Appendix D

Yang-Mills and Gauge Theory

think about gauge theory geometrically - to understand the gauge field as a connection on the principal bundle.

We have already encountered one fibre bundle, from general relativity: the tangent space at a point in spacetime is a fibre, with the fibre bundle

D.1 Summary

The origin of such functions could be diffrent. For example, it could be a awve function $\psi(x)$ from quantum mechanics. Recall that its absolute value $|\psi(x)|$ can be interpreted as the distribution density for the probability of finding the particle at the point x.

we may think of values of ψ as vectors in a 2-plane $\mathbb{R}^2$ which can be intepreted as the internal space of the particle. No particular direction in this plane has special meaning.

A particle is moving from point to point and carrying its internal spaqce with it. This has a geometric structure - a fibre bundle. The disjoint union of the internal spaces forms a fibre bundle $\pi : S \to \mathcal{M}$. Its fibre over a point $x \in \mathcal{M}$ is the internal space S_x. It could be a vector space (then we speak about a vector bundle) or a group (this leads to a principal bundle). For example, we can interpret the phase angle θ as an example of the group $U(1)$.

There is no common internal space in general, so one cannot identify the all internal spcaes with the spaces with the same space F. Or the fibre bundle is not trival, in general. However, we allow to identify the fibres along paths in $\mathcal{M}$. Thus, if $x(t)$ depends on t, the internal state $\overline{x}(t) \in S_{x(t)}$ describes a path in S lying over the original path. This leads to the notion of paralle transport in the fibre, or equivalently, a connection. In general, there is no reasonto expect that diferent paths from x to y lead to the same parallel transport of the internal state. They could differ by application of a symmetry group acting on the fibres (the structure grouop of fibre bundle). Physically, this is viewed as a phase shift. It is produced by the external field. Quantatively, the phase shift is described by the "curvative" of the connection.

The phase shift is determined by the commutator of the coovariant derivative ∇_μ:

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

Here the group G is a Lie group, and the commutator takes its value in the Lie algebra **g** of G.

In general our state bundle $S \to \mathcal{M}$ is only locally trivial. This means that, for any point $x \in \mathcal{M}$, one can find a coordinate system in its neighbourhod U such that $\pi^{-1}(U_i)$ can be identified with $U \times F$, and the connection is given as above. If V is another neighbourhood, thwn we asssume that the new identification $\pi^{-1}(V) = V \times F$ differs over $U < capV$ from the old one by the "gauge transformation" $g(x) \in G$ so that $(x, s) \in U \times F$ corresponds to $(x, g(x)s)$ in $V \times F$. We shall see that the connection changes by the formula

$$A_\mu \to g^{-1} A_\mu + g^{-1} \partial_\mu g.$$

Note that we may take $V = U$ and change the trivialization by a function $g : U \to G$. The set of such functions forms a group (infinite-dminsional). It is called the gauge group.

D.2 Maxwell Equations and Gauge Theory

D.2.1 Gauge Symmetry

From Global to local symmetries. We have a single, free non-relativistic particle described by a wavefuncion $\psi(x)$. Multiplying the wavefunction by a complex number with unit yields a wavefunction that is physically equivalent to the original.

Consider a *local* transformation, that is , a transformation that depends on $\vec{r}$ and t,

$$\psi'(\vec{r}, t) = e^{i\alpha(\vec{r}, t)} \psi(\vec{r}, t) \tag{D.1}$$

where $\alpha(\vec{r}, t)$ a real-valued function. The Schrödinger equation

for although they have the same probability distribution for position, they will have different momentum probability distribution

$$\partial_\mu \psi' = e^{i\alpha} [\partial_\mu \psi + i(\partial_\mu \alpha) \psi] \tag{D.2}$$

and the term in $\partial_\mu \alpha$. We want a the gauge transformation to be a symmetry of the theory, we have to modify

At the end of this chapter we will place these ideas in the more general mathematical context of what are known as fibre bundles.

The demand for gauge invariance does not give us the equation of motion for the electromagnetic field - it only gives us the interaction between the charged particle and the gauge potential. The simplest non-trivial gauge invariant Lagraigian

$$\mathcal{L}(A) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x). \tag{D.3}$$

$$\begin{aligned}\mathcal{L}(A) \to \mathcal{L}(A') &= \partial_\mu (A_\nu(x) + \partial_\nu \alpha(x)) - \partial_\nu (A_\mu(x) + \partial_\mu \alpha(x)) \\ &= \mathcal{L}(A) + \partial_\mu \partial_\nu \alpha(x) - \partial_\nu \partial_\mu \alpha(x) \\ &= \mathcal{L}(A).\end{aligned} \tag{D.4}$$

In the presence of no charges the system is completely described by the. A point in phase space represents the state of a particular system at time t. As time goes on, the state of the system changes according to Maxwell's equations. The entire history, past, present and future, of the system comes to be represented by a certain trajectory in phase space. When a system has gauge symmetries there is more than one trajectory in phase space that represents a physically equivalent system. The collection of such trajectories forms a (hyper-)surface in phase space. If the gauge symmetry is . A particular trajectory is called a gauge slice. Two distinct gauge slices are related to each other through a gauge transformation; there is a trajectory along which the gauge transformation drags the first gauge slice onto the other - this called a *gauge orbit*.

In the case of spacetime in GR, an action is invariant under general coordinate transformations could be built from tensors by contracting all their indices, so that the Jacobian (and inverse Jacobian) transformation matrices cancel.

D.2.2 Simple example

consider the integral

$$Z = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; e^{(x-y)^2} \tag{D.5}$$

$$\begin{aligned} x &\to x + a \\ y &\to y + a \end{aligned} \tag{D.6}$$

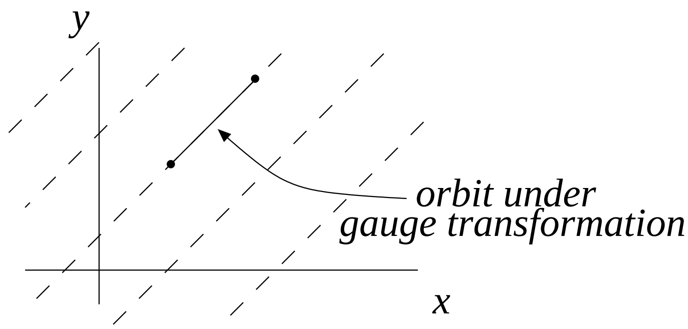

Figure D.1: simplegauge. The motion of a "configuration," (x, y), in the configuration space under the "gauge" transformation defined in (D.6). The path is called a gauge orbit. In the simple example here, the gauge orbits are lines of constant $x - y$.

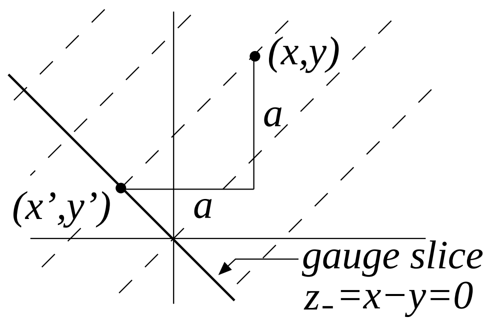

Figure D.2: simplegauge2. The gauge choice $x + y = 0$ defines a "gauge slice" through the configuration space. (x', y') is a configuration on the slice, that is, it satisfies the gauge condition. (x, y) is a gauge equivalent configuration, since both (x, y) and (x', y') reside on the same gauge orbit. a is the gauge transformation that takes us from the slice (x, y).

the gauge orbits are lines of constant $x - y$. The "action", $(x - y)^2$ is a gauge invariant.

$$\overset{-1}{\det} \left(\frac{\partial f}{\partial x} \right) \bigg|_{f=a} = \int dx \delta(f(x) - a) \tag{D.7}$$

The degeneracy is caused by the fact that we integrate over a redundant set of integration variables which results in an infinite volume factor. This situation occurs because of the way we formulate the theory as based on the principle of a local gauge invariance. The complete physical content is contained by one contribution out of each equivalence class. One selects to one such member by imposing a condition called a *gauge fixing condition.*

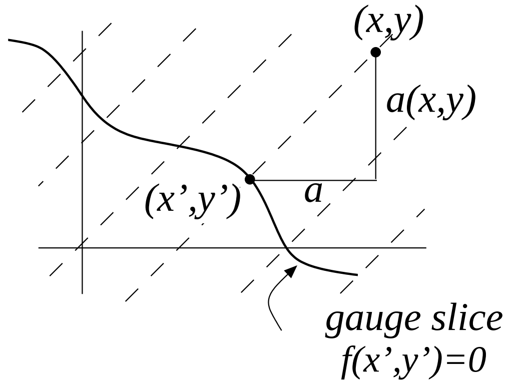

Figure D.3: simplegauge3. Illustration of a general choice of gauge, $f(x,y) = 0$. The desired change of coordinates is from (x,y) to (s,y), s being a variable that runs along the slice, and a doing the gauge transformation that runs from the slice (x,y).

D.2.3 Gauge Constraints

In the case of constrained coherent state path integral, we will end up integrating over the gauge orbits in effect averaging over all possible gauge orbits. Because of this, we will not have to fix a gauge and will not encounter the Gribov problem.(quant-ph/9611026)

$$\partial_a E^a(x) \approx 0 \tag{D.8}$$

which is the Gauss's law in the absence of charge.

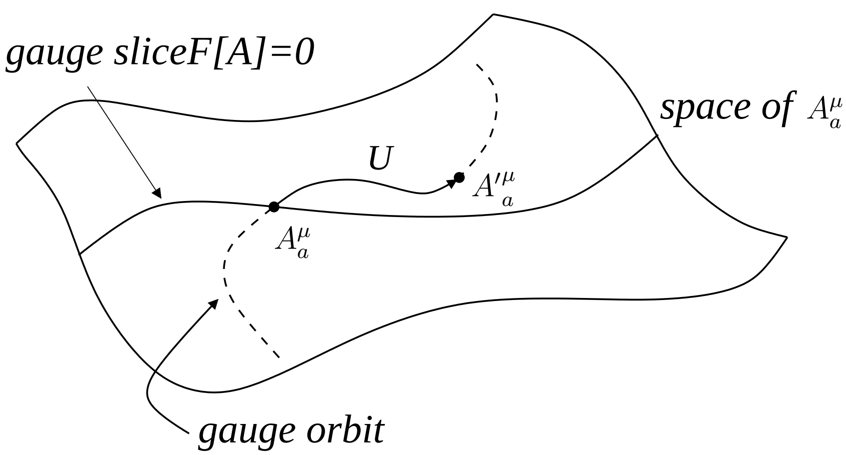

Figure D.4: orbitgenerator. A gauge choice is made. Configurations A'^μ_a that satisfy the gauge condition, $F[A] = 0$, fall on the gauge slice. Two configurations that are gauge equivalent lie on the same gauge orbit. U is the gauge transformation relating A'^μ_a to A^μ_a.

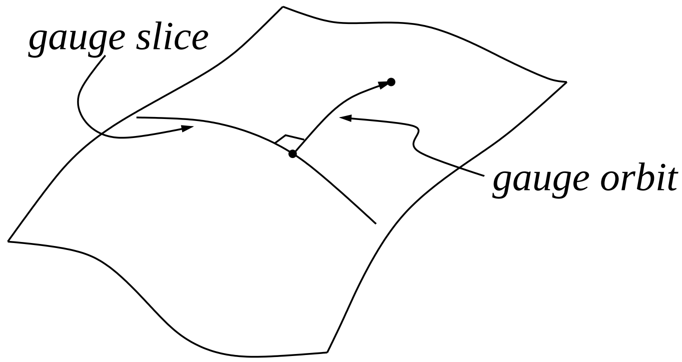

Figure D.5: gaugeslice.

D.2.4 Gauge Fixing in Electrodynamics

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \tag{D.9}$$

and

$$\vec{E} = -\vec{\nabla} A^0 - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A} \tag{D.10}$$

where

$$\vec{E} = (F^{10}, F^{20}, F^{30}), \quad \vec{B} = (-F^{23}, -F^{31}, -F^{12}).$$

Equation (D.9) together with

$$\Box A^\alpha - \partial^\alpha(\partial_\beta A^\beta) = j^\alpha. \tag{D.11}$$

are equivalent to Maxwell's equations.

We have residual gauge invariance. Given any A^α, choosing ϕ obeying

$$\Box \phi + \partial_\alpha A^\alpha = 0 \tag{D.12}$$

$A'^\alpha = A^\alpha + \partial^\alpha \phi$ will lead to $\partial_\alpha A'^\alpha = 0$

$$\Box A'^\alpha - \partial^\alpha(\partial_\beta A'^\beta) = \Box A^\alpha - \partial^\alpha(\partial_\beta A^\beta) + \Box\phi + \partial_\alpha A^\alpha = j^\alpha$$

and (D.11) is unchanged.

Substitution of $A_\mu \to A'_\mu = A_\mu + \partial_\mu \phi$ and doesn't change the form of the equations of motion and give rise to the same electromagnetic field.

It is possible to use the gauge invariance to require some particular condition of the field A_μ. For instance, we can perform a gauge transformation in such a way that the transformed field satisfies

$$\partial_\mu A^\mu = 0. \tag{D.13}$$

This is called the Lorentz gauge giving the wave equation for the potential

$$\Box A^\alpha = j^\alpha. \tag{D.14}$$

Or we can choose a gauge such that

$$A_0 = 0. \tag{D.15}$$

This is called the temporal gauge, or

$$\vec{\nabla} \cdot \vec{A} = 0. \tag{D.16}$$

which is called the Coulomb gauge. Each time we fix the constraint by restricting the gauge field A_μ, we need to check that there exists a choice of ϕ such that the gauge condition is possible.

The Wave Equation

Coulomb gauge

With the choice of the Coulomb gauge, time component of Maxwell equation (M.-19) becomes

$$-\nabla^2 A^0 = e^2 j^0. \tag{D.17}$$

We see that A^0 is not a dynamical variable, but rather is determined by the charge density a the same time,

$$A^0(\vec{x}, t) = \int d^3x' \frac{1}{4\pi|\vec{x} - \vec{x}'|} e^2 j^0. \tag{D.18}$$

Writing the electric field

the longitudinal piece

$$
\begin{aligned}
F^{ok(L)} &= -\partial^k A^0 \\
&= \int d^3x' \frac{x^k - x'^k}{4\pi|\vec{x} - \vec{x}'|^{3/2}} e^2 j^0.
\end{aligned}
\tag{D.19}
$$

Lorentz Gauge

• From the wave equation $k \cdot k = 0$ hence $k^2 = \omega^2$, i.e. $E^2 = p^2c^2$ (massless photons).

• From rom Lorenz gauge condition $\epsilon \cdot \mathbf{k} = 0$ implies $\epsilon^0 = \epsilon \cdot \mathbf{k}/\omega$.

• Polarization 4-vector $\epsilon'^\mu = \epsilon^\mu + ak^\mu$ is equivalent to ϵ^μ for any constant a. Hence we can always choose $\epsilon^0 = 0$. Then Lorenz condition becomes the transverstity condition: $\epsilon \cdot \mathbf{k} = 0$.

• For k along z-axis -we can express ' in terms of plane polarization states ,

$$
\epsilon_x^\mu = (0,1,0,0), \qquad \epsilon_y^\mu = (0,0,1,0) \tag{D.20}
$$

or circular polarization states $\epsilon_{R,L}^\mu = (0,1,\pm i,0)/\sqrt{2}$. Note only two polarization states for real photons.

