Appendix G

Spin Networks

Mostly follows "A Spin Network Primer" [56]

G.1 Diagrammatic Mathematics

Diagrammatic algebra designed too handle the combinatorics of irreducible representations, all the familiar results of representation theory have diagrammatical form.

G.1.1 Line, Bend ad Loop

Consider the tensor

$$(\delta^A_B) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which can be represented diagramatically as in fig G.1

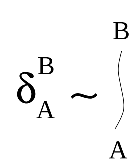


Figure G.1: Diagramatically representation of δ^A_B

Consider the two antisymmetric tensors

$$(\epsilon_{AB}) = (\epsilon^{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{G.1}$$

We associate a curve with a matrix with two upper (lower) indices. The first trial for ϵ_{AB} we look at is in fig (G.2) and for ϵ^{AB} fig (G.3)

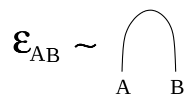


Figure G.2: epdown

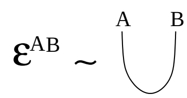


Figure G.3: epup

This fits well with the diagramatitics of $\delta_A^C \epsilon_{CB} = \epsilon_{AB}$. We soon find trouble with this choose however. Firstly:

$$\delta_A^C \epsilon_{CD} \epsilon^{DE} \delta_E^B = -\delta_A^B$$

and straightening a line yields a minus sign:

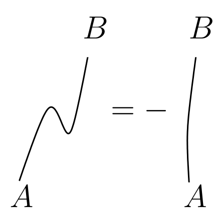


Figure G.4: firstprob

Secondaly, as a conseqence of

$$\epsilon_{AD} \epsilon_{BC} \epsilon^{CD} = -\epsilon_{AB},$$

topological difficulties can be fixed by modifying the definition

$$\epsilon_{AB} \rightarrow \tilde{\epsilon}_{AB} = i\epsilon_{AB}.$$

As the indices take two values, we have the identity

$$\epsilon_{[EB}\epsilon_{C]F} = 0 \tag{G.2}$$

which reduces to

$$\epsilon_{EB}\epsilon_{CF} + \epsilon_{BC}\epsilon_{EF} + \epsilon_{CE}\epsilon_{BF} = 0 \tag{G.3}$$

Contracting this with ϵ^{EA} and ϵ^{FD}, then using $\epsilon_{EB}\epsilon^{EA} = \delta_B^A$, $\epsilon_{CF}\epsilon^{FD} = -\delta_C^D$ and $\epsilon_{EF}\epsilon^{EA}\epsilon^{FD} = \epsilon^{AD}$ etc we obtain the so-called binor identity:

$$\epsilon_{AC}\epsilon^{BD} = \delta_A^B\delta_C^D - \delta_A^D\delta_C^B \tag{G.4}$$

Using the definitions of the $\tilde{\epsilon}$ matrices, the binor identity becomes

$$\tilde{\epsilon}_{AC}\tilde{\epsilon}^{BD} - \delta_A^D\delta_C^B + \delta_A^B\delta_C^D = 0. \tag{G.5}$$

Then introducing the rule that we assign a minus sign to each crossing, equation (G.5) can be diagrammatically, represented as in fig G.5.

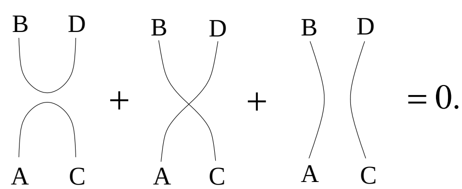


Figure G.5: The binor indentity $\tilde{\epsilon}_{AC}\tilde{\epsilon}^{BD} - \delta_A^D\delta_C^B + \delta_A^B\delta_C^D = 0$

For more than

$\delta_A^C\epsilon_{CD} = \epsilon_{AB}$.

$$\delta_A^C\epsilon_{CD}\epsilon^{DE}\delta_E^B = \epsilon_{AD}\epsilon^{DB} = -\delta_A^B, \tag{G.6}$$

$$\epsilon_{AD}\epsilon_{BC}\epsilon^{CD} = -\epsilon_{AB} \tag{G.7}$$

$$\epsilon_{AB} \to \tilde{\epsilon}_{AB} = i\epsilon_{AB} \tag{G.8}$$

$$\delta_A^D \tag{G.9}$$

Using these rules, we can show that these strands behave as would thin strings in the plane; one can arbitrary deform a graphical expression without changing its meaning.

In translating a diagram into tensor notation, we use

1. assign a minus sign to each

2. assign a minus sign to each crossing

G.1.2 Symmetrizing Products of Delta Functions

Define the $D^{A'B'}_{(A\,B)}$ as the symmetric product of two delta functions:

$$D^{A'B'}_{(A\,B)} := \frac{1}{2!}\left(\delta^{A'}_{A}\delta^{B'}_{B} + \delta^{A'}_{B}\delta^{B'}_{A}\right) \tag{G.10}$$

$D^{A'B'}_{(A\,B)}$ are projectors i.e.

$$D^{A'B'}_{(C\,D)}D^{C\,D}_{(A\,B)} = \frac{1}{2!}\left(\delta^{A'}_{C}\delta^{B'}_{D} + \delta^{A'}_{D}\delta^{B'}_{C}\right)D^{C\,D}_{(A\,B)} = D^{(A'B')}_{(A\,B)} = D^{A'B'}_{(A\,B)} \tag{G.11}$$

More generally $D^{A'B'...D'}_{(A\,B\,...D)}$, the symmetric product of n delta functions, is a projector:

$$\begin{aligned} D^{A'B'...D'}_{(E\,F\,...H)}D^{EF...H}_{(A\,B\,...D)} &= \frac{1}{n!}\left(\delta^{A'}_{E}\delta^{B'}_{F}\ldots\delta^{H'}_{H} + \ldots\right)D^{EF...H}_{(A\,B\,...D)} \\ &= D^{(EF...H)}_{(A\,B\,...D)} = D^{EF...H}_{(A\,B\,...D)}. \end{aligned} \tag{G.12}$$

This general result can be represented diagramatically as in fig G.1.2.

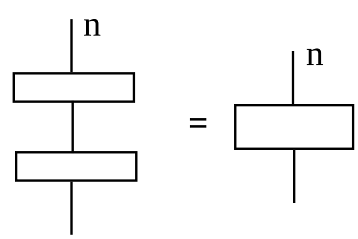


Figure G.6: Projector. The symmetric product of n delta functions, is a projector

G.1.3 Jones-Wenzl Projectors

Starting from the binor identity

$$-\tilde{\epsilon}^{A'B'}\tilde{\epsilon}_{AB} = \delta_A^{A'}\delta_B^{B'} - \delta_B^{A'}\delta_A^{B'}, \tag{G.13}$$

a simple rearrangement gives

$$\frac{1}{2}(\delta_A^{A'}\delta_B^{B'} + \delta_A^{B'}\delta_B^{A'}) = \delta_A^{A'}\delta_B^{B'} + \frac{1}{2}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'B'} \tag{G.14}$$

Written in tha stanard form (see in a momemt)

$$\delta_{(A}^{A'}\delta_{B)}^{B'} = \delta_A^{A'}\delta_B^{B'} - \mu_1\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'B'} \tag{G.15}$$

where $\mu_1 = -1/2$. Which is diagramatically represented in fig G.1.3.

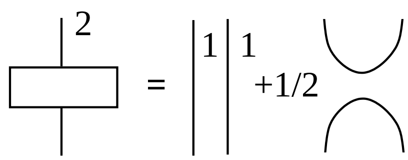


Figure G.7: Diagrammatical representation of equation (G.15) with $\mu_1 = -1/2$.

Jones-Wenzl Projectors for $n = 3$

We can rearange the symmetric product of the three deltas as follows

$$\begin{aligned} 3\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} &= \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \delta_B^{A'}\delta_A^{(B'}\delta_C^{C')} + \delta_C^{A'}\delta_{(B}^{B'}\delta_{A)}^{C'} \\ &= 3\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C')}\right) - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \delta_C^{A'}\delta_B^{(B'}\delta_A^{C')}\right) \end{aligned} \tag{G.16}$$

This rearangment (G.16) can be represented diagramatically as in fig (G.1.3)

Multiply (G.13) by $\delta_C^{C'}$ and symmetrize over the upper indices B' and C' to get

$$\delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C')} = -\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C')}, \tag{G.17}$$

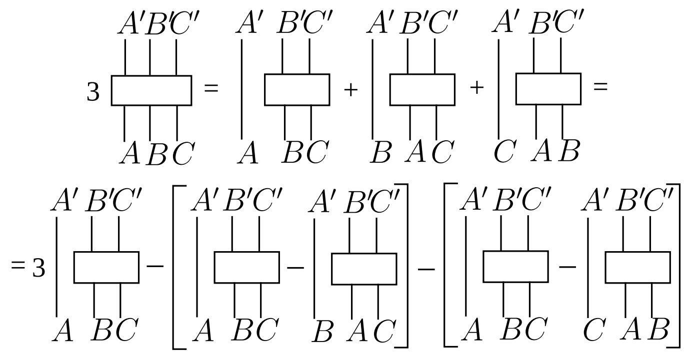


Figure G.8: graphmath7. Diagramatical representation of (G.16).

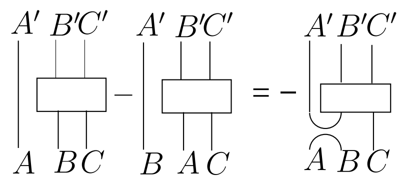


Figure G.9: graphmath9. Diagramatical representation of (G.17)

which is expressed diagramatically as fig.(G.1.3)

Using this in fig.(graphmath7) we obtain

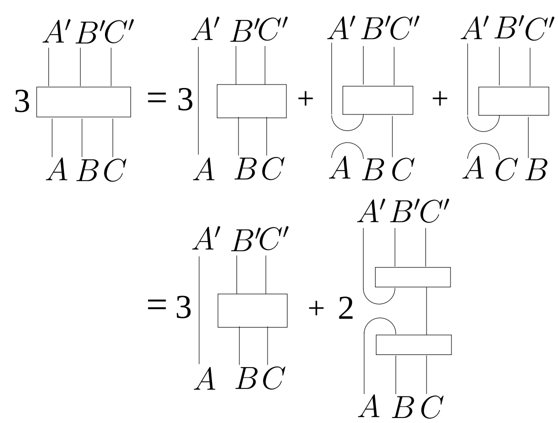


Figure G.10: graphmath8.

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \frac{1}{3}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C')} + \frac{1}{3}\tilde{\epsilon}_{AC}\tilde{\epsilon}^{A'(C'}\delta_B^{B')} \tag{G.18}$$

we obtain

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} + \frac{2}{3}\tilde{\epsilon}_{A(B}\delta_{C)}^{(C'}\tilde{\epsilon}^{B')A'} \tag{G.19}$$

Or

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C')} - \mu_1\tilde{\epsilon}_{A(B}\delta_{C)}^{(C'}\tilde{\epsilon}^{B')A'} \tag{G.20}$$

where

$$\mu_1 = -2/3. \tag{G.21}$$

This is represented in fig (Diagrelfor3)

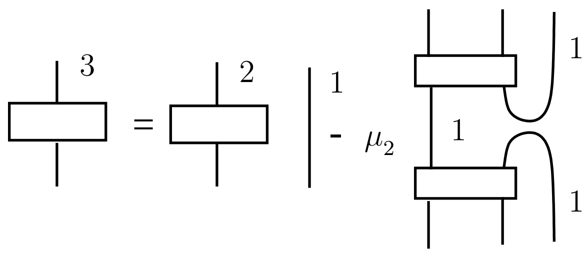


Figure G.11: Diagrelfor3. Compact diagrammatical representation of (G.20)

Jones-Wenzl Projectors for Arbitrary n

We now consider the symmetric product of n δ's. We have:

$$\begin{aligned} n\delta_A^{(A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_F^{F')} &= \delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \dots + \delta_F^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_A^{F')} \\ &= n\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')}\right) - \dots \\ &\quad \dots - \left(\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_F^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_A^{F')}\right) \end{aligned} \tag{G.22}$$

This is represented by diagram (graphmath11)

$$\delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} - \delta_B^{A'}\delta_A^{(B'}\delta_C^{C'}\dots\delta_F^{F')} = -\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} \tag{G.23}$$

This is represented by diagram graphmath12

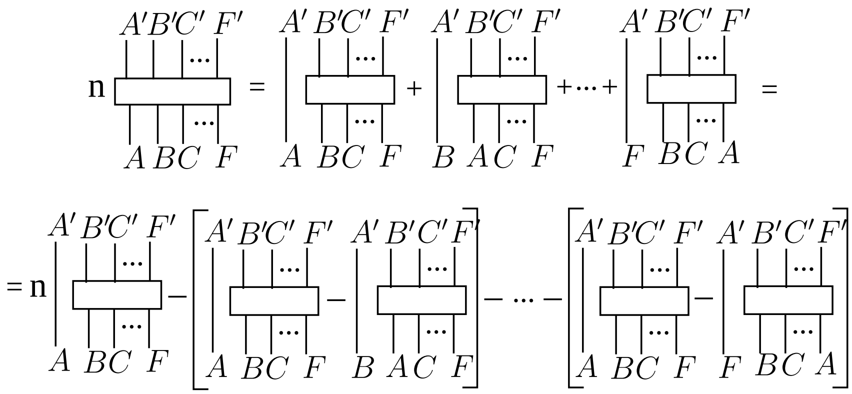


Figure G.12: graphmath11. Representing equation (G.22).

