep R constant.

Usefull in experiments to use the same liquid for example but to change the Reynolds number by varying other characteristic quantities.

8.2 Stokes Flow

These equations

$$
\nabla p = \mu\nabla^2\vec{u} \tag{8.7}
$$

are called the Stokes flow equations where viscous effects totally dominate over inertial forces. The flow is characterised by either extreme viscosities or microscopic length scales and velodcities.

Chapter 9

Solutions of the Navier-Stoke's Equations

(a) Unidirectional (or rectilinear) flow.

As we shall see here the nonlinear term $(\vec{u} \cdot \nabla)\vec{u}$ of the Navier-Stoke's equations vanishes. This has the significant advantage that the equation is linear allowing the use of powerfull mathematical techniques for linear analysis.

(b) Circular streamlines. Example of where the nonlinear term decouples from the equation which determines the velocity vector field.

(c) Two-dimensional stagnation point flow of viscous fluid.

Recall the continuity equation in two dimensions

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

$$\frac{D}{Dt}\left(\nabla^2 \Psi\right) = \nu \nabla^4 \Psi \tag{9.1}$$

expanding the substantial derivative

$$\begin{aligned}
\frac{D}{Dt}\left(\nabla^2 \Psi\right) &= \frac{\partial}{\partial t}\left(\nabla^2 \Psi\right) + u\frac{\partial}{\partial x}\left(\nabla^2 \Psi\right) + v\frac{\partial}{\partial y}\left(\nabla^2 \Psi\right) \\
&= \frac{\partial}{\partial t}\left(\nabla^2 \Psi\right) + \frac{\partial \Psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2 \Psi\right) - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2 \Psi\right) \\
&= \frac{\partial(\nabla^2 \Psi, \Psi)}{\partial(x, y)} \\
&= \nu \nabla^4 \Psi
\end{aligned} \tag{9.2}$$

where the Jacobian is defined by

$$\frac{\partial(\nabla^2\Psi, \Psi)}{\partial(x, y)} = \frac{\partial\Psi}{\partial y}\nabla^2\left(\frac{\partial\Psi}{\partial x}\right) - \frac{\partial\Psi}{\partial x}\nabla^2\left(\frac{\partial\Psi}{\partial y}\right). \tag{9.3}$$

The RHS of this is nonlinear and again is a majour source of difficulty when trying to solve the Navier-Stokes equations.

9.1 Simplification of the Navier-Stokes Equations with Unidirectional Flow

Unidirectional (or rectilinear) flow is when the velocity vector field is given by $\vec{u} = (u, 0, 0)$, that is

$$\vec{u} = u(\vec{x}, t)$$

The continuity equation $\nabla \cdot \vec{u} = 0$ implies that

$$\frac{\partial u}{\partial x} = 0 \tag{9.4}$$

Thus u can only be a function of y, z and t, that is,

$$u = u(y, z, t).$$

The advective nonlinear term in the Navier-Stokes equation $\vec{u} \cdot \nabla\vec{u}$ is then

$$\vec{u} \cdot \nabla\vec{u} = u\frac{\partial u}{\partial x} = 0$$

because (9.4).

As there are no y and z components of velocity there can be no forces on the fluid in the the y or z direction, implying

$$\frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial z} = 0. \tag{9.5}$$

Therefore the pressure p can only be a function of x and t, that is,

$$p = p(x,t).$$

The Navier-Stokes equation reduces to

$$\underbrace{\frac{\partial u}{\partial y}}_{(1)} = \underbrace{-\frac{1}{\varrho}\frac{\partial p}{\partial x}}_{(2)} + \underbrace{\nu\nabla^2 u}_{(3)} \tag{9.6}$$

for unidirectional flows. It has the significant advantage that the equation is linear allowing the use of powerfull mathematical techniques for linear analysis. If we look more closely at (9.6), we see that parts (1) and (3) are functions of y, z and t whereas (2) is a function of x and t. As x is an independent variable $\partial p/\partial x$ cannot be a function of x because neither of the other two terms are. It can be at most a function of t. Therefore the pressure term can be at most be a function of t, that is

$$-\frac{1}{\varrho}\frac{\partial p}{\partial x} = G(t). \tag{9.7}$$

This futher simplification now allows us to write (9.6) as follows

$$\frac{\partial u}{\partial t} = G(t) + \nu\nabla^2 u. \tag{9.8}$$

In practice the pressure gradient is usually prescribed and the problem is to determine u, subject to specified boundary conditions.

9.2 Steady Unidirectional Velocity fields

(i) 2-D Poiseuille flow - the flow between two flat plates a distance b apart.

(ii) Circular cylinder

(iii) Flow in an annulus

(iv) Flow due to motion of boundaries (Couette flow)

9.2.1 2-D Poiseuille Flow

9.2.2 Circular Cylinder

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{G}{\nu}. \tag{9.9}$$

The no-slip boundary condition at the surface of the cylinder $r = a$ provides us with one boundary condition

$$u = 0 \quad \text{on} \quad r = a.$$

A "smoothness" condition at $r = 0$ of

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad r = 0.$$

This says that there is no "stress jump" along the axis of symmetry.

Integrating (9.9) once

$$\frac{\partial u}{\partial r} = -\frac{G}{\nu}\frac{1}{2}r + \frac{A}{r}$$

Using the boundary condition at $r = 0$ implies that $A = 0$. Integrating again

$$u(r) = -\frac{G}{4\nu}r^2 + B \tag{9.10}$$

The non-slip condition implies $B = Ga^2/4$, so that we finally have

$$u(r) = \frac{G}{4\nu}(a^2 - r^2) \tag{9.11}$$

9.2.3 Flow in an Annulus

Same as with (9.9) but now the non-slip conditions aplly to both cylinders, that is

$$u = 0 \quad \text{on} \quad r = a, b.$$

$$u(r) = -\frac{G}{4\nu}r^2 + A\ln r + B \tag{9.12}$$

Using the non-slip boundary conditions (where a is the radius of the outer cylinder and b is the radius of the inner cylinder)

$$0 = -\frac{G}{4\nu}a^2 + A\ln a + B \tag{9.13}$$
$$0 = -\frac{G}{4\nu}b^2 + A\ln b + B \tag{9.14}$$

Subtracting we get

$$A = \frac{G}{4\nu}\frac{(a^2 - b^2)}{\ln a/b}$$

Substituting this into (9.13) gives

$$B = \frac{G}{4\nu}a^2 - \frac{G}{4\nu}\frac{(a^2 - b^2)}{\ln a/b}\ln a$$

The final solution is

$$u(r) = \frac{G}{4\nu}\left[a^2 - r^2 + \frac{(a^2 - b^2)\ln r/a}{\ln a/b}\right]. \tag{9.15}$$

In the limit $b \to 0$, $\ln a/b \to -\infty$ and we recover the parabolic profile given in the previous section.

9.2.4 Couette Flow

Here there is no pressure gradient so $G = 0$. The problem reduces to solving

$$\frac{\partial^2 u}{\partial y^2} = 0$$

subject to the no-slip boundary conditions

$$u = 0 \quad \text{on} \quad y = 0$$
$$u = U \quad \text{on} \quad y = b$$

The solution is

$$u = \frac{Uy}{b} \tag{9.16}$$

a linear velocity profile.

The vorticity in this case is

$$\vec{\omega} = \nabla \times \vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{Uy}{b} & 0 & 0 \end{vmatrix} = -\frac{U}{b}\vec{k}. \tag{9.17}$$

This simple flow is exploited in the design of the Couette viscometer which is used to measure the viscocity of a liquid.

9.3 Unsteady Unidirectional Velocity fields

time dependence of flow have two sources

(a) through the pressure gradient $G(t)$ or

(b) through time dependence of the boundary conditions.

9.3.1 Impulsively Started Plate

$$u = \begin{cases} 0 & t < 0 \\ U_0 & t > 0 \end{cases} \tag{9.18}$$

As no pressure gradient is acting, the equations to be solved are

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

with boundary conditions on $y = 0$. There is no flow at infinity so

$$u(\infty, t) = 0$$

while the no slip boundary condition for flow at the plate for $t > 0$ implies

$$u(0, t > 0) = U_0 \tag{9.19}$$

The equation is the diffusion equation which is known to have a similarity solution.

$$u(y, t) = U_0 f(\eta) \quad \text{where } \eta = y/(2(\nu t)^{1/2})$$

It is easy

$$\frac{\partial u}{\partial t} = U_0 f'(\eta) \frac{\partial \eta}{\partial t} = U_0 f'(\eta) \left(-\frac{1}{2t} \eta \right)$$

and

$$\frac{\partial^2 u}{\partial y^2} = U_0 f''(\eta) \frac{1}{4 \nu t}$$

We obtain the ODE

$$f'' + 2\eta f' = 0 \tag{9.20}$$

or

$$(e^{-\eta^2} f)' = 0$$

so

$$f' = A e^{-\eta^2}$$

and when integrated again we obtain

$$f(\eta) = C erf(\eta) + D \tag{9.21}$$

where $erf(\eta)$ is defined as follows

$$erf(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2} d\xi. \tag{9.22}$$

We now find C and D from the boundary conditions, recalling that $u(y,t) = U_0 f(\eta)$ and $\eta = y/(2(\nu t)^{1/2}$. From no flow at infinity $u(\infty, t) = 0$ we get $C = -D$ (where we have used $\int_0^\infty e^{-\xi^2} d\xi = \sqrt{\pi}/2$). From $u(0, t > 0) = U_0$ we get $D = U_0$. Finally we have

$$u(y,t) = U_0(1 - erf(\eta))\Theta(t) \tag{9.23}$$

9.3.2 General Solution for Infinite Plate

The solution can be generalised to a general velocity $U(t)$, $t > 0$, of the infinite plate. We begin with a couple of trivial cases that will instructive in devloping the general solution.

Impulsively started plate at time t_1

First consier the case where we have the impulsively started plate that moves at speed $(U_1 - U_0)$ after $t = t_1$, then write

$$u(y,t) = (U_1 - U_0)f(\eta) \quad \text{where} \quad \eta = y/(2(t - t_1)^{1/2})$$

then

$$\frac{\partial u}{\partial t} = (U_1 - U_0)f'(\eta)\frac{\partial \eta}{\partial t} = (U_1 - U_0)f'(\eta)\left(-\frac{1}{2(t - t_1)}\eta\right)$$

and

$$\frac{\partial^2 u}{\partial y^2} = (U_1 - U_0)f''(\eta)\frac{1}{4\nu(t - t_1)}$$

We obtain $f(\eta) = Cerf(\eta) + D$ where $\eta = y/(2(t - t_1)^{1/2})$. No flow at infinity implies again that $C = -D$. And $u(0, t > t_1) = U_1 - U_0$ implies $D = U_1 - U_0$

Twice impulsively motioned plate

Second consider the case where we have the impulsively started plate that moves at speed U_0 after $t = 0$ and then later at speed U_1 for $t > t_1$ then the we have for the plate

$$u(t) = U_0 \Theta(t) + (U_1 - U_0)\Theta(t - t_1) \tag{9.24}$$

As the differential equation is linear we may add the solutions to obtain

$$u(y, t) = U_0(1 - erf(\eta^0))\Theta(t) + (U_1 - U_0)(1 - erf(\eta^{t_1}))\Theta(t - t_1) \tag{9.25}$$

where $\eta^{t_1} = y/(2\sqrt{\nu(t - t_1)})$

Let $t > t_1$

General Solution for Infinite Plate

The motion of the plate can be written

$$\begin{aligned} u(t) &= U(0)\Theta(t) + \sum_{\tau=0} d\tau (\frac{U(\tau + d\tau) - U(\tau)}{d\tau})\Theta(t - \tau - d\tau) \\ &= U(0)\Theta(t) + \int_0^\infty d\tau \frac{dU}{d\tau}\Theta(t - \tau) \end{aligned} \tag{9.26}$$

By linearity we can write for the solution

$$\begin{aligned} u(y, t) &= U(0)(1 - erf(\eta^0))\Theta(t) + \int_0^\infty \frac{dU(\tau)}{d\tau}\Theta(t - \tau)(1 - erf(\eta^\tau))d\tau \\ &= U(0)(1 - erf(\eta^0))\Theta(t) + [U(\tau)\Theta(t - \tau)(1 - erf(\eta^\tau))]_0^\infty \\ &\quad - \int_0^\infty U(\tau)\frac{d}{d\tau}[\Theta(t - \tau)(1 - erf(\eta^\tau))]d\tau \\ &= \int_0^\infty U(\tau)\frac{\partial}{\partial t}[\Theta(t - \tau)(1 - erf(\eta^\tau))]d\tau \end{aligned} \tag{9.27}$$

where

$$\eta^\tau = y/(2\sqrt{\nu(t - \tau)}).$$

The final solution for an infinite plate.

$$u(y,t) = \int_0^\infty U(\tau)\frac{\partial}{\partial t}[\Theta(t-\tau)(1-erf(\eta^\tau))]d\tau. \tag{9.28}$$

9.3.3 Oscillating Plane Boundary (Stokes Boundary Layer)

Consider the sinusoidal oscillations of an infinite plate.

$$u(0,t) = V\cos\sigma t \tag{9.29}$$

Assume the solution is of the form

$$u(y,t) = U_1(y)\cos\sigma t + U_2(y)\sin\sigma t. \tag{9.30}$$

Substituting this into

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$$

we obtain

$$-U_1(y)\sigma\sin\sigma t + U_2(y)\sigma\cos\sigma t = \nu\left[\cos\sigma t\frac{d^2U_1}{dy^2} + \sin\sigma t\frac{d^2U_2}{dy^2}\right]. \tag{9.31}$$

Implying

$$\frac{d^2U_1}{dy^2} = \frac{\sigma}{\nu}U_2(y), \qquad \frac{d^2U_2}{dy^2} = -\frac{\sigma}{\nu}U_1(y) \tag{9.32}$$

Or in matrix form

$$\frac{d^2}{dy^2}\begin{pmatrix} U_1(y) \\ U_2(y) \end{pmatrix} = \begin{pmatrix} 0 & \sigma/\nu \\ -\sigma/\nu & 0 \end{pmatrix}\begin{pmatrix} U_1(y) \\ U_2(y) \end{pmatrix}. \tag{9.33}$$

Use the trial solution

$$\begin{pmatrix} a \\ b \end{pmatrix} e^{\alpha y} \tag{9.34}$$

in (9.33)

$$\begin{pmatrix} a \\ b \end{pmatrix} \alpha^2 = \begin{pmatrix} b\nu/\sigma \\ -a\nu/\sigma \end{pmatrix} \tag{9.35}$$

So $a\alpha^2 = b\nu/\sigma = -a(\nu/\sigma)^2/\alpha^2$. Or $\alpha^4 = -(\nu/\sigma)^2$, implying $\alpha^2 = \pm i\nu/\sigma$, implying that

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad (\text{for } \alpha^2 = +i\nu/\sigma), \quad \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad (\text{for } \alpha^2 = -i\nu/\sigma)$$

and

$$\alpha = \pm\frac{1+i}{\sqrt{2}}, \quad \alpha = \pm\frac{1-i}{\sqrt{2}}.$$

The general solution is then

$$\begin{aligned} \begin{pmatrix} U_1(y) \\ U_2(y) \end{pmatrix} &= A\begin{pmatrix} 1 \\ i \end{pmatrix} \exp\left((1+i)\sqrt{\frac{\sigma}{2\nu}}y\right) + B\begin{pmatrix} 1 \\ i \end{pmatrix} \exp\left(-(1+i)\sqrt{\frac{\sigma}{2\nu}}y\right) \\ &+ C\begin{pmatrix} 1 \\ -i \end{pmatrix} \exp\left((1-i)\sqrt{\frac{\sigma}{2\nu}}y\right) + D\begin{pmatrix} 1 \\ -i \end{pmatrix} \exp\left(-(1-i)\sqrt{\frac{\sigma}{2\nu}}y\right) \end{aligned} \tag{9.36}$$