Temporal Gauge

D.2.5 Gauge Invariance Noether

$$
\int_{\partial R} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - T^\mu_\lambda X^\lambda_\nu \right] \delta\omega^\nu d\sigma_\mu = 0
$$

since $\delta\omega^\nu$ is arbitrary,

$$
\int_{\partial R} J^\mu_\nu d\sigma_\mu = 0 \tag{D.21}
$$

where

$$
J^\mu_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Phi_\nu - T^\mu_\lambda X^\lambda_\nu. \tag{D.22}
$$

by Gauss's theorem that $\int_R \partial_\mu J^\mu_\nu d^4 = 0$, and since R is arbitrary,

$$\partial_\mu J^\mu_\nu = 0. \tag{D.23}$$

We therefore have a conserved current J^μ_ν because of gauge invariance of the action under the transformations (M.-19). This gives rise to a conserved charge Q_ν defined by

$$Q_\nu = \int_\sigma J^\mu_\nu d\sigma_\mu \tag{D.24}$$

where the integral is taken over a spacelike hypersurface σ_μ.

$$\int_V \partial_0 J^0_\nu d^3x + \int_V \partial_i J^i_\nu d^3x = 0. \tag{D.25}$$

The second term is transformed into a surface integral by Gauss's theorem. By taking the surface far enough away we see that this term vanishes. So we have

$$\frac{d}{dt}\int J^0_\nu d^3x = \frac{dQ_\nu}{dt} = 0. \tag{D.26}$$

D.3 Canonical Formulation of Yang-Mills

$$\mathcal{L} = \frac{1}{4}F^I_{\mu\nu}F^{I\mu\nu} - \frac{1}{2}F^{I\mu\nu}(\partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g\epsilon^{IJK}A^J_\mu A^K_\mu) \tag{D.27}$$

Palatini method can be applied to Yang-Mills theory. A^I_μ and $F^I_{\mu\nu}$ are taken to be independent fields; they are totally unrelated. However, by eliminating the $F^I_{\mu\nu}$ field by its equations of motion, we can show the equivalence with the standard Yang-Mills action,

$$\mathcal{L}[A] = -\frac{1}{4}F^I_{\mu\nu}F^{I\mu\nu} \tag{D.28}$$

We show this in the following. We take the variation of Eq.(D.27) with with respect to $F^I_{\mu\nu}$ as well as respect to A^I_a. The two equations of motion yield:

$$\frac{\delta S}{\delta F^I_{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial F^i_{\mu\nu}} = \frac{\partial}{\partial F^i_{\mu\nu}}\frac{1}{4}\left[\eta^{ce}\eta^{df}F^i_{cd}F^i_{ef} + \eta^{ce}\eta^{df}(\partial_\mu A^i_\nu - \partial_\nu A^i_\mu)F^i_{fe}\right] \tag{D.29}$$

$$F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + g\epsilon^{IJK}A^J_\mu A^K_\mu \tag{D.30}$$

$$0 = \partial_\mu F^{I\mu\nu} + g\epsilon^{IJK}A^{I\mu}F^I_{\mu\nu}K \tag{D.31}$$

inserting Eq.(D.31) into Eq.(D.27) we recover the standard action of Eq.(D.28).

First, we eliminate F^i_{ab} in terms of A^i_a fields, keeping F^i_{0a} an independent field. Written in these variables, we find that the Lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}F^i_{ab}F^I_{iab} + \frac{1}{2}(F^i_{0a})^2 - F^{a,0i}(\partial_i A^a_0 - \partial_0 A^a_i + gf^{abc}A^b_0 A^c_i) \tag{D.32}$$

Let us define,

$$E^a_i := F^a_{0i} \tag{D.33}$$
$$B^a_i := \frac{1}{2}\epsilon_{ijk}F^{iab} \tag{D.34}$$

In terms of these fields, we now have,

$$E^a_i \dot{A}^a_i - \mathcal{H} \tag{D.35}$$

where,

$$\boxed{\mathcal{H} = \frac{1}{2}[(E^a_i)^2 + (B^a_i)^2] + A^i_0(\partial_a E^a_i + g\epsilon_{ijk}A^k_a E^{aj})} \tag{D.36}$$

$$\mathcal{D}_a E^a_i = \partial_a E^a_i + g\epsilon_{ijk}A^k_a E^{aj} \tag{D.37}$$

$$\begin{aligned} F_{ab} &= \partial_a A_b - \partial_b A_a - i[A_a, A_b] \\ &= \partial_a A^i_b T_i - \partial_b A^j_a T_j - i[A^i_a T_i, A^j_b T_j] \\ &= (\partial_a A^i_b - \partial_b A^i_a)T_i - iA^i_a A^j_b [T_i, T_j] \\ &= (\partial_a A^i_b - \partial_b A^i_a) \\ &= (\partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b f_{ijk})T_i. \end{aligned} \tag{D.38}$$

$$\boxed{F_{ab} = (\partial_a A^i_b - \partial_b A^i_a + A^j_a A^k_b f_{ijk})T_i} \tag{D.39}$$

D.3.1 The Geometry of Gauge Invariance - Gauge Field as a Connection

How do you compare Ψ on one fibre with Ψ another? we can choose coordinates for one fibre that are different than others, that is, apply a rotation of one fibre without rotating others.

the gauge fields play the same role in gauge theory as the Christoffel symbols play in general relativity.

In electrodynamics

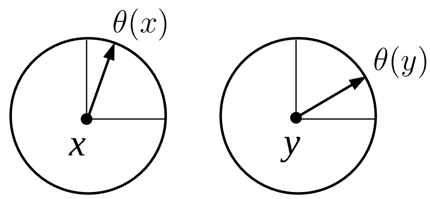

Figure D.6: gaugeElectric. The wavefunction at point x is $\psi(x) = |\psi(x)|e^{i\theta(x)}$ and the wavefunction at a nearby point y is $\psi(y) = |\psi(y)|e^{i\theta(y)}$.

For the sake of simplicity we consider wavefunction combined as a vector

$$\Psi = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{pmatrix} \tag{D.40}$$

We assume a local symmetry such that

$$\Psi'(x) = \exp(-ig\theta(x))\Psi(x) \tag{D.41}$$

is physically equivalent to $\Psi(x)$.

The situation is somewhat analogous to an arbitrary contravariant vector and its Lorentz transformation

$$V^\mu(x), \quad V'^\mu(x) = \Lambda^\mu{}_\nu(x)V^\nu(x) \tag{D.42}$$

The local Lorentz transformation $\Lambda^\mu{}_\nu(x)$ corresponds to the gauge transformation $\exp(-ig\theta(x))$.

$$\frac{V^\mu(x+\delta x) - V^\mu(x)}{\delta x} \tag{D.43}$$

yields additional terms owing to the dependence of the metric tensor $g_{\mu\nu}(x)$ on the position.

In a completely analogous manner we can write down () for the wavefunction $\Psi(x)$:

$$\partial_\mu \Psi(x) \to (\partial_\mu + \tilde{\Gamma}_\mu)\Psi(x), \tag{D.44}$$

$$D_\mu \Psi(x) = (\partial_\mu - igA_\mu)\Psi(x) \tag{D.45}$$

$\Psi(x) \to e^{-ig\theta(x)}\Psi(x)$

Introduce a mathematical notion we consider and electric field in a spacial slice in Minkowskian spacetime. The structure is $\mathcal{M}_{Mink} \times R^3$ They can taken together is a single entity. It is an example of what mathematicians call a ***vector bundle***.

Let G be a Lie group and $\mathcal{M}$ a smooth manifold.

A ***principle G bundle*** is over $\mathcal{M}$ is a manifold which locally looks like $\mathcal{M} \times G$.

The connection tells us how points on one fibre, that is the value of Ψ at one point, are mapped into points on another fibre. If we rotate one fibre (or simply change the coordinate basis we are using to describe it), the rotated points should still map to the same points on the neighbouring fibre

The geometry of the $U(1)$ fibre bundle of electromagnetism is the relationships between phases of wavefunctions at different points in spacetime is determined by the gauge potential $A_\mu(x)$, to be compared with the case of GR the spacetime geometry is determined by the connection $\Gamma^c_{ab}(x)$.

In the case of spacetime in GR, an action is invariant under general coordinate transformations could be built from tensors by contracting all their indices, so that the Jacobian (and inverse Jacobian) transformation matrices cancel.

D.4 Summary of Yang Mills

A Lie group is a group object G in the category of smooth manifolds. For any $g \in G$ we denote by L_g (resp. R_g) the left (resp. right) transaltion action of the group G on itself.

The differnetials of the maps R_g, L_g transform a vecter field η to the vector field

$$(L_g)_*(\eta), (R_g)_*(\eta),$$

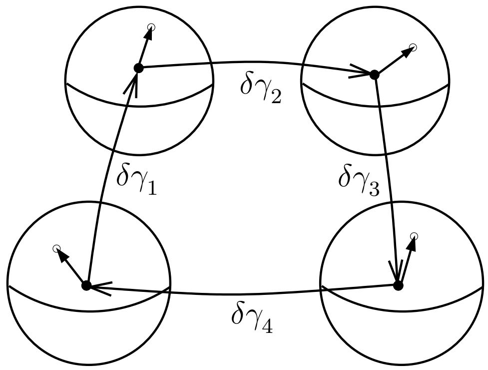

Figure D.7: ConnCurvat. $F_{ab} \approx$

respectively. The Lie algerbra of G is the vector space $\mathbf{g}$ of vector fields which are invariant with respect to left translations. Its Lie algerba structure is the bracket operation of vector fields.

For any $\eta \in \mathbf{g}$, the vector

$$(dL_{g^{-1}})_g(\eta_g)$$

belongs to the vector space

$$T(G)_{\mathbf{1}}$$

and $\tilde{v}$ be the corresponding vectotr field. We have

$$\tilde{v}_g = (dL_g)_{\mathbf{1}}(v),$$

so that

$$(dR_{g^{-1}})_g(\tilde{v}) = dc(g)_1(v)$$

where $c(g) : G \to G$ is the conjugacy action

$$h \to g \cdot h \cdot g^{-1}.$$

D.5 Propagation Kernel Yang-Mills

Comparison Between Gauge Yang-Mills Theories and GR

GR is invariant under diffeomorphisms, namely under the pull back $q \rightarrow g^{\xi} = \xi_{*}g$ of the gravitational field by a map $\xi : \mathcal{M} \rightarrow \mathcal{M}$ from the spacetime to itself. An active diffeormorphism can be thought of as simultaneously dragging the metric and matter fields over the manifold, or keeping the fields "where" they are and mapping the points of the manifold to other points of the manifold, i.e., $\xi : \mathcal{M} \rightarrow \mathcal{M}$. Should not be confused with the freedom of choosing coordinates on $\mathcal{M}$: once coordinates are fixed, one still gets physically identical fields of g and g^{ξ}.

$$g_{ab}(x) = \frac{\partial \xi^{c}(x)}{\partial x^{a}} \frac{\partial \xi^{d}(x)}{\partial x^{b}} g^{\xi}_{cd}(\xi(x)) \tag{D.46}$$

or

$$g^{\xi}_{ab}(x) = \frac{\partial \phi^{c}(x)}{\partial x^{a}} \frac{\partial \phi^{d}(x)}{\partial x^{b}} g_{cd}(\phi(x)) \tag{D.47}$$

This is the change in the metric under an infinitesimal active diffeomorphism along the vector field ξ^{a}.

In electromagnetism the invariance in A_{μ} comes about because the field strength $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is left unchanged by the gauge transformations $A_{\mu} \rightarrow A'_{\mu} = A_{\mu} + \partial_{\mu}\phi$.

Gauge transformation	Active diffeomorphism
Gauge group	Group of all active diffeomorphism
Gauge potential, A_{a}	Connection coefficient, Γ^{c}_{ab}
Field strength, F_{ab}	Curvature tensor, R^{a}_{bcd}
Bianchi identity:	Bianchi identity:
$\mathcal{D}_{a}F_{bc} = 0$	$\mathcal{D}_{a}R^{b}_{cde} = 0$

Faddeev-Popov

$$A^{\Omega}_{\mu} = A_{\mu} + \partial_{\mu}\Omega \tag{D.48}$$

$$A^{\Omega}_{\mu} = \Omega A_{\mu}\Omega^{-1} + i\Omega(\partial_{\mu})\Omega^{-1} \tag{D.49}$$

it has the invariance property:

$$\mathcal{D}\Omega = \mathcal{D}(\Omega'\Omega) \tag{D.50}$$

$$\begin{aligned}
\Delta_{FP}^{-1}(A_\mu^{\Omega'}) &= \int \mathcal{D}\Omega' \; \delta(F(A_\mu^{\Omega'\Omega})) \\
&= \int \mathcal{D}[\Omega'\Omega] \; \delta(F(A_\mu^{\Omega'\Omega})) \\
&= \int \mathcal{D}\Omega'' \; \delta(F(A_\mu^{\Omega''})) \\
&= \Delta_{FP}^{-1}(A_\mu)
\end{aligned} \tag{D.51}$$

$$\Delta_{FP}(A_\mu) = \Delta_{FP}(A_\mu^{\Omega}) \tag{D.52}$$

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}\Omega\delta(F(A_\mu^{\Omega})) \tag{D.53}$$

inserting and then making gauge transformation

$$\int \mathcal{D}A_\mu \left(\Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \; \delta(F(A_\mu^{\Omega})) \right) e^{i \int d^4x \mathcal{L}[A]} \tag{D.54}$$

and then make the replacement followed by gauge transformation on the entire functional integral, so that $A_\mu \to A_\mu^{-\Omega}$

$$\int \mathcal{D}\Omega \; \int \mathcal{D}A_\mu \; \Delta_{FP}(A_\mu) \; \delta(F(A_\mu)) e^{i \int d^4x \mathcal{L}[A]} \tag{D.55}$$

Temporal gauge:

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \; \delta(A_0^{\Omega}) \tag{D.56}$$

The YM gauge transformation (Eq. D.49) does not mix different components of the 4-vector A_μ. The integral (Eq. D.56) reads explicitly

$$1 = \Delta_{FP}(A_\mu) \int \mathcal{D}\Omega \; \delta(\Omega A_0 \Omega^{-1} + i\Omega(\partial_0)\Omega^{-1}), \tag{D.57}$$

hence, Δ_{FP} only depends on the component A_0, i.e., $\Delta_{FP}(A_\mu) = \Delta_{FP}(A_0)$; but this is, in turn, fixed to be zero by the $\delta-$function appearing in the integral:

$$\begin{aligned} 1 &= \int \mathcal{D}\Omega \, \Delta_{FP}(A_0)\delta(A_0^\Omega) \\ &= \int \mathcal{D}\Omega \, \Delta_{FP}(A_0^\Omega)\delta(A_0^\Omega) \\ &= \int \mathcal{D}\Omega \, \Delta_{FP}(A_0^\Omega = 0)\delta(A_0^\Omega) \\ &= \Delta_{FP}(A_0 = 0) \int \mathcal{D}\Omega \, \delta(A_0^\Omega) \end{aligned} \tag{D.58}$$

where we used the gauge invariance of Δ_{FP} to write the second line. It turns out that Δ_{FP} is just a constant.

$$W[A_i', A_i'', T] = \Delta_{FP} \int_{A_i'}^{A_i''} \mathcal{D}A_\mu \int \mathcal{D}\Omega \, \delta(A_0^\Omega) e^{iS[A_\mu]} \tag{D.59}$$

changing the order of the two integrations, changing variables $A_\mu \to A_\mu^\Omega$ and integrating out A_0, we obtain

$$W[A_i', A_i'', T] = \Delta_{FP} \int \mathcal{D}\Omega \int_{A_i'^{\Omega(0)}}^{A_i''^{\Omega(T)}} \mathcal{D}A_i e^{S[A_i;A_0=0]} \tag{D.60}$$

Bulk integration on $\Omega(x,t), 0 > t > T$, factors out leaving

$$W[A_i', A_i'', T] = \Delta_{FP} \int \mathcal{D}\Omega(0) \; \mathcal{D}\Omega(T) \int_{A_i'^{\Omega(0)}}^{A_i''^{\Omega(T)}} \mathcal{D}A_i e^{iS[A_i;A_0=0]} \tag{D.61}$$

That $\mathcal{D}A_i$ and $S[A_i; A_0 = 0]$ are invariant under a time-independent gauge transformation will mean that one of the integrals $\Lambda(\vec{x}, 0)$ and $\Lambda(\vec{x}, T)$ is redundant. $A_\mu \to A_\mu^\Lambda$, replacing $S[A_i; A_0 = 0]$ with $S[A_i^\Lambda; A_0 = 0]$ and $\mathcal{D}[A_i]$ with $\mathcal{D}[A_i^\Lambda]$

$$\begin{aligned} \int_{A_i'^{\Omega(0)}}^{A_i''^{\Omega(T)}} \mathcal{D}A_i \; e^{S[A_i;A_0=0]} &= \int_{A_i'^{\Lambda(0)\Omega^{-1}(T)}}^{A_i''} \mathcal{D}(A_i^{\Omega(T)}) \; e^{iS[A_i^{\Omega(T)};A_0=0]} \\ &= \int_{A_i'^{\Lambda(0)\Omega^{-1}(T)}}^{A_i''} \mathcal{D}A_i e^{S[A_i;A_0=0]} \end{aligned} \tag{D.62}$$

$\mathcal{D}\Omega(0) \to \mathcal{D}(\Omega(0)\Omega^{-1}(T)) = \mathcal{D}\lambda$

$$W[A_i', A_i'', T] = \left(\int \mathcal{D}\Omega(T)\right) \Delta_{FP} \int \mathcal{D}\lambda \int_{A_i'^{\lambda}}^{A_i''} \mathcal{D}A_i \; e^{iS[A_i; A_0=0]} \tag{D.63}$$

we are able to drop the second integral, we therefore have

$$W[A_i', A_i'', T] = \int \mathcal{D}\lambda \tilde{W}[A_i'^{\lambda}, A_i'', T] \tag{D.64}$$

where

$$\tilde{W}[A_i', A_i'', T] = \Delta_{FP} \int \mathcal{D}A_i \; e^{iS[A_i; A_0=0]} \tag{D.65}$$

it is equal to the integration over all gauge factors λ, and hence independent of the gauge.