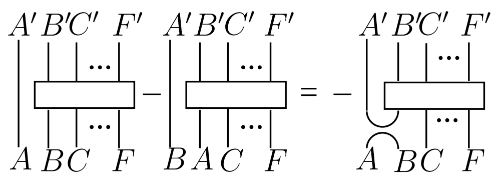


Figure G.13: graphmath12. (G.23)

$$\delta_A^{(A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_F^{F')} = \delta_A^{A'}\delta_B^{(B'}\delta_C^{C'}\dots\delta_F^{F')} + \frac{1}{n}\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} + \dots + \frac{1}{n}\tilde{\epsilon}_{AF}\tilde{\epsilon}^{A'(F'}\delta_C^{C'}\dots\delta_B^{B')} \tag{G.24}$$

$$\begin{aligned}
&\underbrace{\tilde{\epsilon}_{AB}\tilde{\epsilon}^{A'(B'}\delta_C^{C'}\dots\delta_F^{F')} + \tilde{\epsilon}_{AC}\tilde{\epsilon}^{A'(C'}\delta_B^{B'}\dots\delta_F^{F')}\dots + \tilde{\epsilon}_{AF}\tilde{\epsilon}^{A'(F'}\delta_C^{C'}\dots\delta_B^{B')}}_{n-1 \text{ terms}} \\
&\quad = +(n-1)\tilde{\epsilon}_{A(B}\delta_C^{C'}\dots\delta_{F)}^{(F'}\tilde{\epsilon}^{B')A'}.
\end{aligned} \tag{G.25}$$

we obtain

$$\delta_{(A}^{A'}\delta_B^{B'}\delta_C^{C'}\dots\delta_{F)}^{F'} = \delta_A^{A'}\delta_{(B}^{B'}\delta_C^{C'}\dots\delta_{F)}^{F'} + (n-1)\epsilon_{A(B}\delta_C^{C'}\dots\delta_{F)}^{(F'}\epsilon^{B')A'} \tag{G.26}$$

G.1.4 Contractions of Symmetrised Lines

We perform a contraction the symmetrised lines as given by fig (G.1.4), and we denote the resulting value Δ_n. For example $\Delta_1 = -2$:

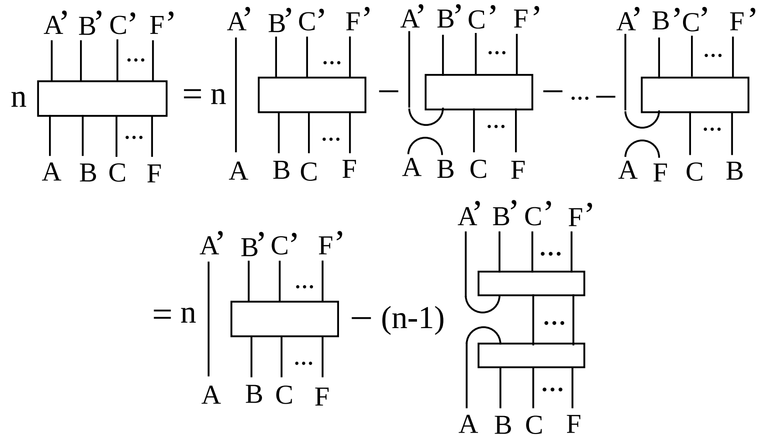


Figure G.14: graphmath13.

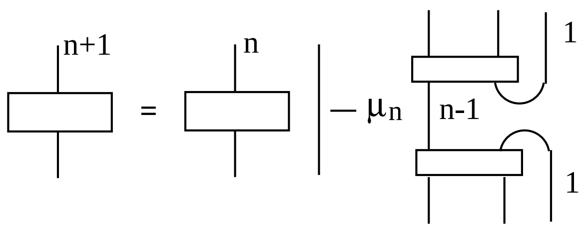


Figure G.15: This one.

$$\Delta_1 = \delta_A^{A'} \left(\tilde{\epsilon}_{A'C} \delta_D^C \tilde{\epsilon}^{DA} \right) = -\delta_A^{A'} \delta_{A'}^{A} = -2.$$

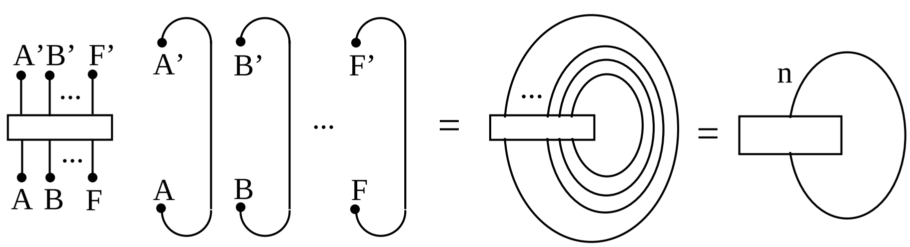


Figure G.16: graphmath2. The contraction of the symmetric product of deltas, the resulting values denoted Δ_n.

As an example, we explicitly work out the value of Δ_2 using the graphical method as shown in fig (graphmath2A). We find that the result is $\Delta_2 = 3$.

In order to find the value of Δ_n for $n > 2$ we derive recursive relations.

a relation between $\Delta_{n+2}, \Delta_{n+1}$ and Δ_n.

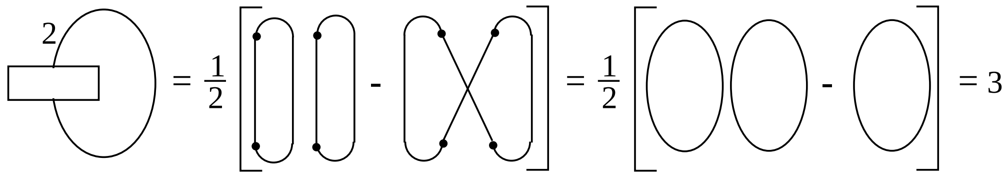


Figure G.17: graphmath2A. Calculation of Δ_2.

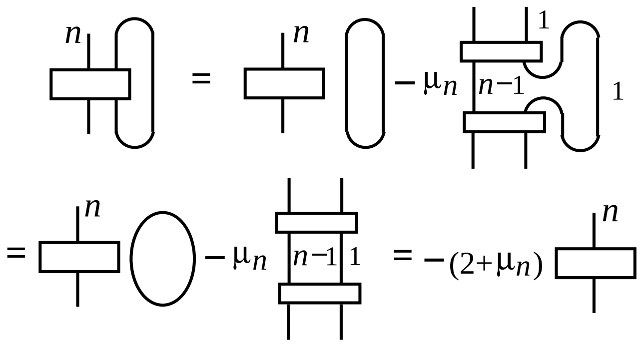


Figure G.18: graphmath3A .

$$\Delta_{n+1} = -(2+\mu_n)\Delta_n$$

$$\mu_n = -\frac{\Delta_{n+1}}{\Delta_n} - 2$$

$$\begin{aligned} \Delta_n &= -2\Delta_{n-1} + \frac{n-1}{n}\Delta_{n-1} \\ &= \Delta_{n-1}[\frac{n-1}{n} - 2] \\ &= -\Delta_{n-1}[\frac{n+1}{n}] \end{aligned} \tag{G.27}$$

Employing this recursive relation we obtain,

$$
\begin{aligned}
\Delta_n &= -\Delta_{n-1}[\frac{n+1}{n}] \\
&= (-1)^2\Delta_{n-2}[\frac{n}{n-1}][\frac{n+1}{n}] \\
&= (-1)^2\Delta_{n-2}[\frac{n+1}{n-1}] \\
&= (-1)^3\Delta_{n-3}[\frac{n+1}{n-2}] \\
&\vdots \\
&= (-1)^{n-1}\frac{(n+1)}{2}\Delta_1 \\
&= (-1)^n(n+1) \qquad \text{(G.28)}
\end{aligned}
$$

where we have used $\Delta_1 = -2$.

So that

$$
\Delta_n = (-1)^n(n+1). \qquad \text{(G.29)}
$$

The one contraction of an $(n+1)-$symmetrised prodct is proportional to an $n-$symmetrised product, as shown in fig graphmath3

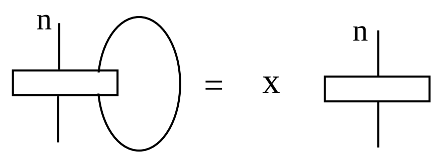


Figure G.19: graphmath3 .

By definition of Δ_n, we see that x is given by Δ_{n+1}/Δ_n. (see fig graphmath1)

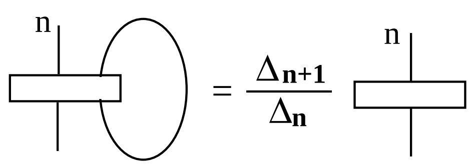


Figure G.20: graphmath1.

Now, if

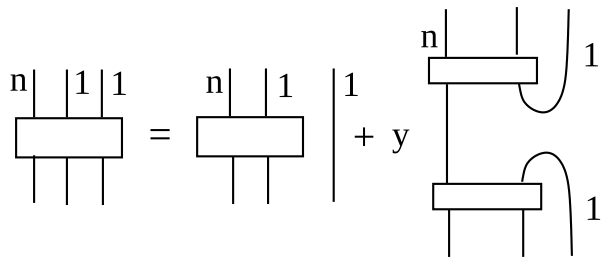


Figure G.21: P math4.

it follows that

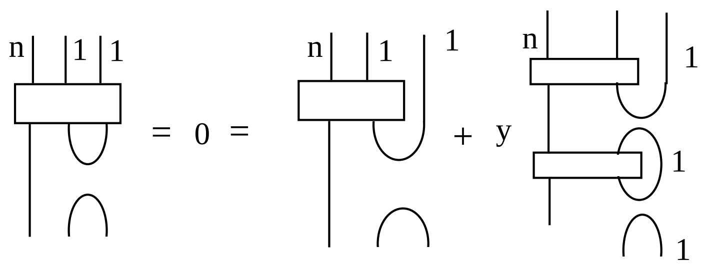


Figure G.22: graphmath5.

Hence

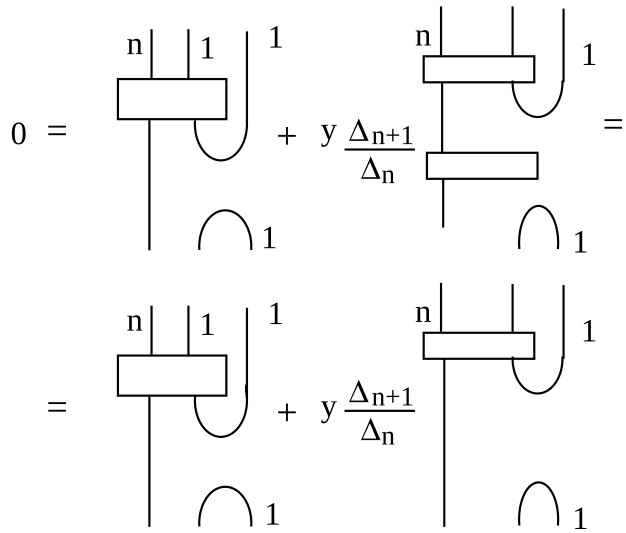


Figure G.23: graphmath6.

Therefore $y = \Delta_n/\Delta_{n+1}$ and the recursion takes the form

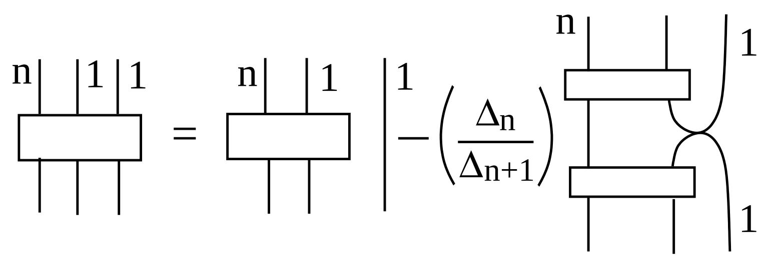


Figure G.24: P math4.

It follows that

$$\Delta_{n+2} = -2\Delta_{n+1} - \Delta_n \tag{G.30}$$

with $\Delta_1 = -2$ and $\Delta_2 = 3$. This obviously has a unique solution which is

$$\Delta_n = (-1)^n(n+1), \tag{G.31}$$

as is easily checked:

$$
\begin{aligned}
-2\Delta_{n+2} - \Delta_n &= -2(-1)^{n+1}(n+2) - (-1)^n(n+1) \\
&= (-1)^{n+2}[2(n+2) - (n+1)] \\
&= (-1)^{n+2}(n+3),
\end{aligned} \tag{G.32}
$$

$$\Delta_1 = (-1)^1(1+1)$$

$$\Delta_2 = (-1)^2(2+1).$$

$$
\begin{aligned}
\Delta_{n+1} + \Delta_n &= (-1)(\Delta_n + \Delta_{n-1}) \\
&= (-1)^2(\Delta_{n-1} + \Delta_{n-2}) \\
&\quad \cdots \\
&= (-1)^{n-1}(\Delta_2 + \Delta_1) \\
&= (-1)^{n-1}(3-2) = (-1)^{n-1}
\end{aligned} \tag{G.33}
$$

each containing a turn back.

G.1.5 3-Vertices

We define a 3-vertex as in fig...

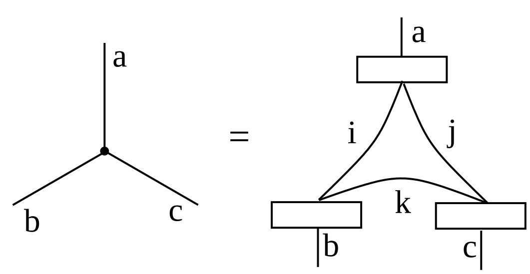


Figure G.25: P.

$$
\begin{aligned}
i &= \frac{a+b-c}{2} \\
j &= \frac{a+c-b}{2} \\
k &= \frac{b+c-a}{2}.
\end{aligned} \tag{G.34}
$$

We consider the "bubble" diagram.

Lemma G.1.1 *The network is zero if $a \neq b$. If $a = b$, then*

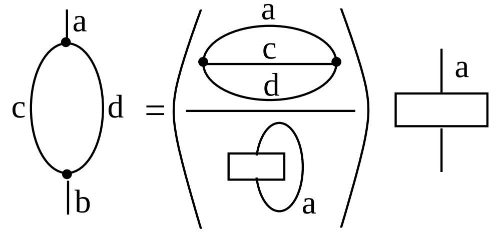


Figure G.26: P.

Proof:

Assume that $a > b$.

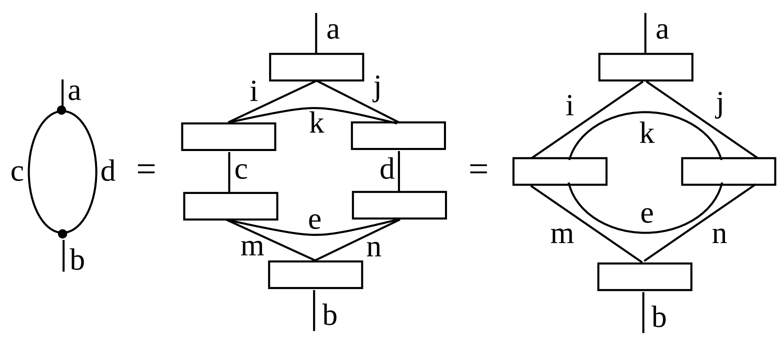


Figure G.27: graphmath15.

where

$$
\begin{aligned}
i &= \frac{a+c-d}{2} \qquad l = \frac{c+d-b}{2} \\
j &= \frac{a+d-c}{2} \qquad m = \frac{c+b-d}{2} \\
k &= \frac{c+d-a}{2} \qquad n = \frac{b+d-c}{2}.
\end{aligned}
\tag{G.35}
$$

Rewritting, we find $e = (a - b)/2$

Consider expanding each of the two middle projectors into their sum of products of δ's. It followes that each term will contain a turn-back with respect to the $a-$projector above and give zero.