At this point we can use the boundary condition at infinity which is that there is no flow ther to get $A = C = 0$, so that

$$\begin{pmatrix} U_1(y) \\ U_2(y) \end{pmatrix} = B\begin{pmatrix} 1 \\ i \end{pmatrix} \exp\left(-(1+i)\sqrt{\frac{\sigma}{2\nu}}y\right) + D\begin{pmatrix} 1 \\ -i \end{pmatrix} \exp\left(-(1-i)\sqrt{\frac{\sigma}{2\nu}}y\right) \tag{9.37}$$

We get

$$
\begin{aligned}
\begin{pmatrix} U_1(y) \\ U_2(y) \end{pmatrix} &= \left\{ B \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-i\sqrt{\frac{\sigma}{2\nu}}y} + D \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{i\sqrt{\frac{\sigma}{2\nu}}y} \right\} e^{-\sqrt{\frac{\sigma}{2\nu}}y} \\
&= [B \begin{pmatrix} 1 \\ i \end{pmatrix} (\cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right) - i\sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right)) \\
&\qquad + D \begin{pmatrix} 1 \\ -i \end{pmatrix} (\cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right) + i\sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right))]e^{-\sqrt{\frac{\sigma}{2\nu}}y} \\
&= [\begin{pmatrix} B+D \\ 0 \end{pmatrix} \cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right) + \begin{pmatrix} 0 \\ B+D \end{pmatrix} \sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right)]e^{-\sqrt{\frac{\sigma}{2\nu}}y} \\
&+ \ i[\begin{pmatrix} 0 \\ B-D \end{pmatrix} \cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right) + \begin{pmatrix} -B+D \\ 0 \end{pmatrix} \sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right)]e^{-\sqrt{\frac{\sigma}{2\nu}}y}.
\end{aligned}
\tag{9.38}
$$

For the solution to be real we must have $B = D$, and then we have

$$
\begin{pmatrix} U_1(y) \\ U_2(y) \end{pmatrix} = [\begin{pmatrix} 2B \\ 0 \end{pmatrix} \cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right) + \begin{pmatrix} 0 \\ 2B \end{pmatrix} \sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right)] \exp\left(-\sqrt{\frac{\sigma}{2\nu}}y\right) \tag{9.39}
$$

Substituting this result into (9.30) we find for $u(y,t)$ we obtain

$$
u(y,t) = [2B\cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right)\cos\sigma t + 2B\sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right)\sin\sigma t]\exp\left(-\sqrt{\frac{\sigma}{2\nu}}y\right) \tag{9.40}
$$

Using the boundary condition that $u(0,0) = V$ implies $2B = V$. So that

$$
u(y,t) = V(\cos\left(\sqrt{\frac{\sigma}{2\nu}}y\right)\sin\sigma t + \sin\left(\sqrt{\frac{\sigma}{2\nu}}y\right)\sin\sigma t)\exp\left(-\sqrt{\frac{\sigma}{2\nu}}y\right) \tag{9.41}
$$

Using $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, the solution can be written in the final form

$$
u(y,t) = V\cos\left(\sigma t - \sqrt{\frac{\sigma}{2\nu}}y\right)\exp\left(-\sqrt{\frac{\sigma}{2\nu}}y\right). \tag{9.42}
$$

9.3.4 Oscillating Pressure Gradiant - Circular Cylinder

We define the oscillatory gradient by

$$G(t) = G \cos \sigma t \tag{9.43}$$

while requiring the non-slip condition to be satisfied on the surface of the cylinder $r = a$. The equation to be solved becomes

$$\frac{\partial u}{\partial t} = G \cos \sigma t + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{9.44}$$

Assume the solution has the form

$$u(r,t) = U_1(r) \cos \sigma t + U_2(r) \sin \sigma t \tag{9.45}$$

which leads to

$$\begin{aligned} -U_1(r)\sigma \sin \sigma t + U_2(r)\sigma \cos \sigma t \quad = \quad & G \cos \sigma t + \nu \left(\frac{d^2 U_1(r)}{dr^2} + \frac{1}{r} \frac{dU_1(r)}{dr} \right) \cos \sigma t \\ & + \nu \left(\frac{d^2 U_2(r)}{dr^2} + \frac{1}{r} \frac{dU_2(r)}{dr} \right) \sin \sigma t \end{aligned} \tag{9.46}$$

which gives the coupled differential equations

$$\begin{aligned} r^2 \frac{d^2 U_1(r)}{dr^2} + r \frac{dU_1(r)}{dr} - \frac{\sigma}{\nu} r^2 U_2(r) \quad &= \quad -r^2 \frac{G}{\nu} \\ r^2 \frac{d^2 U_2(r)}{dr^2} + r \frac{dU_2(r)}{dr} + \frac{\sigma}{\nu} r^2 U_1(r) \quad &= \quad 0 \end{aligned} \tag{9.47}$$

The non-slip condition satisfied at the surface of the cylinder $r = a$ reads

$$u(a,t) = U_1(a) \cos \sigma t + U_2(a) \sin \sigma t = 0. \tag{9.48}$$

which implies

$$U_1(a) = U_2(a) = 0 \tag{9.49}$$

Defining $\tilde{U}_2(r) = U_2(r) - G/\sigma$, then we can write (9.47) as

$$\begin{aligned} r^2\frac{d^2U_1(r)}{dr^2}+r\frac{dU_1(r)}{dr}-\frac{\sigma}{\nu}r^2\tilde{U}_2(r) &= 0 \\ r^2\frac{d^2\tilde{U}_2(r)}{dr^2}+r\frac{d\tilde{U}_2(r)}{dr}+\frac{\sigma}{\nu}r^2U_1(r) &= 0 \end{aligned} \tag{9.50}$$

If we write

$$z(r) = U_1(r) - i\tilde{U}_2(r) \tag{9.51}$$

Then equations (9.50) can be rewritten

$$r^2\frac{d^2z(r)}{dr^2}+r\frac{dz(r)}{dr}-i\frac{\sigma}{\nu}r^2z(r)=0 \tag{9.52}$$

and has the solution

$$z(r) = CI_0(i^{1/2}\sqrt{\frac{\sigma}{\nu}}r) \tag{9.53}$$

where I_0 is the modified Bessel function of the first kind which satisfies the differential equation

$$x^2\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dr}-x^2y(x)=0.$$

From $I_0(x) = J_0(ix)$ we have $I_0(i^{1/2}x) = J_0(i^{3/2}x)$ (where J_0 is the Bessel function of the first kind). The real and imaginary parts of $J_0(i^{3/2}x)$ are the so-called Kelvin functions,

$$ber_0(x) = Re\ J_0(i^{3/2}x), \qquad bei_0(x) = Im\ J_0(i^{3/2}x) \tag{9.54}$$

The solution is then

$$z(r) = C[ber_0(\sqrt{\frac{\sigma}{\nu}}r) + ibei_0(\sqrt{\frac{\sigma}{\nu}}r)]$$

or

$$U_1(r) - i\tilde{U}_2(r) = (C_R + iC_I)[ber_0(\sqrt{\frac{\sigma}{\nu}}r) + ibei_0(\sqrt{\frac{\sigma}{\nu}}r)].$$

Written out

$$
\begin{aligned}
& U_1(r) - i(U_2(r) - \frac{G}{\sigma}) \\
= \;& (C_R + iC_I)(ber_0(\sqrt{\frac{\sigma}{\nu}}r) + i\; bei_0(\sqrt{\frac{\sigma}{\nu}}r)) \\
= \;& C_R ber_0(\sqrt{\frac{\sigma}{\nu}}r) - C_I bei_0(\sqrt{\frac{\sigma}{\nu}}r) + i[C_I ber_0(\sqrt{\frac{\sigma}{\nu}}r) + C_R bei_0(\sqrt{\frac{\sigma}{\nu}}r)]
\end{aligned}
\tag{9.55}
$$

The boundary conditions (9.49) imply

$$
\begin{aligned}
U_1(a) &= C_R ber_0(\sqrt{\frac{\sigma}{\nu}}a) - C_I bei_0(\sqrt{\frac{\sigma}{\nu}}a) = 0 \\
U_2(a) - \frac{G}{\sigma} &= -C_I ber_0(\sqrt{\frac{\sigma}{\nu}}a) - C_R bei_0(\sqrt{\frac{\sigma}{\nu}}a) = -\frac{G}{\sigma}
\end{aligned}
\tag{9.56}
$$

The first condition is solved by

$$
C_R = A\; bei_0(\sqrt{\frac{\sigma}{\nu}}a), \qquad C_I = A\; ber_0(\sqrt{\frac{\sigma}{\nu}}a)
$$

inserting this into the second condition gives

$$
A\left(ber_0^2(\sqrt{\frac{\sigma}{\nu}}a) + bei_0^2(\sqrt{\frac{\sigma}{\nu}}a)\right) = \frac{G}{\sigma}.
$$

Therefore we have:

$$
U_1(r) = \frac{G}{\sigma}\; \frac{bei_0(\sqrt{\frac{\sigma}{\nu}}a)ber_0(\sqrt{\frac{\sigma}{\nu}}r) - ber_0(\sqrt{\frac{\sigma}{\nu}}a)bei_0(\sqrt{\frac{\sigma}{\nu}}r)}{ber_0^2(\sqrt{\frac{\sigma}{\nu}}a) + bei_0^2(\sqrt{\frac{\sigma}{\nu}}a)} \tag{9.57}
$$

and

$$
U_2(r) = -\frac{G}{\sigma}\; \frac{[ber_0(\sqrt{\frac{\sigma}{\nu}}r)ber_0(\sqrt{\frac{\sigma}{\nu}}a) + bei_0(\sqrt{\frac{\sigma}{\nu}}r)bei_0(\sqrt{\frac{\sigma}{\nu}}a)]}{ber_0^2(\sqrt{\frac{\sigma}{\nu}}a) + bei_0^2(\sqrt{\frac{\sigma}{\nu}}a)} + \frac{G}{\sigma} \tag{9.58}
$$

Let us introduce

$$\alpha = \sqrt{\frac{\sigma}{\nu}} a \tag{9.59}$$

the well known Womersley parameter of blood control. If we write

$$B_r = \frac{ber_0(\alpha)}{ber_0^2(\alpha) + bei_0^2(\alpha)} \quad \text{and} \quad B_i = \frac{bei_0(\alpha)}{ber_0^2(\alpha) + bei_0^2(\alpha)}$$

then (9.45) can be written as

$$\begin{aligned} u(r,t) &= \frac{G}{\sigma}\left[(B_i ber_0(\alpha r/a) - B_r bei_0\,(\alpha r/a)\right] \cos\sigma t + \\ &\quad + \frac{G}{\sigma}\left[1 - B_r ber_0(\alpha r/a) + B_i bei_0(\alpha r/a)\right] \sin\sigma t. \end{aligned} \tag{9.60}$$

This can be written as

$$u(r,t) = Re\left[\frac{G}{i\sigma}\left(1 - \frac{I_0(\sqrt{i}\alpha r/a)}{I_0(\sqrt{i}\alpha)}\right) e^{i\sigma t}\right] \tag{9.61}$$

as can be seen from writting out

$$\begin{aligned} & \frac{G}{\sigma}\frac{1}{i}\left(1 - \frac{I_0(\sqrt{i}\alpha r/a)}{I_0(\sqrt{i}\alpha)}\right) \\ = & \frac{G}{\sigma}\frac{1}{i}\left(1 - \frac{ber_0(\alpha r/a) + i bei_0(\alpha r/a)}{ber_0(\alpha) + i bei_0(\alpha)}\right) \\ = & \frac{G}{\sigma}\frac{1}{i}\left(1 - \frac{[ber_0(\alpha r/a) + i bei_0(\alpha r/a)][ber_0(\alpha) - i bei_0(\alpha)]}{ber_0^2(\alpha) + bei_0^2(\alpha)}\right) \\ = & \frac{G}{\sigma}\frac{1}{i}\left(1 - [ber_0(\alpha r/a) + i bei_0(\alpha r/a)][B_r - iB_i]\right) \\ = & \frac{G}{\sigma}\frac{1}{i}(1 - B_r ber_0(\alpha r/a) - B_i bei_0(\alpha r/a) + i\left[-B_r bei_0(\alpha r/a) + B_i ber_0(\alpha r/a)\right]) \\ = & \frac{G}{\sigma}\left[B_i ber_0(\alpha r/a) - B_r bei_0(\alpha r/a)\right] - i\frac{G}{\sigma}\left[1 - B_r ber_0(\alpha r/a) + B_i bei_0(\alpha r/a)\right]. \end{aligned} \tag{9.62}$$

9.4 Circular Streamlines

Let us consider steady flow between two rotating cylinders. The inner cylinder has radius r_1 and angular velocity Ω_1 whereas the outer cylinder has radius r_2 and angular velocity Ω_2.

The boundary conditions for the velocity are

$$\begin{aligned} v = \Omega_1 r_1 \quad & \text{on} \quad r = r_1 \\ v = \Omega_2 r_2 \quad & \text{on} \quad r = r_2. \end{aligned} \tag{9.63}$$

$$\vec{u} = u_r \vec{e}_r + u_\phi \vec{e}_\phi + u_z \vec{e}_z \tag{9.64}$$

$$\frac{\partial}{\partial \phi} \vec{e}_\phi = -\vec{e}_r, \qquad \frac{\partial}{\partial \phi} \vec{e}_r = \vec{e}_\phi \tag{9.65}$$

The Laplace of a scalar in cylindrical polar coordinates (with no z dependence and $u_z = 0$)

$$\nabla^2 F = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial F}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 F}{\partial \phi^2}.$$

The advective term becomes

$$\begin{aligned} \vec{u} \cdot \nabla \vec{u} &= (u_\phi \vec{e}_\phi) \cdot (\vec{r} \frac{\partial}{\partial r} + \vec{e}_\phi \frac{1}{r} \frac{\partial}{\partial \phi})(u_\phi \vec{e}_\phi) \\ &= \frac{v(r)}{r} \frac{\partial}{\partial \phi} (v(r) \vec{e}_\phi) \\ &= -\frac{v^2}{r} \vec{e}_r \end{aligned} \tag{9.66}$$

We inspect the effect of the Laplacian on the vector $v\vec{e}_\phi$ employing (9.65) and using $v = v(r)$,

$$\begin{aligned} \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right) (v\vec{\phi}) &= \vec{e}_\phi \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) \right) v + \frac{1}{r^2} \frac{\partial}{\partial \phi} (-v \vec{e}_r) \\ &= \vec{e}_\phi \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} \right) \right) v - \frac{v}{r^2} \vec{e}_\phi \\ &= \vec{e}_\phi \left(\nabla^2 - \frac{1}{r^2} \right) v \end{aligned} \tag{9.67}$$

(this illustrates the fact that in non-Cartesian coordinates the vector Laplacian resolved in these non-Cartesian coordinates does not coincide with the Laplacian of a scalar in the non-Cartesian coordinates).

The Navier-Stokes equation

$$\vec{u} \cdot \nabla \vec{u} = -\frac{1}{\varrho}\frac{dp}{dr} + \nu \nabla^2 \vec{u}$$

becomes

$$-\frac{v^2}{r}\vec{e}_r = -\frac{1}{\varrho}\frac{dp}{dr}\vec{e}_r + \nu \vec{e}_\phi \left(\nabla^2 - \frac{1}{r^2} \right) v$$

and decouple to yield

$$\frac{1}{\varrho}\frac{dp}{dr} = \frac{v^2}{r} \tag{9.68}$$

and

$$\left(\nabla^2 - \frac{1}{r^2} \right) v(r). \tag{9.69}$$

Rewriting (9.69)

$$\left(r^2 \frac{d^2}{dr^2} + r\frac{d}{dr} - 1 \right) v(r) = 0$$

and trying the trial solution r^λ gives

$$(\lambda(\lambda - 1) + \lambda - 1)r^\lambda = 0$$

or $(\lambda - 1)(\lambda + 1) = 0$. The general solution is then

$$v(r) = Ar + \frac{B}{r} \tag{9.70}$$

The boundary conditions (9.63) read

$$\Omega_1 r_1 = Ar_1 + \frac{B}{r_1} \quad \text{and} \quad \Omega_2 r_2 = Ar_2 + \frac{B}{r_2}$$

implying $\Omega_2 r_2^2 - \Omega_1 r_1^2 = A(r_2^2 - r_1^2)$ and $\Omega_1 r_1(r_1 r_2^2) - \Omega_2 r_2(r_2 r_1^2) = B(r_2^2 - r_1^2)$, so that

$$\begin{aligned} A &= \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} \\ B &= -\frac{(\Omega_2 - \Omega_1) r_1^2 r_2^2}{r_2^2 - r_1^2}. \end{aligned} \tag{9.71}$$

Special cases:

(i) $B = 0$ is obtained if $\Omega_2 = \Omega_1 = \Omega$. The solution is then $v = \Omega r$ and this corresponds to ridgid body rotation.