$$\Psi_{t+T}[A_i] = \mathcal{D}A_i' \int W[A_i, A_i', T]\Psi_t[A_i']. \tag{D.66}$$

$\Psi_t[A_i] = \Psi_t[A_i^{\lambda}]$

$$W[A_i, A_i', T] = \sum_n e^{-iE_n T} \overline{\Psi_n[A_i']} \Psi_n[A_i]. \tag{D.67}$$

$$\int_{-\infty}^{+\infty} dT \; W[A_i, A_i' = 0, T] = const. \; \Psi_0[A_i]. \tag{D.68}$$

$$W[A_i', A_i'', T] = \mathcal{N}(T) \exp\left\{ \frac{i}{2} \int \frac{d^3p}{(2\pi)^3} p \frac{(|A'^T|^2 + |A''^T|^2)\cos T - 2A'^T \cdot A''^T}{\sin T} \right\}. \tag{D.69}$$

$$D_{ij} = \delta_{ij} - \frac{p_i p_j}{p^2}. \tag{D.70}$$

$$W[A_i, A_i', T] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\epsilon_1 \ldots \epsilon_n} \int \frac{d^3p_1}{(2\pi)^3} \cdots \frac{d^3p_n}{(2\pi)^3} e^{-i\sum_{\alpha=1}^{n} E_\alpha T} \; \overline{\psi_{p_1\epsilon_1,\ldots,p_n\epsilon_n}[A_i]} \; \psi_{p_1\epsilon_1,\ldots,p_n\epsilon_n}[A_i'] \tag{D.71}$$

Using $\Psi_0[A_i] = \lim_{T\to\infty} W[A_i, A_i' = 0, T]$ the (non-normalized) vacuum state can be read from the zero'th order of (M.-19)

D.5.1 n Uncoupled Harmonic Oscillators

The Hamiltonian of a general one-dimensional harmonic oscillator is

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}k(\hat{x} - x_0)^2$$

if we measure the energy H we observe that the outcome is quantized. We can interpret as the number of quanta in x. we could call these quanta particles (Fock states - more on that later - also what Rovelli terms *global* particles.)

$$W[x, x', T] =< x'|e^{-iHT}|x > \tag{D.72}$$

$$W[x, x', T] = \mathcal{N}(T) \exp \frac{i}{2}\left(\frac{(x^2 + x'^2)\cos T - 2xx'}{\sin T}\right) \tag{D.73}$$

$$W[x_1, x_2, T] = \sum_n e^{-iE_nT}\, \overline{\psi_n(x_2)}\psi_n(x_1) \tag{D.74}$$

we expand (M.-19) in a power series in e^{-iT} by writing the sines and cosines in terms of exponentials and expanding.

$$\begin{aligned} W[x_1, x_2, T] &= \mathcal{N}(T) \exp \frac{i}{4}\left(\frac{(x^2 + x'^2)\cos T - 2xx'}{\sin T}\right) \\ &= \mathcal{N}(T) \exp -\frac{1}{4}\left((x^2 + x'^2)\frac{(1 + e^{-2iT})}{1 - e^{-2iT}} - \frac{2xx'e^{-iT}}{1 - e^{-2iT}}\right) \end{aligned} \tag{D.75}$$

$$\frac{(1 + e^{-2iT})}{1 - e^{-2iT}} = (1 + e^{-2iT})(1 + e^{-2iT} + e^{-4iT} + \dots) = 1 + 2e^{-2iT} + 2e^{-4iT} + \dots$$

$\mathcal{N}(T)$ part is

$$\mathcal{N}(T) = \left(\frac{\omega}{2\pi i \sin T}\right)^{1/2} = \left(\frac{\omega}{2\pi}\right)^{1/2} e^{-T/2}(1-e^{-2iT})^{-1/2} = \left(\frac{\omega}{2\pi}\right)^{1/2} e^{-T/2}\left(1+\frac{1}{2}e^{-2iT}+\dots\right).$$

Putting it together

$$W[x_1, x_2, T] = \left(\frac{\omega}{\pi}\right)^{1/2} e^{-iT/2} \exp\left(-\frac{\omega}{2}(x^2 + x'^2)\right)\left(1 + \frac{1}{2}e^{-2iT} + \dots\right)$$
$$(1 - \omega(x^2 + x'^2)e^{-2iT} + \dots)(1 + 2\omega x x' e^{-iT} + \dots). \qquad \text{(D.76)}$$

the leading term in the $iT \to \infty$ limit,

$$\lim_{iT\to\infty} W[x_1, x_2, T] = \lim_{iT\to\infty} e^{-i\omega T/2}\left(\frac{\omega}{\pi}\right)^{1/2} \exp\left(-\frac{\omega}{2}(x^2 + x'^2)\right)$$

Ground state is

$$\Psi_0(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} \exp(-y^2/2) \qquad \text{(D.77)}$$

and first three excited states are

$$\begin{aligned} \Psi_1 &= \left(\frac{\alpha}{\pi}\right)^{1/4} \sqrt{2} y \exp(-y^2/2) \\ \Psi_2 &= \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{2}}(2y^2 - 1)\exp(-y^2/2) \\ \Psi_3 &= \left(\frac{\alpha}{\pi}\right)^{1/4} \frac{1}{\sqrt{3}}(2y^3 - 3y)\exp(-y^2/2) \end{aligned} \qquad \text{(D.78)}$$

where $\alpha = m\omega/\hbar$ and $y = \sqrt{\alpha}x$

the wavefunctions $\psi_n(x)$ can be identified as a Hermite polynomial times $\exp\left(-\frac{1}{2}m\omega x^2\right)$, as is well known.

Now, if we have n uncoupled harmonic oscillators:

$$\begin{aligned} & \exp\left(\sum_{m=1}^{n} m\frac{H(m)\cos mT - H(m)}{\sin mT}\right) \\ = & \\ = & \left[1 + \frac{1}{2}\sum_m mH(m)e^{-imT} + \dots\right] \end{aligned} \qquad \text{(D.79)}$$

there are first excited states

$$
\begin{aligned}
\Psi_{1,0,0,\ldots,0} &= \left(\frac{\alpha}{\pi}\right)^{n/4}\sqrt{2}y_1\exp(-\sum_{m=1}^{n}y_m^2/2)\\
\Psi_{0,1,0,\ldots,0} &= \left(\frac{\alpha}{\pi}\right)^{n/4}\sqrt{2}y_2\exp(-\sum_{m=1}^{n}y_m^2/2)\\
&\ldots\\
\Psi_{0,0,0,\ldots,1} &= \left(\frac{\alpha}{\pi}\right)^{n/4}\sqrt{2}y_n\exp(-\sum_{m=1}^{n}y_m^2/2)
\end{aligned} \tag{D.80}
$$

And higher excited states, for example

$$
\Psi_{1,0,1,\ldots,0} = \left(\frac{\alpha}{\pi}\right)^{n/4}(\sqrt{2}y_1)(\sqrt{2}y_3)\exp(-\sum_{m=1}^{n}y_m^2/2)
$$

$$
\Psi_{1,2,0,\ldots,0} = \left(\frac{\alpha}{\pi}\right)^{n/4}(\sqrt{2}y_1)\left(\frac{1}{\sqrt{2}}(2y_2^2-1)\right)\exp(-\sum_{m=1}^{n}y_m^2/2)
$$

and

$$
\Psi_{3,2,0,\ldots,0} = \left(\frac{\alpha}{\pi}\right)^{n/4}\left(\frac{1}{\sqrt{3}}(2y_1^3-3y_1)\right)\left(\frac{1}{\sqrt{2}}(2y_2^2-1)\right)\exp(-\sum_{m=1}^{n}y_m^2/2)
$$

multi-particle propagator

$$
\begin{aligned}
W[T] &= \sum_m < x'|e^{-iE_mT}|x> + \frac{1}{2!}\sum_{m_1,m_2} < x_1',x_2'|e^{-i(E_{m_1}+E_{m_2})T}|x_1,x_2> +\\
&+ \frac{1}{2!}\sum_{m_1,m_2,m_3} < x_1',x_2',x_3'|e^{-i(E_{m_1}+E_{m_2}+E_{m_3})T}|x_1,x_2,x_3> + \ldots
\end{aligned} \tag{D.81}
$$

$$
W[T] = \sum_{n=0}^{\infty}\frac{1}{n!}e^{-i\sum_{m=1}^{n}E_mT}\,\overline{\psi(x_1,x_2,\ldots,x_n)}\,\psi(x_1,x_2,\ldots,x_n) \tag{D.82}
$$

$n-$**photon states**

(D.71) lends itself to a particle interpretation. The ground state is the absence of particles:

$$\Psi_0[A^T] = \mathcal{N} \exp\left(-\frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\omega_p \, A^{T\prime} A^{T\prime\prime}\right) \tag{D.83}$$

A wave function representing a state with one photon of momentum $\vec{p}_1$ is obtained from (D.83) by replacing the ground state wave function with the mode $\vec{p}_1$ with the harmonic oscillator wave function for the first excited state,

$$\Psi_1[A^T] = \sqrt{2} A^{T\prime}(p_1) \; \Psi_0[A^T]. \tag{D.84}$$

A wave function with two photons of momentum $\vec{p}_1$, then we replace (D.83) the mode $\vec{p}_1$ with the second excited wave function,

$$\Psi_2[A^T] = \frac{1}{\sqrt{2}}(A^{T\prime}(p_1)\cdot A^{T\prime}(p_1) - 1) \; \Psi_0[A^T]. \tag{D.85}$$

A wave function with two photons of momentum $\vec{p}_1$ and momentum $\vec{p}_2$, then we replace (D.83) the modes $\vec{p}_1$ and $\vec{p}_2$ with the product of two first excited wave function,

$$\Psi_{11}[A^T] = 2(A^{T\prime}(p_1))(A^{T\prime}(p_2)) \; \Psi_0[A^T], \tag{D.86}$$

and so on.

D.6 Propagation Kernel in General Relativity

$$W[g'_{ij}, g''_{ij}, T] = \int_{g'_{ij}}^{g''_{ij}} \mathcal{D}g_{ab} \; e^{iS[g_{ab}]} \tag{D.87}$$

The action

We start with the covariant Lorentzian action for a metric g and generic matter fields ϕ:

$$S[g_{ab}] = \int_0^T \int_{\Sigma_t} d^3x \, \sqrt{-g} g^{ab} R_{ab} + \int_{\Sigma_0 \cup \Sigma_T} d^3x \; \mathcal{K} \equiv \int_0^T \int_{\Sigma_t} d^3x \; \mathcal{L} \tag{D.88}$$

where R_{ab} is the Ricci tensor and g is the determinant of the metric, and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary. In the path-integral formulation, the surface term, sometimes called the *Gibbons-Hawking* term, is needed in order to have only first-order time derivatives in the action, so that the convolution property of the propagator kernel

$$W[g'''_{ij}, g'_{ij}] = \int_{\Sigma_{t_2}} \mathcal{D}g''_{ij}\; W[g'''_{ij}, g''_{ij}]W[g''_{ij}, g'_{ij}]$$

is guaranteed [269].

The surface term is also required so that the action yields the correct equations of motion subject only to the condition that the induced three metric on the boundary is held fixed.

The invariance under active diffeomorphisms of GR makes the integral (D.87) infinite. This situation is analogous to the YM case, and can be cured with a gauge-fixing.

D.6.1 Gauge-Fixing

For the action (D.88) to be covariant, ξ must not change the boundaries of the spacetime region considered, that is

$$\xi^0(0, \vec{x}) = 0, \qquad \xi^0(T, \vec{x}) = T. \tag{D.89}$$

The GR analogue of the temporal gauge

$$g_{00} = -1, \qquad g_{0i} = 0. \tag{D.90}$$

was introduced in section **??**.

As for YM, this is not a complete gauge fixing, but we expect that additional gauge fixing is not required in the path integral. In the linearized case we shall explicitly see that the remaining part of the gauge is taken care by the integration over the gauge parameters. Thus, we gauge fix the path integral by inserting in (M.-19) the FP identity

$$1 = \Delta_{\text{FP}} \int \mathcal{D}\xi\; \delta(g^{\xi}_{00} + 1)\delta(g^{\xi}_{01}), \tag{D.91}$$

where $\mathcal{D}\xi$ is a formal measure over the active diffeomorphisms.

$$W[g'_{ij}, g''_{ij}, T] = \int \mathcal{D}\xi^i_{fin}\mathcal{D}\xi^0_{fin}\mathcal{D}\xi^0_{ini} \int_{g'_{ij}{}^{\xi^0}}^{g''_{ij}{}^{\xi^\mu}} \mathcal{D}g_{ij}\Delta_{\text{FP}} \exp\left\{ i \int d^3x \int_{\xi^0(\vec{x},0)}^{\xi^0(\vec{x},T)} dt\mathcal{L}[g_{ij}] \right\} \tag{D.92}$$

D.6.2 The Disappearance of Time

The scalar constraint can then be written in the form

$$\left(g^{ik}g^{jl} - \frac{1}{2}g^{ij}g^{kl}\right)\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{kl}}{\partial t} - g_{00}\det g_{ij}\; R[g_{ij}] = 0. \tag{D.93}$$

$$T(\vec{x}) = \int_0^1 \sqrt{g_{00}(\vec{x},t)}dt = \int_0^1 \sqrt{\frac{(g^{ij}g^{kl} - \frac{1}{2}g^{ij}g^{kl})\frac{\partial g_{ij}}{\partial t}\frac{\partial g_{kl}}{\partial t}}{\det g_{ij}\; R[g_{ij}]}} \tag{D.94}$$

between initial and final surface, along the $\vec{x} = const.$ lines, which in this gauge are the geodesics normal to the initial surface.

D.6.3 Linearized Theory

$$g_{ab}(x) = \eta_{ab} + h_{ab}(x) \tag{D.95}$$

in the temporal gauge the action reads

$$\int d^3x \int_0^T dt\; \mathcal{L} = \int d^3x \int_0^T dt\; \Big\{\partial_i\partial_j h^{ij} - \nabla^2 h + \\ + \frac{1}{4}\left[\partial_a h^i{}_i \partial^a h^j{}_j - \partial^a h^{ij}\partial_a h_{ij} + \partial^k h^{ij}\partial_i h_{jk} + \partial^k h^{ij}\partial_j h_{ik} - \partial^j h_{ij}\partial^i h - \partial_i h^{ij}\partial_j h^k{}_k\right]\Big\}. \tag{D.96}$$

Writing

$$\begin{aligned} H^{TT}(\vec{p}) &= \overline{h^{TT\prime\prime}_{ij}(\vec{p})}h^{TT\prime\prime}_{ij}(\vec{p}) + \overline{h^{TT\prime}_{ij}(\vec{p})}h^{TT\prime}_{ij}(\vec{p}) \\ \tilde{H}^{TT}(\vec{p}) &= \overline{h^{TT\prime\prime}_{ij}(\vec{p})}h^{TT\prime}_{ij}(\vec{p}) + \overline{h^{TT\prime}_{ij}(\vec{p})}h^{TT\prime\prime}_{ij}(\vec{p}) \end{aligned} \tag{D.97}$$

$$W[A'_i, A''_i, T] = \mathcal{N}(T)\; \delta(h'_0)\delta(h''_0)\exp\left\{\frac{i}{2}\int \frac{d^3p}{(2\pi)^3}p\; \frac{H^{TT}(\vec{p})\cos T - \tilde{H}^{TT}(\vec{p})}{\sin T}\right\}. \tag{D.98}$$

where the function $\mathcal{N}(T)$ is a normalization factor. This is the field propagation kernel of linearized GR.