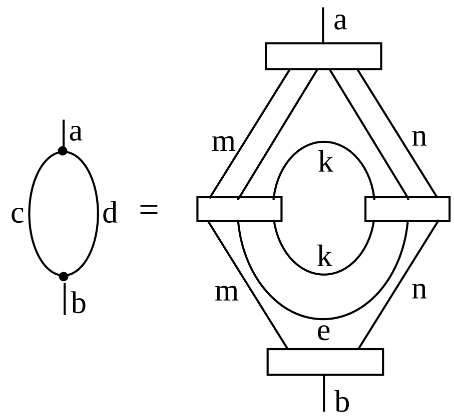


Figure G.28: graphmath16.

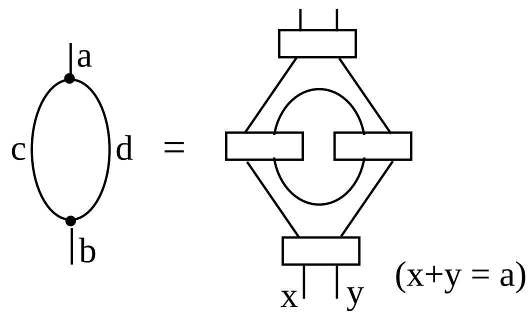


Figure G.29: graphmath17.

Now assume that $a = b$. Consider

Hence

and

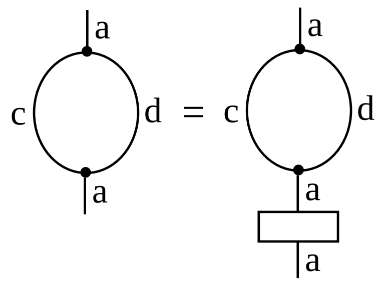


Figure G.30: graphmath18.

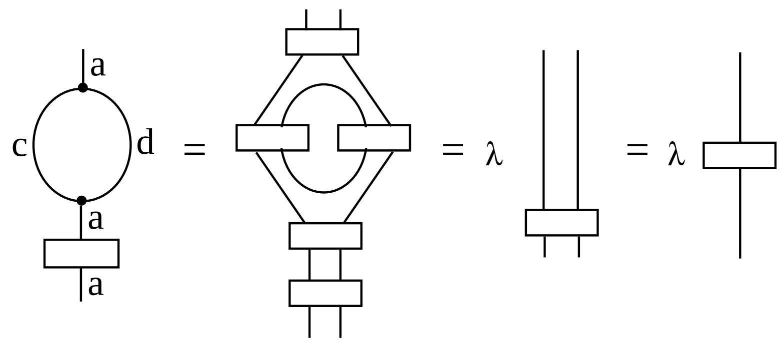


Figure G.31: graphmath18A.

Consider expanding each of the two middle projectors into their sum of products of δ's. Only straight-ahead terms survive the extra projector at the bottom. Thus

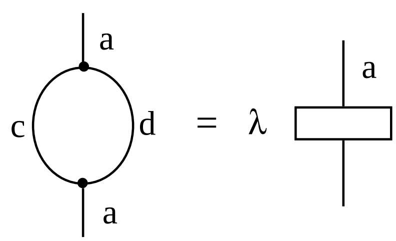


Figure G.32: graphmath19.

Hence

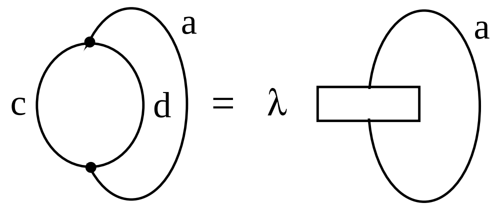


Figure G.33: graphmath20.

$\square$

$$N_{abc}\begin{pmatrix} a & b & c \\ m_a & m_b & m_c \end{pmatrix} \tag{G.36}$$

$$N_{abc} = \left[\frac{(a+b-c)!(b+c-a)!(c+a-b)!}{2^2(a+b+c+2)!}\right]^{1/2} \tag{G.37}$$

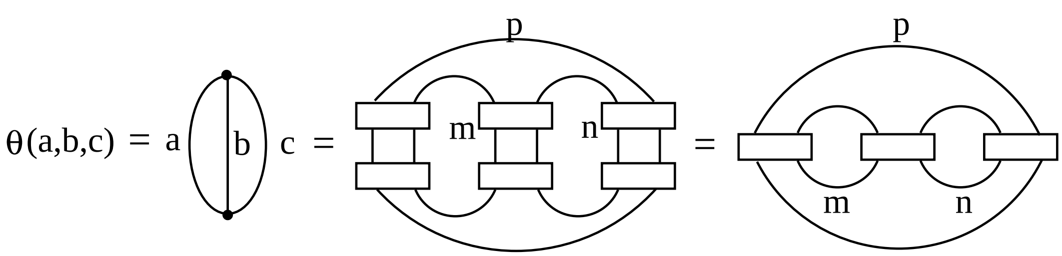


Figure G.34: graphmath21.

where

$$\begin{aligned} m &= \frac{a+b-c}{2}, \\ n &= \frac{b+c-a}{2}, \\ p &= \frac{a+c-b}{2}. \end{aligned} \tag{G.38}$$

In the case $p = 0$, we get

$$Net(m, n, 0) \tag{G.39}$$

$$\Delta_{m+n}. \tag{G.40}$$

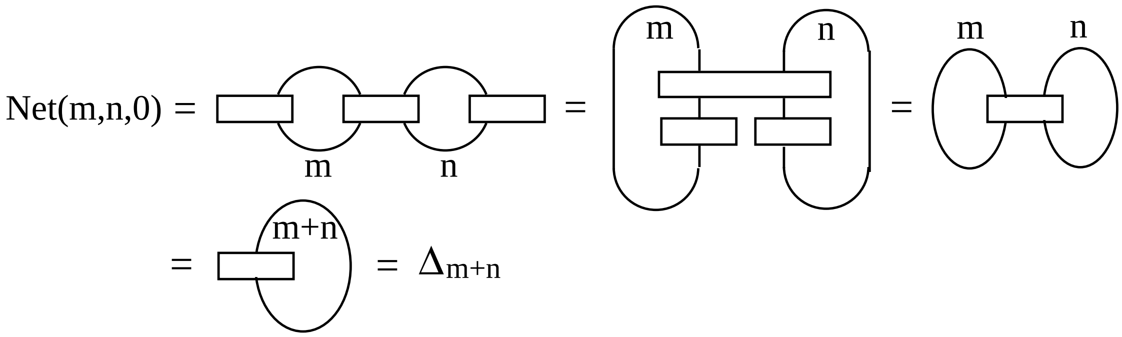


Figure G.35: graphmath22.

$Net(n-1,1,1)$ we will need to get the eigenvalue of the area operator. $Net(m,n,1) =$ is easy to deal with.

$$Net(m,n,1) = -(2 + \mu_m + \mu_n)\Delta_{m+n}. \tag{G.41}$$

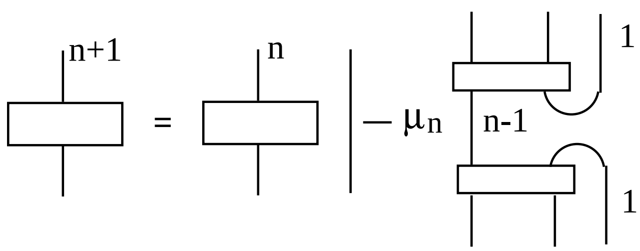


Figure G.36: This one.

Applying this to $Net(m,n,1)$ as shown below.

We see that the last term is equivalent to (G.1.5) and so is zero.

The first network is $-2Net(m,n,0)$, where $Net(m,n,0)$ has already been calculated in the previous Lemma as Δ_{m+n}. The second and third nets are each equivalent to $Net(m,n,0)$. The forth network vanishes. Thus

$$Net(m,n,1) = -(2\Delta_{m+n} + \mu_m\Delta_{m+n} + \mu_n\Delta_{m+n}).$$

Definition Let $Net(m,n,p_e,p_i)$, for $p_e + p_i = p - 1 \geq 1$

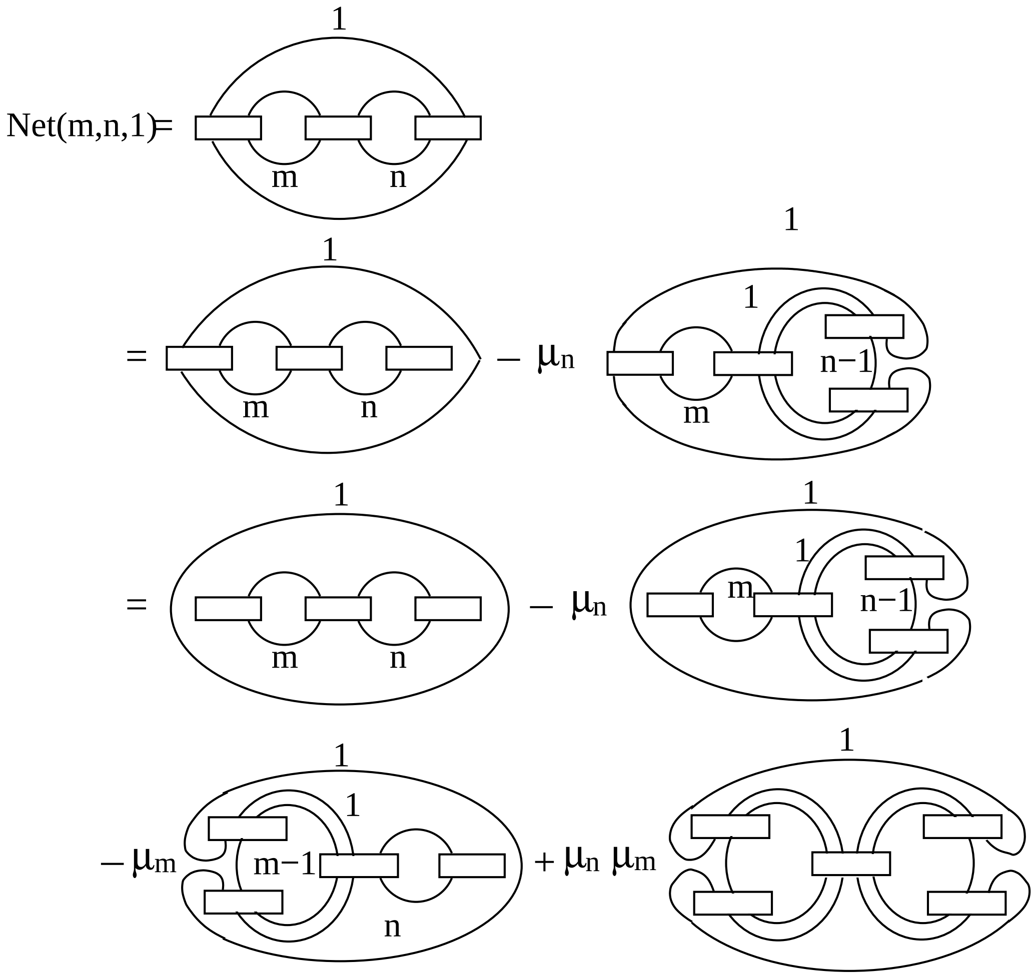


Figure G.37: graphmath23.

Similarly,

$$
\begin{aligned}
Net(m,n,p) &= (-2-\mu_{m+p-1}-\mu_{n+p-1})Net(m,n,p-1) \\
&\quad + \mu_{m+p-1}\mu_{n+p-1}Net(m,n,1,p-2)
\end{aligned} \tag{G.42}
$$

Recursion relation for Net

(a) $Net(m,n,p-1,0)=(-2-\mu_m-\mu_n)Net(m,n,p-1)$

(b) $Net(m,n,p_e,p_i)=(-2-\mu_{m+p_i}-\mu_{n+p_i})Net(m,n,p-1)+\mu_{m+p_i}\mu_{n+p_i}Net(m,n,p_e+1,p_i-1)$

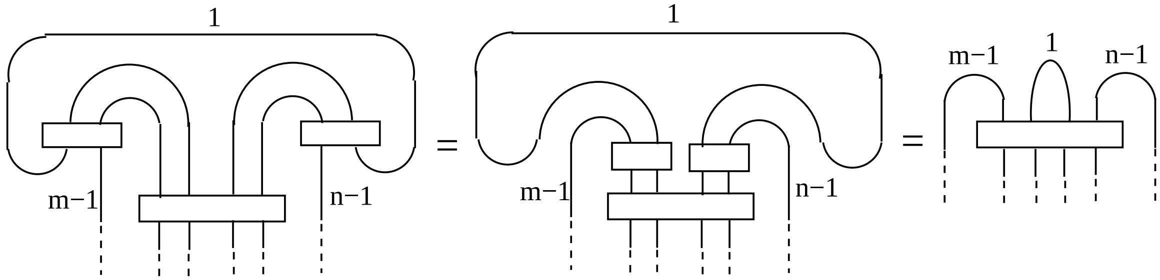


Figure G.38: graphmath25.

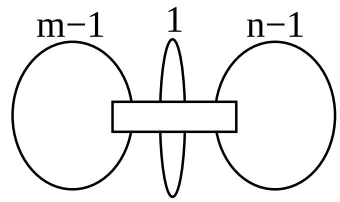


Figure G.39: graphmath24.