(ii) $A = 0$ occurs when $\Omega_2 = 0$ and $r_2 \to \infty$. In this case $v = \Omega_1 r_1^2/r$ which corresponds to the motion due to an irrotational vortex: the circulation around a circle centered at the origin of radius $r \geq r_1$ is

$$\Gamma = \int_0^{2\pi} \frac{\Omega_1 r_1^2}{r}(r d\theta) = 2\pi \Omega_1 r_1^2 \tag{9.72}$$

so no-zero, yet the vorticity vanises everywhere where in the fluid,

$$\vec{\omega} = \nabla \times \vec{u} = \frac{1}{r} \begin{vmatrix} \vec{e}_r & r\vec{e}_\phi & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ 0 & \Omega_1 r_1^2 & 0 \end{vmatrix} = 0. \tag{9.73}$$

Consider the situation when $r_1 \to 0$ while $\Omega_1 r_1^2 = Const.$ and invoke Stokes theorem

$$\oint_C \vec{u} \cdot d\vec{x} = \int_S \vec{\omega} \cdot \vec{n} dS,$$

this implies a delta function of vorticity at the origin.

9.5 Two Dimensional Stagnation Point flow

The streamfunction was

$$\psi = Axy$$

The velocities were

$$u = \frac{\partial \psi}{\partial y} = Ax$$

$$v = -\frac{\partial \psi}{\partial x} = -Ay$$

This will not be a solution for viscous flow because $u \neq 0$ on the boundary $y = 0$.

We require the far-field to look like the above solution whereas on the plane $y = 0$, $u = v = 0$. In summary

$$\begin{aligned} u = v = 0 \quad &\text{on} \quad y = 0 \\ \psi = Axy \quad &\text{as} \quad y \to \infty. \end{aligned} \tag{9.74}$$

The steady, two-dimensional Navier-Stokes equations become

$$\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = \nu \nabla^4 \psi \tag{9.75}$$

Let us consider a solution of the form

$$\psi = xF(y) \tag{9.76}$$

The velocities are then

$$\begin{aligned} u &= \frac{\partial \psi}{\partial y} = xF'(y) \\ v &= -\frac{\partial \psi}{\partial x} = F(y). \end{aligned} \tag{9.77}$$

The assumption (9.76) leads to

$$\begin{aligned} &\nabla^2\psi = xF'' \qquad \nabla^4\psi = xF'''' \\ &\frac{\partial(\nabla^2\psi, \psi)}{\partial(x, y)} = x(F''F' - F'''F). \end{aligned} \tag{9.78}$$

where we used (9.3). The equation reduces to

$$F''F' - F'''F = \nu F''''$$

This can be written

$$((F')^2 - FF'')' = \nu F''''$$

upon integrating once gives

$$(F')^2 - FF'' = \nu F''' + c$$

where c is the constant of integration. It is clear that as $y \to \infty$ that $F \to Ay$, $F' \to A$ etc so that $c = A^2$. Then

$$(F')^2 - FF'' = \nu F''' + A^2$$

This equation has two parameters ν and A. If we apply the following scaling

$$\eta = \sqrt{\frac{A}{\nu}}y, \qquad F(y) = \sqrt{A\nu}G(\eta)$$

the equation reduces to

$$(G')^2 - GG'' = G''' + 1$$

with boundary conditions $u = v = 0$ at $y = 0$ which imply

$$G(0) = G'(0) = 0$$

and boundary condition $\psi = Axy$ as $y \to \infty$ which implies

$$\frac{dF}{dy}(\infty) = A \quad \text{or} \quad \sqrt{\frac{A}{\nu}}\frac{d\sqrt{A\nu}G)}{d\eta}(\infty) = A$$

or

$$G'(\infty) = 1.$$

This may be solved numerically. The solution of the equation is now of the form

$$\psi = \sqrt{A\nu}\, x G\left(\sqrt{\frac{A}{\nu}}\, y\right)$$

Chapter 10

Mathematical Theorems

The theorems derived here pertain to the approximations to the Navier-Stokes equation given by

$$\nabla p = \mu \nabla^2 \vec{u}$$

and the continuity equationd

$$\nabla \cdot \vec{u} = 0.$$

These are the Stokes flow equations for slow viscous flow of an incompressible fluid.

Recall the rate of strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and that the incompressibilty condition $\nabla \cdot \vec{u} = 0$ is equivalent to $e_{kk} = 0$. Also that the constituative relation then for incompressible flow is

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$$

It is easy to see that an alternative form of Stokes flow equations are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0 \tag{10.1}$$

using the continuity equation in component form

$$\frac{\partial u_j}{\partial x_j} = 0.$$

10.1 Uniqueness Theorem

Theorem Let $(\vec{u}, p)$ satisfy

$$\nabla p = \mu \nabla^2 \vec{u} \qquad \nabla \cdot \vec{u} = 0 \tag{10.2}$$

in a volume V with boundary conditions

$$\vec{u} = \vec{U}(x) \quad \text{on} \quad S$$

($x \in S$, S may consist of interior as well as exterior boundaries). Let $(\vec{u}', p')$ also satisfy (10.2); then

$$\vec{u} \equiv \vec{u}' \quad \text{in} \quad V. \tag{10.3}$$

Proof

Let

$$\begin{aligned} \tilde{\vec{u}} &= \vec{u} - \vec{u}' \\ \tilde{e}_{ij} &= e_{ij} - e'_{ij} \\ \tilde{\sigma}_{ij} &= \sigma_{ij} - \sigma'_{ij} \end{aligned} \tag{10.4}$$

Consider the following integral,

$$\begin{aligned} \int_V \tilde{e}_{ij} \tilde{\sigma}_{ij} dV &= \int_V \frac{1}{2} \left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i} \right) \tilde{\sigma}_{ij} dV \\ &= \int_V \frac{\partial \tilde{u}_i}{\partial x_j} \tilde{\sigma}_{ij} dV \end{aligned} \tag{10.5}$$

The last line on the RHS is obtained because of the symmetry of $\tilde{\sigma}_{ij}$ namely $\tilde{\sigma}_{ij} = \tilde{\sigma}_{ji}$. Using the alternative form of Stokes flow equation, (10.1), we can write (10.5) as

$$\int_V \frac{\partial}{\partial x_j}(\tilde{u}_i \tilde{\sigma}_{ij}) dV$$

which becomes

$$\int_S \tilde{u}_i \tilde{\sigma}_{ij} n_j dS = 0$$

after employing the divergence theorem and noting that

$$\tilde{\vec{u}} = \vec{u} - \vec{u}' = 0$$

on S, because both the velocity fields satisfy the boundary conditions. We have thus shown that (10.5) vanishes.

Now because

$$\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + 2\mu \tilde{e}_{ij}$$

and

$$\tilde{e}_{ij}\delta_{ij} = \tilde{e}_{kk} = 0$$

it is easy to show that the LHS of (10.5) is equivalent to

$$2\mu \int_V \tilde{e}_{ij}\tilde{e}_{ij} dV$$

which we have shown is zero. Clearly the integrand must be zero, that is

$$\tilde{e}_{ij} = 0$$

everywhere in V which tells us that

$$\tilde{\vec{u}} := \vec{u} - \vec{u}'$$

induces at most a uniform translation or rotation. However since $\vec{u} = \vec{u}'$ on S it implies that

$$\vec{u} = \vec{u}' = 0 \quad \text{in} \quad V.$$

This proves the theorem.

10.2 Minimum Dissipation Theorem

10.2.1 Energy Balance and Dissipation of Energy

$$E_{kin} = \frac{1}{2}\int_V \varrho\vec{u}\cdot\vec{u}dV$$

The rate of change of Kinetic Energy is

$$\frac{\partial E_{kin}}{\partial t} = \int_V \frac{1}{2}\frac{\partial(\varrho\vec{u}\cdot\vec{u})}{\partial t} \tag{10.6}$$

In the absense of any forces an kinetic Energy change within the volume V can occur by Energy flowing across the boundary,

$$\frac{\partial E_{kin}}{\partial t} = -\int_S (\frac{1}{2}\varrho\vec{u}\cdot\vec{u})\vec{u}\cdot\vec{n}dS \tag{10.7}$$

In the pressence of viscous and body forces the rate of change of energy is

$$\frac{\partial E_{kin}}{\partial t} = -\int_S (\frac{1}{2}\varrho\vec{u}\cdot\vec{u})\vec{u}\cdot\vec{n}dS = \int_S u_i\sigma_{ij}n_j dS + \int_V \varrho u_i f_i dV \tag{10.8}$$

Using the divergence theorem

$$
\begin{aligned}
\int_V \left[\frac{\partial E}{\partial t} + \nabla\cdot(E\,\vec{u})\right] dV &= \int_V \left[\frac{\partial}{\partial x_j}(u_i\sigma_{ij}) + \varrho u_i f_i\right] dV \\
&= \int_V \left[\frac{\partial u_i}{\partial x_j}\sigma_{ij} + u_i\frac{\partial \sigma_{ij}}{\partial x_j} + \varrho u_i f_i\right] dV \\
&= \int_V \left[e_{ij}\sigma_{ij} + u_i\frac{\partial \sigma_{ij}}{\partial x_j} + \varrho u_i f_i\right] dV \\
&= \int_V \left[e_{ij}(-p\delta_{ij} + 2\mu e_{ij}) + u_i\frac{\partial}{\partial x_j}(-p\delta_{ij} + 2\mu e_{ij}) + \varrho u_i f_i\right] dV \\
&= \int_V \left[2\mu e_{ij}e_{ij} - \vec{u}\cdot\nabla p + \mu u_i\nabla^2 u_i + \varrho u_i f_i\right] dV.
\end{aligned} \tag{10.9}
$$

The LHS can be rewritten

$$
\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}(\varrho u^2) + \frac{1}{2}\nabla\cdot(\varrho u^2\vec{u}) &= \varrho\frac{\partial}{\partial t}(\frac{1}{2}u^2) + \frac{1}{2}u^2\frac{\partial\varrho}{\partial t} + \varrho\vec{u}\cdot\nabla(\frac{1}{2}u^2) + \frac{1}{2}u^2\nabla\cdot(\varrho\vec{u}) \\
&= \varrho\left[\frac{\partial}{\partial t}(\frac{1}{2}u^2) + \vec{u}\cdot\nabla(\frac{1}{2}u^2)\right] + \frac{1}{2}u^2\left[\frac{\partial\varrho}{\partial t} + \nabla\cdot(\varrho\vec{u})\right] \\
&= \varrho\left[\frac{\partial}{\partial t}(\frac{1}{2}u^2) + \vec{u}\cdot\nabla(\frac{1}{2}u^2)\right] \\
&= \varrho\frac{D}{Dt}(\frac{1}{2}u^2).
\end{aligned} \tag{10.10}
$$

As we are assuming an incompressible fluid we can bring the density ϱ inside the substantial derivative. We obtain an Energy balance equation, Energy is not conserved and there is a viscous dissipation of Energy on the RHS

$$
\frac{D}{Dt}(\frac{1}{2}\varrho u^2) + \vec{u}\cdot\nabla p - \varrho\vec{u}\cdot\vec{f} - \mu\vec{u}\cdot\nabla^2\vec{u} = 2\mu e_{ij}e_{ij}. \tag{10.11}
$$

10.2.2 Proof of Minimum Dissipation Theorem

Viscous dissipation is given

$$
2\mu\int_V e_{ij}e_{ij}dV. \tag{10.12}
$$

Theorem Let $(\vec{u}, p)$ be the unique flow satisfying

$$\begin{aligned} \nabla p = \mu \nabla^2 \vec{u}, \qquad \nabla \cdot \vec{u} = 0 \\ \vec{u} = \vec{U} \quad \text{on} \quad S \end{aligned} \tag{10.13}$$

and let $\vec{u}'$ be any other vector field satisfying

$$\nabla \cdot \vec{u}' = 0 \qquad \text{and} \qquad \vec{u}' = \vec{U} \quad \text{on} \quad S \tag{10.14}$$

then the rate of dissipation of energy is least in the flow satisfying (10.13).

Proof

Let

$$\begin{aligned} \tilde{\vec{u}} &= \vec{u} - \vec{u}' \\ \tilde{e}_{ij} &= e_{ij} - e'_{ij} \end{aligned} \tag{10.15}$$

where the $\vec{u}'$ is the one defined in this theorem and e'_{ij} corresponds to this $\vec{u}'$. We do a caluculation similar to one done in the previous theorem,

$$\begin{aligned} \int_V \tilde{e}_{ij} \sigma_{ij} dV &= \int_V \frac{\partial \tilde{u}_i}{\partial x_j} \sigma_{ij} dV \\ &= \int_V \frac{\partial}{\partial x_j} (\tilde{u}_i \sigma_{ij}) dV \\ &= \int_S \tilde{u}_i \sigma_{ij} n_j dS = 0. \end{aligned} \tag{10.16}$$

As $\vec{u} - \vec{u}' = 0$ on S. Substituiting $\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij}$ into the first integral implies

$$2\mu \int \tilde{e}_{ij} e_{ij} dV = 0. \tag{10.17}$$

$$\begin{aligned} 2\mu \int_V e'_{ij} e'_{ij} dV &= 2\mu \int_V (e_{ij} - \tilde{e}_{ij})^2 dV \\ &= 2\mu \int_V e_{ij} e_{ij} dV + 2\mu \int_V \tilde{e}_{ij} \tilde{e}_{ij} dV - 4\mu \int_V e_{ij} \tilde{e}_{ij} dV \end{aligned} \tag{10.18}$$

Since the last term is zero by (10.17) and $\tilde{e}_{ij}\tilde{e}_{ij} \geq 0$ we must have

$$2\mu \int_V e'_{ij}e'_{ij}dV \geq 2\mu \int_V e_{ij}e_{ij}dV \tag{10.19}$$

which proves the theorem. That is the Stokes flow is the one of minimum dissipation for given boundary conditions.

10.2.3 Vorticity Minimum Dissipation Theorem

Assume $\vec{u}$ is an irrotaional velocity field field described by a velocity potential, ϕ, and $\vec{u}'$ is an arbitrary solenoidal rotational velocity field. The velocity field corresponding to an irrotational flow has a least amount of kinetic energy. We impose the boundary conitions on these velocity fields

$$\vec{u} \cdot \vec{n} = \vec{u}' \cdot \vec{n}. \tag{10.20}$$

Assuming, the same, constant density ϱ the continuity equation for both these velocity fields are $\nabla \cdot \vec{u} = \nabla \cdot \vec{u}' = 0$. The difference in kinetic energies of the two flows, $\Delta E_{kin} = E_{kin}(\vec{u}') - E_{kin}(\vec{u})$, is

$$\begin{aligned}\Delta E_{kin} &= \frac{1}{2}\varrho \int_V (\vec{u}' \cdot \vec{u}' - \vec{u} \cdot \vec{u})dV \\ &= \frac{1}{2}\varrho \int_V (\vec{u}' - \vec{u}) \cdot (\vec{u}' - \vec{u}) + \varrho \int_V (\vec{u}' - \vec{u}) \cdot \vec{u}dV.\end{aligned} \tag{10.21}$$

Using $\vec{u} = \nabla\phi$ and the divergence theorem on the last term on the RHS gives

$$\begin{aligned}\int_V (\vec{u}' - \vec{u}) \cdot \nabla\phi dV &= \int_V \nabla \cdot [(\vec{u}' - \vec{u})\phi]dV \\ &= \int_S \phi(\vec{u}' - \vec{u}) \cdot \vec{n}dS,\end{aligned} \tag{10.22}$$

which is zero by the boundary condition (10.20). This implies that the RHS of (10.21) is positive and therefore the kinetic energy of rotational flow with velocity $\vec{u}'$ is greater than the kinetic energy of the corresponding irrotational flow with velocity $\vec{u}$.