D.6.4 Ground-State and Graviton States

$$W[h'_{ij}, h''_{ij}, T] = \mathcal{N}(T)\ \delta(h'_o)\delta(h''_0)e^{-\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}p\ H^{TT}(\vec{p})}\left[1 + \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}p\ H^{TT}(\vec{p}) + \ldots\right]. \tag{D.99}$$

Using $\Psi_0[h_{ij}] = \lim_{T\to\infty} W[h_{ij}, h'_{ij} = 0, T]$ the (non-normalized) vacuum state can be read from the zero'th order of (D.99)

$$\overline{\Psi_0[h''_{ij}]}\Psi_0[h'_{ij}] = \delta(h'_0)\delta(h''_0)\ \exp\left\{-\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}p\ H^{TT}(\vec{p})\right\}, \tag{D.100}$$

and therefore

$$\Psi_0[h_{ij}] = \delta(h_0)\ \exp\left\{-\frac{1}{4}\int \frac{d^3p}{(2\pi)^3}p\ h^{TT}_{ij}(\vec{p})h^{TT}_{ij}(-\vec{p})\right\}. \tag{D.101}$$

The graviton states can be obtained from the analog of (M.-19).

$$W[h_{ij}, h'_{ij}, T] = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\epsilon_1\ldots\epsilon_n}\int \frac{d^3p_1}{(2\pi)^3}\cdots\frac{d^3p_n}{(2\pi)^3}\ e^{-i\sum_{\alpha=1}^n E_\alpha T}\ \overline{\psi_{p_1\epsilon_1,\ldots,p_n\epsilon_n}[h_{ij}]}\ \psi_{p_1\epsilon_1,\ldots,p_n\epsilon_n}[h'_{ij}] \tag{D.102}$$

This expression can be matched with (D.99) to extract the n-graviton $n-$states. The (non-normalized) wave functional of the one-graviton state with momentum p and polarization ϵ, for example, reads

$$\Psi_{p,\epsilon}[h_{ij}] = \delta(h_0)\ \sqrt{p}\epsilon^{ij}\ h^{TT}_{ij}(\vec{p})\ \Psi_0[h_{ij}]. \tag{D.103}$$

D.6.5 Newton Potential from the Propagation Kernel

$$E_0 = -\frac{1}{32\pi}\int d^3x\int d^3y\frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}, \qquad m = \int \rho(\vec{x})\ d^3x. \tag{D.104}$$

D.7 Holonomies

$$\begin{aligned}\Delta\phi &= \frac{e}{\hbar c}\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S}\\ &= \frac{e\Phi}{\hbar c}\end{aligned} \tag{D.105}$$

$\Phi = \int \mathbf{B} \cdot d\mathbf{S}$ is the magnetic flux through the surface

$$A \to A' = A - A \tag{D.106}$$

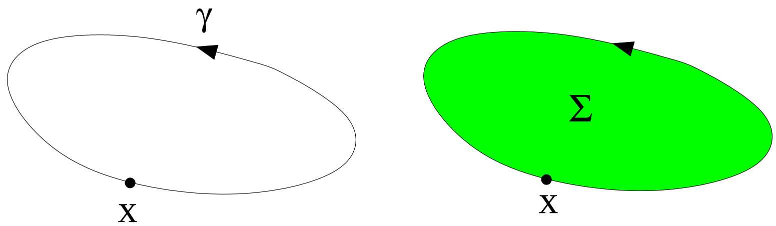

Figure D.8: The Wilson loop integral is taken around a closed loop. It can be expressed in terms of the flux integral of the field strength over a surface bounded by the loop γ.

$$A'_a = U A_a U^{-1} + iU(\partial_a)U^{-1}. \tag{D.107}$$

whereas the holonomy transforms homogeneously

$$H(\gamma; A') = e^{-i\Lambda(P_2)} H(\gamma; A) e^{-i\Lambda(P_1)}. \tag{D.108}$$

Suppose that we consider the displacement

$$P_1 = x, \quad P_2 = x + dx. \tag{D.109}$$

$$H(\gamma; A) = 1 + iA \cdot dx. \tag{D.110}$$

$$\prod_{l=1}^{N} (1 - i\Delta x_a(l) A^a(x(l))). \tag{D.111}$$

The combined effect of many infinitesimal parallel transports leads to the integral

$$H(\gamma, A) = \mathcal{P} \exp\left\{ \int_\gamma A_a(\gamma(s)) \frac{d\gamma^a}{ds} ds \right\} \tag{D.112}$$

Hence, the Wilson loop is the power-series expansion of the exponential, with matrices in each term ordered so that higher values of s stand to the left. This prescription is called *path-ordering* and is denoted by the symbol $\mathcal{P}$. This expression is similar to the time-ordered exponential for the interaction-picture perturbation expansion. If we consider U to be a continuous function of the parameter s, rather than fixing $s = 1$ at the end point we can write the analogous differential equation,

$$\frac{d}{ds}H(x(s),y) = \left(g\frac{dx^a}{ds}A^i_a(x(s))T^i\right)H(x(s),y). \tag{D.113}$$

$$U(x)(1 - idx^a A_a)U^{-1}(x + dx) = 1 - idx^a A'_a. \tag{D.114}$$

$$\begin{aligned} U(x)(1 - idx^a A_a)U^{-1}(x + dx) &= U(x)U^{-1}(x) + dx^a U(x)\partial_a U^{-1}(x) - idx^a[U(x)A_a U^{-1}(x)]. \\ &= 1 - idx^a[iU(\partial_a)U^{-1} + UA_a U^{-1}] \\ &= 1 - idx^a A'_a. \end{aligned} \tag{D.115}$$

where we substituted (D.107) in the final step.

$$\begin{aligned} U(P_1)H(\gamma,A)U(P_N) &= U(P_1)\,[1 - idx^a A_a(P_2)]\,U(P_2)U^{-1}(P_2)\ldots \\ &\quad U(P_{N-2})^{-1}\,U(P_{N-2})[1 - idx^a A_a(P_{N-1})]U^{-1}(P_{N-1}) \\ &\quad U(P_{N-1})[1 - idx^a A_a(P_N)]U^{-1}(P_N) \\ &= H(\gamma, A') \end{aligned} \tag{D.116}$$

$$U(P_1)H(\gamma,A)U^{-1}(P_N) = H(\gamma,A') \tag{D.117}$$

We now consider the case where γ is a closed loop i.e. $P_N = P_1 = P$

$$U(P)H(\gamma,A)U^{-1}(P) = H(\gamma,A') \tag{D.118}$$

taking the trace of the above equation and use the cyclic property of matrices that $\mathrm{Tr}AB = \mathrm{Tr}BA$,

$$\begin{aligned} \mathrm{Tr}H(\gamma,A') &= \mathrm{Tr}\{U(P)H(\gamma,A)U^{-1}(P)\} \\ &= \mathrm{Tr}\{U^{-1}(P)U(P)H(\gamma,A)\} = \mathrm{Tr}H(\gamma,A). \end{aligned} \tag{D.119}$$

We find that the trace of the holonomy around a closed loop is gauge invariant.

$$\mathrm{Tr}(\phi^i\tau_i\phi'^j\tau_j) = \phi^i\phi'^j\mathrm{Tr}(\tau_i\tau_j) = \phi^i\phi'^j\frac{1}{2}\mathrm{Tr}(\tau_i\tau_j + \tau_j\tau_i) = \phi^i\phi'^j\delta_{ij} = \tag{D.120}$$

Geometric interpretation in internal space.

$$H(\gamma,A) = \phi^a\tau_a = \begin{pmatrix} \phi^0 + \phi^3 & \phi^1 - i\phi^2 \\ \phi^1 + i\phi^2 & \phi^0 - \phi^3 \end{pmatrix} \tag{D.121}$$

The trace and determinant of

$$\mathrm{Tr} H(\gamma, A) = 2\phi^0, \quad \det H(\gamma, A) = \phi^0 - (\phi^1)^2 - (\phi^2)^2 - (\phi^3)^2 \tag{D.122}$$

The two above relations tell one that the similarity transformation in (D.119) conserves the dot product $\phi^i \phi^i$. This suggests that the similarity transformation effects a rotation of ϕ^i. In fact

$$\exp\left(-i\frac{\mathbf{a}^i}{2}\tau_i\right)\phi^i \tau_i \exp\left(i\frac{\mathbf{a}^i}{2}\tau_i\right) = \phi'^i \tau_i \tag{D.123}$$

this effects a rotation on ϕ^i by an angle of $|\mathbf{a}|/2$ around the axis whose unit vector is $\mathbf{a}$.

$$A_a = A_a(x) = A_a^I(x) T_I \tag{D.124}$$

$$F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b] \tag{D.125}$$

where $[A_a, A_b] = A_a A_b - A_b A_a$. Consider an infinitesimal circuit

$$= \exp\{iF_{ab} dx^a dx^b\} \tag{D.126}$$

D.8 Topological Field Theories

Topological quantum field theories have revealed the existence of deep connections between 3-dimensional topology, complex analysis, and algebra, particularly the algebra of quantum groups.

One of the most interesting topological quantum field theory is Chern-Simons theory. This a field theory in 3-dimensions, and the reason it's called "topological" is that you don't need any metric or other geometrical structure on your 3d spacetime manifold for this theory to make sense. Thus it admits all diffeomorphisms as symmetries.

D.8.1 $U(1)$ Chern-Simons theory

$$S = \int_\Sigma d^3x \epsilon^{\mu\nu\gamma} A_\mu \partial_\nu A_\gamma \tag{D.127}$$

$$\begin{aligned}\epsilon^{\mu\nu\gamma}\, A_\mu(x) B_\nu(x) C_\gamma(x) &= \frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial x'^\nu}{\partial x^\tau}\frac{\partial x'^\gamma}{\partial x^\rho}\,\epsilon^{\sigma\tau\gamma}\, A'_\mu(x') B'_\nu(x') C'_\gamma(x') \\ &= \det\left(\frac{\partial x'}{\partial x}\right)\epsilon^{\sigma\tau\gamma}[A'_\mu B'_\nu C'_\gamma](x')\end{aligned} \tag{D.128}$$

On the other hand $d^3x' = d^3x \det(\partial x'/\partial x)$.

$$d^3x A_\mu(x) B_\nu(x) C_\gamma(x) = d^3x' A'_\mu(x') B'_\nu(x') C'_\gamma(x') \tag{D.129}$$

which is invariant without the benefit of $\sqrt{-g}$.

Let's take a look at this again but this time in the language of exterior calculus:

$$S = \int_\Sigma A \wedge dA \tag{D.130}$$

where $dA = A_\gamma\, dx^\gamma$

$$dA = d(A_\gamma\, dx^\gamma) = \frac{\partial A_\gamma}{\partial x^\nu}\, dx^\nu \wedge dx^\gamma \tag{D.131}$$

$$\begin{aligned}A \wedge dA &= (A_\mu\, dx^\mu) \wedge \frac{\partial A_\nu}{\partial x^\mu}\, dx^\nu \wedge dx^\gamma \\ &= A_\mu \frac{\partial A_\nu}{\partial x^\mu}\, dx^\mu \wedge dx^\nu \wedge dx^\gamma\end{aligned} \tag{D.132}$$

D.8.2 Canonical Structure

We recall the Hamiltonian formulation of Maxwell theory. In the gauge $A_0 = 0$ the spacial components of the gauge field $\vec{A}$ are canonically conjugate to the electric field components $\vec{E}$, and Gauss's law $\vec{\nabla} \cdot \vec{E} = \rho$ appears as a constraint, for which the non-dynamical field A_0 is a Lagrange multiplier.

Now let us consider the canonical structure of the Maxwell Chern-Simons theory with Lagrangian,

$$\mathcal{L}_{MCS} = \frac{1}{2e^2} \tag{D.133}$$

In the A^0 gauge we identify the A_i as "coordinate" fields, with corresponding "momentum" fields

$$\Gamma^i \equiv \frac{1}{e^2}\dot{A}_i + \frac{\kappa}{2}\epsilon^{ij}A_j \tag{D.134}$$

Pure Chern-Simons Canonical Theory

$$\boxed{\mathcal{L}_{CS} = \frac{\kappa}{2}\epsilon^{ij}\dot{A}_i A_j + \kappa A_0 B} \tag{D.135}$$

Once again, A_0 is a Lagrange multiplier field, imposing the Gauss law: $B = 0$. The Lagrangian is first order in time derivatives, so it is already in the Legendre transformed form $\mathcal{L} = p\dot{(x)} - \mathcal{H}$, with $\mathcal{H} = 0$.The only dynamics would be inherited from coupling matter fields.

At first sight, pure Chern-Simmons theory looks rather boring, because the source-free classical equations of motion (M.-19) reduce to $F_{\mu\nu} = 0$, the solutions of which are just flat connections (this is saying that in Chern-Simons theory the analog of the magnetic field vanishes). This is in contrast to source free Maxwell theory

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{\det g}}\frac{\partial S_{CS}}{\partial q_{\mu\nu}} \tag{D.136}$$

The components of the gauge field are canonically conjugate to each other,

$$\{A_i(x), A_j(y)\} = \frac{i}{\kappa}\epsilon_{ij}\delta(\vec{x} - \vec{y}) \tag{D.137}$$

$$\prod^{i} = \frac{\kappa}{2}\epsilon^{ij}A_j \tag{D.138}$$

The canonical commutation relations arise because of the constraints, noting that these are second class constraints so we must use Dirac brackets to find the canonical relations between A_i and A_J.

U(1)

$$\tilde{A}_i = \oint_{C_i} \tilde{A}_j(x) dx^j \tag{D.139}$$

of any connection i.e. that can be reduced to a constant one by a homotopically trivial gauge transformation ??? . A homotopically non-trivial gauge transformation

$$g(x) = e^{2\pi i(m_1 x_1 + m_2 x_2)} \tag{D.140}$$

Eq (connection transform) show that $\tilde{A}_1$ and $\tilde{A}_2$ remain constant under this transformation, but their values are shifted:

$$\tilde{A}_i \to \tilde{A}_i + 2\pi m_i. \tag{D.141}$$

The phase space $S_! \times S_1$ has volume $(2\pi)^2$. The WKB approximation tells us that the dimension of the Hilbert space is approximately

$$\dim \mathcal{H}^{U(1)}_{T^2} \approx \frac{(2\pi)^2}{2\pi\hbar} \equiv 2k \tag{D.142}$$

in the limit of large k.

D.8.3 Chern-Simons with Magnetic Flux Lines

The charge density

$$\rho(\vec{x}, t) = e \sum_{n=1}^{N} \delta(\vec{x} - \vec{x}_n(t)) \tag{D.143}$$

describes N such particles, with the n^{th} particle following the trajectory $\vec{x}_n(t)$. The corresponding current density is

$$\vec{j}(\vec{x}, t) = e \sum_{n=1}^{N} \dot{\vec{x}}_n(t) \delta(\vec{x} - \vec{x}_n(t)). \tag{D.144}$$

$$B(\vec{x}, t) = \frac{1}{\kappa} e \sum_{n=1}^{N} \delta(\vec{x} - \vec{x}_n(t)) \tag{D.145}$$

which follows each point particle through out its motion.

If each particle has mass m, the total action is

$$S = \frac{m}{2}\sum_{n=1}^{N}\int dt\dot{\vec{x}}^2 + \frac{\kappa}{2}\int d^3x\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \int d^3xA_\mu J^\mu. \tag{D.146}$$

$$A^i(x_n) = \frac{e}{2\pi\kappa}\sum_{m\neq n}\epsilon^{ij}\frac{(x_n^j - x_m^j)}{|\vec{x}_n - \vec{x}_m|^2} \tag{D.147}$$

D.8.4 $SU(2)$ Chern-Simons theory

$$\begin{aligned}
&\delta\frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A\wedge\, dA + \frac{2}{3}A\wedge A\wedge A) = \\
&\quad = \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(\delta A\wedge\, dA + A\wedge\delta\, dA + 2A\wedge A\wedge\delta A) \\
&\quad = \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(-\, dA\wedge\delta A + A\wedge\delta\, dA + 2A\wedge A\wedge\delta A) \\
&\quad = \frac{\mathcal{A}}{4\pi}\int_H 2\mathrm{tr}[(A\wedge A -\, dA)\wedge\delta A] \\
&\quad = \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(F\wedge\delta A)
\end{aligned} \tag{D.148}$$

$$\int_H \mathrm{tr}A\wedge\delta\, dA \tag{D.149}$$

$$\mathcal{CS} = \tag{D.150}$$

$$A = A^i_\gamma T_i dx^\gamma \tag{D.151}$$

$$dA = \partial_\nu A^i_\gamma T_i dx^\nu\wedge dx^\gamma \tag{D.152}$$

$$A\wedge\, dA = A^\mu_i\partial_\nu A^j_\gamma T_i T_j dx^\mu\wedge dx^\nu\wedge dx^\gamma \tag{D.153}$$

$$A \wedge dA = \epsilon^{\mu\nu\gamma} A^i_\mu \partial_\nu A^j_\gamma T_i T_j dx^1 \wedge dx^2 \wedge dx^3 \qquad \text{(D.154)}$$

Similarly $A \wedge A \wedge A = \epsilon^{\mu\nu\gamma} A^i_\mu A^j_\nu A^k_\gamma T_i T_j T_k dx^1 \wedge dx^2 \wedge dx^3$

$$\mathcal{CS} = \epsilon^{\mu\nu\gamma} \left(A^\mu_i \partial_\nu A^j_\gamma T_i T_j - \frac{2}{3} A^\mu_i A^j_\nu A^k_\gamma T_i T_j T_k \right) dv \qquad \text{(D.155)}$$

$$dv = dx^1 \wedge dx^2 \wedge dx^3 \qquad \text{(D.156)}$$

$$\frac{\mathcal{A}}{4\pi} \int_H \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \qquad \text{(D.157)}$$

$$\int_{\mathcal{M}} \mathrm{tr}(\mathcal{CS}^g) = \int_{\mathcal{M}} \mathrm{tr}(\mathcal{CS}) + 8\pi^2 n(g) \qquad \text{(D.158)}$$

where $n(g)$ is the degree of the mapping

by exponentiating the integral we obtain a gauge invariant quantity

$$\exp\left(\frac{ik}{4\pi} \mathrm{tr}\mathcal{CS} \right). \qquad \text{(D.159)}$$

$$K_\mu = \frac{1}{6} \epsilon^{\mu\nu\gamma\delta} \mathrm{tr}(\Omega^{-1}\partial_\nu \Omega)(\Omega^{-1}\partial_\gamma \Omega)(\Omega^{-1}\partial_\delta \Omega) \qquad \text{(D.160)}$$

we parameterize the invariant $SU(2)$ group measure dU (which we introduced in appendix A) as follows:

$$dU = \rho(\sigma_1, \sigma_2, \sigma_3)\, d\sigma_1\, d\sigma_2\, d\sigma_3 \qquad \text{(D.161)}$$

where U is an element of $SU(2)$ parameterized by some coordinates σ_i. Let U_0 be a fixed element of $SU(2)$ and $U' = U_0 U$. If $\{\sigma\}$ are the coordinates that parameterize U and $\{\sigma\}$ the coordinates that parameterize U', then the group measure obeys:

$$dU = dU' = \rho({\sigma'}_1, {\sigma'}_2, {\sigma'}_3)\, d{\sigma'}_1\, d{\sigma'}_2\, d{\sigma'}_3 \qquad \text{(D.162)}$$

that is, the group measure obeys $dU = dU_0 U)$ for fixed U_0 - the rearrangement theorem for continuous groups.