Starting with (G.42) and using (b) over again $p-2$ times, and then finally using (a) we can obtain a realtion:

$$Net(m, n, p) = \rho(m, n, p) Net(m, n, p-1). \tag{G.43}$$

To simplify the analysis, we introduce the following. Since $\mu_{m+j} = \Delta_{m-1+j}/\Delta_{m+j}$,

$$-2 - \mu_{m+j} - \mu_{n+j} = \frac{-2\Delta_{m+j}\Delta_{n+j} - \Delta_{m-1+j}\Delta_{n+j} - \Delta_{m+j}\Delta_{n-1+j}}{\Delta_{m+j}\Delta_{n+j}}$$

Write

$$\alpha_j = -2\Delta_{m+j}\Delta_{n+j} - \Delta_{m-1+j}\Delta_{n+j} - \Delta_{m+j}\Delta_{n-1+j}$$

and

$$\beta_j = \Delta_{m+j}\Delta_{n+j}$$

First (G.42) becomes

$$Net(m, n, p) = \frac{\alpha_{p-1}}{\beta_{p-1}} Net(m, n, p-1) + \frac{\beta_{p-2}}{\beta_{p-1}} Net(m, n, 1, p-2) \tag{G.44}$$

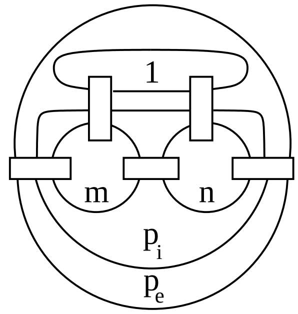
Net(m,n,p_e,p_i) = 

Figure G.40: graphmath26.

then we would use (b) with $p_e = 1$ and $p_i = p - 2$,

$$Net(m,n,1,p-2) = \frac{\alpha_{p-2}}{\beta_{p-2}} Net(m,n,p-1) + \frac{\beta_{p-3}}{\beta_{p-2}} Net(m,n,2,p-3) \tag{G.45}$$

and next we would use (b) with $p_e = 2$ and $p_i = p - 3$,

$$Net(m,n,2,p-3) = \frac{\alpha_{p-3}}{\beta_{p-3}} Net(m,n,p-1) + \frac{\beta_{p-4}}{\beta_{p-3}} Net(m,n,3,p-4) \tag{G.46}$$

and so on until

$$\begin{aligned} Net(m,n,p-2,1) &= \frac{\alpha_1}{\beta_1} Net(m,n,p-1) + \frac{\beta_0}{\beta_1} Net(m,n,p-1,0) \\ &= \frac{\alpha_1}{\beta_1} Net(m,n,p-1) + \frac{\beta_0}{\beta_1}\frac{\alpha_0}{\beta_0} Net(m,n,p-1) \end{aligned} \tag{G.47}$$

where in the last line we used (a). Putting it together,

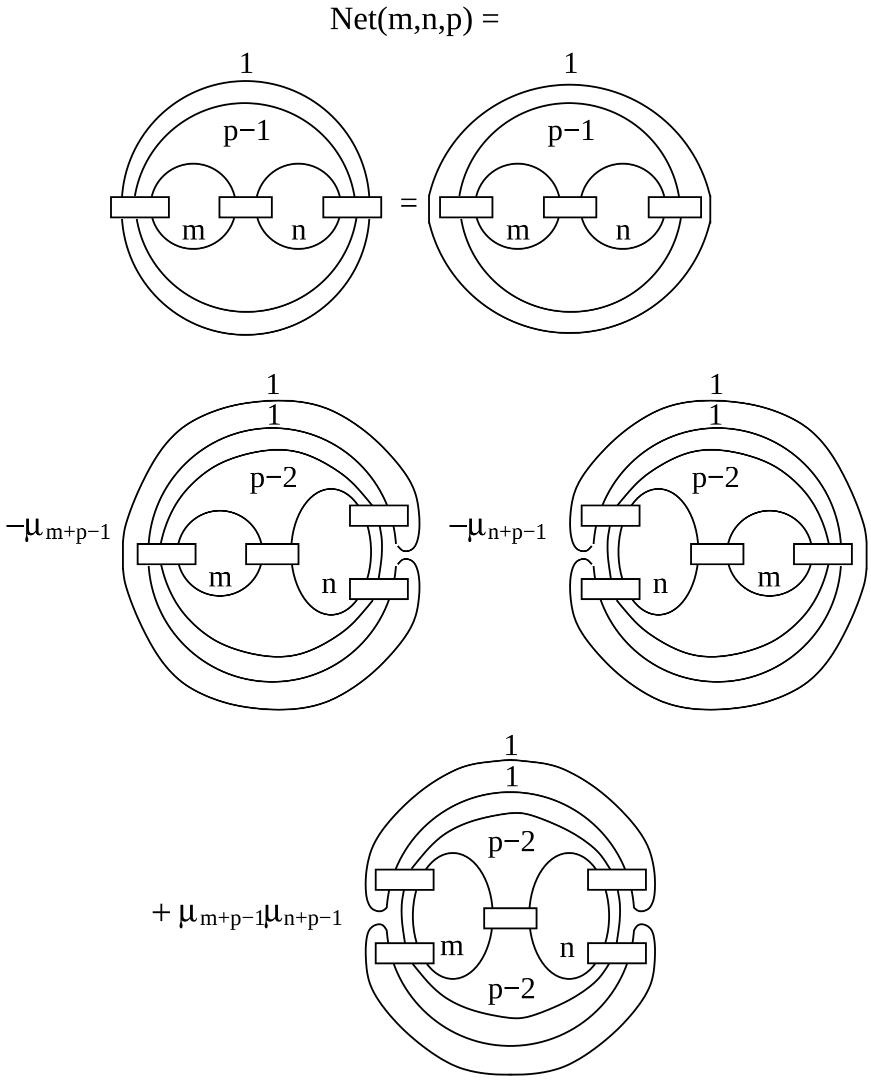


Figure G.41: Netmnp.

$$\begin{aligned}
Net(m,n,p) &= \frac{\alpha_{p-1}}{\beta_{p-1}}Net(m,n,p-1) + \frac{\beta_{p-2}}{\beta_{p-1}}Net(m,n,1,p-2) \\
&= (\frac{\alpha_{p-1}+\alpha_{p-2}}{\beta_{p-1}})Net(m,n,p-1) + \frac{\beta_{p-3}}{\beta_{p-1}}Net(m,n,2,p-3) \\
&= (\frac{\alpha_{p-1}+\alpha_{p-2}+\alpha_{p-3}}{\beta_{p-1}})Net(m,n,p-1) + \frac{\beta_{p-4}}{\beta_{p-1}}Net(m,n,2,p-4) \\
&= \cdots \\
&= \frac{1}{\beta_{p-1}}\left(\sum_{j=0}^{p-1}\alpha_j\right)Net(m,n,p-1) \qquad \text{(G.48)}
\end{aligned}$$

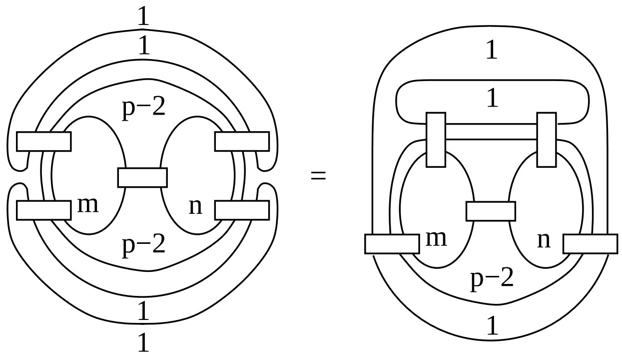


Figure G.42: Netmn1p-2. Equivalence of last network in fig (G.1.5) with $Net(m, n, 1, p-2)$.

Therefore

$$\rho(m,n,p) = \frac{1}{\beta_{p-1}} \sum_{j=0}^{p-1} \alpha_j. \tag{G.49}$$

Or

$$\begin{aligned}
\rho(m,n,p) &= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \big(-2\Delta_{m+j}\Delta_{n+j} - \Delta_{m+j-1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \Big((-2\Delta_{m+j} - \Delta_{m+j-1})\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\Big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \sum_{j=0}^{p-1} \Big(\Delta_{m+j+1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1}\Big) \\
&= \frac{1}{\Delta_{m+p-1}\Delta_{n+p-1}} \Big(\Delta_{m+1}\Delta_{n} - \Delta_{m}\Delta_{n-1} \\
&\qquad \Delta_{m+2}\Delta_{n+1} - \Delta_{m+1}\Delta_{n} \\
&\qquad \Delta_{m+3}\Delta_{n+2} - \Delta_{m+2}\Delta_{n+1} \\
&\qquad \cdots \\
&\qquad \Delta_{m+j+1}\Delta_{n+j} - \Delta_{m+j}\Delta_{n+j-1} \\
&\qquad \cdots \\
&\qquad \Delta_{m+p-1}\Delta_{n+p-2} - \Delta_{m+p-2}\Delta_{n+p-3} \\
&\qquad \Delta_{m+p}\Delta_{n+p-1} - \Delta_{m+p-1}\Delta_{n+p-2}\Big) \\
&= \frac{\Delta_{m+p}\Delta_{n+p-1} - \Delta_{m}\Delta_{n-1}}{\Delta_{m+p-1}\Delta_{n+p-1}}
\end{aligned} \tag{G.50}$$

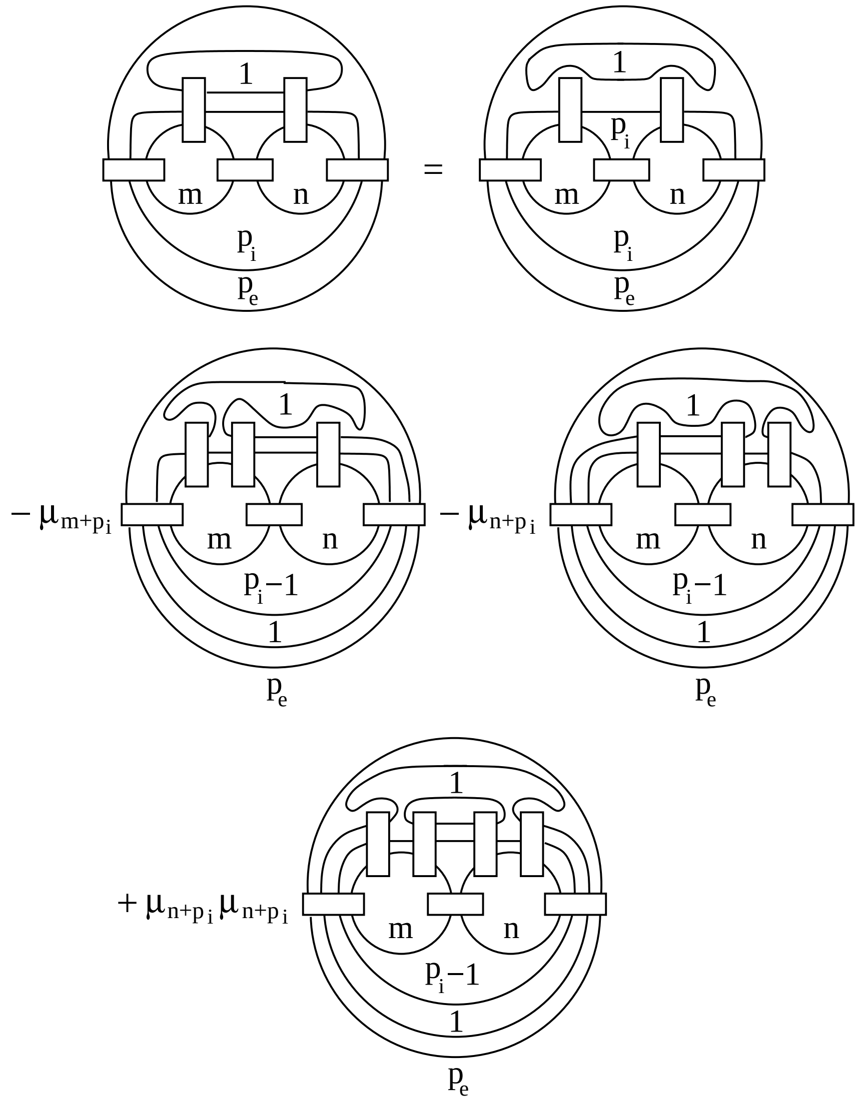


Figure G.43: Netmnpepi.

where we used $\Delta_{k+2} = -2\Delta_{k+1} - \Delta_k$. We can simplify futher,

$$
\begin{aligned}
\Delta_{m+p}\Delta_{n+p-1} - \Delta_m\Delta_{n-1} &= (-1)^{m+p}(m+p+1)(-1)^{n+p-1}(n+p) - (-1)^m(m+1)(-1)^{n-1}n \\
&= (-1)^{m+n+2p-1}[(m+p+1)(n+p) - (m+1)n] \\
&= (-1)^{m+n+2p-1}[np + (m+p+1)p] \\
&= (-1)^{m+n+p}(m+n+p+1)(-1)^{p-1}p \\
&= \Delta_{m+n+p}\Delta_{p-1} \qquad \text{(G.51)}
\end{aligned}
$$

Therefore

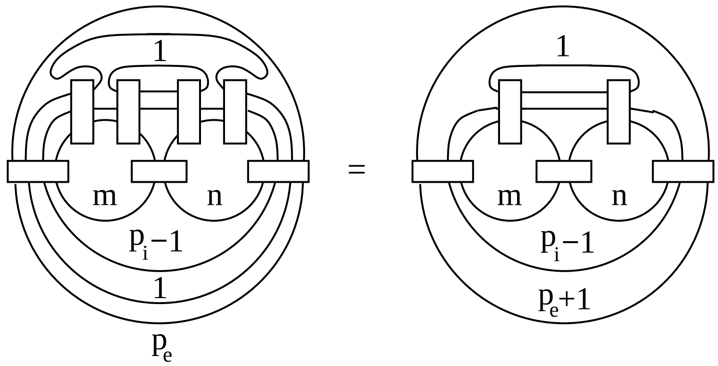


Figure G.44: Netmnpe+1pi-1.

$$\rho(m,n,p) = \frac{\Delta_{m+n+p}\Delta_{p-1}}{\Delta_{m+p-1}\Delta_{n+p-1}} \tag{G.52}$$

Denote

$$\Delta_n! := \Delta_n\Delta_{n-1}\Delta_{n-2}\cdots\Delta_1.$$

$$\begin{aligned}
\theta(a,b,c) &= \rho(m,n,p)Net(m,n,p-1) \\
&= \left(\prod_{j=1}^{p}\rho(m,n,j)\right)Net(m,n,0) \\
&= \left(\prod_{j=1}^{p}\rho(m,n,j)\right)\Delta_{m+n}
\end{aligned} \tag{G.53}$$