Chapter 11

Stoke's Streamfunction

Three-dimensional axisymmetric flows.

$$\nabla \cdot \vec{u} = 0$$

implies

$$\vec{u} = \nabla \times \vec{A} \tag{11.1}$$

11.1 Streamfunction in Cylindrical and Spherical Polar Coordinates

11.1.1 Streamfunction in Cylindrical Polar Coordinates

The velocity field is independent of ϕ and only have

$$\vec{u} = u_\rho \vec{e}_\rho + u_z \vec{e}_z$$

$$\vec{A} = \frac{\Psi(\rho, z)}{\rho} \vec{e}_\varphi \tag{11.2}$$

$$
\begin{aligned}
\vec{u} &= \nabla \times (A_\rho \vec{e}_\rho + A_\varphi \vec{e}_\varphi + A_z \vec{e}_z) \\
&= \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix} \\
&= \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & \Psi(\rho, z) & 0 \end{vmatrix} \\
&= -\frac{1}{\rho}\frac{\partial \Psi}{\partial z}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho}\vec{e}_z
\end{aligned} \tag{11.3}
$$

Reading off the velocity components,

$$
u_r = -\frac{1}{\rho}\frac{\partial \Psi}{\partial z}, \qquad u_z = \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho} \tag{11.4}
$$

We confirm that Ψ is indeed constant along streamlines, which are defined by

$$
\frac{d\vec{x}}{ds} = \lambda \vec{u}
$$

Streamlines given by $d\rho = \lambda u_\rho ds$ and $dz = \lambda u_z ds$, so

$$
\begin{aligned}
d\Psi &= \frac{\partial \Psi}{\partial r}d\rho + \frac{\partial \Psi}{\partial z}dz \\
&= \rho u_z d\rho - \rho u_\rho dz \\
&= \rho u_z \lambda u_\rho ds - \rho u_\rho \lambda u_z ds = 0.
\end{aligned} \tag{11.5}
$$

Therefore $\Psi = Const.$ on streamlines (actually stream surfaces in 3D flow).

By using the formula for the curl in cylindrical polar coordinates (an example of an orthogonal curvilinear coordinate system)

$$
\nabla \times \vec{u} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}.
$$

It turns out that the term $\nabla(\nabla \cdot \vec{A})$ vanishes under certain circumstances e.g., when either $\nabla \times \vec{u}$ is bounded in space or vanishes more rapidily than $1/\rho$ for large ρ. Then

$$
\nabla \times \vec{u} = -\nabla^2 \vec{A}
$$

$$\begin{aligned}
\nabla \times \vec{u} &= \nabla \times (u_\rho \vec{e}_\rho + u_\theta \vec{e}_\theta + u_\varphi \vec{e}_\varphi) \\
&= \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho \vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ u_\rho & \rho u_\varphi & u_z \end{vmatrix} \\
&= \frac{1}{\rho} \begin{vmatrix} \vec{e}_\rho & \rho \vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ -\frac{1}{\rho}\frac{\partial \Psi}{\partial z} & 0 & \frac{1}{\rho}\frac{\partial \Psi}{\partial \rho} \end{vmatrix} \\
&= \left(-\frac{\partial}{\partial \rho} \left(\frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} \right) - \frac{1}{\rho} \frac{\partial^2 \Psi}{\partial z^2} \right) \vec{e}_\varphi \\
&= -\frac{1}{\rho} \left(\frac{\partial^2 \Psi}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial \Psi}{\partial \rho} + \frac{\partial^2 \Psi}{\partial z^2} \right) \vec{e}_\varphi \qquad (11.6)
\end{aligned}$$

We define the operator D via

$$\nabla \times \vec{u} = -\frac{1}{\rho} D^2 \Psi \vec{e}_\varphi. \qquad (11.7)$$

hence

$$D^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \qquad (11.8)$$

That $\nabla p = -\mu \nabla^2 \vec{u}$ and from the identity $\nabla \times (\nabla p) = 0$, we have that $\nabla \times (\nabla \times (\nabla \times \vec{u})) = 0$ but

$$\begin{aligned}
\nabla \times (\nabla \times (\nabla \times \vec{u})) &= \nabla(\nabla \cdot (\nabla \times \vec{u})) - \nabla^2 (\nabla \times \vec{u}) \\
&= -\nabla^2 (\nabla \times \vec{u}) \qquad (11.9)
\end{aligned}$$

where we have used the identity $\nabla \cdot (\nabla \times \vec{u}) = 0$. We have already deomstrated that

$$\nabla \times (\nabla \times \vec{A}) = \nabla \times (\nabla \times \frac{\Psi}{\rho} \vec{e}_\varphi) = \frac{(-D^2 \Psi)}{\rho} \vec{e}_\varphi.$$

Following the same steps as before we find

$$\nabla \times (\nabla \times (\nabla \times \vec{u})) = \nabla \times (\nabla \times \frac{(-D^2 \Psi)}{\rho} \vec{e}_\varphi) = \frac{(-D^4 \Psi)}{\rho} \vec{e}_\varphi.$$

The streamfucntion equation is then

$$D^4\Psi = 0. \tag{11.10}$$

Pressure

From $\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2\vec{u}$ and using $\nabla \cdot \vec{u} = 0$ we can rewrite $\nabla p = -\mu\nabla^2\vec{u}$ as

$$\nabla p = -\mu\nabla \times (\nabla \times \vec{u}) \tag{11.11}$$

The usual formula for the grad operator in cylindrical polar coordinate basis (here no ϕ component)

$$\nabla p = \frac{\partial p}{\partial \rho}\vec{e}_\rho + \frac{\partial p}{\partial z}\vec{e}_z \tag{11.12}$$

and

$$\begin{aligned}
\nabla \times (\nabla \times \vec{u}) &= \nabla \times (-\frac{1}{r}D^2\Psi\vec{e}_\varphi) \\
&= \frac{1}{\rho}\begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \varphi} & \frac{\partial}{\partial z} \\ 0 & -D^2\Psi & 0 \end{vmatrix} \\
&= \frac{1}{\rho}\left(-\frac{\partial}{\partial z}(-D^2\Psi)\vec{e}_\rho + \frac{\partial}{\partial \rho}(-D^2\Psi)\vec{e}_z\right)
\end{aligned} \tag{11.13}$$

Using (11.11), (11.12) and (11.13) we obtain equations for the pressure in terms of the stream function,

$$\frac{\partial p}{\partial \rho} = \mu\frac{\partial}{\partial z}(\frac{-D^2\Psi}{\rho}) \tag{11.14}$$

and

$$\frac{\partial p}{\partial z} = -\mu\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho(\frac{-D^2\Psi}{\rho}). \tag{11.15}$$

11.1.2 Streamfunction in Spherical Polar Coordinates

The velocity field is independent of ϕ and only have

$$\vec{u} = u_r \vec{e}_r + u_\theta \vec{e}_\theta$$

implying that $\vec{A}$ must be int the $\vec{\varphi}$-direction:

$$\vec{A} = \frac{\Psi(r,\theta)}{r\sin\theta}\vec{e}_\varphi \tag{11.16}$$

Computing $\nabla \times \vec{A}$ we find

$$\begin{aligned}
\vec{u} &= \nabla \times (A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\varphi\vec{e}_\varphi) \\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ 0 & 0 & \Psi(r,\theta) \end{vmatrix} \\
&= \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\vec{e}_r - \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}\vec{e}_\theta.
\end{aligned} \tag{11.17}$$

Reading off the velocity components,

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}, \qquad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \tag{11.18}$$

We confirm that Ψ is indeed constant along streamlines, which are defined by

$$\frac{d\vec{x}}{ds} = \lambda\vec{u}$$

Streamlines given by $dr = \lambda u_r ds$ and $rd\theta = \lambda u_\theta ds$, so

$$\begin{aligned}
d\Psi &= \frac{\partial\Psi}{\partial r}dr + \frac{\partial\Psi}{\partial\theta}d\theta \\
&= -r\sin\theta u_\theta dr + r^2\sin\theta u_r d\theta \\
&= -r\sin\theta u_\theta\lambda u_r ds + r\sin\theta u_r\lambda u_\theta ds = 0.
\end{aligned} \tag{11.19}$$

Therefore $\Psi = Const.$ on streamlines (again stream surfaces as it is 3D flow).

By using the formula for the curl in spherical polar coordinates

$$\begin{aligned}
\nabla \times \vec{u} &= \nabla \times (u_r \vec{e}_r + u_\theta \vec{e}_\theta + u_\varphi \vec{e}_\varphi) \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta \vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ u_r & r u_\theta & r\sin\theta u_\varphi \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta \vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ \frac{1}{r^2 \sin\theta}\frac{\partial \Psi}{\partial \theta} & -\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial r} & 0 \end{vmatrix} \\
&= \frac{1}{r}\left(-\frac{1}{\sin\theta}\frac{\partial^2 \Psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\right)\right)\vec{e}_\varphi \\
&= -\frac{1}{r\sin\theta}\left(\frac{\partial^2 \Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \Psi}{\partial \theta}\right)\right)\vec{e}_\varphi
\end{aligned} \tag{11.20}$$

We define an operator D via

$$\nabla \times \vec{u} = -\frac{1}{r\sin\theta} D^2 \Psi \vec{e}_\varphi. \tag{11.21}$$

hence

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right) \tag{11.22}$$

The streamfucntion equation is

$$D^4 \Psi = 0. \tag{11.23}$$

Pressure

Again from $\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$ and using $\nabla \cdot \vec{u} = 0$ we can rewrite $\nabla p = -\mu \nabla^2 \vec{u}$ as

$$\nabla p = -\mu \nabla \times (\nabla \times \vec{u}) \tag{11.24}$$

The usual formula for the grad operator in spherical polar coordinate basis (here no φ component)

$$\nabla p = \frac{\partial p}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\vec{e}_\theta \tag{11.25}$$

and

$$\begin{aligned}
\nabla \times (\nabla \times \vec{u}) &= \nabla \times (-\frac{1}{r \sin^2\theta} D^2\Psi \vec{e}_\varphi) \\
&= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta \vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 0 & 0 & -\frac{1}{\sin\theta} D^2\Psi \end{vmatrix} \\
&= \frac{1}{r^2 \sin\theta}\left(-\frac{\partial}{\partial\theta}\frac{1}{\sin\theta} D^2\Psi\vec{e}_r + r\frac{\partial}{\partial r}\frac{1}{\sin\theta} D^2\Psi \vec{e}_\theta\right)
\end{aligned} \tag{11.26}$$

Using (11.24), (11.25) and (11.26) we obtain equations for the pressure in terms of the stream function,

$$\frac{\partial p}{\partial r} = -\mu \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\frac{-D^2\Psi}{r\sin^2\theta})\sin^2\theta \tag{11.27}$$

and

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \mu\frac{1}{r}\frac{\partial}{\partial r} r(\frac{-D^2\Psi}{r\sin^2\theta})\sin\theta. \tag{11.28}$$

11.2 Flow Around a Corner

Suppose the rigid boundary OA is scraped along the plane OB at a constant angle α with velocity V Relative to O the flow is steady.

Near the origin the equation of motion will be

$$\nabla^4\psi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2 \psi = 0$$

The boundary conditions on the rigid boundaries are $\psi = c$, a constant on $\theta = 0$, α,

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = -V \quad \text{on} \quad \theta = 0 \tag{11.29}$$

and

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0 \quad \text{on} \quad \theta = \alpha. \tag{11.30}$$

From the first of the u_r boundary conditions it is clear that the solution is of the form

$$\psi = rVf(\theta) \tag{11.31}$$

where

$$f(0) = f(\alpha) = 0 \tag{11.32}$$

and the boundary conditions (11.29) and (11.30) become

$$f'(0) = -1 \quad \text{and} \quad f'(\alpha) = 0. \tag{11.33}$$

Now we have

$$\begin{aligned} \nabla^2\psi &= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) rVf(\theta) \\ &= \frac{V}{r}(f + f'') =: \frac{VF(\theta)}{r} \end{aligned} \tag{11.34}$$

and

$$\begin{aligned} \nabla^4\psi &= \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \frac{VF(\theta)}{r} \\ &= \frac{V}{r^3}(F + F'') = 0. \end{aligned} \tag{11.35}$$

The solution to (11.35) is

$$F = \tilde{A}\cos\theta + \tilde{B}\sin\theta$$

which substituted into (11.34) gives the differential equation

$$f + f'' = \tilde{A}\cos\theta + \tilde{B}\sin\theta$$

The remainder of the solution is found from considering $f + f'' = 0$, this as we already know is of the form $A\cos\theta + B\sin\theta$. This leads to the general solution

$$f(\theta) = A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\sin\theta. \tag{11.36}$$

Substitution of the boundary conditions (11.32) into this gives

$$A = 0, \quad \text{and} \quad B\sin\alpha + C\alpha\cos\alpha + D\alpha\sin\alpha = 0.$$

Substitution of the boundary conditions (11.33) gives

$$B + C = -1, \quad \text{and} \quad B\cos\alpha + C\cos\alpha + D\sin\alpha - C\alpha\sin\alpha + D\alpha\cos\alpha = 0.$$

Eliminating B using $B = -C - 1$ from the equations above we obtain,

$$\begin{aligned} (-\alpha\sin\alpha)C + (\alpha\cos\alpha + \sin\alpha)D &= \cos\alpha \\ (\alpha\cos\alpha - \sin\alpha)C + (\alpha\sin\alpha)D &= \sin\alpha. \end{aligned} \tag{11.37}$$

Multiplying the first by $\alpha\sin\alpha$ and the second by $\alpha\cos\alpha + \sin\alpha$ gives

$$\begin{aligned} (-\alpha^2\sin^2\alpha)C + (\alpha\cos\alpha + \sin\alpha)(\alpha\sin\alpha)D &= \alpha\cos\alpha\sin\alpha \\ (\alpha^2\cos^2\alpha - \sin^2\alpha)C + (\alpha\sin\alpha)(\alpha\cos\alpha + \sin\alpha)D &= (\alpha\cos\alpha\sin\alpha + \sin^2\alpha) \end{aligned}$$

subtracting the first from the second and simplifying gives

$$C = \frac{-\sin^2\alpha}{\sin^2\alpha - \alpha^2}.$$

Substituting this into the second equation of (11.37) and dividing both sides by $-\sin\alpha$ gives

$$\frac{(\alpha\cos\alpha\sin\alpha - \sin^2\alpha)}{\sin^2\alpha - \alpha^2} - D = -1.$$

From which we easily get the expression for D. We easily find the expression for B by substituting the expression for C into $B = -C - 1$. Altogather we have,

$$B = \frac{\alpha^2}{\sin^2\alpha - \alpha^2} \qquad C = \frac{-\sin^2\alpha}{\sin^2\alpha - \alpha^2} \qquad D = \frac{\cos\alpha\sin\alpha - \alpha}{\sin^2\alpha - \alpha^2}$$

Substituting these into (11.36) gives

$$f(\theta) = \frac{(\alpha^2\sin\theta + [-\sin^2\alpha]\theta\cos\theta + [\cos\alpha\sin\alpha - \alpha]\theta\sin\theta)}{\sin^2\alpha - \alpha^2}.$$

Simplifying we obtain

$$f(\theta) = \frac{(\theta\sin(\theta-\alpha)\sin\alpha - \alpha(\theta-\alpha)\sin\theta)}{\sin^2\alpha - \alpha^2}.$$

Substituting this into (11.31) gives

$$\psi = rV\frac{(\theta\sin(\theta-\alpha)\sin\alpha - \alpha(\theta-\alpha)\sin\theta)}{\sin^2\alpha - \alpha^2} \tag{11.38}$$

11.3 Drag on Sphere

Let us consider Stokes flow around a sphere

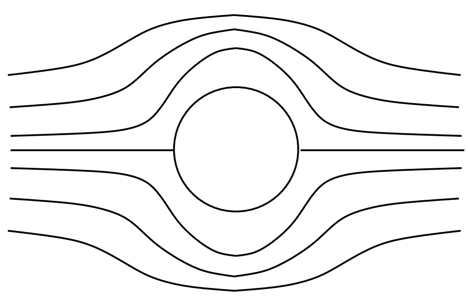

Figure 11.1: Streamlines for Stokes flow around a sphere.