The invariant measure is given by

$$\boxed{\rho(\sigma_1, \sigma_2, \sigma_3) = \epsilon^{ijk} \mathrm{tr} \left(U^{-1} \frac{\partial U}{\partial \sigma_i} U^{-1} \frac{\partial U}{\partial \sigma_j} U^{-1} \frac{\partial U}{\partial \sigma_k} \right)} \qquad \text{(D.163)}$$

With this expression one can check explicitly that:

$$\rho(\sigma_i) = \rho(\sigma'_i) \det \left(\frac{\partial \sigma'}{\partial \sigma} \right) \qquad \text{(D.164)}$$

$$\begin{aligned} n &= \frac{1}{4\pi^2} \int d^4x \partial_\mu K^\mu \\ &= \frac{1}{24\pi^2} \oint_{S^3} d^3\sigma \epsilon^{\mu\nu\gamma\delta} \vec{n}_\mu \mathrm{tr}(\Omega^{-1}\partial_\nu \Omega)(\Omega^{-1}\partial_\gamma \Omega)(\Omega^{-1}\partial_\delta \Omega) \\ &= \frac{1}{24\pi^2} \int_G dU. \end{aligned} \qquad \text{(D.165)}$$

Let $\phi(\theta)$ map one circle S^1 $(0 \leq \theta \leq 2\pi)$ onto another circle S^1 such that it satisfies the boundary condition $\phi(0) = \phi(2\pi) + N2\pi$ where N can be any negative or positive integer i.e. the set $\mathbf{Z}$.

for example $\phi(\theta) = 2N\pi\theta$

$$\mathcal{Q} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \frac{d\phi(\theta)}{d\theta} = \frac{1}{2\pi} [\phi(2\pi) - \phi(0)] \qquad \text{(D.166)}$$

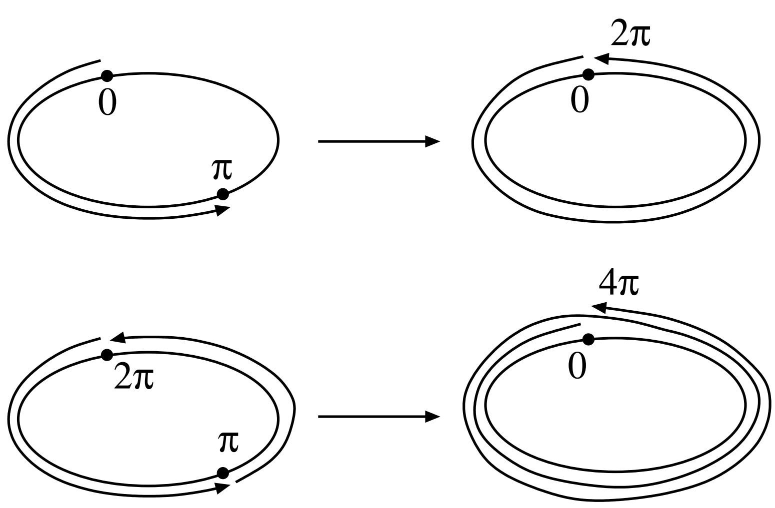

Figure D.9: $\phi(\theta) = 2\theta$ as an example of a mapping with winding number = 2.

It does not change if we smoothly deform the function $\phi(\theta)$ while keeping the boundary conditions the same. $\mathcal{Q}$ is known as the "winding number" and is a topological invariant.

Two functions are said to be in the same equivalence or “homotopy class” if they have the same N.

$$\pi_1(S^1) = \mathbb{Z} \tag{D.167}$$

Symplectic Structure algebra

Note that in terms of the Poisson bracket $\{\ ,\ \}$ defined by ${}^G\Omega_{CS}$, we have

$$\{A^i_a(x), A^j_b(y)\} = (undretilde)\eta_{ab}k^{ij}\delta(x,y), \tag{D.168}$$

where $(undertilde)\eta_{ab}$ and k^{ij} denote the inverse of $\tilde{\eta}^{ab}$ and k_{ij}. This result follows from the fact that for any $f : \Gamma_{CS} \to R$, the Hamilton vector field X_f is given by

$$X_f = \int_\Sigma (undretilde)\eta_{ab}k^{ij}\ \frac{\delta f}{\delta A^j_b}\frac{\delta}{\delta A^i_a}.$$

Hence the Poisson bracket of any two functions f, g is

$$\{f, g\} = \int_\Sigma (undretilde)\eta_{ab}k^{ij}\ \frac{\delta f}{\delta A^j_b}\frac{\delta g}{\delta A^i_a}$$

Constraint algebra

we construct a constraint function associated with

$$k_{ij}\tilde{\eta}^{ab}F^i_{ab} = 0.$$

given test field v^i (which takes values in the Lie algebra $\mathcal{G}$), we define

$$G(v) := \frac{1}{2}\int_\Sigma v^i k_{ij}\tilde{\eta}^{ab}F^i_{ab} \tag{D.169}$$

$$\frac{\delta G(v)}{\delta A^i_a} = k_{ij}\tilde{\eta}^{ab}\mathcal{D}_b v^i \tag{D.170}$$

where $\mathcal{D}_a$ is any torsion free (compatible) extension of the generalized derivative operator associated with A^i_a, (so that $\mathcal{D}_b v^j = \partial_b v^j + C_n m^j A^n_a v^m$). From this it follows that the Hamiltonian vector field $X_{G(v)}$ is given by

$$X_{G(v)} = \int_\Sigma -(\mathcal{D}_a v^i) \frac{\delta}{\delta A_a^i} \tag{D.171}$$

so that

$$A_a^i \mapsto A_a^i - \epsilon \mathcal{D}_a v^i + \mathcal{O}(\epsilon^2) \tag{D.172}$$

under the 1-parameter family of diffeomorphisms on the Chern-Simon's phase space ${}^G\Gamma_{CS}$ associated with $X_{G(v)}$. This is the usual gauge transformation of the connection $A_a^i(x)$ that we find in Yang-mills theory, (also compare with section **??**). Thus, $G(v)$ can be appropriately called a Gauss constraint function. (Relation to interpretation of the isolated quantum constraint in Black hole entropy calculation???)

D.8.5 The Quantum Hall Effect

$$\begin{aligned} S_{bulk} &= \int_{\mathcal{M}} d^3x \left[-\frac{t}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\sigma_H}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right], \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu. \end{aligned} \tag{D.173}$$

while our metric is $(-1, +1, +1)_{diag}$

under a gauge transformation the surface term

$$-\frac{\sigma_H}{2} \int_{\partial\mathcal{M}} d^2x \epsilon^{\mu\nu} (\partial_\mu \alpha) A_\nu. \tag{D.174}$$

$$S_{tot} = S_{bulk} + \frac{\sigma_H}{2} \int_{\partial\mathcal{M}} d^2x \epsilon^{\mu\nu} (\partial_\mu \phi) A_\nu - \frac{\sigma_H}{4} \int_{\partial\mathcal{M}} d^2x \epsilon^{\mu\nu} D_\mu \phi D^\mu \phi, \tag{D.175}$$

$$D_\mu \phi = \partial_\mu \phi - A_\mu. \tag{D.176}$$

The field transforms as

$$\phi \to \phi + \alpha \tag{D.177}$$

so that

$$D\phi := d\phi - A \to D\phi. \tag{D.178}$$

As we will now demonstrate, it is the second term in which restores gauge invariance.

$$\begin{aligned} \frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\epsilon^{\mu\nu}\partial_\mu\phi A_\nu &\rightarrow \frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\epsilon^{\mu\nu}(\partial_\mu\phi+\partial_\mu\alpha)(A_\nu+\partial_\nu\alpha) \\ &= \frac{\sigma_H}{2}\int_{\partial\mathcal{M}} d^2x\left[\epsilon^{\mu\nu}(\partial_\mu\phi)A_\nu+\epsilon^{\mu\nu}(\partial_\mu\alpha)A_\nu+\epsilon^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\alpha)\right] \end{aligned} \tag{D.179}$$

D.9 How to Quantize 2+1 Gravity and Solve it Exactly *a la* Witten

Witten [387] was able to show that the 2+1 theory of gravity simplifies considerably when expressed in Palantini form. in fact, Witten demonstrated the the 2+1 Palantini Tetrad theory was equivalent to Chern-Simons thoery based on the inhomogeneous Lie group $ISO(2,1)$ and thus could be explicitly canonically quantized.

$$S_{EH}(e) = \frac{1}{2}\int \tilde{\eta}^{abc}\epsilon_{IJK}\, e_a^I\, R_{bc}{}^{JK} \tag{D.180}$$

$$S_{EH}(e,\omega) = \frac{1}{2}\int \tilde{\eta}^{abc}\epsilon_{IJK}\, e_a^I\, F_{bc}{}^{JK} \tag{D.181}$$

where

$$F_{abI}{}^J = 2\partial_{[a}\omega_{b]I}{}^J + [\omega_a,\omega_b]_I{}^J$$

Since the 2+1 Palantini action is a functional of both a co-triad and a connection 1-form, we will obtain two Euler-Lagrange equations of motion.

$$e^a = e^a_\mu dx^\mu, \quad \omega^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu bc}dx^\mu, \tag{D.182}$$

The first order action takes the form

$$T^a_{\mu\nu} = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \epsilon^a{}_{bc}(\omega_\mu{}^b e_\nu{}^c + \omega_\nu{}^b e_\mu{}^c) = 0 \tag{D.183}$$

$$R^a_{\mu\nu} = \partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon_{abc}\omega_\mu{}^b\omega_\nu{}^c = 0 \tag{D.184}$$

The first of these implies that the connection is torsion-free, and, if e is invertible, that ω has the standard expression in terms of the triad. Given such a spin connection, (D.184) is then equivalent to the standard Einstein field equations.

In a vacuum in three-dimensions the vanishing of the Ricci tensor implies the Reinmann tensor vanishes

$$R_{ab} = 0 \implies R^a{}_{bcd} = 0 \tag{D.185}$$

so the solution of the equations of motion is that curvature vanishes.

$$S_{CS} = \frac{k}{4\pi} tr \int (A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \tag{D.186}$$

This condition, which is imposed *a priori* in Einstein's formulation of general relativity, is seen to be part of the equations of motion.

The local trivialiaation of the gauge field gives rise to a parametrization of the phase space in terms of the holonomies along a set of generators of the fundamental group $\pi_1(S_g)$ and ...

Using the explicit algebra basis, the commutation relations

$$[T_a, T_b] = C_{ab}{}^c T_c.$$

and the inner product $< T_a, T_b >$, we can obtain a more explicit expression for the Cher-Simons action. First we consider the term $A \wedge A \wedge A$

$$\begin{aligned} \epsilon^{ijk} A_i{}^a A_j{}^b A_k{}^c \, T_a T_b T_c &= \frac{1}{2} \epsilon^{ijk} (A_i{}^a A_j{}^b A_k{}^c - A_j{}^a A_i{}^b A_k{}^c) \, T_a T_b T_c \\ &= \frac{1}{2} \epsilon^{ijk} (T_a T_b T_c - T_b T_a T_c) A_i{}^a A_j{}^b A_k{}^c \\ &= \frac{1}{2} \epsilon^{ijk} \, [T_a, T_b] T_c \, A_i{}^a A_j{}^b A_k{}^c \end{aligned} \tag{D.187}$$

with the notation $tr(\cdot, \cdot) =< \cdot, \cdot >$ we write

$$\begin{aligned} S_{CS} = k \int_{\mathcal{M}} d^3x \Big\{ &< T_a, T_b > \epsilon^{ijk} A_i{}^a (\partial_j A_k{}^b - \partial_k A_j{}^b) + \\ &+ \frac{2}{3} < \frac{1}{2} [T_a, T_b], T_c > \epsilon^{ijk} A_i{}^a A_j{}^b A_k{}^d \Big\}, \end{aligned} \tag{D.188}$$

which on substituting the commutation relations becomes

$$
\begin{aligned}
S_{CS} &= < T_c, T_d > k \int_{\mathcal{M}} d^3x \epsilon^{ijk} \Big(A_i{}^c (\partial_j A_k{}^d - \partial_k A_j{}^d) + \frac{1}{3} C_{ab}{}^c A_i{}^a A_j{}^b A_k{}^d \Big) \\
&= k \int_{\mathcal{M}} d^3x \epsilon^{ijk} k_{cd} \left(A_i{}^c (\partial_j A_k{}^d - \partial_k A_j{}^d) + \frac{1}{3} C_{ab}{}^c A_i{}^a A_j{}^b A_k{}^d \right) \qquad \text{(D.189)}
\end{aligned}
$$

It is important to note that Chern-Simons theory is not defined for arbitrary Lie groups - we need the additional structureprovided by the invariant , nondegenerate bilinear form k_{ab}.

$$
< T_a, T_b >= k_{ab} \qquad \text{(D.190)}
$$

For the Lie group $SU(2)$ we have

$$
k_{ab} = \frac{\delta_{ab}}{2} \qquad \text{(D.191)}
$$

??

semi-direct products

N is the group of all translations in three-space and H is the group of all rotations about some fixed origin. $N \otimes_S H$ is the group of all, the group generated by all translations and rotations.

N is the group of all translations in spacetime and H is the group of all homogeneous Lorentz transformations. $N \otimes_S H$ is called the Poincarë group. the symmetry group of spacetime special relativity.