Hence, by (G.52)

$$\begin{aligned}
\theta(a,b,c) &= \prod_{j=1}^{p}\left[\frac{\Delta_{m+n+j}\Delta_{j-1}}{\Delta_{m+j-1}\Delta_{n+j-1}}\right]\Delta_{m+n} \\
&= \frac{\left(\Delta_{m+n+p}\Delta_{m+n+p-1}\cdots\Delta_{m+n}\right)\Delta_{p-1}!}{\left(\Delta_{m+p-1}\Delta_{m+p-2}\cdots\Delta_{m}\right)\left(\Delta_{n+p-1}\Delta_{n+p-2}\cdots\Delta_{n}\right)} \\
&= \frac{\Delta_{m+n+p}!\Delta_{n-1}!\Delta_{m-1}!\Delta_{p-1}!}{\Delta_{m+p-1}!\Delta_{n+p-1}!\Delta_{m+n-1}!}
\end{aligned} \tag{G.54}$$

The minus signs in the factorial

$$
\begin{aligned}
\Delta_{m+n+p}! &= (-1)^{m+n+p}(m+n+p+1)(-1)^{m+n+p-1}(m+n+p)\cdots(-1)^{1}2! \\
&= (-1)^{(m+n+p)+(m+n+p-1)+\cdots+1}(m+n+p+1)! \\
&= (-1)^{(m+n+p)(m+n+p+1)/2}(m+n+p+1)! \qquad (G.55)
\end{aligned}
$$

So that we get

$$
\begin{aligned}
\Delta_{m+n+p}! &= (-1)^{(m+n+p)(m+n+p+1)/2}(m+n+p+1)! \\
\Delta_{m+n-1}! &= (-1)^{(m+n-1)(m+n)/2}(m+n)! \\
\Delta_{m-1}! &= (-1)^{(m-1)m/2}m! \qquad (G.56)
\end{aligned}
$$

Collecting the exponents of (-1) in (G.54) is

$$
\begin{aligned}
&\frac{1}{2}\big[(m+n+p)(m+n+p+1)+(n-1)n+(m-1)m+(p-1)p+ \\
&+(m+p-1)(m+p)+(n+p-1)(n+p)+(m+n-1)(m+n)\big] \\
&=\frac{1}{2}\big[(m+n+p)^2+n^2+m^2+p^2 \\
&+(m+p)^2-(m+p)+(n+p)^2-(n+p)+(m+n)^2-(m+n)\big] \\
&=\frac{1}{2}\big[(m^2+n^2+p^2+2mn+2mp+2np)+n^2+m^2+p^2+ \\
&+2m^2+2n^2+2p^2+2mp+2np+2mn-2(m+n+p)\big] \\
&=2m^2+2n^2+2p^2+2mn+2np+2pm-m-n-p\equiv m+n+p\ (mod\ 2). \qquad (G.57)
\end{aligned}
$$

Therefore,

$$
\theta(a,b,c)=\frac{(-1)^{m+n+p}(m+n+p+1)!m!n!p!}{(m+n)!(n+p)!(m+p)!} \qquad (G.58)
$$

where

$$
\begin{aligned}
m &= \frac{a+b-c}{2}, \\
n &= \frac{b+c-a}{2}, \\
p &= \frac{a+c-b}{2}. \qquad (G.59)
\end{aligned}
$$

$$
\begin{aligned}
m + p &= 2a \\
m + n &= 2b \\
n + p &= 2c \\
m + n + p &= 2a + 2b + 2c.
\end{aligned}
\tag{G.60}
$$

TET

Recoupling formula

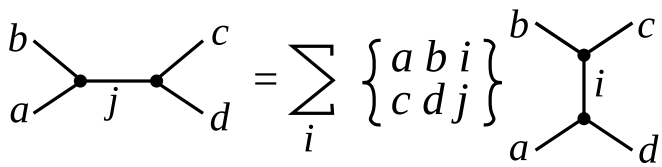


Figure G.45: recouplefig. The recoupling equation

The tetrahedron network.

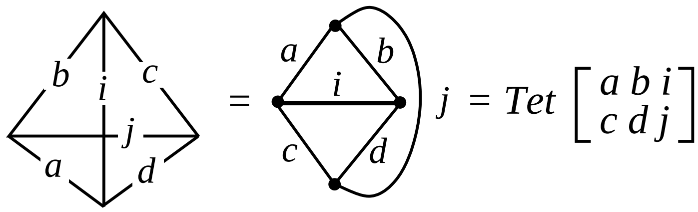


Figure G.46: TetDef.

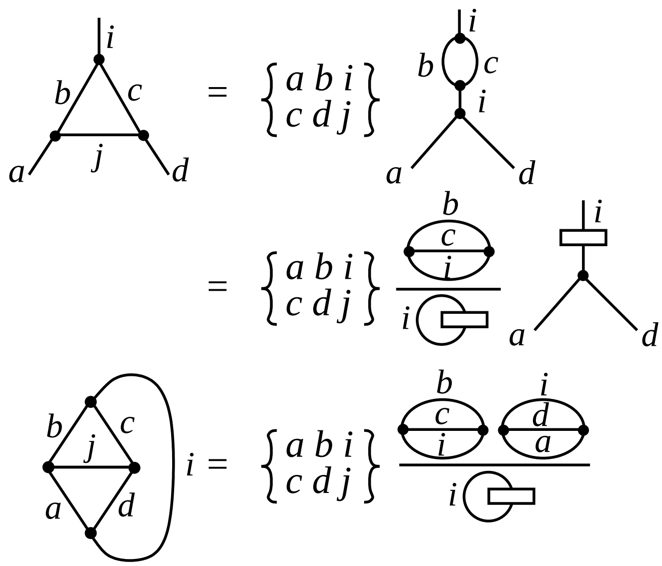


Figure G.47: 6jandTET.

The tetrahedron formula for recoupling theoery.

The evaluation of the tetrahedron network.

$$\left\{ \begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right\} = \frac{Tet \left[\begin{array}{ccc} a & b & i \\ c & d & j \end{array} \right] \Delta_i}{\theta(a,d,i)\theta(b,c,j)} \tag{G.61}$$

$$Tet \left[\begin{array}{ccc} A & B & E \\ C & D & F \end{array} \right] = \frac{\mathcal{I}}{\mathcal{E}} \sum_{m \le S \le M} \frac{(-1)^S (S+1)!}{\prod_i (S - a_i)! \prod_j (b_j - S)!} \tag{G.62}$$

where

$$
\begin{aligned}
a_1 &= \frac{A+D+E}{2}, & b_1 &= \frac{B+D+E+F}{2} \\
a_2 &= \frac{B+C+E}{2}, & b_2 &= \frac{A+C+E+F}{2} \\
a_3 &= \frac{A+B+F}{2}, & b_3 &= \frac{A+B+C+D}{2} \\
a_4 &= \frac{C+D+F}{2}, & & \\
m &= \max\{a_i\}, & M &= \min\{b_j\} \\
\mathcal{E} &= A!B!C!D!E!F!, & \mathcal{I} &= \prod_{ij}(b_j - a_i)!.
\end{aligned}
\tag{G.63}
$$

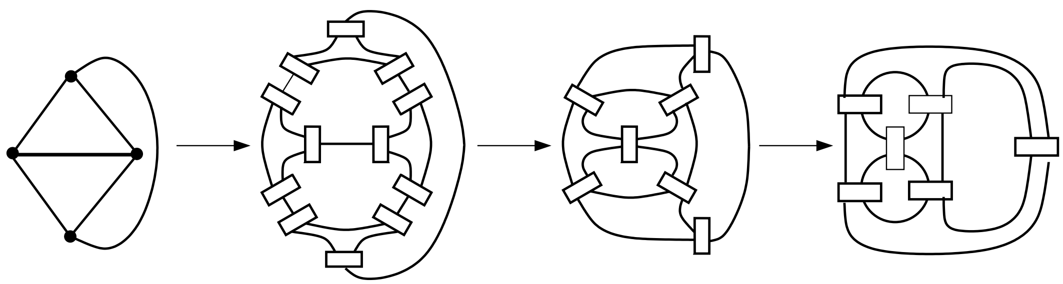


Figure G.48: Tetfig2.

The $6j-$sybols have a number of properties including the orthogonal identity

$$
\sum_l \left\{ \begin{array}{ccc} a & b & l \\ c & d & j \end{array} \right\} \left\{ \begin{array}{ccc} d & a & i \\ b & c & l \end{array} \right\} = \delta_i^j \tag{G.64}
$$

and the Biedenharn-Elliot or Pentagon identity

$$
\sum_l \left\{ \begin{array}{ccc} d & i & l \\ e & m & c \end{array} \right\} \left\{ \begin{array}{ccc} a & b & f \\ e & l & i \end{array} \right\} \left\{ \begin{array}{ccc} a & f & k \\ d & d & l \end{array} \right\} = \left\{ \begin{array}{ccc} a & b & k \\ c & d & i \end{array} \right\} \left\{ \begin{array}{ccc} k & b & f \\ e & m & c \end{array} \right\} \tag{G.65}
$$

The reduction formula

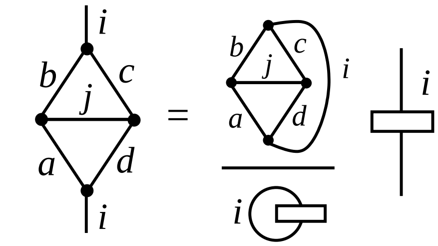


Figure G.49: reductfigs.

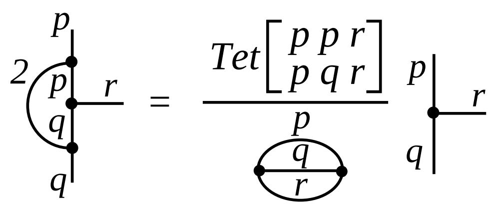


Figure G.50: reductfigs2.

Change of basis for 4-valent spin networks.

(1)

Answers:

Rotate the network on the RHS by clockwise and apply the recoupling identity again.

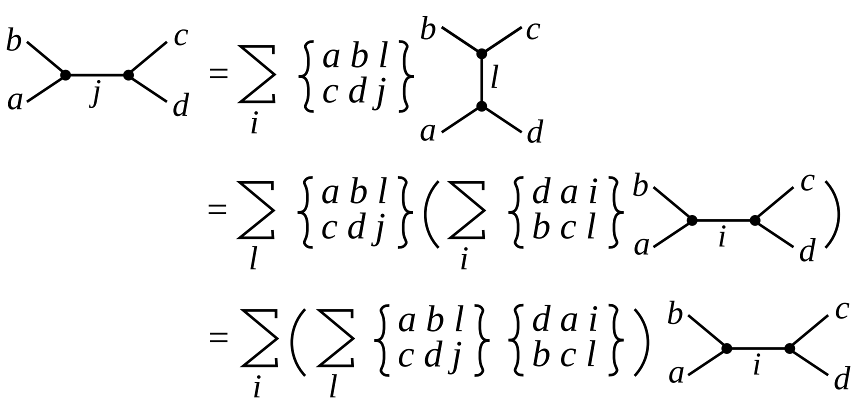


Figure G.51: recoupfig2. Proof of the orthogonality identity.

G.1.6 Angular Momentum Representation

The Pauli matrices are:

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{G.66}$$

The three matrices

$$\tau^{\mathbf{i}} = -\frac{i}{2}\hat{\sigma}^i \tag{G.67}$$

are the generators of $SU(2)$ in its fundamental representation.

$$\tau^{\mathbf{i}}\tau^{\mathbf{j}} - \tau^{\mathbf{j}}\tau^{\mathbf{i}} = \epsilon_{ijk}\tau^{\mathbf{k}} \tag{G.68}$$

Higher-order representations are generated from

$$\tau^{\mathbf{i}}_{(\mathbf{j})} = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2}\right) \otimes \cdots \otimes 1 \tag{G.69}$$

as these can be shown to satisfy

$$\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})} = \epsilon_{ijk}\tau^{\mathbf{k}}_{(\mathbf{j})} \tag{G.70}$$

$$\left| \frac{1}{2}\ \frac{1}{2} \right\rangle = u^A \quad \text{and} \quad d\left| \frac{1}{2}\ -\frac{1}{2} \right\rangle = d^A, \tag{G.71}$$

which diagrammatically represented

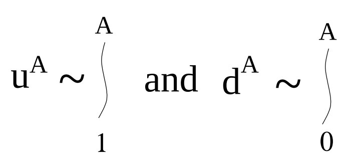


Figure G.52: The "u" stands for up and corresponds to index value $A = 1$. Likewise the "d" for down and corresponds to index value $A = 0$.

The "u" stands for up and corresponds to index value $A = 1$. Likewise the "d" for down and corresponds to index value $A = 0$. The inner product is given by linking upper and lower indices, for instance

For higher representations

$$| \, j \; m\rangle := | \, r \; s\rangle = N_{rs} u^{(A} u^{B} \ldots u^{C} \;\; d^{D} d^{E} \ldots d^{F)} \tag{G.72}$$

in which

$$N_{rs} \left(\frac{1}{r!s!(r+s)!} \right)^{1/2} \;, \quad j = \frac{r+s}{2}, \quad \text{and} \quad m = \frac{r-s}{2} \tag{G.73}$$

The parentheses in (G.73) around the indices indicate symmetrization, e.g. $u^{(A}u^{B)} = u^{A}u^{B} + u^{B}u^{A}$. The normalization ensures that the states are orthonormal in the usual inner product.

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \;\; \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \;\; \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{G.74}$$

with

$$\hat{S}_i = \frac{\hbar}{2} \hat{\sigma}^i \tag{G.75}$$

for $i = 1, 2, 3$. One has

$$\frac{\hat{\sigma}^3}{2} \left| \, \frac{1}{2} \; \frac{1}{2} \right\rangle = \frac{1}{2} \left| \, \frac{1}{2} \; \frac{1}{2} \right\rangle, \tag{G.76}$$

$$(\tau_{AB}) := (\sigma_A) \otimes (\rho_B) \tag{G.77}$$

$$(\tau_{AB}) = (\eta_A) \otimes (\rho_B) \neq (\rho_B) \otimes (\eta_A) = (\tau_{BA}) \tag{G.78}$$

$$(\hat{\sigma}_{AC} \otimes \hat{\sigma}_{BD})((\eta_C) \otimes (\rho_D)) = (\hat{\sigma}_{AC}\eta_C) \otimes (\hat{\sigma}_{BD}\rho_D) \tag{G.79}$$

The antisymmetric tensor is invariant under the action of $SU(2)$

$$U^A_{\;C} U^B_{\;D} \epsilon^{CD} = \epsilon^{AB} \tag{G.80}$$

(analogous to how the 3×3 tensor (δ_{ab} is invariant under the action of $SO(3)$ i.e. $O^c_{\;a} O^d_{\;b} \delta_{cd} = \delta_{ab}$, which preserves the scalar product between two vectors under rotation).