Here we present the classic solution of the Stokes equations representing uniform motion of a sphere of radius a in an infinitge expanse of fluid.

The force on a sphere is

$$F_i = \int_{S\ r=a} \sigma_{ij} n_j dS \tag{11.39}$$

The net force F_D on a sphere is in the $z-$direction and given by $\vec{F} \cdot \vec{z}$. Using have $\vec{e}_r \cdot \vec{e}_z = \cos\theta$ and $\vec{e}_\theta \cdot \vec{e}_z = -\sin\theta$ we have

$$F_D = a^2 \int_0^{2\pi} \int_0^{\pi} [\sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta] \sin\theta d\theta \tag{11.40}$$

11.3.1 Boundary conditions

At infinity

$$u_r = U\cos\theta, \qquad u_\theta = -U\sin\theta \tag{11.41}$$

Non-slip conditions on the shpere

$$u_r = 0, \qquad u_\theta = 0 \tag{11.42}$$

We seek a separable solution of the form

$$\Psi = f(r)\sin^2\theta. \tag{11.43}$$

11.3.2 Calculating the Streamfunction

Using separation of variables and from the behaviour at infinity we try the solution of the form $\Psi(r,\theta) = f(r)\sin^2\theta$. Calculating $D^2\Psi$ first,

$$\begin{aligned} D^2[f(r)\sin^2\theta] &= \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right] f(r)\sin^2\theta \\ &= \sin^2\theta\frac{\partial^2}{\partial r^2}f(r) + \frac{f(r)}{r^2}\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\sin^2\theta \\ &= \sin^2\theta\frac{\partial^2}{\partial r^2}f(r) - \sin^2\theta\frac{f(r)}{r^2} \\ &= \left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right) f(r)\sin^2\theta. \end{aligned} \tag{11.44}$$

It is obvious then that

$$D^4[f(r)\sin^2\theta] \;=\; \sin^2\theta\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)^2 f(r) \tag{11.45}$$

The equation for $f(r)$ is then

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)^2 f(r) = 0. \tag{11.46}$$

Trying $f = r^\lambda$ we get $(\lambda^2-1)(\lambda-2)(\lambda-4) = 0$ and therefore

$$f(r) = \frac{A}{r} + Br + Cr^2 + Dr^4. \tag{11.47}$$

We can express the velocity components in terms of Ψ via (11.4). Using the above boundary conditions on the velocity components allows us to determine the constants A, B, C and D.

$$\begin{aligned} u_r &= \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta} \\ &= \frac{2\cos\theta}{r^2}\left(\frac{A}{r}+Br+Cr^2+Dr^4\right) \\ &= \cos\theta\left(\frac{2A}{r^3}+\frac{2B}{r}+2C+2Dr^2\right) \end{aligned} \tag{11.48}$$

$$\begin{aligned} u_\theta &= -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r} \\ &= -\sin\theta\left(-\frac{A}{r^3}+\frac{B}{r}+2C+4Dr^2\right) \end{aligned} \tag{11.49}$$

$D = 0$ otherwise flow at infinity diverges. From (11.41) we have $2C = U$. Non-slip conditions at $r = a$ give

$$\begin{aligned} u_r|_{r=a} &= \cos\theta\left(\frac{2A}{a^3}+\frac{2B}{a}+U\right) = 0 \\ u_\theta|_{r=a} &= -\sin\theta\left(-\frac{A}{a^3}+\frac{B}{a}+U\right) = 0. \end{aligned} \tag{11.50}$$

Solving for A and B gives

$$A = \frac{1}{4}Ua^3, \qquad B = -\frac{3}{4}Ua.$$

So the streamline functions is

$$\Psi = U(\frac{1}{2}r^2 + \frac{1}{4}\frac{a^3}{r} - \frac{3}{4}ar)\sin^2\theta \tag{11.51}$$

Velocities

The velocities are then

$$u_r = U\cos\theta\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3 - \frac{3}{2}\left(\frac{a}{r}\right)\right) \tag{11.52}$$

and

$$u_\theta = -U\sin\theta\left(1 - \frac{1}{4}\left(\frac{a}{r}\right)^3 - \frac{3}{4}\left(\frac{a}{r}\right)\right). \tag{11.53}$$

Rate of strain tensor

Expressions for the rate of strain tensor in spherical polar coordinates are given in the appendix. We will need the components ϵ_{rr} and $\epsilon_{r\theta}$.

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{11.54}$$

$$\begin{aligned}\epsilon_{rr} &= U\cos\theta\frac{\partial}{\partial r}\left(1 + \frac{1}{2}\left(\frac{a}{r}\right)^3 - \frac{3}{2}\left(\frac{a}{r}\right)\right) \\ &= \frac{U}{2}\cos\theta\left(\frac{-3a^3}{r^4} + 3\frac{a}{r^2}\right) \qquad (11.55)\end{aligned}$$

$$\epsilon_{r\theta} = \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} \tag{11.56}$$

$$
\begin{aligned}
\epsilon_{r\theta} &= -U\sin\theta\frac{r}{2}\frac{\partial}{\partial r}\left(\frac{1}{r}-\frac{1}{4}\frac{a^3}{r^4}-\frac{3}{4}\frac{a}{r^2}\right)-U\sin\theta\frac{1}{2r}u_r \\
&= -\frac{U\sin\theta}{2}\left[\left(-\frac{1}{r}+\frac{a^3}{r^4}+\frac{3}{2}\frac{a}{r^2}\right)+\left(\frac{1}{r}+\frac{1}{2}\frac{a^3}{r^4}-\frac{3}{2}\frac{a}{r^2}\right)\right] \\
&= -\frac{3U\sin\theta}{4}\frac{a^3}{r^4} \qquad (11.57)
\end{aligned}
$$

11.3.3 Pressure

Differential equations for the pressure p come from (11.27) and (11.28). We need to calculate $\frac{D^2[\Psi]}{r\sin^2\theta}$ from (11.51). Using (11.44) we can write

$$
\begin{aligned}
\frac{D^2[\Psi]}{r\sin^2\theta} &= \frac{1}{r}\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f(r) \\
&= \frac{U}{r}\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)(\frac{1}{2}r^2+\frac{1}{4}\frac{a^3}{r}-\frac{3}{4}ar) \\
&= \frac{U}{r}\left[(1+\frac{1}{2}\frac{a^3}{r^3})-(1+\frac{1}{2}\frac{a^3}{r^3}-\frac{3}{2}\frac{a}{r})\right] \\
&= U\frac{3}{2}\frac{a}{r^2} \qquad (11.58)
\end{aligned}
$$

From (11.27) we can write

$$
\begin{aligned}
\frac{\partial p}{\partial r} &= -\mu\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(-U\frac{3}{2}\frac{a}{r^2})\sin^2\theta \\
&= 3\mu Ua\frac{1}{r^3}\cos\theta. \qquad (11.59)
\end{aligned}
$$

This implies

$$
p = -\frac{3}{2}\mu Ua\frac{\cos\theta}{r^2}+g(\theta) \qquad (11.60)
$$

where $g(\theta)$ is an arbitray function of θ. From (11.28) we can write

$$
\begin{aligned}
\frac{1}{r}\frac{\partial p}{\partial\theta} &= \mu\frac{1}{r}\frac{\partial}{\partial r}r(-U\frac{3}{2}\frac{a}{r^2})\sin\theta. \\
&= \frac{3}{2}\mu Ua\frac{1}{r^3}\sin\theta. \qquad (11.61)
\end{aligned}
$$

This implies

$$p = -\frac{3}{2}\mu U a \frac{\cos\theta}{r^2} + h(r) \tag{11.62}$$

where $h(r)$ is an arbitray function of r. Comparing (11.60) and (11.62) noting that we must have that $g(\theta) = h(r) = Const. = p_\infty$ we obtain the final result

$$p = -\frac{3}{2}\mu U a \frac{\cos\theta}{r^2} + p_\infty \tag{11.63}$$

11.3.4 Stress tensor

The stress tensor is related to the strain tensor.

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \tag{11.64}$$

$$\sigma_{rr} = -p + 2\mu\epsilon_{rr} \tag{11.65}$$

$$\sigma_{r\theta} = 2\mu\epsilon_{r\theta} \tag{11.66}$$

$$\sigma_{rr} = -p + \mu U \cos\theta \left(\frac{-3a^3}{r^4} + 3\frac{a}{r^2} \right) \tag{11.67}$$

$$\sigma_{r\theta} = -\mu U \sin\theta \frac{3}{2}\frac{a^3}{r^4} \tag{11.68}$$

11.3.5 Force on the sphere

The force on a sphere is

$$F_D = a^2 \int_0^{2\pi} \int_0^{\pi} [\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta] \sin\theta d\theta \tag{11.69}$$

$$\begin{aligned} F_D &= 2\pi a^2 \int_0^{\pi} \left(\left[\frac{3}{2}\mu U a \frac{\cos\theta}{r^2} - p_\infty + \mu U \cos\theta \left(\frac{-3a^3}{r^4} + 3\frac{a}{r^2} \right) \right]_{r=a} \cos\theta \right. \\ &\quad \left. + \left[\frac{3}{2}\mu U \sin\theta \frac{a^3}{r^4} \right]_{r=a} \sin\theta \right) \sin\theta d\theta \end{aligned} \tag{11.70}$$

The p_∞ term vanishes because of the integral

$$\int_0^\pi \cos\theta \sin\theta d\theta = -\int_1^{-1} u du = \left[\frac{u^2}{2}\right]_1^{-1} = 0.$$

where we used substitution $u = \cos\theta$. The drag force is then given by

$$F_D = \underbrace{3\pi\mu a U \int_0^\pi \cos^2\theta \sin\theta d\theta}_{pressure} + \underbrace{3\pi\mu a U \int_0^\pi \sin^3\theta d\theta}_{viscous} \tag{11.71}$$

The integrals are easy

$$\int_0^\pi \cos^2\theta \sin\theta d\theta = -\int_1^{-1} u^2 du = \left[\frac{u^3}{3}\right]_1^{-1} = \frac{2}{3}$$

$$\int_0^\pi \sin^3\theta d\theta = \int_0^\pi \sin\theta(1-\cos^2\theta) d\theta = [-\cos\theta]_0^\pi - \frac{2}{3} = \frac{4}{3}.$$

The final result is

$$F_D = \underbrace{2\pi\mu a U}_{pressure} + \underbrace{4\pi\mu a U}_{viscous} = 6\pi\mu a U. \tag{11.72}$$

That is, one third of the drag is due to pressure forces, two-thirds are due to viscous forces.

11.3.6 Calculating the force on the sphere by doing an integral at infinity

$$F_i = \int_{S\ r=a} \sigma_{ij} n_j dS \tag{11.73}$$

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Taking the derivative

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial}{\partial x_i} p + \mu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial}{\partial x_i} \frac{\partial u_j}{\partial x_j} \right)$$

From the continuity equation $\frac{\partial u_j}{\partial x_j} = 0$, so

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial}{\partial x_i} p + \mu \frac{\partial^2 u_i}{\partial x_j^2}$$

But as $\nabla p = \mu \nabla^2 \vec{u}$, we have

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0.$$

By Gauss theorem

$$\int_{S\ r=a} \sigma_{ij} n_j dS + \int_{S\ r=\infty} \sigma_{ij} n_j dS = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV = 0.$$

Implying that

$$F_i = -\int_{S\ r=\infty} \sigma_{ij} n_j dS. \tag{11.74}$$

Appendix A

Glossary

• Body force: Force acting on an infintesimal volume of fluid per unit mass.

• Boundary layer flow: There will be a layer close to the solid wall where the viscous terms are important even for very high Reynolds number flow. The non-slip condition holds at the surface while a height above the fluid motion is uneffected. This layer will be very thin and the flow in that region is called the boundary layer flow.

• Cauchy's equation: $\varrho \frac{\partial u_i}{\partial t} + (\vec{u} \cdot \nabla) u_i = \varrho f_i + \frac{\partial \sigma_{ij}}{\partial x_j}$ where σ_{ij} is the stress tensor.

• Cavitation: Cavitation is when a liquid vaporizes because the pressure is sufficiently low. It occurs whenever the cavitation number σ, defined by

$$\sigma = \frac{p_\infty - p_v}{\frac{1}{2}\varrho V^2}$$

is less than the critical cavitation number σ_{crit}, which depends on the geometry and the Reynolds number.

• Continuity equation: The continuity equation is $\frac{\partial \varrho}{\partial t} + \nabla \cdot (\varrho \vec{u}) = 0$ where ϱ is the density and $\vec{u}$ is the fluid velocity. The continuity equation expressing the conservation of mass. For constant density, or incompressible flow the continuity equation reduces to $\nabla \cdot \vec{u} = 0$.

• Conservative force: $\vec{F} = -\nabla \Omega$

• Density: Denoted ϱ here, is the mass of fluid per unit volume. It can be a function fo space and time, but oftent the assumption that it is constant is made.

• Euler equations of motion: For flows with high Reynolds number R, where the viscous effects are small, most of the flow can be considered inviscid and a simpler set of equations can be solved, which corresponds to the $R \to \infty$ limit in the Navier-Stokes equations. They read $\rho \frac{\partial}{\partial t}\vec{u} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \rho \vec{f}$.

• Guass Divergence Theorem: $\int_V \nabla \cdot \vec{A} dV = \oint_S \vec{A} \cdot \vec{n} dS$ where V is a volume, S is the surface bounding the volume V, and $\vec{n}$ is an outward pointing vector normal to the surface S.

• Ideal (or perfect) fluid: This is a fluid which can exert no shearing stress across any surface $\sigma_{ij} = -p\delta_{ij}$ that is the stress tensor for a perfect fluid is diagonal. This is done by ignoring viscous effects.

• Incompressible fluid is a fluid which always has constant density. The continuity equation is then $\nabla \cdot \vec{u} = 0$.

• Inviscid: An inviscid fluid has zero viscocity.

• Kinematic viscocity: The kinematic viscocity is the ratio of viscocity and the density: $\nu = \mu/\rho$.

• Lagrangian description:

• Navier-Stokes equation: The Navier-Stokes equation is $\rho \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \mu \nabla^2 \vec{u} + \rho \vec{f}$ together with the continuity equation $\nabla \cdot \vec{u} = 0$. Another form of the Navier-Stokes equation is $\frac{\partial \vec{u}}{\partial t} - (\vec{u} \times \omega) = -\nabla \left(\frac{1}{2}|\vec{u}|^2 + \frac{p}{\rho} \right) - \nu \nabla \times \vec{\omega} + \vec{f}$ where $\vec{\omega}$ is the vorticity.

• Non-slip condition: Dictates that the speed of the fluid at a solid boundary is zero relative to the boundary.

• Newtonian fluid:

• Pathline: A pathline is the locus of a particle.

• Rate of strain tensor: $e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$

• Reynolds number: $R = \frac{UL}{\nu}$ where U is the characterisic velocity, L the characteristic length sclae and ν the viscocity. The smallness of R can be achieved by considering the extremely small length scales, or dealing with a very viscous liquid, or treating flows with very small velocity, so-called creeping flows.

• Slender body theory: Methodology that can be used to take advantage of the slenderness of a body to obtain an approximation to the fluid field surrounding it and/or the net effect of the field on the body.

• Stagnation point: A point in the flow field where the local velocity of the fluid is zero.