??

$$
\mathbb{R}^3 \otimes_S SO(3). \qquad \text{(D.192)}
$$

We introduce translation generators P_a, $a = 1, 2, 3$, which satisfy

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{ab}{}^c J_c, \\
[J_a, P_b] &= \epsilon_{ab}{}^c P_c, \\
[P_a, P_b] &= 0 \qquad \text{(D.193)}
\end{aligned}
$$

The group has a nondegenerate invariant bilinear form, a form that commutes with all algebra elements. The expression $J^a P_a$ satisfies these conditions as we can check:

$$
\begin{aligned}
[J^a P_a, J_b] &= J^a[P_a, J_b] + [J^a, J_b]P_a = \epsilon_{ab}{}^c(J^a P_c + \eta^{ad} J_c P_d) = 0 \\
[J^a P_a, P_b] &= J^a[P_a, P_b] + [J^a, P_b]P_a = \epsilon_{ab}{}^c \eta^{ad} P_c P_d = 0
\end{aligned} \tag{D.194}
$$

$$
Tr(T_a T_b) = \sum_{\alpha\beta} C_{a\alpha\beta} C_{b\alpha\beta} = k_{ab} \tag{D.195}
$$

Lie algebra $su(2)$ and can be represented by the matrices

$$
J_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad J_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \qquad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{D.196}
$$

in (D.188)

$$
\begin{aligned}
T_1 &= J_1 \qquad & T_4 &= P_1 \\
T_2 &= J_2 \qquad & T_5 &= P_2 \\
T_3 &= J_3 \qquad & T_6 &= P_3
\end{aligned} \tag{D.197}
$$

$[T_a, T_b] = C_{ab}{}^c T_c$ and $\eta = (-1, 1, 1)$

$$
\begin{aligned}
[T_1, T_2] &= C_{12}{}^a T_a = C_{12}{}^3 T_3 = \epsilon_{123} \eta^{33} T_3, \\
[T_1, T_3] &= C_{13}{}^a T_a = \epsilon_{12}{}^3 T_2 = C_{13}{}^2 T_2, \\
[T_2, T_3] &= C_{23}{}^a T_a = \epsilon_{23}{}^1 T_1 = C_{23}{}^1 T_1 \\
\\
[T_1, T_1] &= C_{11}{}^a T_a = 0, \\
[T_2, T_2] &= C_{11}{}^a T_a = 0, \quad etc... \\
\\
[T_1, T_4] &= [J_1, P_1] = C_{14}{}^a T_a = 0 \\
[T_1, T_5] &= [J_1, P_2] = C_{15}{}^a T_a = \epsilon_{12}{}^3 T_6 \\
[T_1, T_6] &= [J_1, P_3] = C_{16}{}^a T_a = \epsilon_{13}{}^2 T_5 \\
\\
[T_4, T_5] &= [P_1, P_2] = C_{45}{}^a T_a = 0, \quad etc...
\end{aligned} \tag{D.198}
$$

$$(C_{ab}{}^{1}) = \begin{pmatrix} 0 & 0 & 0 & 0 & & \\ 0 & 0 & -1 & & & \\ 0 & 1 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \quad (C_{ab}{}^{2}) = \begin{pmatrix} 0 & -1 & 0 & 0 & & \\ 0 & 0 & 0 & & & \\ 1 & 0 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \tag{D.199}$$

$$(C_{ab}{}^{3}) = \begin{pmatrix} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \quad (C_{ab}{}^{4}) = \begin{pmatrix} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & & & \\ -1 & 0 & 0 & & & \\ 0 & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{pmatrix} \tag{D.200}$$

An element of the semi-direct product $\mathbb{R}^3 \otimes_S SO(3)$ is

$$\exp\left\{\alpha_c (C_{ab})^c\right\}$$

$$\begin{aligned} k_{11} &= Tr(T_1 T_1) = \sum_{\alpha,\beta=1}^{6} C_{1\alpha\beta} C_{1\alpha\beta} = \\ &= \end{aligned} \tag{D.201}$$

The inner product

$$< \cdot\,, \,\cdot >= k_{ab} \cdot \; \cdot \tag{D.202}$$

The inner product can be identified:

$$\begin{aligned} < J_a, P_b > &= \eta_{ab}, \\ < J_a, J_b > &= 0, \\ < P_a, P_b > &= 0 \end{aligned} \tag{D.203}$$

the Cartan connection can be written as

$$A_\mu(x) = \sum_{a'=1}^{6} A_\mu^{a'}(x) T_{a'} = \sum_{a=1}^{3} (\omega_{a\mu}(x) J^a + e_{a\mu}(x) P^a) \tag{D.204}$$

It is a connection on a principle bundle $SO(3)$ bundle, but takes values in $iso(3)$.

the terms $< \frac{1}{2}[T_a, T_b], T_c >$ are

$$
\begin{aligned}
< [J_a, P_b], J_c > &= \epsilon_{ab}{}^{d} < P_d, J_c >= \epsilon_{ab}{}^{d}\eta_{cd} = -\epsilon_{abc} \\
< [J_a, J_b], P_c > &= \epsilon_{ab}{}^{d} < J_d, P_c >= \epsilon_{abc} \\
\\
< [P_a, P_b], P_c > &= 0, \quad < [P_a, P_b], J_c >= 0 \\
< [J_a, P_b], P_c > &= \epsilon_{ab}{}^{d} < P_d, P_c >= 0 \\
< [J_a, J_b], J_c > &= \epsilon_{ab}{}^{d} < J_d, J_c >= 0
\end{aligned} \tag{D.205}
$$

Substituting these into (D.188) we get

$$
\begin{aligned}
S_{CS} &= \frac{k}{2}\int d^3x\epsilon^{\mu\nu\rho}\Big\{ < J_a, P_b > \omega_\mu{}^a\partial_\nu e_\rho{}^b + < P_a, J_b > e_\mu{}^a\partial_\nu\omega_\rho{}^b + \\
&\quad \frac{2}{3}\Big(\frac{1}{2} < [P_a, J_b], J_c > e^a_\mu\omega_\nu{}^b\omega_\rho{}^c \\
&\quad +\frac{1}{2} < [J_a, P_b], J_c > \omega_\mu{}^a e_\nu{}^b\omega_\rho{}^c + \frac{1}{2} < [J_a, J_b], P_c > \omega_\mu{}^a\omega_\nu{}^b e_\rho{}^c\Big)\Big\} \\
&= \frac{k}{2}\int d^3x\epsilon^{\mu\nu\rho}\Big(\omega_\mu{}^a\partial_\nu e_{a\rho} + e^a_\mu\partial_\nu\omega_\rho{}^a + \epsilon_{abc}e^a_\mu\omega_\nu{}^b\omega_\rho{}^c\Big) \\
&= \frac{k}{2}\int \epsilon^{\mu\nu\rho}\Big(e_{\mu a}(\partial_\nu\omega_\rho{}^a - \partial_\rho\omega_\nu{}^a + \epsilon^a{}_{bc}\omega_\nu{}^b\omega_\rho{}^c)\Big) + \text{ (surface term)} \\
&= \int d^3x\epsilon^{\mu\nu\rho}e_{\rho a}R^a{}_{\mu\nu}
\end{aligned} \tag{D.206}
$$

D.9.1 Gauge transformations

gauge transformation generated by $\epsilon = \rho^a P_a + \tau^a J_a$, where ρ^a and τ^a are infinitesimally small.

$$
\begin{aligned}
\delta A_\mu &= -D_\mu\epsilon = -\partial_\mu\epsilon - [A_\mu, \epsilon] \\
&= -(\partial_\mu\rho^a + \epsilon^a{}_{bc}\omega_\mu{}^b\rho^c + \epsilon^a{}_{bc}e_\mu{}^b\tau^c)P_a - (\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega_\mu{}^b\tau^c)J_a
\end{aligned} \tag{D.207}
$$

so that

$$
\delta\omega_\mu{}^a = -(\partial_\mu\tau^a + \epsilon^a{}_{bc}\omega_\mu{}^b\tau^c) \tag{D.208}
$$

$$
\delta e_\mu{}^a = -(\partial_\mu\rho^a + \epsilon^a{}_{bc}\omega_\mu{}^b\rho^c + \epsilon^a{}_{bc}e_\mu{}^b\tau^c) \tag{D.209}
$$

Now a diffeomorphism generated by a vector field $-v^\mu$

$$\delta_{Diff}\omega_\mu{}^a = -v^\nu(\partial_\nu\omega_\mu{}^a - \partial_\mu\omega_\nu{}^a) - \partial_\mu(v^\nu\omega_\nu{}^a) \tag{D.210}$$
$$\delta_{Diff}e_\mu{}^a\delta = -v^\nu(\partial_\nu e_\mu{}^a - \partial_\mu e_\nu{}^a) - \partial_\mu(v^\nu e_\nu{}^a) \tag{D.211}$$

If we take

$$\rho^a = v^\mu e_\mu{}^a$$

and compare with (D.211) with (D.209)

$$\delta_{Diff}e_\mu{}^a - \delta e_\mu{}^a = -v^\nu(D_\nu e_\mu{}^a - D_\mu e_\nu{}^a) + \epsilon^{abc}v^\nu\omega_{\nu b}e_{\mu c}. \tag{D.212}$$

The expression

$$D_\nu e_\mu{}^a - D_\mu e_\nu{}^a$$

is just the torsion, which should vanish as a consequence of the equations of motion. The remaining term $\epsilon^{abc}v^\nu\omega_{\nu b}e_{\mu c}$ corresponds to a local Lorentz transformation with parameter $\tau^a = v^\mu\omega_{\mu a}$.

Similarly, the difference between the transformations of $\omega_{\mu a}$ is

$$\delta_{Diff}\omega_\mu{}^a - \delta\omega_\mu{}^a = -v^\nu(D_\nu\omega_\mu{}^a - D_\mu\omega_\nu{}^a) + \epsilon^{abc}v^\nu\omega_{\nu b}\omega_{\mu c}. \tag{D.213}$$

Here we have chosen again $\tau^a = v^\mu\omega_{\mu a}$. Part of the difference is given by the curvature $R^a = D_\nu\omega_\mu{}^a - D_\mu\omega_\nu{}^a$, which upon imposing the equations of motion vanish es. The remaining term is again a local Lorentz transformation.

So we see that the Lorentz transformations and diffeomorphisms agree with the gauge transformations of $ISO(2,1)$ as long as we impose the equations of motion following from the Chern-Simons action. The space of metrics solving Einstein's equation up to infinitesimal active diffeomorphisms is therefor isomorphic to the space of flat Chern-Simons gauge fields modulo infinitesimal Chern-Simons gauge transformations.

Inhomogeneous Lie groups

$\mathcal{L}_G \oplus \mathcal{L}^*_G$

The Lie bracket

$$[v, w] := C^I{}_{JK} v^J w^K$$

and the co-adjoint bracket

$$\{v, \beta\} := C^K{}_{JI} v^J \beta_K$$

We can define a bracket on $\mathcal{L}_G \oplus \mathcal{L}_G^*$

$$[(\alpha, v), (\beta, w)]^i := (-\{w, \alpha\} + \{w, \alpha\}, [v, w])^i, \tag{D.214}$$

$$\{[v, w], \alpha\}_I = -\{v, \{w, \alpha\}\}_I + \{w, \{v, \alpha\}\}_I$$

use this to show that (D.214) satisfies the Jacobi identity.

$$\{C^L{}_{JK} v^J w^K, \alpha\}_I = -\{v$$

the inhomogeneous Lie group IG is obtained by exponentiating the Lie algebra $\mathcal{L}_{IG}$.

D.9.2 Cosmological Constant

$$S_{CS} = \frac{k}{2} \int \epsilon^{\mu\nu\rho} \Big(e_{\mu a} (\partial_\nu \omega_\rho{}^a - \partial_\rho \omega_\nu{}^a) + \epsilon_{abc} e_\mu{}^a (\omega_\nu{}^b \omega_\rho{}^c - \frac{\lambda}{3} e_\nu{}^b e_\rho{}^c) \Big) \tag{D.215}$$

$-\lambda$ is the cosmological constant. The curvature tensor is here

$$R^a{}_{\mu\nu} = \partial_\mu \omega_\nu{}^a - \partial_\nu \omega_\mu{}^a + \epsilon_{abc} \omega_\mu{}^b \omega_\nu{}^c - \frac{\lambda}{3} e_\nu{}^b e_\rho{}^c \tag{D.216}$$

The torsion $T_{\mu\nu}{}^a$ is the same. If we don't change the inner product on the algebra, this term can come from the term

$$\frac{k}{2} \int d^3x \epsilon^{\mu\nu\rho} \frac{2}{3} \big(\frac{1}{2} < [P_a, P_b], P_c > e_\mu{}^a e_\nu{}^b e_\rho{}^c \big) \tag{D.217}$$

if we take $[P_a, P_b]$ equal to $-\lambda \epsilon_{ab}{}^c J_c$ instead of zero. This changes the gauge group:

$$
\begin{aligned}
[J_a, J_b] &= \epsilon_{ab}{}^c J_c, \\
[J_a, P_b] &= \epsilon_{ab}{}^c P_c, \\
[P_a, P_b] &= -\lambda \epsilon_{ab}{}^c J_c
\end{aligned}
\tag{D.218}
$$

D.10 Witten and Link Polynomials

A knot is a non-intersecting smooth closed curve in a three-manifold (see fig.(D.10)). It is said to be oriented if there is a direction...

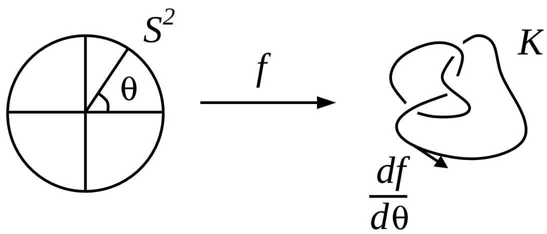

Figure D.10: knotDef.

With a given knot we associate a *knot diagram* obtained by projecting the knot on to a plane.

D.10.1 Knot and Link Polynomials

Two shadows of the same knot are related by a sequence of Reidermeister moves.

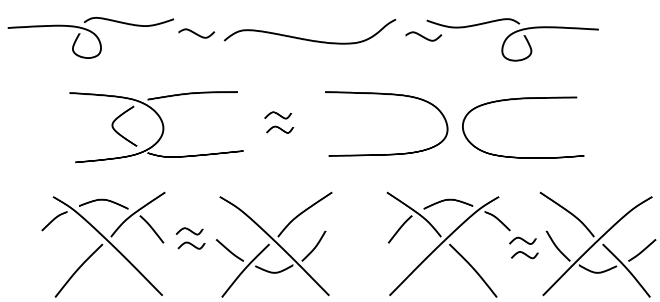

Figure D.11: Reidemeister moves.

The bracket polynomial

Try to form a invriant under Reidermeister moves - then if two shadow representations of the same knot will share this invariant.

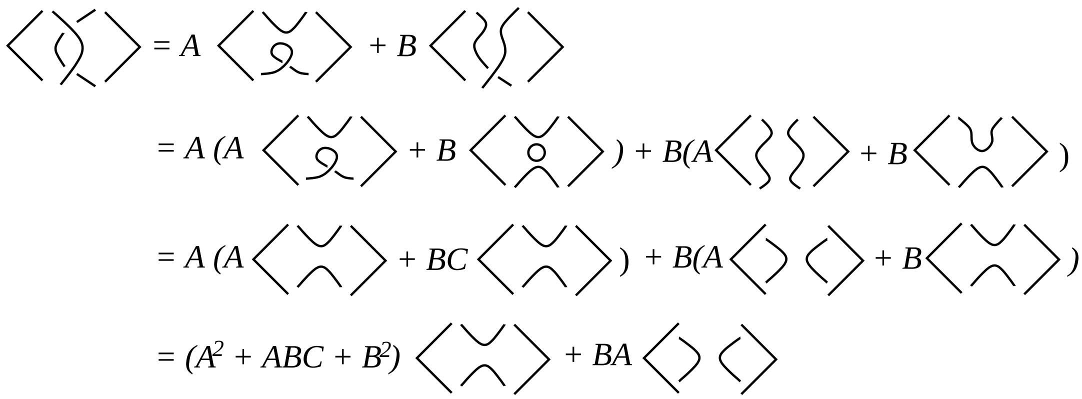

Figure D.12: Bracketpoly1.

$$AB = 1 \quad \Rightarrow B = A^{-1} \qquad A^2 + ABC + B^2 = 0 \quad \Rightarrow C = -A^2 - A^{-2} \tag{D.219}$$

Then if two shadows have different polynomials they are topologically inequivalent.

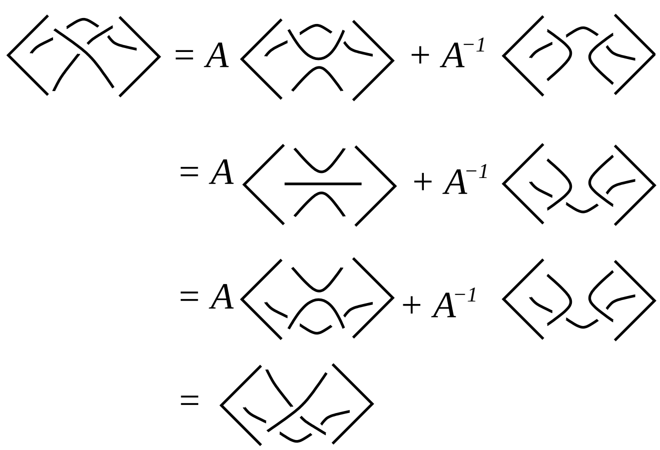

Figure D.13: Bracketpoly2.

Where we used move II a couple of times. So move III follows from move RII!

So not an invariant of ambeint isotopy. But we can fix it

The bracket polynomial is an invariant of the regular isotopy - the framing is not irrelevant.

$$\mathcal{L}_K(A) = (-A^3)^{-w(K)} < K > \tag{D.220}$$

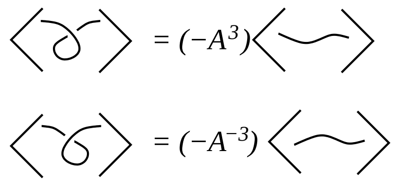

Figure D.14: Bracketpoly4.