Contracting this equation with ϵ_{AB} we obtain the condition that the determinate of U is one

$$\det U = \frac{1}{2}\epsilon_{AC}\epsilon^{BD}U^A{}_B U^C{}_D = 1 \tag{G.81}$$

since

$$\epsilon_{AB}\epsilon^{AB} = 2. \tag{G.82}$$

The inverse of an $SU(2)$ matrix can be written as

$$(U^{-1})^A{}_B = -\epsilon_{BD}U^D{}_C\epsilon^{CA}. \tag{G.83}$$

Recall the anti-symmetric tensors (G.1) are used for raising or lowering but must be careful about the down-left-up-right rule:

$$\begin{aligned} \eta^A &= \epsilon^{AB}\eta_B \\ \zeta_A &= \zeta^B\epsilon_{BA}. \end{aligned} \tag{G.84}$$

We have the identity

$$\frac{1}{2}\sum_{i=1}^{3}\hat{\sigma}_i{}^B_A\hat{\sigma}_i{}^D_C = \frac{1}{2}(\epsilon_{AC}\epsilon^{BD} - \delta^D_A\delta^B_C) \tag{G.85}$$

We check this by direct calculation, in reference to (G.74). There are 16 possibilities in total.

$A = C$, $B \neq D$:

$$\begin{aligned} \delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^0{}_1 &= 0 \\ \delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^0{}_0 &= 0 \\ \delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^1{}_1 &= 0 \\ \delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^1{}_0 &= 0 \end{aligned} \tag{G.86}$$

$B = D$, $A \neq C$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^1{}_0 &= 0\\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^0{}_0 &= 0\\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^1{}_1 &= 0\\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^0{}_1 &= 0
\end{aligned}
\tag{G.87}
$$

$A = C$ and $B = D$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^0{}_0 &= 1\\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^0{}_1 &= 0\\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^1{}_0 &= 0\\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^1{}_1 &= 1
\end{aligned}
\tag{G.88}
$$

$A \neq C$ and $B \neq D$:

$$
\begin{aligned}
\delta^{ij}\sigma_i{}^0{}_0\sigma_j{}^1{}_1 &= -1\\
\delta^{ij}\sigma_i{}^0{}_1\sigma_j{}^1{}_0 &= 2\\
\delta^{ij}\sigma_i{}^1{}_0\sigma_j{}^0{}_1 &= 2\\
\delta^{ij}\sigma_i{}^1{}_1\sigma_j{}^0{}_0 &= -1
\end{aligned}
\tag{G.89}
$$

All 16 possible answers have been shown to be in accordance with (G.85).

The identity (G.85) is expressed diagrammatically as in fig G.53

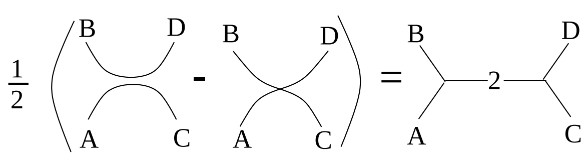
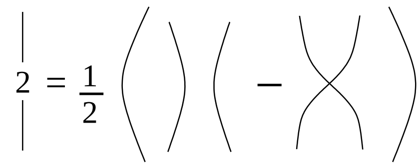


Figure G.53: The identity: $\frac{1}{2}(\epsilon_{AC}\epsilon^{BD} - \delta_A^D\delta_C^B) = \frac{1}{2}\sum_{i=1}^3 \hat{\sigma}_i{}^B{}_A\hat{\sigma}_i{}^D{}_C$.

Edges may be further joined into networks by making use of internal trivalent vertices

G.1.7 Recoupling Theory: Combinatorics of Angular Momentum

The rules of addition of angular momentum are known as recoupling theory



G.1.8 Loop states and the spin network states

$$\psi_\alpha(A) = -\mathrm{Tr}U_\alpha(A) \tag{G.90}$$

Proof: The orthogonality relations for loop-network states on a given graph γ follow from basic group integration theory. By the Peter-Weyl theorem we have

$$< S, S >= \tag{G.91}$$

where we have used that the non-equivalent irreducible - as well as our choice of equivalent - representations of a compact group are orthogonal, that is, $\pi(1)$ is a projector.

We can thus follow the contraction along the graph, obtaining a sequence of the edges $(e_1, e_2, \dots)$. Since the graph is finite, the sequence must close on itself.

The completeness of these states for $L_2(\mathcal{A}/\mathcal{G}, d\mu_{0,\gamma})$ follows also from the Peter-Weyl theorem together with a gauge-invariance argument:

hep-th/9601105 19th Jan 1996

$$|\Psi_\alpha|^2 = \int dU |\mathrm{Tr}U|^2 = 1 \tag{G.92}$$

$$\Psi_{[\alpha]}[A] = \Psi_{\alpha_1}[A] \dots \Psi_{\alpha_n}[A] = \mathrm{Tr}H(A, \alpha_1) \dots \mathrm{Tr}H(A, \alpha_n) \tag{G.93}$$

The function

$$\Psi(\alpha) = \int d\mu_0[A] \mathrm{Tr}\mathcal{P}e^{\oint_\alpha A} \Psi[A] \tag{G.94}$$

G.1.9 Summary of binary calculus and recoupling theory

The dashed circle is a magnification of the dot in the diagram on the left. Such dashed lines indicate spin network structures at a point. The internal labels i, j, k are positive integers determined by the external labels a, b, c via

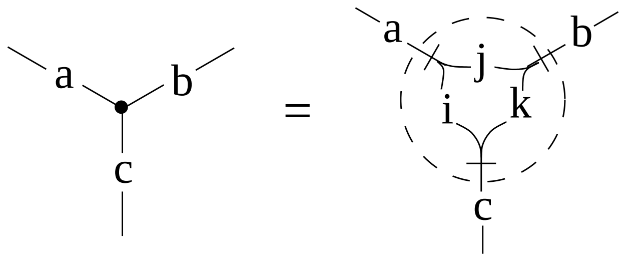
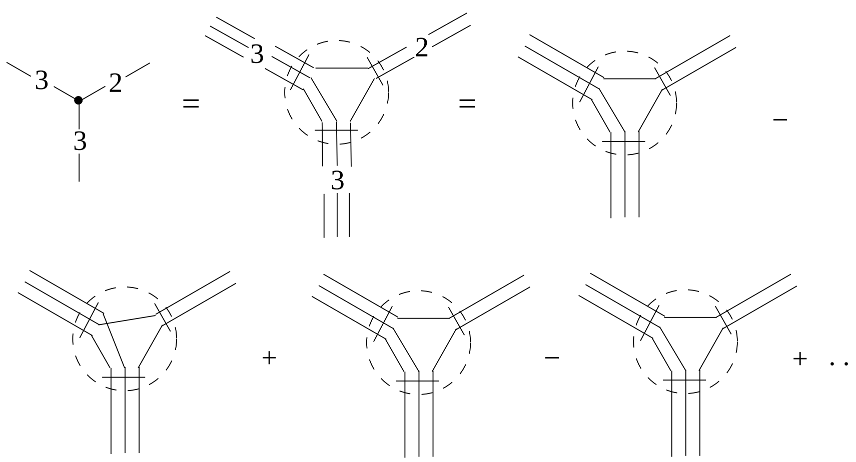


Figure G.54: Elementary recoupling.

$$a = i + j, \quad b = j + k, \quad c = i + k \tag{G.95}$$

or

$$i = (a + c - b)/2, \quad j = (a + b - c)/2, \quad k = (b + c - a)/2. \tag{G.96}$$



as can be seen by solving the simultaneous equations (G.95), or by drawing the strands through the vertex (see fig()). As in quantum mechanics the external labels satisfy the triangular inequalities

$$a + b \geq c, \quad b + c \geq a, \quad a + c \geq b \tag{G.97}$$

and the sum $a + b + c$ is an even integer (see Eq.(G.95)).

• Jones-Wenzl projectors.

• q-deformed binary calculus.

G.1.10 The Spectrum of the Area Operator

$$\mathcal{A}[S] = \int_S d^2\sigma \sqrt{g^S} = \int_S d^2\sigma \sqrt{n_a n_b \tilde{E}^a_i \tilde{E}^b_j} \tag{G.98}$$

We need to turn express $\mathcal{A}[S]$ in terms of loop variables, which can be accomplished through a limiting procedure.

$$U(\gamma, A) = \mathcal{P} \exp \left\{ \int_\gamma A_a(\gamma(s)) \frac{d\gamma^a}{ds} ds \right\} \tag{G.99}$$

$$\frac{\delta}{\delta A^i_a(x)} U(\gamma, A) = \int_\gamma ds\ \dot{x}^a(s)\ \delta^3(\gamma(s), x)\ U(\gamma(0,s), A))\ \tau^i\ U(\gamma(s,1), A)). \tag{G.100}$$

$$\tilde{E}(S) = \sqrt{\tilde{E}(S)^i \tilde{E}(S)^i}. \tag{G.101}$$

Acting on a state $\mathbf{\Psi_s}$, that intersects Σ only once, it gives

$$\begin{aligned} E^2(\Sigma)\mathbf{\Psi_s}(A) &= -\mathbf{\Psi_{s-\gamma}}^{lm}(A) \Big(j(U[\gamma(0,s), A)]\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)} j(U[\gamma(s,\gamma), A)) \Big)_{lm} \\ &= j(j+1)\mathbf{\Psi_s}(A) \end{aligned} \tag{G.102}$$

Since one has for the Casimir operator

$$\tau^{(j)}_i \tau^{(j)}_i = -j(j+1)\mathbf{1} \tag{G.103}$$

Thus,

$$E(\Sigma)\mathbf{\Psi_s}(A) = \sqrt{j(j+1)}\mathbf{\Psi_s}(A) \tag{G.104}$$

$$\hat{A}_S|s> \frac{G}{4c^3} \sum_i \hat{O}_i^{1/2}|s> \tag{G.105}$$

$$\begin{aligned} \hat{O}_e|s> &= -\hat{J}^2|s> \\ &= -\hbar^2 \frac{n^2}{2} \end{aligned} \tag{G.106}$$

$$\begin{aligned}\theta(n,n,1) = Net(n-1,1,1) &= (d-\mu_{n-1}-\mu_1)\Delta_n \\ &= (-2+\frac{n}{n+1}\frac{1}{2})(-1)^n(n+1) \qquad \text{(G.107)}\end{aligned}$$

$$\frac{\theta(n,n,1)}{\Delta_n} = (-2+\frac{n}{n+1}\frac{1}{2}) = \left(-\frac{n+2}{2n}\right) \qquad \text{(G.108)}$$

Now the diagram may be reduced using the recoupling identities. The bubble may be extracted with ()

$$\begin{aligned}\hat{O}_e|s> &= -\hbar^2\frac{n^2}{2}|(s-e)> \\ &= -\hbar^2\frac{n^2}{2}\frac{\theta(n,n,2)}{\Delta_n}|(s-e)> \\ &= -\hbar^2\frac{n^2}{2}\left(-\frac{n+2}{2n}\right)|s> \\ &= -\hbar^2\frac{n(n+1)}{4}|s> \qquad \text{(G.109)}\end{aligned}$$

Details J

$$j(U[\gamma(s,\gamma)])^{A'B'\ldots D'}_{AB\ldots D} = U[\gamma(s,\gamma)]^{(A'}_{A}U[\gamma(s,\gamma)]^{B'}_{B}\ldots U[\gamma(s,\gamma)]^{C')}_{C} \qquad \text{(G.110)}$$

$$\delta^D_{A'}\delta^E_{B'}\ldots\delta^F_{C'}\; j(U[\gamma(s,\gamma)])^{A'B'\ldots D'}_{AB\ldots C} = \delta^D_{(A'}\delta^E_{B'}\ldots\delta^F_{C')}\; j(U[\gamma(s,\gamma)])^{A'B'\ldots D'}_{AB\ldots D} \qquad \text{(G.111)}$$

$$\begin{aligned}j(U[\gamma(s,\gamma)])^{DE\ldots F}_{AB\ldots D} &= \delta^D_{(0}\delta^E_{0}\ldots\delta^F_{0)}\; j(U[\gamma(s,\gamma)])^{00\ldots 0}_{AB\ldots C} + \delta^D_{(1}\delta^E_{0}\ldots\delta^F_{0)}\; j(U[\gamma(s,\gamma)])^{10\ldots 0}_{AB\ldots C} + \ldots \\ &+ \delta^D_{(1}\delta^E_{1}\ldots\delta^F_{1)}\; j(U[\gamma(s,\gamma)])^{11\ldots 1}_{AB\ldots C} \qquad \text{(G.112)}\end{aligned}$$

or

$$\begin{aligned}j(U[\gamma(s,\gamma)])^{DE\ldots F}_{AB\ldots D} &= \omega^{DE\ldots F}(i=0)\; j(U[\gamma(s,\gamma)])^{00\ldots 0}_{AB\ldots C} + \omega^{DE\ldots F}(i=1)\; j(U[\gamma(s,\gamma)])^{10\ldots 0}_{AB\ldots C} + \ldots \\ &+ \omega^{DE\ldots F}(i=2j)\; j(U[\gamma(s,\gamma)])^{11\ldots 1}_{AB\ldots C} \qquad \text{(G.113)}\end{aligned}$$

$$\left(\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}\right)^{AB...C}_{DE...F}\ \omega^{DE...F}(i=0,1,\ldots,2j) = j(j+1)\omega^{AB...C}(i=0,1,\ldots,2j) \tag{G.114}$$

Hence

$$\left(\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}\right)^{AB...C}_{DE...F}\ j(U[\gamma(s,\gamma)])^{DE...F}_{A'B'...C'} = j(j+1)\ j(U[\gamma(s,\gamma)])^{AB...C}_{A'B'...C'} \tag{G.115}$$

or

$$\tau^{\mathbf{i}}{}_{(j)}\tau^{\mathbf{i}}{}_{(j)}\ j(U[\gamma(s,\gamma)]) = j(j+1)\ j(U[\gamma(s,\gamma)]) \tag{G.116}$$

Quotations to do with important conceptual points in Rovelli's paper which I have attempted to cover in this last section.

gr-qc/9806079 Loop quantum gravity and quanta of space: a primer

$$\theta(a,b,c) = \frac{(-1)^{m+n+p}(m+n+p+1)m!n!p!}{a!b!c!} \tag{G.117}$$

where $m=(a+b-c)/2$, $m=(b+c-a)/2$, $p=(c+a-b)/2$,

G.1.11 The Spectrum of the Volume Operator

$$V[\sigma]_\epsilon = \int_\sigma \sqrt{g} = \int \sqrt{\epsilon_{abc}\epsilon_{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k} \tag{G.118}$$

$\hat{W}_{[rst]} =$

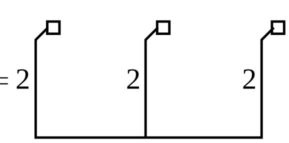


Figure G.55: grasp3edge.