• Stokes equation: For Stoke's flow the Navier-Stokes equation reduces to Stoke's equation $\nabla p = \mu \nabla^2 \vec{u}$.

• Stokes flow: A type of flow where advective inertial forces are small compared with viscous forces. The Reynolds number is low.

• Stokes Theorem: Stayes that for any surface S encloed by the the closed curve C $\int_S \nabla \times \vec{A} = \oint_C \vec{A} \cdot d\vec{l}$ holds.

• Streamline: A line that is everywhere tangent to the velocity vector of a flow.

• Stress tensor: σ_{ij}.

• Substantial derivative: $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + (\vec{u} \cdot \nabla)$

• Surface forces:

• Viscocity: Viscocity is a measure of the fluid's resistance to deformation by shear stress.

• Vorticity: $\vec{\omega} = \nabla \times \vec{u}$

Appendix B

Vector Calculus, Divergence Theorem and Stoke's Theorem

B.1 Vector Notions

An important vector operation is the cross product defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{B.1}$$

which means

$$\vec{a} \times \vec{b} = (a_2 b_3 - b_2 a_3)\vec{e}_x + (a_1 b_3 - b_1 a_3)\vec{e}_y + (a_1 b_2 - b_1 a_2)\vec{e}_z \tag{B.2}$$

The dot product of $\vec{a}$ and $\vec{a} \times \vec{b}$ is

$$\vec{a} \cdot (\vec{a} \times \vec{b}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

Similarly $\vec{b} \cdot (\vec{a} \times \vec{b}) = 0$ therefore $\vec{a} \times \vec{b}$ is a vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$. Also note

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

We consider the length of a cross product. We can simplify things by writing $\vec{a} = |\vec{a}|\vec{e}_x$ and $\vec{b} = |\vec{b}|\cos\theta\vec{e}_x + |\vec{b}|\sin\theta\vec{e}_y$, then

$$|\vec{a}\times\vec{b}| = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ |\vec{a}| & 0 & 0 \\ |\vec{b}|\cos\theta & |\vec{b}|\sin\theta & 0 \end{vmatrix} = |\vec{a}||\vec{b}|\sin\theta\vec{e}_z.$$

Thus $\vec{a}\times\vec{b}$ has length $|\vec{a}||\vec{b}|\sin\theta$ and a direction $\vec{e}_z$ perpedicular to the plane containing $\vec{a}$ and $\vec{b}$.

B.2 Vector Calculus

B.2.1 Gradient of a Scalar

The difference in ϕ is a scalar. However

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

As the the LHS is a scalr and dx, dy and dz are components of a vector the three numbers

$$\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}$$

are aslo components of a vector. This vector is often denoted as $\nabla\phi$ and so

$$\nabla\phi = \vec{e}_x\frac{\partial\phi}{\partial x} + \vec{e}_y\frac{\partial\phi}{\partial y} + \vec{e}_z\frac{\partial\phi}{\partial z}$$

We call $\nabla\phi$ the gradient of a scalar. Now the argument that $\nabla\phi$ is a vector does not depend on what scalar field we differentiated. As the trasformation as a vector is independent of what scalar field we are differentiatinf we could just as well omit the ϕ.

Since the differential operators themselves transform as the components of a vector, we call them components of a vector operator. We can write

$$\nabla = \vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z} \tag{B.3}$$

The gradient ∇ is an operator as it must act upon something.

B.2.2 Divergence

An obvious object the gradient ∇ could operate on would be a vector field $\vec{v}$,

$$\begin{aligned} \nabla \cdot \vec{v} &= \nabla_x v_x + \nabla_y v_y + \nabla_z v_z \\ &= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \end{aligned} \tag{B.4}$$

This is called the divergence and obviously transforms as a scalar.

Another way to combine a vector $\vec{v}$ with ∇ via a dot product is

$$\vec{v} \cdot \nabla$$

This is a scalar operator and does appear in fluid mechanics, in particular the advective term of the substantial derivative.

B.2.3 Laplacian

It is possibly to combine two ∇? Take $\nabla \cdot (\nabla\phi)$,

$$\nabla \cdot (\nabla\phi) = (\nabla \cdot \nabla)\phi = \nabla^2\phi.$$

The ∇^2 is a scalar operator that often appears in physics and is called the Laplacian,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{B.5}$$

How about acting ∇^2 on a vector (we will at times need to do this as this terms appears in the /navier-Stokes equations and the Stokes flow equations). In Cartesian coordinates it is straightforward. For example the x-component of $\nabla^2\vec{v}$ is simply

$$(\nabla^2\vec{v})_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) v_x = \nabla^2 v_x. \tag{B.6}$$

The situation is not so simply in coordinates systems other than Cartesian coordinates. We will later derive formula for spherical polar and cylindrical polar coordinates.

B.2.4 Curl

There is another object we can come uo with from the operator ∇ and a vector field $\vec{v}$ using the cross product. The curl denoted $\nabla \times \vec{v}$ can be defined via the cross product,

$$\nabla \times \vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\ \vec{v}_x & \vec{v}_y & \vec{v}_z \end{vmatrix} \tag{B.7}$$

The curl can be written in component form

$$[\nabla \times \vec{v}]_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} v_k \tag{B.8}$$

B.2.5 Identities

$$\nabla \times \nabla \phi = \epsilon_{ijk} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} \phi = 0$$

as partial derivatives commute, hence

$$\nabla \times \nabla \phi = 0. \tag{B.9}$$

$$\nabla \cdot (\nabla \times \vec{v}) = \frac{\partial}{\partial x_i} \epsilon_{ijk} \frac{\partial}{\partial x_j} \vec{v}_k = 0$$

again as partial derivatives commute, hence

$$\nabla \cdot (\nabla \times \vec{v}) = 0. \tag{B.10}$$

We can derive an identity for

$$\nabla \times (\nabla \times \vec{v})$$

with the usefull identity

$$\epsilon_{ijk} \epsilon_{ki'j'} = (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}) \tag{B.11}$$

$$
\begin{aligned}
[\nabla \times (\nabla \times \vec{v})]_i &= \epsilon_{ijk}\partial_j[\nabla \times \vec{v}]_k \\
&= \epsilon_{ijk}\partial_j(\epsilon_{ki'j'}\partial_{i'}v_{j'}) \\
&= \epsilon_{ijk}\epsilon_{ki'j'}\partial_j\partial_{i'}v_{j'} \\
&= (\delta_{ii'}\delta_{jj'} - \delta_{ij'}\delta_{ji'})\partial_j\partial_{i'}v_{j'} \\
&= \partial_i(\partial_j v_j) - \partial_j\partial_j v_i. \qquad \text{(B.12)}
\end{aligned}
$$

In vector notation this reads

$$
\nabla \times (\nabla \times \vec{v}) = \nabla(\nabla \cdot \vec{v}) - \nabla^2\vec{v}. \qquad \text{(B.13)}
$$

B.3 Definitions and Statement of Line Integral Theorem, Gauss's Divergence Theorem and Stoke's Theorem

B.3.1 Defintion of a Line Integral

The first type of integral will need to introduce is a line integral of a scalar function f, denoted

$$
\underbrace{\int_P^Q}_{along\ \Gamma} f dl \qquad \text{(B.14)}
$$

We define the line integral as the limit of a sum,

$$
\sum_n f_n \Delta l_n \qquad \text{(B.15)}
$$

where f_n is the value of the fucntion at the nth segment. The integral (B.14) comes from increasing the number of segments so that the largest $\Delta l_n \to 0$

The particular type of line integral that we will be interested in is of the form

$$
\underbrace{\int_P^Q}_{along\ \Gamma} \vec{v} \cdot d\vec{l} \qquad \text{(B.16)}
$$

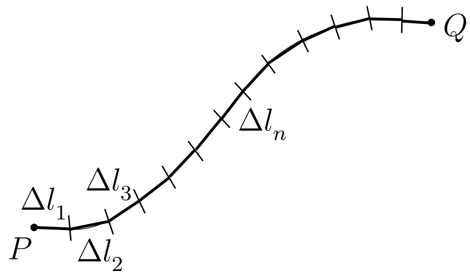

Figure B.1: .

where $\vec{v}$ is a vector field and $d\vec{l}$ is an infintesimal displacement vector along the line Γ. Hence it is of the form (B.14) where instead of f we have another scalar - the component of $\vec{v}$ in the direction of $d\vec{l}$,

$$(\vec{v} \cdot \vec{t})\, dl = \vec{v} \cdot d\vec{l}.$$

where $\vec{t}$ is the unit tangent vector to the curve ($\vec{t}$ is the unit vector parallel to $d\vec{l}$). The integral (B.16) is the sum of such terms.

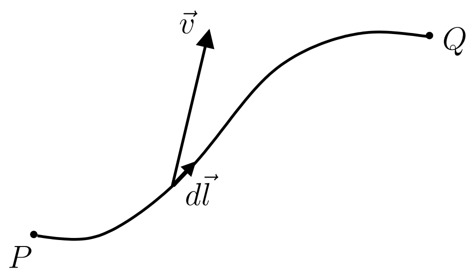

Figure B.2: .

B.3.2 Theorem for a Line Integral

Our first theorem comes from using $\vec{v} = \nabla\psi$ in the line integral.

Theorem: If Γ is any curve joining point P and point Q, the following is true,

$$\psi(Q) - \psi(P) = \underbrace{\int_P^Q}_{\substack{any\ curve\ from \\ P\ to\ Q}} (\nabla\psi) \cdot d\vec{l}. \tag{B.17}$$

This will be proved in section B.4.

B.3.3 Definition of Flux and Statement Gauss's Divergence Theorem

Given a surface S with outward unit-normal vector field $\vec{n}$ and a vector field $\vec{u}$, the flux is defined as

$$\int_S \vec{u} \cdot \vec{n} dS.$$

Gauss's theorem pertains to a closed surface S that encloses a volume V, and relates the flux through the surface to the behaviour of a vector field inside the surface.

Gauss's Divergence theorem:

Let V be a closed volume enclosed by the surface S, then

$$\int_V \nabla \cdot \vec{A} dV = \oint_S \vec{A} \cdot \vec{n} dS \tag{B.18}$$

This will be proved in section B.5.

B.3.4 Statement of Stokes' Theorem

Stokes' theorem Let C be any simple closed path (i.e.,a path that starts and ends at the same point and has no intersections), and consider any surface S of which C is the boundary.Then stokes'theorem says that

$$\int_S \nabla \times \vec{A} \cdot \vec{n} dS = \oint_C \vec{A} \cdot d\vec{l} \tag{B.19}$$

This will be proved in section B.6.

B.3.5 Conservative Forces

A conservative force is a force with the property that the work done, W, in moving a particle between two points

$$W = \int_P^Q \vec{F} \cdot d\vec{l}$$

is independent of the path taken. Equivalently, if a particle travels around a closed loop, the net work done by a conservative force is zero. If a force $\vec{F}$ can be written in the form

$$\vec{F} = -\nabla\Phi$$

where Φ is a scalar field, the potential, the above theorem guarantees the force is conservative. An obvious example of a conservative force is the gravitational force.

B.4 Proof of Theorem for Line Integral of $\nabla\psi$

We now prove the theorem staed in section B.3.2. Consider the curve Γ split up into segments as in fig (B1) where

$$\Delta\vec{l}_n = \vec{p}_{n+1} - \vec{p}_n.$$

Then from multi-variable calculus

$$\begin{aligned}(\nabla\psi)\cdot\Delta\vec{l}_n &= \frac{\partial\psi}{\partial x}\Delta x_n + \frac{\partial\psi}{\partial y}\Delta y_n + \frac{\partial\psi}{\partial y}\Delta z_n \\ &= (\Delta\psi)_n \\ &= \psi(\vec{p}_{n+1}) - \psi(\vec{p}_n). \end{aligned} \tag{B.20}$$

Now with $\psi(\vec{p}_0) = \psi(P)$ and $\psi(\vec{p}_{N+1}) = \psi(Q)$, the approximation to the line integral is the summation

$$\begin{aligned}\sum_{n=0}^{N}(\nabla\psi)\cdot\Delta\vec{l}_n &= [\psi(\vec{p}_1) - \psi(\vec{p}_0)] + [\psi(\vec{p}_2) - \psi(\vec{p}_1)] + [\psi(\vec{p}_3) - \psi(\vec{p}_2)] + \cdots + \\ &\quad + \cdots + [\psi(\vec{p}_N) - \psi(\vec{p}_{N-1})] + [\psi(\vec{p}_{N+1}) - \psi(\vec{p}_N)] \\ &= \psi(\vec{p}_{N+1}) - \psi(\vec{p}_0) \\ &= \psi(Q) - \psi(P). \end{aligned} \tag{B.21}$$

where all but two terms cancel. Taking the continuum limit does not change the last line on the RHS. Also it is obvious that the answer will be the same regardless which curve we use between P and Q.

B.5 Proof of Gauss's Divergence Theorem

Say we have a closed surface S that encloses a volume V. Separate the volume into two parts by taking a slice resulting it two closed surfaces and volumes as in fig .. The volume V_1 is enclosed by the closed surface S_1, which is made up of part of the original surface S_1' and of the surface of the slice S_{12}. The volume V_2 is enclosed by S_2, which is made up of the rest of the original surface S_2' and the surface of the slice S_{12}. The flux through S_1 is

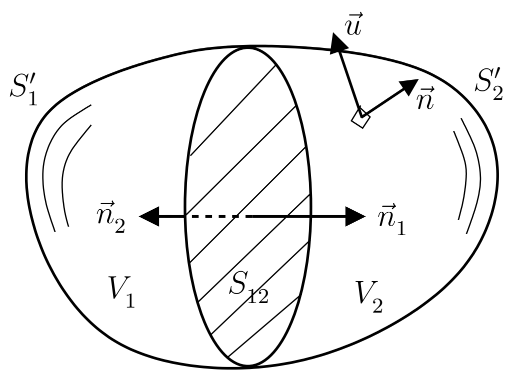

Figure B.3: .

We then have $S_1 = S_1' + S_{12}$ and $S_2 = S_2' + S_{12}$. The flux through S_1 can be written as the sum of two parts

$$\int_{S_1'} \vec{A} \cdot \vec{n} dS + \int_{S_{12}} \vec{A} \cdot \vec{n}_1 dS \tag{B.22}$$

and similarly for the flux through S_2,

$$\int_{S_2'} \vec{A} \cdot \vec{n} dS + \int_{S_{12}} \vec{A} \cdot \vec{n}_2 dS \tag{B.23}$$

As $\vec{n}_1 = -\vec{n}_2$, adding the fluxes of each of these surfaces gives

$$\int_{S_1'} \vec{A} \cdot \vec{n} dS + \int_{S_2'} \vec{A} \cdot \vec{n} dS \tag{B.24}$$

which is just to the flux through the original surface $S = S_1 + S_2$. We can subdivide the volume again and again and it will generally be true that the flux through the outer surface will be equal to the sum of the fluxes out of all the smaller interior pieces.

We now consider the special case of the flux out of a small cube.

$$\left[v_x + \frac{\partial v_x}{\partial x}dx\right]dydz - v_x dydz = \frac{\partial v_x}{\partial x}dxdydz \tag{B.25}$$

Similar contributions come from the other two pairs of faces, adding together their contributions the total flux through all faces is

$$\int_{cube} \vec{v}\cdot\vec{n}dS = \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)dxdydz \tag{B.26}$$

or

$$\int_{cube} \vec{v}\cdot\vec{n}dS = (\nabla\cdot\vec{v})dxdydz \tag{B.27}$$

Splitting the volume V enclosed by a closed surface S into infintesimally small cubes, summing the LHS of the above equation gives the total flux out of the closed surface and summing over the RHS gives the volume integral

$$\int_V \nabla\cdot\vec{v}dV$$

resulting in

$$\int_S \vec{v}\cdot\vec{n}dS = \int_V \nabla\cdot\vec{v}dV \tag{B.28}$$

B.6 Proof of Stokes' Theorem

Let us find the circulation around an infintesimal square.