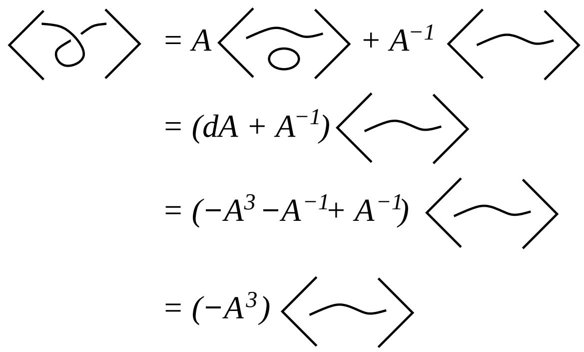

Figure D.15: Brackpoly4Pr. Proof of bracket of curl

The Jones polynomial

$$V_K(t) = \mathcal{L}_K(t^{-1/4}) \tag{D.221}$$

The Jones polynomial is an invariant of oriented links - the framing is irrelevant.

$$A(-A^3) <> (-A^3)^{-(w(K)+1)} - A^{-1}(A^3)^{-1} <> (-A^3)^{-(w(K)-1)} = (A^2 - A^{-1}) <> (-A^3)^{-w(K)} \tag{D.222}$$

$$\begin{aligned} A(-A^3)\mathcal{L} + A^{-1}(A^3)^{-1}\mathcal{L} &= (A^2 - A^{-1})\mathcal{L} \\ -A^4\mathcal{L} + A^{-4}\mathcal{L} &= (A^2 - A^{-2})\mathcal{L} \end{aligned} \tag{D.223}$$

Letting $A = t^{-1/4}$, we finally arrive at

$$t^{-1}\mathcal{L}X - t\mathcal{L}X = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)\mathcal{L}X \qquad \mathcal{L}\, O = 1 \tag{D.224}$$

HOMFLY Polynomial

The first discovery was a direct generalization of the original Jones polynomial to an invariant $P_K(a;z)$ in two variables a and z such that

$$aP_{K_+} - a^{-1}P_{K_-} = zP_{K_0}. \tag{D.225}$$

PK is called the HOMFLY polynomial after the different people that discovered it and proved its properties. They are Hoste, Ocneanu, Millett, Freyd, Lickorish, Yetter, (Przytcki and Trawzyk) in the order of the acronym.

D.10.2 Link Invariants from Chern-Simons Field Theory

$$W(\gamma) = Tr\ \mathcal{P} \oint ds\ \dot{\gamma}^a(s) A_a(\gamma(s)) \tag{D.226}$$

Smooth deformations in the ambient space do not modify the topological properties of the link.

$$< W(\gamma) > = \int \mathcal{D}[A]\ e^{CS}\ W(A,\gamma) \tag{D.227}$$

Now this expectation value should be invariant under diffeomorphisms, i.e. it should be a knot invariant, parameterized by the coupling constant k.

D.10.3 Witten On the Jones Polynomial

$$W_\gamma[A] = 1 + \int \mathcal{L}_{CS} ds + \int_{CS} ds \int_{CS} \mathcal{L}_{CS} dt + \dots \tag{D.228}$$

Figure D.16: Chern-Simons regularization.

Perturbation theory

Smooth deformations in the ambient space do not modify the topological properties of the link defined by $\{C, C_f\}$. An ambient isotopy invariant of the framed knot C.

If we expand this out, remembering the expressions for X and g we get,

$$< W(A,\gamma) >^{(1)} = \oint ds \oint dt \dot{\gamma}^a(s)\dot{\gamma}^b(t)\epsilon_{abc}\frac{\gamma^c(s) - \gamma^c(t)}{|\gamma^c(s) - \gamma^c(t)|^3} = \mathrm{GSL}\gamma) \tag{D.229}$$

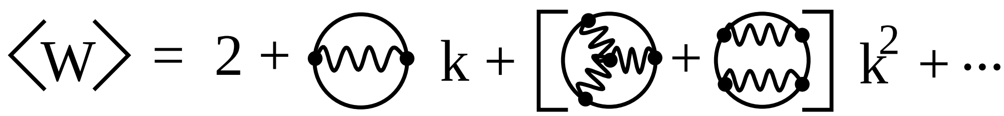

Figure D.17: DiagExpanW. Chern-Simons.

$$\text{Kauffmann bracket}(\gamma)[k] = e^{kGSL(\gamma)}\text{Jones polynomial}(\gamma)[k] \tag{D.230}$$

so we see that all the framing dependence can be concentrated in the "phase factor" $\exp(kGSL(\gamma))$.

This translates into a recursion relation for the expectation values of Wilson loops given by

$$\alpha\mathcal{Z}(\mathcal{L}) + \beta\mathcal{Z}(\mathcal{L}_1) + \alpha\mathcal{Z}(\mathcal{L}_2) = 0. \tag{D.231}$$

Diagrammatically this is

$$\begin{aligned} \alpha &= -\exp\left(\frac{2\pi i}{N(N+k)}\right) \\ \beta &= -\exp\left(\frac{i\pi(2-N-N^2)}{N(N+k)}\right) + \exp\left(\frac{i\pi(2+N-N^2)}{N(N+k)}\right) \\ \gamma &= \exp\left(\frac{2\pi i(1-N^2)}{N(N+k)}\right) \end{aligned} \tag{D.232}$$

The generalisation of the Jones polynomial to the gauge group $G = SU(N)$ is the HOMFLY polynomial. In the case where $G = SU(2)$ we get the Jones polynomial.

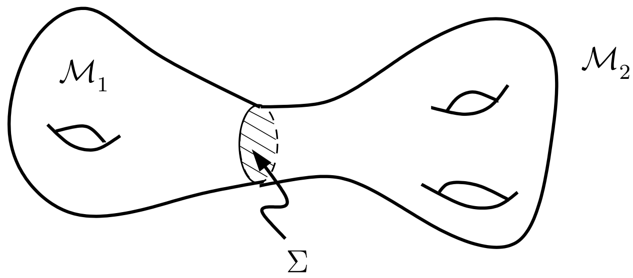

Figure D.18: Heegard. Heegard spilting of a three manifold $\mathcal{M}$ into two three manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ with common boundary Σ.

Witten' paper

The boundary of $\mathcal{M}_2$ is also Σ, but with the opposite orientation , so its Hilbert space is the dual space $\mathcal{H}(\Sigma)^*$. The homeomorphism $f : \Sigma \rightarrow$ is represented by an operator acting on $\mathcal{H}(\Sigma)$

$$U_f : \mathcal{H}(\Sigma) \rightarrow \mathcal{H}(\Sigma). \tag{D.233}$$

and the partition function is

$$\mathcal{Z}(\mathcal{M}) = < \Psi_{\mathcal{M}_2} | U_f | \Psi_{\mathcal{M}_1} > . \tag{D.234}$$

If we know the wavefunctions and the operators associated to homeomorphisms we can compute the partition function.

For a conformal field theory in two dimensions

$$< \phi(z_1, \bar{z}_1, \ldots, z_n, \bar{z}_n) >= \sum_{ab} C^{ab} \mathcal{F}_a(z_1, \ldots, z_n) \overline{\mathcal{F}}_b(\bar{z}_1, \ldots, \bar{z}_n) \tag{D.235}$$

$\mathcal{F}_a(z_1, \ldots, z_n)$ are called the conformal blocks or chiral blocks.

The physical Hilbert space of a three-dimensional topological Chern-Simons theory can be interpreted as the space of conformal blocks of the corresponding Wess-Zumino-Witten model of two dimensions.

The space of conformal blocks of a WZW model on Σ with gauge group G and level k.

The connection between Chern-Simons theory and Wess-Zumino-Witten conformal field theory, the functional integrals of these three-balls correspond to states in the space of four point correlator conformal blocks of the WZW conformal theory.

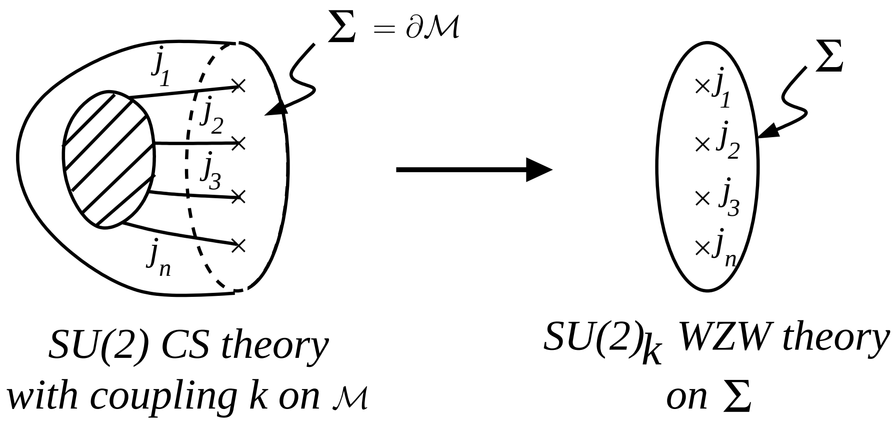

Figure D.19: ChernStoWZW. (a) $SU(2)$ CS theory with coupling k with Wilson lines carrying representations $j_1, j_2, \ldots, j_n$, ending at n points in the boundary Σ. (b) $SU(2)_k$ WZW theory on Σ with n punctures carrying primary fields in representations $j_1, j_2, \ldots, j_n$.

The dimension of the space of conformal blocks depends on a simply connected complex Lie grouop and an integer called the level.

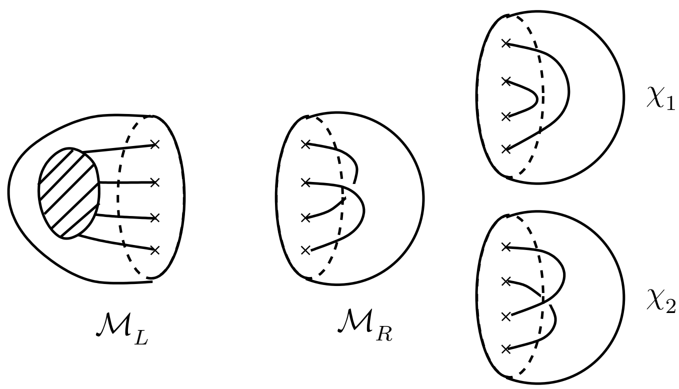

Figure D.20: CuttingMan.

D.10.4 Dehn Surgery

There is a procedure, called Dehn surgery, by which all closed (compact and without boundary), orientable, connected three-dimensional manifolds can be constructed []. One begins by drawing a knot or link (a link is a set of knotted loops all tangled up) on a given manifold. One then cuts out a tubular neighbourhood around each of the knotted lines a glues them back in differently.

We can see the number of times a knot wraps meridionally around the torus and the

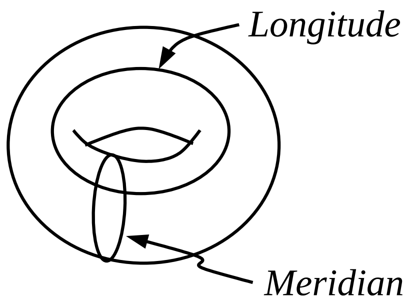

Figure D.21: TorusLongMer. Ameridian and longitude on a torus.

number of times it wraps longitudinally by counting the number of times the knot crosses a meridian and the number of times it crosses a longitude curve respectively.

A three-sphere S^3 can be constructed by gluing together two solid tori.

“The Knot Book, An Elementary Introduction to the Mathematical Theory of Knots”, Colin C. Adams.

“Intuitive Topology”, V. V. Prasolov.

gluing the two boundaries

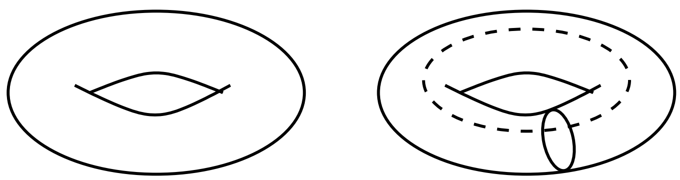

Figure D.22: gluingTori. tori .

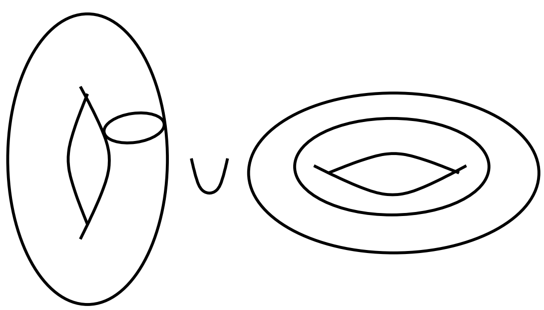

Figure D.23: gluingTori2. Gluing together two tori to get a 3-sphere S^3.

the complement of a knot

that two Dehn surgery descriptions of a three-manifold they must be related by Kirby moves. these are like the Reidemeister. instead of going from one projection of a knot onto another through a sequence of Reidermeister moves, Kirby moves take us from one Dehn surgery description of three-manifold to another through a sequence of Dehn descriptions of the same three-manifold.

Kerby calculus forms a basis for new invariants distinguishing three-manifolds

D.10.5 Chern-Simons Theory on a Torus

$SL(2,\mathbb{Z})$ transformation that maps the meridian M' of T' onto $nM + mL$, where M and L are a meridian and a longitude on T and m, n are respectively prime.

D.10.6 Chern-Simons Quantum Field Theory and Dehn Surgery

The Jones Polynomial Vaughan F.R. Jones Department of Mathematics, University of California at Berkeley, Berkeley CA 94720, U.S.A. 18 August 2005

The expectation value $< W(L) >$ can be inetrpreted as a scalar product between two states. Given a surface which separates $\mathbb{R}^3$ into two parts P_1 and P_2, the computation of the fucntional intergral can be divided into two parts. The integral is performed in P_1 and P_2 with given boundary conditions on the surface for the fields . These can be thought of as the "bra" and "ket" state vectors. The scalar product between these two states gives the result of the whole functional integral in $\mathbb{R}^3$.

D.10.7 Three-Manifold Invariants

Witten showed how the quantum field theory context defined many invariants of three dimensional manifolds and how these invariants could be calculated (by a surgery description) through invariants of knots and links in the three dimensional sphere.

Framed unoriented links in S^3 modulo equivalence under Kirby moves $\leftrightarrow$ Closed (compact without boundary), orientable, connected three-manifolds manifolds modulo homeomorphisms.

Combination of framed link invariants which do not change under Kirby moves = Invariants of associated three-manifold.

Quantum gravity vacuum and invariants of embedded spin networks

arXiv:gr-qc/0301047 v2 24 Jun 2003 *Quantum gravity vacuum and invariants of embedded spin networks* A. Mikovic

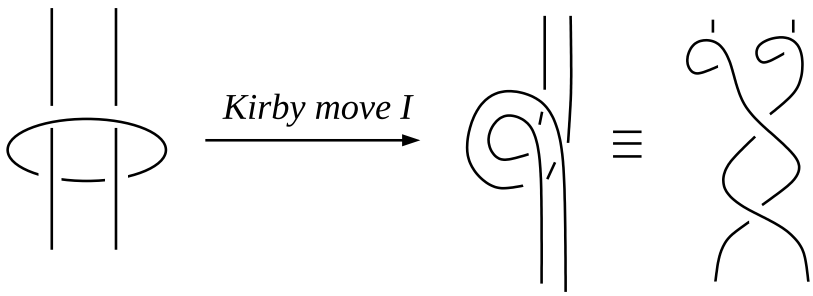

Figure D.24: KirbyMIfigs2.

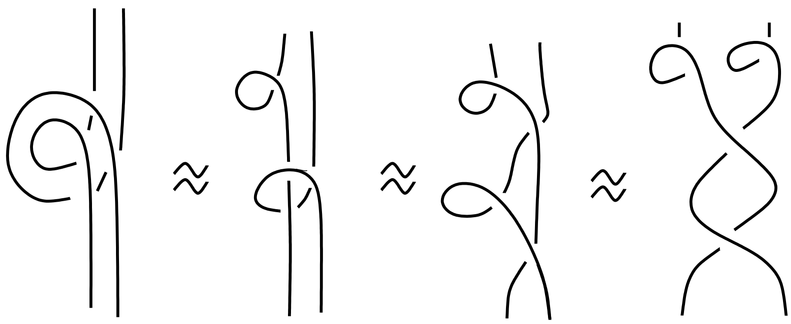

Figure D.25: KirbyMIfigs3.

Abstract:

We show that the path integral for the three-dimensional $SU(2)$ BF theory with a Wilson loop or a spin network function inserted can be understood as the Rovelli-Smolin loop transform of a wavefunction in the Ashtekar connection representation, where the wavefunction satisfies the constraints of quantum general relativity with zero cosmological constant. This wavefunction is given as a product of the delta functions of the $SU(2)$ field strength and therefore it can be naturally associated to a flat connection spacetime. The loop transform can be defined rigorously via the quantum $SU(2)$ group, as a spin foam state sum model, so that one obtains invariants of spin networks embedded in a three-manifold. These invariants define a flat connection vacuum state in the q-deformed spin network basis. We then propose a modification of this construction in order to obtain a vacuum state corresponding to the flat metric spacetime.