Since the "comb" basis spans the space of all intertwiners, we can write the action of the W operator as sending the original vertex to a superposition of other vertices in the same basis:

$$\sum_{k_2,\ldots,k_{n-2}} \hat{W}_{[rst]i_2,\ldots,i_{n-1}}^{\quad k_2,\ldots,k_{n-1}}$$

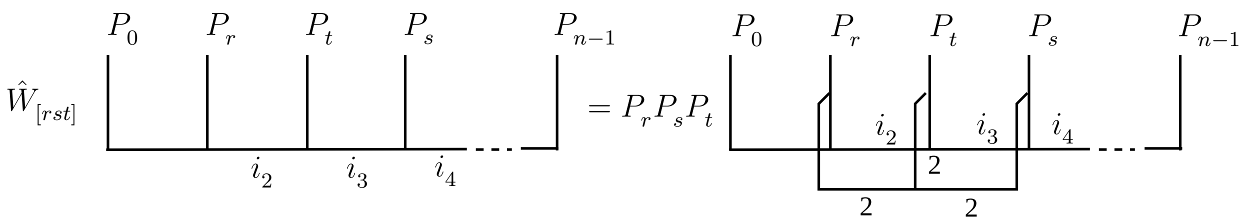


Figure G.56: grasp3edge2.

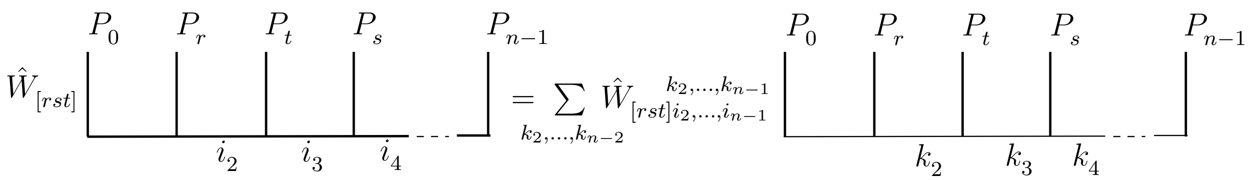


Figure G.57: grasp3edge3.

$$W^{(n)\ k_2\cdots k_{n-2}}_{[rst]\ i_2\cdots i_{n-2}} = -P_r P_s P_t \begin{Bmatrix} k_2 & P_t & k_3 \\ i_2 & P_t & i_3 \\ 2 & 2 & 2 \end{Bmatrix} (\lambda^{i_2}_{k_2})^2 \delta^{k_4}_{i_4} \cdots \delta^{k_{n-2}}_{i_{n-2}}$$
$$\times \frac{Tet \begin{bmatrix} P_r & P_r & P_0 \\ k_2 & i_2 & 2 \end{bmatrix} Tet \begin{bmatrix} P_s & P_s & k_4 \\ k_3 & i_3 & 2 \end{bmatrix} \Delta_{k_2}\Delta_{k_3}}{\theta(k_2,i_2,2)\theta(k_3,i_3,2)\theta(P_0,P_r,k_2)\theta(k_2,k_3,P_t)\theta(k_3,k_4,P_s)} \quad \text{(G.119)}$$

Using (G.61), this can be expressed in terms of the Kauffman-Lins 6-j symbols

$$W^{(n)\ k_2\cdots k_{n-2}}_{[rst]\ i_2\cdots i_{n-2}} = -P_r P_s P_t \begin{Bmatrix} k_2 & P_t & k_3 \\ i_2 & P_t & i_3 \\ 2 & 2 & 2 \end{Bmatrix} (\lambda^{i_2}_{k_2})^2 \delta^{k_4}_{i_4} \cdots \delta^{k_{n-2}}_{i_{n-2}}$$
$$\frac{\begin{Bmatrix} P_r & P_r & P_0 \\ k_2 & i_2 & 2 \end{Bmatrix} \begin{Bmatrix} P_s & P_s & k_4 \\ k_3 & i_3 & 2 \end{Bmatrix}}{\theta(P_0,P_r,k_2)} \quad \text{(G.120)}$$

In both these formulas we have used the 9-j symbol, which is given by the spin network fig.(G.58)

The eigenvalues of the volume operator are then proportional to the sum of the absolute values of the W-eigenvalues:

$$\hat{V} = \sqrt{\sum_{0\le r<s<t\le n-1} \left|\frac{i}{16}\hat{W}_{[rst]}\right|} \quad \text{(G.121)}$$

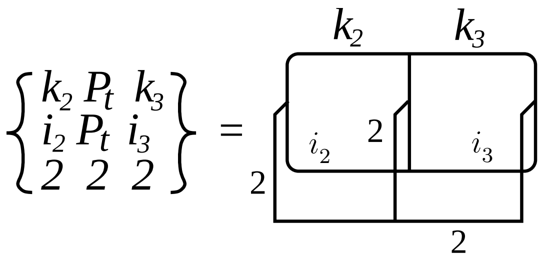


Figure G.58: 9jsymbolFig.

$$W_{[012]i}^{\;k} = \frac{P_0 P_1 P_2 \begin{Bmatrix} P_0 & P_1 & k \\ P_0 & P_1 & i \\ 2 & 2 & 2 \end{Bmatrix} \Delta_j}{\theta(P_0, P_1, j)\theta(P_2, P_3, j)\theta(i, j, 2)} \tag{G.122}$$

G.1.12 Reidermeister Moves

Remarkably, a knot in three dimensional space can be continuously deformed into another knot, if and only if, the planar projection of the knots can be transformed into each other via a sequence of four moves called the "Reidermeister moves" [76].

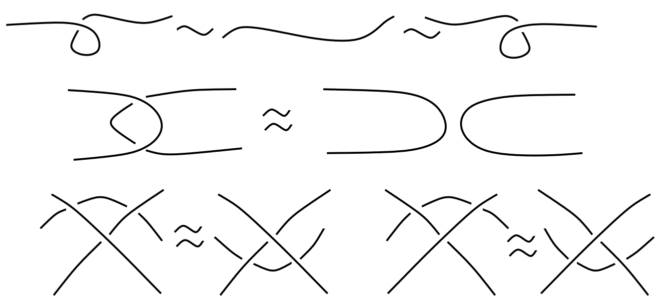


Figure G.59: Reidemeister moves.

G.1.13 Kauffman Bracket

Quantum $SU(2)$ group.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{G.123}$$

$$U \tilde{\epsilon} U^T = \tilde{\epsilon} \tag{G.124}$$

$$
\begin{aligned}
&ba = qab \ \ dc = qcd \\
&ca = qac \ \ db = qbd \\
&bc - cb = 0 \\
&ad - da = -(q - q^{-1})bc \\
&ad - q^{-1}bc = 1
\end{aligned}
\tag{G.125}
$$

where $q = aA^2$. complex non-commuting components a, b, c, d.

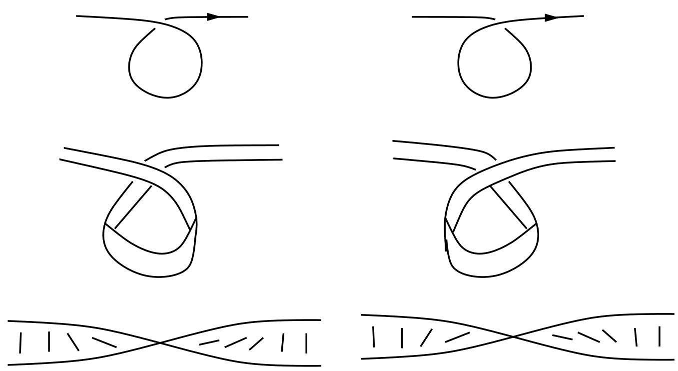


Figure G.60: Twist and q-deformed $su(2)$ - $su_q(2)$ (or quantum group of $su(2)$).

G.1.14 The Braid Group B_N

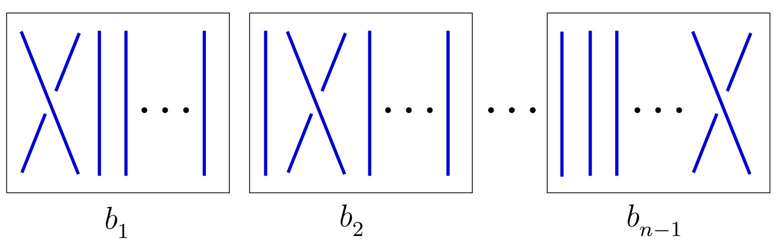


Figure G.61: braidgroup. The generators of the braid group.

The n-stranded braid group

$$
\begin{cases} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & \text{for } \ 1 \leq i < n \\ b_i b_j = b_j b_i & \text{for } \ |i - j| \geq 2 \end{cases}
\tag{G.126}
$$

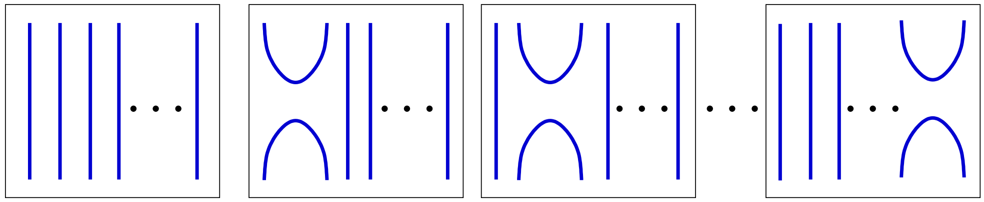


Figure G.62: Tempalgebra. The generators of the Temperley-algebra.

G.1.15 Temperley-Lieb Algebra

d is the value assigned to a closed loop

$$\begin{cases} U_i^2 = dU_i \\ U_i U_{i\pm 1} U_i = U_i \\ U_i U_j = U_J U_i \qquad \text{for } \ |i-j| \geq 1 \end{cases} \tag{G.127}$$

Proposition G.1.2 *If $g_n \in T_n$ denotes the image of ... in the Temperley-Lieb algebra T_n, then*

(i) $g_n^2 = g_n$

(ii) $g_n U_i = U_i g_n = 0$ for all $i = 1, 2, \ldots, n-1$.

The canonical construction

$$g_n = \frac{1}{\{n\}!} \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)} \hat{\sigma} \tag{G.128}$$

$$\{n\}! := \sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)} \tag{G.129}$$

$$\{n\}! := \prod_{k=1}^{n} \frac{1 - A^{-4k}}{1 - A^{-4}} \tag{G.130}$$

$$\{2\}! = 1 + A^{-4} = \frac{1 - (A^{-4})^2}{1 - A^{-4}} \tag{G.131}$$

From fig.(G.65). From the we have first row is $1 + A^{-4k} + (A^{-4k})^2$, while the second row is the same thing but multiplied by A^{-4k}. This is obviously because, while in the first row the blue lines are not crossing, in the second they are crossing once. Hence,

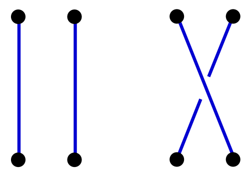


Figure G.63: 2tangEx.

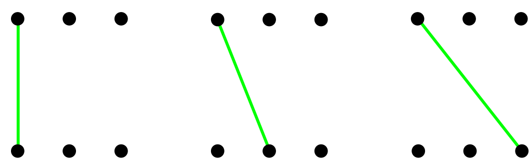


Figure G.64: posBraids.

$$
\begin{aligned}
\{3\}! &= \big(1 + A^{-4k} + (A^{-4k})^2\big) + A^{-4}\big(1 + A^{-4k} + (A^{-4k})^2\big) \\
&= \underbrace{(1 + A^{-4})}_{\text{blue line factor}} \underbrace{\big(1 + A^{-4k} + (A^{-4k})^2\big)}_{\text{green line factor}} \\
&= \{2\}! \cdot \frac{1 - (A^{-4})^3}{1 - A^{-4}}
\end{aligned} \tag{G.132}
$$

Similar reasoning applies to $\{4\}!$,

$$
\begin{aligned}
\{4\}! &= \{3\}! \cdot \big(1 + A^{-4k} + (A^{-4k})^2 + (A^{-4k})^3\big) \\
&= \{3\}! \cdot \frac{1 - (A^{-4})^4}{1 - A^{-4}}
\end{aligned} \tag{G.133}
$$

and so on,

$$
\begin{aligned}
\{r+1\}! &= \{r\}! \cdot \big(1 + A^{-4k} + (A^{-4k})^2 + \cdots + (A^{-4k})^r\big) \\
&= \{r\}! \cdot \frac{1 - (A^{-4})^{r+1}}{1 - A^{-4}}
\end{aligned} \tag{G.134}
$$

□

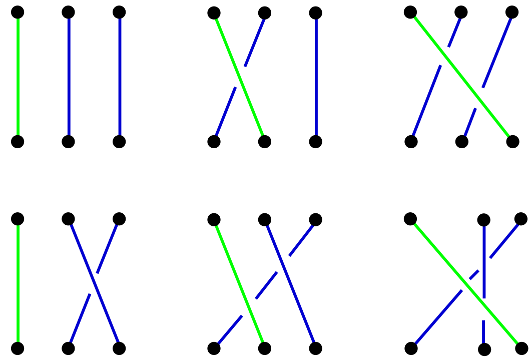


Figure G.65: canconconstr.