The circulation around the square is then

$$\begin{aligned}\oint \vec{v}\cdot d\vec{l} &= v_x dx + (v_y + \frac{\partial v_y}{\partial x})dy - (v_x + \frac{\partial v_x}{\partial y})dx - v_y dy \\ &= \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)dxdy \end{aligned} \tag{B.29}$$

Now say we had a collection of such squares as in fig (B.2) and wished to add up the circulation from each individual square. Interior paths are transversed in opposite directions, thus their contributions to each line integral cancel pairwise. Therefore only

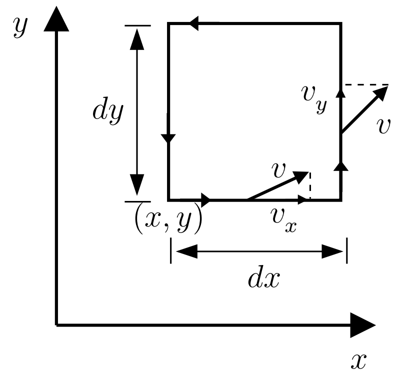

Figure B.4: Finding the circulation around an infintesimal square.

the outside edge contributes. This observation is an underlying principle in the proof of Stoke's theorem.

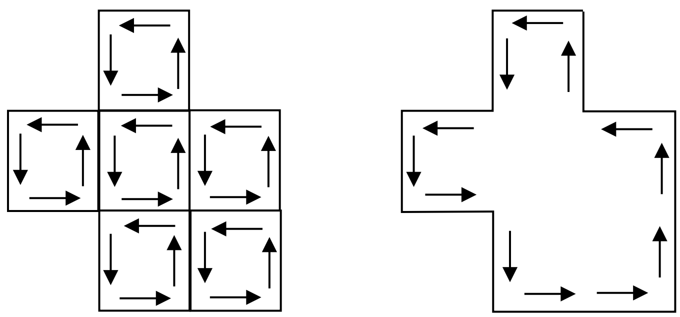

Figure B.5: When adding together the circulation of each loop the only remaing contribution to the line integral comes from the outside edge.

We can write (B.6)

$$(\nabla \times \vec{v})_z dS \tag{B.30}$$

Here the z-component is the normal to the surface. We can therefore write the circulation around a infintesimal square in an invariant vector form,

$$\oint_C \vec{v} \cdot d\vec{l} = (\nabla \times \vec{v}) \cdot \vec{n} dS \tag{B.31}$$

So we have that the circulation of any vector $\vec{v}$ around an infintesimal square is the component of the curl of $\vec{v}$ normal to the surface, times the area of the square. This result is independent on the orientation of the square.

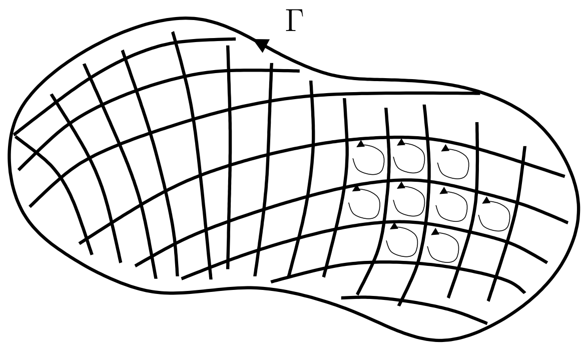

Figure B.6: We have some surface bounded by the loop Γ. The surface is divided into many small areas, each approximately a square.

Now suppose that we have a loop which is the boundary of some surface. There are of course an infinite number of surfaces that have all have this loop as their boundary. However the result does not depend on the particular surface choosen. Let the choosen surface be divided into many small loops. If we take the loops small enough, we can assume that each of the small loops enclose an area which is essentially flat. Also we can choose our small loops so that each is very nearly a square. Combining (B.31) with the fact that when you add up the circulation of each individual loop the only remaining contribution comes from the outside edge, we have

$$\int_S \nabla \times \vec{v} \cdot \vec{n} dS = \oint_\Gamma \vec{v} \cdot d\vec{l}. \tag{B.32}$$

Appendix C

Orthogonal Curvilinear Coordinates

Curvilnear coordinates are a coordinate system for Euclidean space in which the coordinate lines may be curved. We have already met such coordinates - cylindrical and spherical polar coordinates.

We will use Guass's theroem and Stoke's theorem to find formula for the Divergence and Curl respectively in orthogonal curvilinear coordinates. We have already derived formula for the Gradient in the main text,combining this with formula for the Divergence we will obtain a formula for the Laplacian of a scalar field.

The vector Laplacian and the advective term, both of which appear in the Navier-Stokes equations, will be resolved in circular cylindrical and spherical polar coordinates respectively in appendices D and E.

C.1 Gradient of a Scalar Field

We have already derived formula for the Gradient in the main text in section 3.1. The formuala for the grad of a scalar is

$$\nabla \phi = \sum_{i=1}^{3} \vec{e}_i \frac{1}{h_i} \frac{\partial \phi}{\partial q_i}. \tag{C.1}$$

The h_i is defined by

$$h_i = \left| \frac{\partial \vec{r}}{\partial q_i} \right|$$

where $\vec{r}$ is the position vector.

In cylindrical coordinates we have

$$q_1 = \rho, \quad q_2 = \varphi, \quad q_3 = z$$
$$h_1 = 1, \quad h_2 = \rho, \quad h_3 = 1 \tag{C.2}$$

and in spherical coordinates

$$q_1 = r, \quad q_2 = \theta, \quad q_3 = \varphi$$
$$h_1 = 1, \quad h_2 = r, \quad h_3 = r\sin\theta. \tag{C.3}$$

From these examples it is clear that $h_i dq_i$ (no summation over i implied) is the distance change brought about by a coordinate change dq_i.

C.2 The Divergence

We can use the Gauss's Divergence theorem to find a formula for the divergence.

$$\nabla \cdot \vec{v}\,(q_1, q_2, q_3) = \lim_{dV \to 0} \frac{\int \vec{v} \cdot d\vec{S}}{\int dV} \tag{C.4}$$

with differencial volume

$$dV = h_1 h_2 h_3 dq_1 dq_2 dq_3.$$

Define

$$v_1 = \vec{v} \cdot \vec{e}_1, \quad v_2 = \vec{v} \cdot \vec{e}_2, \quad v_1 = \vec{v} \cdot \vec{e}_3.$$

The area integral for the two faces $q_1 = Const.$ is given by

$$\left[v_1 h_2 h_3 + \frac{\partial}{\partial q_1}(v_1 h_2 h_3) dq_1\right] dq_2 dq_3 - v_1 h_2 h_3 dq_2 dq_3 = \frac{\partial}{\partial q_1}(v_1 h_2 h_3) dq_1 dq_2 dq_3. \tag{C.5}$$

Adding in similar results for the other two pair of surfaces, we obtain

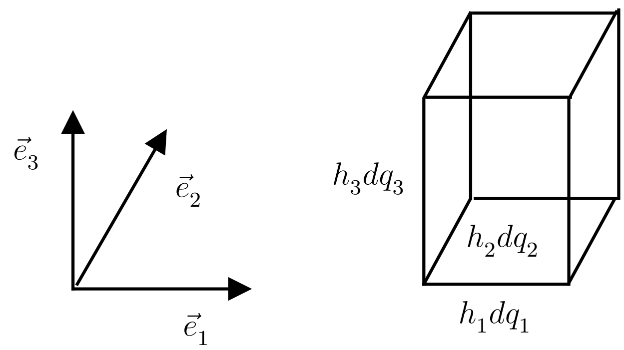

Figure C.1: The gradient of $\vec{v}$.

$$\int \vec{v}\,(q_1,q_2,q_3)\cdot d\vec{S} = \left[\frac{\partial(v_1h_2h_3)}{\partial q_1} + \frac{\partial(v_2h_1h_3)}{\partial q_2} + \frac{\partial(v_3h_1h_2)}{\partial q_3}\right] dq_1dq_2dq_3. \tag{C.6}$$

And division by the diffrential volume (C.4) yields

$$\nabla\cdot\vec{v}\,(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial(v_1h_2h_3)}{\partial q_1} + \frac{\partial(v_2h_1h_3)}{\partial q_2} + \frac{\partial(v_3h_1h_2)}{\partial q_3}\right] \tag{C.7}$$

C.3 Laplacian of a Scalar

Combining (C.1) and (C.7) we obtain the formula for the Laplacian of a scalar field,

$$\nabla^2\phi(q_1,q_2,q_3) = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial\phi}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right] \tag{C.8}$$

C.4 The Curl

Using Stoke's theorem

$$\int \nabla\times\vec{v}\cdot d\vec{S} = \oint_C \vec{v}\cdot d\vec{l}$$

in differential form we calculate the componet $\vec{e}_1\cdot\nabla\times\vec{v}$ from

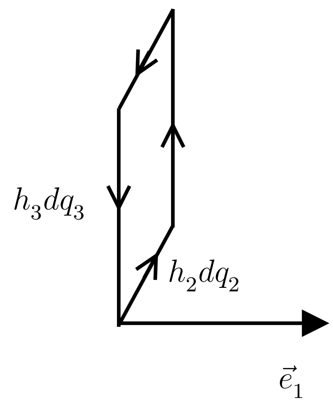

Figure C.2: The $\vec{e}_1$ component of the Curl of an infintesimal loop.

$$\begin{aligned}\vec{e}_1 \cdot \nabla \times \vec{v}\, dq_2 dq_3 h_2 h_3 &= \left(\left[h_3 v_3 + \frac{\partial}{\partial q_2}(h_3 v_3) dq_2\right] - h_3 v_3\right) dq_3 \\ &- \left(\left[h_2 v_2 + \frac{\partial}{\partial q_3}(h_2 v_2) dq_3\right] - h_2 v_2\right) dq_2 \end{aligned} \tag{C.9}$$

and so

$$\vec{e}_1 \cdot \nabla \times \vec{v} = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(h_3 v_3) - \frac{\partial}{\partial q_3}(h_2 v_2)\right] \tag{C.10}$$

Similar results for the other two rectangles, those orthogonal to $\vec{e}_2$ and $\vec{e}_3$, we obtain

$$\begin{aligned}\nabla \times \vec{v} &= \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial q_2}(h_3 v_3) - \frac{\partial}{\partial q_3}(h_2 v_2)\right]\vec{e}_1 \\ &+ \frac{1}{h_1 h_3}\left[\frac{\partial}{\partial q_3}(h_1 v_1) - \frac{\partial}{\partial q_1}(h_3 v_3)\right]\vec{e}_2 \\ &+ \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_3 v_3) - \frac{\partial}{\partial q_3}(h_1 v_1)\right]\vec{e}_3 \end{aligned} \tag{C.11}$$

which is compactly written as

$$\begin{aligned}\nabla \times \vec{v} &= \nabla \times (v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3) \\ &= \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \vec{e}_1 h_1 & \vec{e}_2 h_2 & \vec{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 v_1 & h_2 v_2 & h_3 v_3 \end{vmatrix}. \end{aligned} \tag{C.12}$$

Appendix D

Cylindrical Polar Coordinates

The formuala for

Relation to Cartesian coordinates:

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned} \tag{D.1}$$

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ \varphi &= \tan^{-1}(y/x) \\ z &= z \end{aligned} \tag{D.2}$$

D.1 Vector Differential operators

The non-zero derivatives of the basis vectors are

$$\frac{\partial \vec{e}_\rho}{\partial \varphi} = \vec{e}_\varphi, \qquad \frac{\partial \vec{e}_\varphi}{\partial \varphi} = -\vec{e}_\rho. \tag{D.3}$$

$$\nabla\phi = \frac{\partial \phi}{\partial r}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial \phi}{\partial \varphi}\vec{e}_\varphi + \frac{\partial \phi}{\partial z}\vec{e}_z \tag{D.4}$$

The Laplace operator is obtained from (C.2) and (C.8),

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}. \tag{D.5}$$

The curl of a vector field is obtained from (C.2) and (C.12),

$$\begin{aligned}\nabla \times \vec{u} &= \nabla \times (u_\rho \vec{e}_\rho + u_\varphi \vec{e}_\varphi + u_z \vec{e}_z) \\ &= \frac{1}{\rho}\begin{vmatrix} \vec{e}_\rho & \rho\vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ u_\rho & \rho u_\varphi & u_z \end{vmatrix}\end{aligned} \tag{D.6}$$

The Laplacian of a vector field $\nabla^2\vec{u}$ resolved in cylindrical polar coordinates is

$$\begin{aligned}\nabla^2\vec{u}\big|_\rho &= \nabla^2 u_\rho - \frac{1}{\rho^2}u_\rho - \frac{2}{\rho^2}\frac{\partial u_\varphi}{\partial\varphi} \\ \nabla^2\vec{u}\big|_\varphi &= \nabla^2 u_\varphi - \frac{1}{\rho^2}u_\varphi + \frac{2}{\rho^2}\frac{\partial u_\rho}{\partial\varphi} \\ \nabla^2\vec{u}\big|_z &= \nabla^2 u_z\end{aligned} \tag{D.7}$$

where we have applied (D.5) to $\vec{u} = u_\rho\vec{e}_\rho + u_\varphi\vec{e}_\varphi + u_z\vec{e}_z$ and used (D.3). These would be used in writing the $\mu\nabla^2\vec{u}$ term in the Navier-Stokes equations in cylindrical polar coordinates.

If $\vec{u}$ is the velocity field and if we had incompressible flow, $\nabla\cdot\vec{u} = 0$. Then from the identity $\nabla\times(\nabla\times\vec{u}) = \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u}$ and using $\nabla\cdot\vec{u} = 0$, then we can use the alternative formula $\nabla^2\vec{u} = -\nabla\times(\nabla\times\vec{u})$.

The non-linear term $(\vec{u}\cdot\nabla)\vec{u}$ of the Navier-Stokes equations using (D.3) and (D.4) is

$$\begin{aligned}(\vec{u}\cdot\nabla)\vec{u} &= (u_\rho\frac{\partial}{\partial\rho} + u_\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi} + u_z\frac{\partial}{\partial z})(u_\rho\vec{e}_\rho + u_\varphi\vec{e}_\varphi + u_z\vec{e}_z) \\ &= \left(u_\rho\frac{\partial u_\rho}{\partial\rho} + u_\varphi\frac{1}{\rho}\frac{\partial u_\rho}{\partial\varphi} + u_z\frac{\partial u_\rho}{\partial z} - \frac{u_\varphi^2}{\rho}\right)\vec{e}_\rho \\ &\quad + \left(u_\rho\frac{\partial u_\varphi}{\partial\rho} + u_\varphi\frac{1}{\rho}\frac{\partial u_\varphi}{\partial\varphi} + u_z\frac{\partial u_\varphi}{\partial z} + \frac{u_\rho u_\varphi}{\rho}\right)\vec{e}_\varphi \\ &\quad + \left(u_\rho\frac{\partial u_z}{\partial\rho} + u_\varphi\frac{1}{\rho}\frac{\partial u_z}{\partial\varphi} + u_z\frac{\partial u_z}{\partial z}\right)\vec{e}_z\end{aligned} \tag{D.8}$$

Or

$$
\begin{aligned}
(\vec{u}\cdot\nabla)\vec{u}|_{\rho} &= (\vec{u}\cdot\nabla)u_{\rho} - \frac{u_{\varphi}^2}{\rho} \\
(\vec{u}\cdot\nabla)\vec{u}|_{\varphi} &= (\vec{u}\cdot\nabla)u_{\varphi} + \frac{u_{\rho}u_{\varphi}}{\rho} \\
(\vec{u}\cdot\nabla)\vec{u}|_{z} &= (\vec{u}\cdot\nabla)u_{z}.
\end{aligned} \tag{D.9}
$$