D.11 BF-Theory

Topological field theory:

$$S^{BF} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_i. \tag{D.236}$$

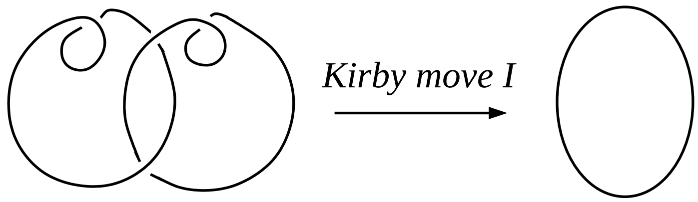

Figure D.26: KirbyMIfigs1.

No local degrees of freedom:

$$F^i = -\Lambda B^i, \qquad \mathcal{D} \wedge B^i = 0 \tag{D.237}$$

We add a quadratic constraint

$$B^{(i} \wedge B^{j)} = \frac{1}{3}\delta^{ij} B^k \wedge B_k \tag{D.238}$$

The result is general relativity:

$$S_{Plebanski} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_I - \frac{1}{2}\phi_{ij} B^i \wedge B^j \tag{D.239}$$

Quantization is determined by the quantization of the topological field theory. As the constraints are non-derivative the gravitational filed has the same commutation relations as the topological theory.

Consider the action

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2) \tag{D.240}$$

which has two constrained degrees of freedom (q_1, q_2), which we assume to live on a circle, with conjugate momentum (p_1, p_2). This theory is completely constrained because both q_1 and q_2 must be zero. There are no degrees of freedom. Now let us impose another constraint with corresponding Lagrange multiplier ξ. The action principle becomes

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2 + \xi(\lambda_1 - \lambda_2)) \tag{D.241}$$

The two original Lagrange multipliers are constrained and we now have one degree of freedom: λ_1 has to be equal to λ_2 and thus only $q_1 + q_2$ has to be zero whereas the difference is free,

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1(q_1 + q_2)). \tag{D.242}$$

This mimics the transition from BF-theory to gravity where additional constraints (the simplicity constraints) reduce the freedom the original Lagrange multipliers of the BF-theory and thereby introduce local degrees of freedom.

FIBRE BUNDLES USE TO BE HERE!!!!

D.12 Characteristic Classes

The information about the topology of the $P(\mathcal{M}, G)$, which is encoded in the transition functions, can be extracted from integrals of polynomials in the curvature, called characteristic classes, of an arbitrary connection, which represent topological invariants.

$$\det(t\mathbf{I} + a_c T^c) \tag{D.243}$$

$$\det(t\mathbf{I} + a_c T^c) = \sum_{i=0}^{m} t^i P_{m-i}(a_c) \tag{D.244}$$

$$\det(t\mathbf{I} + a_c T^c) = \det(t\mathbf{I} + g a_c T^c g^{-1}) = \det(g(t\mathbf{I} + a_c T^c g^{-1}) = \det(t\mathbf{I} + a_c T^c) \tag{D.245}$$

$$a_i(P) = \frac{(-1)^i}{(2\pi)^i} \sum_{j_1 < j_2 < \cdots < j_i \leq n} \prod_j \frac{1}{i_j! j^{i_j}} (TrF^j)^{i_j} \tag{D.246}$$

second *Chern class*

$$c_2(P) = \frac{1}{4\pi^2} \left(\frac{1}{2}(TrF) \wedge (TrF) - \frac{1}{2} Tr(F \wedge F) \right) \tag{D.247}$$

For $SU(n)$

$$c_2(P) = -\frac{1}{8\pi^2} Tr(F \wedge F) \tag{D.248}$$

to do with the calculation of black hole entropy.

D.13 Biblioliographical notes

In this chapter I have relied on the following references:

D.14 Worked Exercises

$$\frac{\partial}{\partial g_{ij}} g_{mn} = \delta_{im}\delta_{jn} \tag{D.249}$$

This implies

$$\frac{\partial}{\partial g_{ij}} (g^{-1})_{mn} = -g^{-1}_{im} g^{-1}_{jn} \tag{D.250}$$

$$0 = \frac{\partial}{\partial g_{ij}} g g^{-1} = \left(\frac{\partial}{\partial g_{ij}} g\right) g^{-1} + g \frac{\partial}{\partial g_{ij}} g^{-1} \tag{D.251}$$

$$\mathbf{d}(ig^{-1}\mathbf{d}g + g^{-1}\mathbf{A}g) = i\mathbf{d}g^{-1} \wedge \mathbf{d}g + (\mathbf{d}g^{-1} \wedge \mathbf{A}g + g^{-1}\mathbf{d}\mathbf{A}g - g^{-1}\mathbf{A} \wedge \mathbf{d}g) \tag{D.252}$$

where we used $d^2 g = 0$ and that, in the last term, in passing through $\mathbf{A}$ we picked up a minus sign.

Proofs

Questions

1.

where Ad is the adjoint representation.

$$[T_a, T_b] = C^{abc} T_c, \tag{D.253}$$

C^{abc} forms a representation of (D.256), the matrix $\mathcal{T}^a$ with components given by

$$(\mathcal{T}^a)_{bc} = iC^{abc} \tag{D.254}$$

$$[[T_a, T_b], T_c] + [[T_b, T_c], T_a] + [[T_c, T_a], T_b] \equiv 0. \tag{D.255}$$

Proof

1. Forms

$$[\mathcal{T}^a, \mathcal{T}^b] = C^{abc}\mathcal{T}^c, \tag{D.256}$$

Which can easily be shown to be so from the Jacobi identity for C_{abc} and its antisymmetry in the first two indices - $C_{abc} = -C_{bac}$,

$$C_{abd}C_{dce} + C_{bcd}C_{dae} + C_{cad}C_{dbe} \equiv 0 \tag{D.257}$$

$$C_{abd}C_{dce} = C_{ade}C_{bcd} - C_{bde}C_{acd} \tag{D.258}$$

$$C^{abd}(\mathcal{T}^d)_{ce} = (\mathcal{T}^a)_{de}(\mathcal{T}^b)_{cd} - (\mathcal{T}^b)_{de}(\mathcal{T}^a)_{cd} \tag{D.259}$$

chern-Simons.

$$\mathcal{L}_{\mathcal{CS}} = \kappa\epsilon^{ijk}\text{tr}\left(A_i\partial_j A_k + \frac{2}{3}A_iA_jA_k\right) \tag{D.260}$$

$$A_i \to A_i^g \equiv g^{-1}A_ig + g^{-1}\partial_i g \tag{D.261}$$

We prove:

$$\mathcal{L}_{\mathcal{CS}} \to \mathcal{L}_{\mathcal{CS}} - \kappa\epsilon^{ijk}\partial_i\text{tr}\left(\partial_j gg^{-1}A_k\right) - \frac{\kappa}{3}\epsilon^{ijk}\text{tr}\left(g^{-1}\partial_i gg^{-1}\partial_j gg^{-1}\partial_k g\right) \tag{D.262}$$

Taking a derivative of the identity $I = g^{-1}g$ we get an identity involving derivatives:

$$\partial_i(I) = 0 = \partial_i(g^{-1}g) = \partial_i g^{-1}g + g^{-1}\partial_i g \tag{D.263}$$

which means that

$$\partial_i g^{-1}g = -g^{-1}\partial_i g \quad \text{or} \quad \partial_i g^{-1} = -g^{-1}\partial_i gg^{-1}. \tag{D.264}$$

we will use this to replace derivatives of the inverse matrix g^{-1} in favour of $-g^{-1}\partial_i gg^{-1}$ which involves the derivative of g. We will also make use of the cyclic symmetry of $\text{tr}(AB) = \text{tr}(BA)$.

$$\boxed{\begin{aligned}\mathcal{L}^g_{\mathcal{CS}} &= \kappa\epsilon^{ijk}\mathrm{tr}\left([g^{-1}A_i g + g^{-1}\partial_i g]\partial_j[g^{-1}A_k g + g^{-1}\partial_k g]\right.\\ &+ \left.\frac{2}{3}[g^{-1}A_i g + g^{-1}\partial_i g][g^{-1}A_j g + g^{-1}\partial_j g][g^{-1}A_k g + g^{-1}\partial_k g]\right) \qquad \text{(D.265)}\end{aligned}}$$

Multiplying out the second term of (D.265) is made easier if we split evaluate the terms with three, two, one and zero A's separately.

3 A**'s**:

$$\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_iA_jA_kg\right) = \frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{A_iA_jA_k\right\} \tag{D.266}$$

0 A**'s**:

$$\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i gg^{-1}\partial_j gg^{-1}\partial_j g\right) \tag{D.267}$$

2 A**'s**:

$$\begin{aligned}&\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{g^{-1}A_iA_j\partial_k g + g^{-1}A_i\partial_j gg^{-1}A_kg + g^{-1}\partial_i gg^{-1}A_jA_kg\right\}\\ &\quad = \frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{A_iA_j\partial_k gg^{-1} + A_kA_i\partial_j gg^{-1} + A_jA_k\partial_i gg^{-1}\right\}\\ &\quad = 2\kappa\epsilon^{ijk}\mathrm{tr}(A_iA_j\partial_k gg^{-1})\end{aligned} \tag{D.268}$$

we can $i \to j$, $j \to k$, $k \to i$ etc and not change the value of this term. We could have written down the last line of (D.268) straight away from noting the from the permutative symmetry of the second term of (D.265)

1 A:

$$\begin{aligned}&\frac{2}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left\{g^{-1}A_i\partial_j gg^{-1}\partial_k g + g^{-1}\partial_j gg^{-1}A_jg + g^{-1}\partial_i gg^{-1}\partial_j gg^{-1}A_kg\right\}\\ &\quad = 2\kappa\epsilon^{ijk}\mathrm{tr}\left\{\partial_i gg^{-1}\partial_j gg^{-1}A_k\right\}\\ &\quad = 2\kappa\epsilon^{ijk}\mathrm{tr}\left\{\partial_j g\partial_i g^{-1}A_k\right\}\end{aligned} \tag{D.269}$$

From the first term of (D.265) for convenience of comparison:

2 A**'s**:

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i g\partial_j(g^{-1}A_k g)\right)\\
&\quad = \kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i g\partial_j g^{-1}A_k g + g^{-1}A_i\partial_j A_k g + g^{-1}A_i A_k\partial_j g\right)\\
&\quad = \kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i(-g^{-1}\partial_j g g^{-1})A_k g + g^{-1}A_i A_k\partial_j g + A_i\partial_j A_k)\right)\\
&\quad = \kappa\epsilon^{ijk}\mathrm{tr}\left(-2A_i\partial_j g g^{-1}A_k + A_i\partial_j A_k\right) \qquad \text{(D.270)}
\end{aligned}
$$

0 A**'s**:

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g\partial_j(g^{-1}\partial_k g)\right)\\
&\quad = \kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g\partial_j g^{-1}\partial_k g + g^{-1}\partial_i g g^{-1}\partial_j\partial_k g\right)\\
&\quad = \kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g\partial_j g^{-1}\partial_k g\right)\\
&\quad = -\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g\right) \qquad \text{(D.271)}
\end{aligned}
$$

because $\epsilon^{ijk}\partial_j\partial_k g = 0$.

1 A:

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}A_i g\partial_j(g^{-1}\partial_k g) + g^{-1}\partial_i g\partial_j(g^{-1}A_k g)\right)\\
&\quad = \kappa\epsilon^{ijk}\mathrm{tr}\left(g\partial_j g^{-1}\partial_k g g^{-1}A_i + \right.\\
&\qquad\qquad \left. + g^{-1}\partial_i g\partial_j g^{-1}A_k g + g^{-1}\partial_i g g^{-1}\partial_j A_k g + g^{-1}\partial_i g g^{-1}A_k\partial_j g\right) \qquad \text{(D.272)}
\end{aligned}
$$

Let us consider the first term in the last line of (D.272)

$$
g\partial_j g^{-1}\partial_k g g^{-1}A_i = (g\partial_j g^{-1}g)(g^{-1}\partial_k g g^{-1})A_i = \partial_j g\partial_k g^{-1}A_i \qquad \text{(D.273)}
$$

$$
\begin{aligned}
&\kappa\epsilon^{ijk}\mathrm{tr}\left(\partial_j g\partial_k g^{-1}A_i + \partial_i g\partial_j g^{-1}A_k + \partial_i g g^{-1}\partial_j A_k - \partial_i g^{-1}A_k\partial_j g\right)\\
&\quad = -\kappa\epsilon^{ijk}\mathrm{tr}\left(3\partial_j g\partial_i g^{-1}A_k + \partial_j g g^{-1}\partial_i A_k\right) \qquad \text{(D.274)}
\end{aligned}
$$

Adding (D.266) and the second term of (D.270) we have the $\mathcal{L}_{\mathcal{CS}}$ term in (D.265)

Adding (D.271) and (D.267)

$$
-\frac{1}{3}\kappa\epsilon^{ijk}\mathrm{tr}\left(g^{-1}\partial_i g g^{-1}\partial_j g g^{-1}\partial_k g\right) \qquad \text{(D.275)}
$$

Adding (D.274) and (D.269)

$$-\kappa\epsilon^{ijk}\mathrm{tr}\left(\partial_i g\partial_j g^{-1}A_k + \partial_i g g^{-1}\partial_j A_k\right) = -\kappa\epsilon^{ijk}\partial_j \mathrm{tr}\left(\partial_i g g^{-1}A_k\right) \qquad \text{(D.276)}$$

And we are done.

Jone's polynomial (satellite formulae)

(1)

$T^a_{(\rho_1)} + T^a_{(\rho_2)} := T^a_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}$

Answers:

$T^a_{(\rho_1)} T^b_{(\rho_1)} = T^c_{(\rho_1)}$

$$\begin{aligned} & \big(T^a_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}\big)\big(T^b_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^b_{(\rho_2)}\big) \\ = & \big(T^a_{(\rho_1)}T^b_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}T^b_{(\rho_2)}\big) + T^a_{(\rho_1)} \otimes T^b_{(\rho_2)} + T^b_{(\rho_1)} \otimes T^a_{(\rho_2)} \\ = & \big(T^c_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)}\big) + T^a_{(\rho_1)} \otimes T^b_{(\rho_2)} + T^b_{(\rho_1)} \otimes T^a_{(\rho_2)} \end{aligned} \qquad \text{(D.277)}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{(D.278)}$$

As the last two terms are symmetric in the indices a and b,

$$\begin{aligned} & \big[T^a_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{(\rho_2)}\ ,\ T^b_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^b_{(\rho_2)}\big] \\ = & [T^a_{(\rho_1)}, T^b_{(\rho_1)}] \otimes \mathbf{1} + \mathbf{1} \otimes [T^a_{(\rho_2)}, T^b_{(\rho_2)}] \\ = & C^{ab}{}_c\big(T^c_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)}\big) \end{aligned} \qquad \text{(D.279)}$$

These are generators of a *reducible* representation of G. That is, there exists a unitary matrix U such that

$$U(T^a_{(\rho_1)} + T^a_{(\rho_2)})U^\dagger = \begin{pmatrix} T^a_{\rho(t_1)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & T^a_{\rho(t_n)} \end{pmatrix} \qquad \text{(D.280)}$$

where $\rho(t_r)$ are irreducible representations. We have

$$Tr_{\rho_1}\, Tr_{\rho_2} = Tr_{\rho_1 \otimes \rho_2}$$

$$W' = W_{\rho_1 \otimes \rho_2}(U_1)$$

$$\begin{aligned}
&\left(T^c_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)} + T^{(a}_{(\rho_1)} \otimes T^{b)}_{(\rho_2)}\right)\left(T^d_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^d_{(\rho_2)}\right) \\
&T^c_{(\rho_1)}T^d_{(\rho_1)} \otimes \mathbf{1} + \mathbf{1} \otimes T^c_{(\rho_2)}T^d_{(\rho_2)} + T^{(c}_{(\rho_1)} \otimes T^{d)}_{(\rho_2)} + \\
&T^a_{(\rho_1)}T^d_{(\rho_1)} \otimes T^b_{(\rho_2)} + T^b_{(\rho_1)}T^d_{(\rho_1)} \otimes T^a_{(\rho_2)} + T^a_{(\rho_1)} \otimes T^b_{(\rho_2)}T^d_{(\rho_2)} + T^b_{(\rho_1)} \otimes T^a_{(\rho_2)}T^d_{(\rho_2)}
\end{aligned} \tag{D.281}$$