Proof of properties

Given any $i \in \{1, 2, \ldots, n-1\}$ choose the set braids W that do not end in σ_i or σ_i^{-1}, as is easy to see from the canonical construction, the remaining braids in $\{n\}!g_n$ are given by the set $W' = \{w\sigma_i \mid w \in W\}$ (note that the choice of the set W ensures the minimality of the braids in W'). So that we have

$$\{n\}!g_n = \sum_{w \in W} (A^{-3})^{t(w)} w + (A^{-3})^{t(w)+1} w\sigma_i, \tag{G.135}$$

we have given examples in fig.(G.66). Since $w\sigma_i U_i = (-A^{-3})wU_i$ in T_n, it follows that $g_n U_i = 0$ for $i = 1, 2, \ldots, n-1$.

$\sum_{w \in W} (A^{-3})^{t(w)} w =$

$\sum_{w \in W} (A^{-3})^{t(w)} w\sigma_1 =$

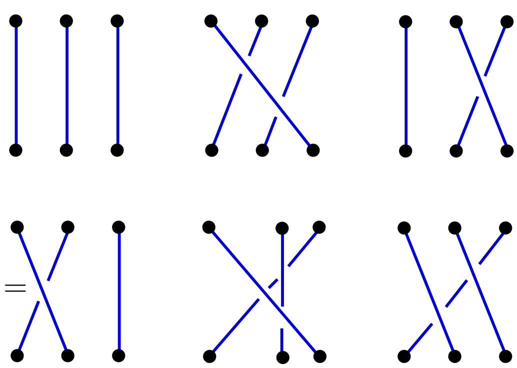


Figure G.66: ProofpropQ1.

it is claimed that the coefficient of 1_n is $\{n\}!$

$$\tilde{g}_n := \sum_{\sigma} (A^{-3})^{t(\sigma)} \hat{\sigma}$$

$\sum_{w \in W} (A^{-3})^{t(w)} w =$

$\sum_{w \in W} (A^{-3})^{t(w)} w \sigma_2 =$

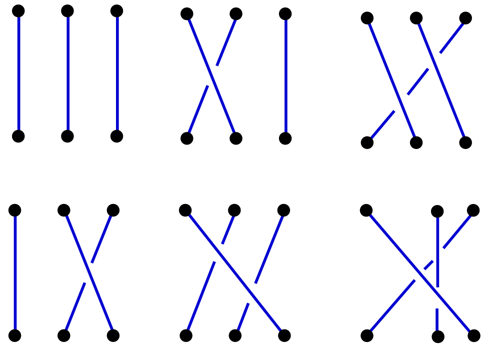


Figure G.67: ProofpropQ2.

$$\hat{\sigma} = (A^{-1})^{t(\sigma)} 1_n + \ldots$$

$$\hat{g}_n := \sum_{\sigma \in S_n} (A^{-3})^{t(\sigma)} \left[(A^{-1})^{t(\sigma)} 1_n + \ldots \right] + \cdots = \{n\}! 1_n + \ldots$$

Hence the coefficient of 1_n in $\tilde{g}_n$ is the sum $\sum_{\sigma \in S_n} (A^{-4})^{t(\sigma)} = \{n\}!$.

□

G.1.16 The twist move

$$\lambda_c^{ab} = (-1)^{(a+b-c)/2} A^{[a(a+2)+(b(b+2)+(c(c+2)]/2} \tag{G.136}$$

G.1.17 q-deformed Recoupling Theory

We will modify our spin network technology slightly to make there be only finitely many vector spaces "j".

q-deformed graphs are ribbon (framed) graphs with braiding. Thus, any undeformed spin network has to be supplemented with information about twists and crossings before evaluation.

[PDF] SPIN NETWORKS AND THE BRACKET POLYNOMIAL File Format: PDF/Adobe Acrobat - View as HTML LC Jeffrey, Chern-Simons-Witten invariants of lens spaces and torus bundles, and the ... E. Witten, Quantum field theory and the Jones polynomial, Comm. ...

We will modify the binor identity.

We would like to pick A and B so that the Reidemeister moves are preserved.

Note: the first Reidemeister move didn't hold before!

We'll call it d. So we want:

$$A^2 + A^{-2} + d = 0 \tag{G.137}$$

This tells us $d = -(A^2 + A^{-2})$.

We want identity:

Which implies $A = B$. $d = -2$

Applications:

(i) Theory with non-zero cosmological constant seem to require the use of q-deformed spin networks.

(ii) q-spin networks are manifold invariants making them usefull in mathematical investigations in topology.

Smolin and Markoppoulou have used this to define abstract states of quantum gravity which encode the topology in the quantum state itself []. Smolin has developed a very tentative formulation of M-theory, arguing that topology change that is needed for mirror symmetry in string theory []. Smolin has also shown that perturbations in the q-deformed theory look very much like propagating strings [].

(iii)

(v) Infrared regularization in .spin foam state sum models.

G.1.18 Jones Polynomial

$B = A^{-1}$ we also need $A^2 + C + A^{-1} = 0$ so

$$C = -A^2 - A^{-2} \tag{G.138}$$

G.2 Spatialy Diffeomorphism Invariant Space

$$f^a(\alpha(t(s))) = \beta^a(s) \tag{G.139}$$

G.3 General Madlestam Identities

Mandelstam identiies of the first kind are a simple consequence of the cyclic property of matrices, ($Tr(\mathbb{A}\mathbb{B}) = Tr(\mathbb{B}\mathbb{A})$ for matrices $\mathbb{A}$ and $\mathbb{B}$),

$$W(\gamma_1 \circ \gamma_2) = W(\gamma_2 \circ \gamma_1). \tag{G.140}$$

This holds for any gauge group of any dimension.

There are various identities of the second kind. Here is an example. First note that the product of $N+1$ δ's of dimension$N$ and anti-symmetrised indices is identically zero,

$$\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_{N+1}}_{B_{N+1}]} = 0. \tag{G.141}$$

Then contract this with

$$H(\gamma_1)_{A_1}{}^{B_1}\ldots H(\gamma_{N+1})_{A_{N+1}}{}^{B_{N+1}}. \tag{G.142}$$

The result is an identically vanishing sum of products of the traces of products of holonomies.

For example for $N=1$

$$\begin{aligned} 0 &\equiv 2\sum_{A_1,B_1,A_2,B_2=1}^{1} \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2]}H(\gamma_1)_{A_1}{}^{B_1}H(\gamma_2)_{A_2}{}^{B_2} \\ &= \sum_{A_1,B_1,A_2,B_2=1}^{1} (\delta^{A_1}_{B_1}\delta^{A_2}_{B_2} - \delta^{A_1}_{B_2}\delta^{A_2}_{B_1})H(\gamma_1)_{A_1}{}^{B_1}H(\gamma_2)_{A_2}{}^{B_2} \\ &= W(\gamma_1)W(\gamma_2) - W(\gamma_1 \circ \gamma_2). \end{aligned} \tag{G.143}$$

There is a compact way of writing this identity for an arbitrary order in terms of the quantities

$$M_K := \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_K]}H(\gamma_1)_{A_1}{}^{B_1}\ldots H(\gamma_K)_{A_K}{}^{B_K}. \tag{G.144}$$

We show below that M_K satifies the following recursive relation

$$
\begin{aligned}
(K+1)M_{K+1}(\gamma_1,\ldots,\gamma_{K+1}) &= W(\gamma_{K+1})M_K(\gamma_1,\gamma_2,\ldots,\gamma_K) - \cdots \\
&\quad \cdots - M_K(\gamma_1 \circ \gamma_{K+1},\gamma_2,\ldots,\gamma_K) \\
&\quad \cdots - M_K(\gamma_1 \ldots \gamma_i \circ \gamma_{K+1},\ldots,\gamma_K) \\
&\quad \cdots - M_K(\gamma_1,\gamma_2,\ldots,\gamma_K \circ \gamma_{K+1})
\end{aligned} \tag{G.145}
$$

with

$$
M_1(\gamma) = W(\gamma). \tag{G.146}
$$

In terms of the Ms, the identity for an $N \times N$ matrix group can be written as

$$
M_{N+1}(\gamma_1,\ldots,\gamma_{N+1}) = 0. \tag{G.147}
$$

Let us now derive the recursive relation. For $K = 2$ this comes from the identity

$$
2\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2]} = 2\frac{1}{2}(\delta^{A_1}_{B_1}\delta^{A_2}_{B_2} - \delta^{A_1}_{B_2}\delta^{A_2}_{B_1}) \tag{G.148}
$$

As can be seen by contracting this with $H(\gamma_1)_{A_1}{}^{B_1}H(\gamma_1)_{A_2}{}^{B_2}$ resulting in

$$
2M_2(\gamma_1,\gamma_2) = W(\gamma_1)M_1(\gamma_2) - M_1(\gamma_1 \circ \gamma_2). \tag{G.149}
$$

For $K = 3$ we have

$$
\begin{aligned}
3\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\delta^{A_3}_{B_3]} &= 3\frac{1}{3!}(\delta^{A_1}_{B_1}\delta^{A_2}_{B_2}\delta^{A_3}_{B_3} - \delta^{A_1}_{B_3}\delta^{A_2}_{B_2}\delta^{A_3}_{B_1} - \delta^{A_1}_{B_1}\delta^{A_2}_{B_3}\delta^{A_3}_{B_2} \\
&\quad - \delta^{A_1}_{B_2}\delta^{A_2}_{B_1}\delta^{A_3}_{B_3} + \delta^{A_1}_{B_2}\delta^{A_3}_{B_3}\delta^{A_3}_{B_1} + \delta^{A_1}_{B_3}\delta^{A_2}_{B_1}\delta^{A_2}_{B_2} \\
&= \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2]}\delta^{A_3}_{B_3} - \delta^{A_1}_{[B_3}\delta^{A_2}_{B_2]}\delta^{A_3}_{B_1} - \delta^{A_1}_{[B_1}\delta^{A_2}_{B_3]}\delta^{A_3}_{B_2})
\end{aligned} \tag{G.150}
$$

Contracting this with $H(\gamma_1)_{A_1}{}^{B_1}H(\gamma_2)_{A_2}{}^{B_2}H(\gamma_3)_{A_3}{}^{B_3}$ gives

$$
3M_3(\gamma_1,\gamma_2,\gamma_3) = W(\gamma_3)M_2(\gamma_1,\gamma_2) - M_2(\gamma_1 \circ \gamma_3,\gamma_2) - M_2(\gamma_1,\gamma_2 \circ \gamma_3) \tag{G.151}
$$

For arbitary K we have

$$
\begin{aligned}
(K+1)\delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_{K+1}}_{B_{K+1}]} &= \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_K]}\delta^{A_{K+1}}_{B_{K+1}} \\
&- \delta^{A_1}_{[B_{k+1}}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_K]}\delta^{A_{K+1}}_{B_1} \\
&- \delta^{A_1}_{[B_{k+1}}\delta^{A_2}_{B_2}\cdots\delta^{A_i}_{B_i}\cdots\delta^{A_K}_{B_K]}\cdots\delta^{A_{K+1}}_{B_i} \\
&- \delta^{A_1}_{[B_1}\delta^{A_2}_{B_2}\cdots\delta^{A_K}_{B_{K+1}]}\delta^{A_{K+1}}_{B_K} \qquad \text{(G.152)}
\end{aligned}
$$

To see this note that the LHS is made up of a total of $(K+1)!$ distinct terms where we have a plus sign when we have an even permutation of $(B_1, B_2, \ldots, B_{K+1})$ and a minus sign when we have an odd permutation of $(B_1, B_2, \ldots, B_{K+1})$. The RHS comprises of $K+1$ collections of terms where each of these comprises of $K!$ terms, and hence there are $(K+1)!$ individual terms altogether. Each of these corresponds to a distinct permutation of $(B_1, B_2, \ldots, B_{K+1})$ and appears with the correct plus sign or minus sign to agree with the LHS. Contracting this with $H(\gamma_1)_{A_1}{}^{B_1} H(\gamma_2)_{A_2}{}^{B_2} \cdots H(\gamma_{K+1})_{A_{K+1}}{}^{B_{K+1}}$ and noting

$$
\delta^{A_{K+1}}_{B_i} H(\gamma_i)_{A_i}{}^{B_i} H(\gamma_i)_{A_{K+1}}{}^{B_{K+1}} = [H(\gamma_i)H(\gamma_{K+1})]_{A_i}{}^{B_{K+1}} \qquad \text{(G.153)}
$$

gives (G.145).

An immediate consequence of the recurrence relation (G.145), obtained indentifying the loop $N+1$ with i (the identity loop), is

$$
(N+1)M_{N+1}(\gamma_1, \ldots, \gamma_N, i) = (W(i) - N)M_N(\gamma_1, \ldots, \gamma_N) = 0 \qquad \text{(G.154)}
$$

from which we see that

$$
W(i) = N. \qquad \text{(G.155)}
$$

Let us consider the identity for 2×2 matrices.

N=2

$$
\begin{aligned}
0 &\equiv 3M_3(\gamma_1, \gamma_2, \gamma_3) \\
&= W(\gamma_3)M_2(\gamma_1, \gamma_2) - M_2(\gamma_1 \circ \gamma_3, \gamma_2) - M_2(\gamma_1, \gamma_2 \circ \gamma_3) \\
&= W(\gamma_3)\frac{1}{2}[W(\gamma_2)M_1(\gamma_1) - M_1(\gamma_1 \circ \gamma_2)] - \frac{1}{2}[W(\gamma_2)M_1(\gamma_1 \circ \gamma_3) - M_1(\gamma_1 \circ \gamma_3 \circ \gamma_2)] \\
&\quad - \frac{1}{2}[W(\gamma_2 \circ \gamma_3)M_1(\gamma_1) - M_1(\gamma_1 \circ \gamma_2 \circ \gamma_3)] \qquad \text{(G.156)}
\end{aligned}
$$

so that

$$\begin{aligned} W(\gamma_1)W(\gamma_2)W(\gamma_3) &= W(\gamma_1 \circ \gamma_2)W(\gamma_3) + W(\gamma_2 \circ \gamma_3)W(\gamma_1) + W(\gamma_3 \circ \gamma_1)W(\gamma_2) \\ &\quad - W(\gamma_1 \circ \gamma_2 \circ \gamma_3) - W(\gamma_1 \circ \gamma_3 \circ \gamma_2) \end{aligned} \tag{G.157}$$

G.4 Summary

G.5 Biblioliographical notes

In this chapter I have relied on the following references: Kauffman and Lins

G.6 Worked Exercises and Details

Change of basis for 4-valent spin networks.