D.2 Rate of strain tensor revisted

Recall the rate of strain tensor in cartesian coordinates

$$
\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{D.10}
$$

Take

$$
\nabla\vec{u} = \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\vec{u}
$$

where

$$
\vec{u} = u_x\vec{e}_x + u_y\vec{e}_y + u_z\vec{e}_z
$$

Write

$$
\begin{aligned}
\nabla\otimes\vec{u} &= \left(\vec{e}_x\frac{\partial}{\partial x} + \vec{e}_y\frac{\partial}{\partial y} + \vec{e}_z\frac{\partial}{\partial z}\right)\otimes\vec{u} \\
&= \frac{\partial}{\partial x}(u_x\vec{e}_x)\otimes\vec{e}_x + \frac{\partial}{\partial y}(u_x\vec{e}_x)\otimes\vec{e}_y + \frac{\partial}{\partial z}(u_x\vec{e}_x)\otimes\vec{e}_z \\
&\quad \frac{\partial}{\partial x}(u_y\vec{e}_y)\otimes\vec{e}_x + \frac{\partial}{\partial y}(u_y\vec{e}_y)\otimes\vec{e}_y + \frac{\partial}{\partial z}(u_y\vec{e}_y)\otimes\vec{e}_z \\
&\quad \frac{\partial}{\partial x}(u_z\vec{e}_z)\otimes\vec{e}_x + \frac{\partial}{\partial y}(u_z\vec{e}_z)\otimes\vec{e}_y + \frac{\partial}{\partial z}(u_z\vec{e}_z)\otimes\vec{e}_z \\
&= \sum_{i,j}\frac{\partial u_i}{\partial x_j}\vec{e}_i\otimes\vec{e}_j
\end{aligned} \tag{D.11}
$$

where we have used that the derivatives do not effect the basis vectors. This gives twice the second part of (D.10). It can be seen that

$$
\begin{aligned}
(\nabla \otimes \vec{u})^T &= \vec{u} \otimes \left(\vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z} \right) \\
&= \frac{\partial}{\partial x}(u_x \vec{e}_x) \otimes \vec{e}_x + \frac{\partial}{\partial x}(u_y \vec{e}_y) \otimes \vec{e}_x + \frac{\partial}{\partial x}(u_z \vec{e}_z) \otimes \vec{e}_x \\
&\quad \frac{\partial}{\partial y}(u_x \vec{e}_x) \otimes \vec{e}_y + \frac{\partial}{\partial y}(u_y \vec{e}_y) \otimes \vec{e}_y + \frac{\partial}{\partial y}(u_z \vec{e}_z) \otimes \vec{e}_y \\
&\quad \frac{\partial}{\partial z}(u_x \vec{e}_x) \otimes \vec{e}_z + \frac{\partial}{\partial z}(u_y \vec{e}_y) \otimes \vec{e}_z + \frac{\partial}{\partial z}(u_z \vec{e}_z) \otimes \vec{e}_z \\
&= \sum_{i,j} \frac{\partial u_j}{\partial x_i} \vec{e}_i \otimes \vec{e}_j
\end{aligned} \tag{D.12}
$$

gives twice the first part of (D.10).

So that

$$
\epsilon = \frac{1}{2} \left((\nabla \otimes \vec{u})^T + \nabla \otimes \vec{u} \right) \tag{D.13}
$$

with matrix components

$$
\epsilon = \begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{pmatrix} \tag{D.14}
$$

which are

$$
\begin{aligned}
\epsilon_{xx} &= \frac{\partial u_x}{\partial x} & \epsilon_{yy} &= \frac{\partial u_y}{\partial y} \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z} & \epsilon_{xy} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right) \\
\epsilon_{yz} &= \frac{1}{2} \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \right) & \epsilon_{xz} &= \frac{1}{2} \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right)
\end{aligned} \tag{D.15}
$$

D.3 Rate of strain tensor

We now consider the rate of strain tensor in cylindrical polar coordinates. Take

$$
(\nabla \otimes \vec{u})^T = \vec{u} \otimes \left(\vec{e}_\rho \frac{\partial}{\partial \rho} + \vec{e}_\varphi \frac{1}{\rho} \frac{\partial}{\partial \varphi} + \vec{e}_z \frac{\partial}{\partial z} \right)
$$

where

$$\vec{u} = u_\rho \vec{e}_\rho + u_\varphi \vec{e}_\varphi + u_z \vec{e}_z.$$

Non-trival terms of this are

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial}{\partial \varphi}(u_\rho \vec{e}_\rho) \otimes \vec{e}_\varphi &= \frac{1}{\rho}\frac{\partial u_\rho}{\partial \varphi}\vec{e}_\rho \otimes \vec{e}_\varphi + \frac{u_\rho}{\rho}\frac{\partial \vec{e}_\rho}{\partial \varphi} \otimes \vec{e}_\varphi \\
&= \frac{1}{\rho}\frac{\partial u_\rho}{\partial \varphi}\vec{e}_\rho \otimes \vec{e}_\varphi + \frac{u_\rho}{\rho}\vec{e}_\varphi \otimes \vec{e}_\varphi
\end{aligned} \tag{D.16}$$

and

$$\begin{aligned}
\frac{1}{\rho}\frac{\partial}{\partial \varphi}(u_\varphi \vec{e}_\varphi) \otimes \vec{e}_\varphi &= \frac{1}{\rho}\frac{\partial u_\varphi}{\partial \varphi}\vec{e}_\varphi \otimes \vec{e}_\varphi + \frac{u_\varphi}{\rho}\frac{\partial \vec{e}_\varphi}{\partial \varphi} \otimes \vec{e}_\varphi \\
&= \frac{1}{\rho}\frac{\partial u_\varphi}{\partial \varphi}\vec{e}_\varphi \otimes \vec{e}_\varphi - \frac{u_\varphi}{\rho}\vec{e}_\rho \otimes \vec{e}_\varphi
\end{aligned} \tag{D.17}$$

All the rest are trivial as the derivatives dont effect the basis vectors. For example

$$\frac{\partial}{\partial \rho}(u_\rho \vec{e}_\rho) \otimes \vec{e}_\rho = \frac{\partial u_\rho}{\partial \rho}\vec{e}_\rho \otimes \vec{e}_\rho, \quad \frac{1}{\rho}\frac{\partial}{\partial \varphi}(u_z \vec{e}_z) \otimes \vec{e}_\varphi = \frac{1}{\rho}\frac{\partial u_z}{\partial \varphi}\vec{e}_z \otimes \vec{e}_\varphi.$$

All these terms sum up to give an expression of the form

$$(\nabla \otimes \vec{u})^T = M_{ij}\vec{e}_i \otimes \vec{e}_j. \tag{D.18}$$

It is then seen that the matrix representing the components is

$$\begin{pmatrix}
\frac{\partial u_\rho}{\partial \rho} & \frac{1}{\rho}\frac{\partial u_\rho}{\partial \varphi} - \frac{u_\varphi}{\rho} & \frac{\partial u_\rho}{\partial z} \\
\frac{\partial u_\varphi}{\partial \rho} & \frac{1}{\rho}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\rho}{\rho} & \frac{\partial u_\varphi}{\partial z} \\
\frac{\partial u_z}{\partial r} & \frac{1}{\rho}\frac{\partial u_z}{\partial \varphi} & \frac{\partial u_z}{\partial z}
\end{pmatrix} \tag{D.19}$$

The rate of strain tensor is obtained by adding the transpose of the velocity gradient tensor to the velocity gradient tensor and dividing by 2:

$$\epsilon = \frac{1}{2}\left((\nabla \otimes \vec{u})^T + \nabla \otimes \vec{u}\right) \tag{D.20}$$

with components

$$\begin{pmatrix} \epsilon_{\rho\rho} & \epsilon_{\rho\varphi} & \epsilon_{\rho z} \\ \epsilon_{\varphi\rho} & \epsilon_{\varphi\varphi} & \epsilon_{\varphi z} \\ \epsilon_{z\rho} & \epsilon_{z\varphi} & \epsilon_{zz} \end{pmatrix} \tag{D.21}$$

which are

$$\begin{aligned} \epsilon_{\rho\rho} &= \frac{\partial u_\rho}{\partial \rho} & \epsilon_{\varphi\varphi} &= \frac{1}{\rho}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\rho}{\rho} \\ \epsilon_{zz} &= \frac{\partial u_z}{\partial z} & \epsilon_{\rho\varphi} &= \frac{\rho}{2}\frac{\partial}{\partial \rho}\left(\frac{u_\varphi}{\rho}\right) + \frac{1}{2\rho}\frac{\partial u_\rho}{\partial \varphi} \\ \epsilon_{\varphi z} &= \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial u_z}{\partial \varphi} + \frac{\partial u_\varphi}{\partial z}\right) & \epsilon_{\rho z} &= \frac{1}{2}\left(\frac{\partial u_\rho}{\partial z} + \frac{\partial u_z}{\partial \rho}\right). \end{aligned} \tag{D.22}$$

Appendix E

Spherical Polar Coordinates

Relation to Cartesian coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{E.1}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}(z/(x^2+y^2+z^2)^{1/2}) \\ \varphi &= \tan^{-1}(y/x) \end{aligned} \tag{E.2}$$

E.1 Vector Differential operators

The derivatives of the basis vectors are

$$\begin{aligned} &\frac{\partial\vec{e}_r}{\partial r} = \frac{\partial\vec{e}_\theta}{\partial r} = \frac{\partial\vec{e}_\varphi}{\partial r} = 0 \quad \frac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta \quad \frac{\partial\vec{e}_\theta}{\partial\theta} = -\vec{e}_r \quad \frac{\partial\vec{e}_\varphi}{\partial\theta} = 0 \\ &\frac{\partial\vec{e}_r}{\partial\varphi} = \sin\theta\vec{e}_\varphi \qquad \frac{\partial\vec{e}_\theta}{\partial\varphi} = \cos\theta\vec{e}_\varphi \quad \frac{\partial\vec{e}_\varphi}{\partial\varphi} = -\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta \end{aligned} \tag{E.3}$$

These can be derived by differentiating

$$\begin{aligned}
\vec{e}_r &= \sin\theta\cos\varphi\vec{e}_x + \sin\theta\sin\varphi\vec{e}_y + \cos\theta\vec{e}_z \\
\vec{e}_\theta &= \cos\theta\cos\varphi\vec{e}_x + \cos\theta\sin\varphi\vec{e}_y - \sin\theta\vec{e}_z \\
\vec{e}_\varphi &= -\sin\theta\vec{e}_r - \cos\theta\vec{e}_\theta
\end{aligned} \tag{E.4}$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\varphi}\vec{e}_\varphi \tag{E.5}$$

The divergence of a vector field $\vec{u}$ is

$$\nabla\cdot\vec{u} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} \tag{E.6}$$

The Laplace operator is (C.3) and (C.8),

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{E.7}$$

The curl of a vector field is given by (C.3) and (C.12),

$$\begin{aligned}
\nabla\times\vec{u} &= \nabla\times(u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_\varphi\vec{e}_\varphi) \\
&= \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{e}_r & r\vec{e}_\theta & r\sin\theta\vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ u_r & ru_\theta & r\sin\theta u_\varphi \end{vmatrix}
\end{aligned} \tag{E.8}$$

The Laplacian of a vector field $\nabla^2\vec{u}$ resolved in spherical polar coordinates is

$$\begin{aligned}
\nabla^2\vec{u}\big|_r &= \nabla^2u_r - \frac{2}{r^2}u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}u_\theta - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} \\
\nabla^2\vec{u}\big|_\theta &= \nabla^2u_\theta - \frac{1}{r^2\sin^2\theta}u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi} \\
\nabla^2\vec{u}\big|_\varphi &= \nabla^2u_\varphi - \frac{1}{r^2\sin^2\theta}u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi}.
\end{aligned} \tag{E.9}$$

where we have applied (E.7) to $\vec{u} = u_r\vec{e}_r + u_\theta\vec{e}_\theta + u_\varphi\vec{e}_\varphi$ and used (E.3). These would be used in writing the $\mu\nabla^2\vec{u}$ term in the Navier-Stokes equations in spherical polar coordinates.

If $\vec{u}$ is the velocity field and if we had incompressible flow, $\nabla \cdot \vec{u} = 0$. Then from the identity $\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$ and using $\nabla \cdot \vec{u} = 0$, then we can use the alternative formula $\nabla^2 \vec{u} = -\nabla \times (\nabla \times \vec{u})$.

The non-linear term $(\vec{u} \cdot \nabla)\vec{u}$ of the Navier-Stokes equations

$$\begin{aligned}
(\vec{u} \cdot \nabla)\vec{u}|_r &= (\vec{u} \cdot \nabla)u_r - \frac{u_\theta^2 + u_\varphi^2}{r} \\
(\vec{u} \cdot \nabla)\vec{u}|_\theta &= (\vec{u} \cdot \nabla)u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2 \cot\theta}{r} \\
(\vec{u} \cdot \nabla)\vec{u}|_\varphi &= (\vec{u} \cdot \nabla)u_\varphi + \frac{u_r u_\varphi}{r} - \frac{u_\theta u_\varphi \cot\theta}{r}.
\end{aligned} \tag{E.10}$$

E.2 Rate of strain tensor

We now consider the rate of strain tensor in spherical polar coordinates. Take

$$(\nabla \otimes \vec{u})^T = \vec{u} \otimes \left(\vec{e}_r \frac{\partial}{\partial r} + \vec{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \vec{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \right)$$

where

$$\vec{u} = u_r \vec{e}_r + u_\theta \vec{e}_\theta + u_\phi \vec{e}_\phi.$$

Non-trivial terms are

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial \theta}(u_r \vec{e}_r) \otimes \vec{e}_\theta &= \frac{1}{r}\frac{\partial u_r}{\partial \theta}\vec{e}_r \otimes \vec{e}_\theta + \frac{u_r}{r}\frac{\partial \vec{e}_r}{\partial \theta} \otimes \vec{e}_\theta \\
&= \frac{1}{r}\frac{\partial u_r}{\partial \theta}\vec{e}_r \otimes \vec{e}_\theta + \frac{u_r}{r}\vec{e}_\theta \otimes \vec{e}_\theta
\end{aligned} \tag{E.11}$$

The matrix elements of $\nabla \vec{u}$ are

$$\begin{pmatrix}
\frac{\partial u_r}{\partial r} & \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r} & \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{u_\phi}{r} \\
\frac{\partial \theta}{\partial r} & \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} & \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \cot\theta \frac{u_\phi}{r} \\
\frac{\partial \phi}{\partial r} & \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} & \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \cot\theta \frac{u_\theta}{r} + \frac{u_r}{r}
\end{pmatrix}. \tag{E.12}$$

The rate of strain tensor is obtained by adding the transpose of the velocity gradient tensor and the velocity gradient tensor and dividing by 2:

$$\epsilon = \frac{1}{2}\left((\nabla \otimes \vec{u})^T + \nabla \otimes \vec{u}\right) \tag{E.13}$$

whose components are the matrix elements are

$$\begin{pmatrix} \epsilon_{rr} & \epsilon_{r\theta} & \epsilon_{r\phi} \\ \epsilon_{\theta r} & \epsilon_{\theta\theta} & \epsilon_{\theta\phi} \\ \epsilon_{\phi r} & \epsilon_{\phi\theta} & \epsilon_{\phi\phi} \end{pmatrix} \tag{E.14}$$

which are

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r} & \epsilon_{\theta\theta} &= \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{u_r}{r} \\ \epsilon_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r} & \epsilon_{r\theta} &= \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) + \frac{1}{2r}\frac{\partial u_r}{\partial \theta} \\ \epsilon_{\theta\phi} &= \frac{\sin\theta}{2r}\frac{\partial}{\partial\theta}\left(\frac{u_\phi}{\sin\theta}\right) + \frac{1}{2r\sin\theta}\frac{\partial u_\theta}{\partial\phi} & \epsilon_{\phi r} &= \frac{1}{2r\sin\theta}\frac{\partial u_r}{\partial\phi} + \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right) \end{aligned} \tag{E.15}$$

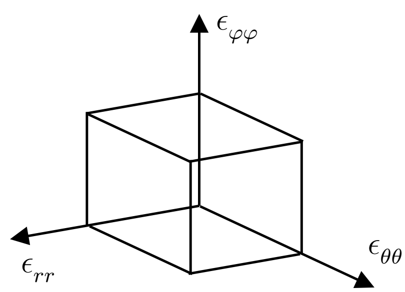

Figure E.1: Stress.