Appendix H

Black Hole Entropy

Compare “ordinary” thermodynamics. The quantum theory of an ideal gas allows us to specify and count states, and the resulting entropy agrees, up to typically small corrections, with the classical prediction. But this is to be expected: the correspondence principle relates the quantum states to the classical phase space, forcing an approximate agreement between the two theories. A classical black hole, on the other hand, has no hair there is no classical phase space to explain the thermodynamics. The states responsible for black hole entropymust be fundamentally quantum mechanical, and there is no obvious reason for them to have any preconceived behavior.

http://math.ucr.edu/home/baez/week148.html

This Week's Finds in Mathematical Physics (Week 148)

H.1 Review of Thermodynamics and Statistical Mechanics

Thermodynamics deals with large systems in terms of macroscopic observables alone.

The system's classical state is described by generalized coordinates $\{q^i\}$ and generalized momenta $\{p_i\}$, where the index j runs from 1 to $n =$ (the number of degrees of freedom). The evolution of q, p is governed by Hamilton's equations

$$\frac{dq^j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = \frac{\partial H}{\partial q^j}, \tag{H.1}$$

in statistical mechanics we shall describe the statistical properties of an ensemble of systems by a distribution function equal to the number of systems per unit volume in a 2n-dimensional phase space that is analogous to kinetic theory's 6-dimensional one.

H.2 Statistical Mechanics of Black Holes

from theomodynamics we se the temperature and entropy arises from underlying statistical mechanics. What microstaes are responsible for black hole thermodynamics?

a classical, stationary black hole is determined completely by its mass, charge, and angular momentum, with no room for additional microscopic states to account for thermal behaviour.

If black hole thermodynamics has a statistical mechanical origin then the relavent states must therefore be non-classical.

A microstate is not given by the Schwarzschild metric, but by some complicated time-dependent non-symmetric metric.

such time-dependent non-symmetric microstates of the geometry into account is essential for a statistical understanding of the thermal behavior of black holes

The interior degrees of freedom of the black hole are indistguishable to an exterior orbserver - classically because there is a causal barrier at horizon stops the interior effecting the exterior - [1], hence these degrees of freedon do not contribute to the entropy and so don't effect the energy exchange between the black hole and the exterior.

Searching for a derivation of black hole thermodynamics from properties of stationary or symmetric metrics alone is like trying to derive the thermodynamics of an ideal gas in a spherical box just from spherically symmetric motions of the molecules.

As Ashetkar notes, *the surface degrees of freedom are born quantum mechanically.*

H.2.1 Isolated Horizons: the Classical Phase Space

In the application of physical theory to model the world, one shapes the theory with boundary conditions to fit the system under study. One does exactly the same thing with gravity. Here we model a spacetime with a black hole as a system with two boundaries, one at asymptotic infinity and one at the apparent horizon fig(**??**). While it satifies these boundary conditions, we demand also that the action princle leads to Einstein's equations outside the black hole horizon, in the bulk of the spacetime. When such conditions are met we say that the action principle satisfies *functional differentiability.*

$$S'[\sigma, A] = -\frac{i}{8\pi}\left[\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) - \int_T \mathrm{tr}(\Sigma \wedge A)\right] \tag{H.2}$$

[1]but must show that in full quantum gravity of checking that this can be assumed

where $F = dA + A \wedge A$.

Variation with respect to A gives rise to a surface term,

$$[\delta S']_H = -\frac{i}{8\pi G}\int_H \mathrm{tr}\Sigma \wedge \delta A \tag{H.3}$$

$$\begin{aligned} S[\sigma, A] &= -\frac{i}{8\pi}\left[\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) - \int_T \mathrm{tr}(\Sigma \wedge A)\right. \\ &\quad \left. + \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A)\right] \end{aligned} \tag{H.4}$$

$U(1)$ Chern-Simons surface term - necessary to ensure functional differentiability of the action.

$$\begin{aligned} \delta\frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) &= \\ &= \frac{\mathcal{A}}{4\pi}\int_H \mathrm{tr}(F \wedge \delta A) \\ &= -\int_H \mathrm{tr}(\Sigma \wedge \delta A) \end{aligned} \tag{H.5}$$

using $F = -\frac{2\pi}{\mathcal{A}}\Sigma$ (on each S^2). Hence, it immediatly follows that the action (H.4) has well defined variation with respect to A and gives rise only to the bulk equations of motion.

Details: $SU(2)$ Chern-Simons theory

$$S'[\sigma, A] = -\frac{i}{8\pi}\left[\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) - \int_T \mathrm{tr}(\Sigma \wedge A)\right] \tag{H.6}$$

Variation with respect to A gives rise to a surface term. Let us specify which part of the action this comes from

$$\begin{aligned} F = dA + A \wedge A &= \partial_\alpha A_\beta dx^\alpha \wedge dx^\beta + (A_\alpha dx^\alpha) \wedge (A_\beta dx^\beta) \\ &= (\partial_\alpha A_\beta + A_\alpha A_\beta) dx^\alpha \wedge dx^\beta \\ &= (\partial_\alpha A_\beta + A_\alpha A_\beta)\epsilon^{\alpha\beta} dx^1 dx^2 \end{aligned} \tag{H.7}$$

$\partial_\alpha A_\beta dx^\alpha \wedge dx^\beta$

$$\int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge F) = \int_{\mathcal{M}} \mathrm{tr}(\Sigma \wedge [\partial_\alpha A_\beta dx^\alpha \wedge dx^\beta]) \tag{H.8}$$

$$\begin{aligned} &= -\int_{\mathcal{M}} \mathrm{tr}(\Sigma_\alpha \partial_\beta A_\gamma) dx^\alpha \wedge dx^\beta \wedge dx^\gamma \\ &= -\int_{\mathcal{M}} \mathrm{tr}(\partial_\beta \Sigma_\alpha A_\gamma)\epsilon^{\alpha\beta\gamma} dv + \int_{\mathcal{M}} \mathrm{tr}\partial_\beta(\Sigma_\alpha A_\gamma)\epsilon^{\alpha\beta\gamma} dv \end{aligned} \tag{H.9}$$

$$[\delta S']_H = -\frac{i}{8\pi G}\int_H \mathrm{tr}(\Sigma_\alpha \delta A_\beta)\epsilon^{\alpha\beta} ds \tag{H.10}$$

$$[\delta S']_H = -\frac{i}{8\pi G}\int_H \mathrm{tr}(\Sigma \wedge \delta A) \tag{H.11}$$

Details: Edge states in $U(1)$ Chern-Simons theory

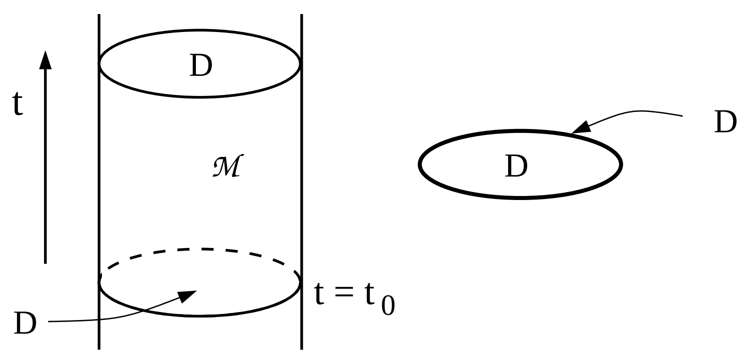

Figure H.1: disc D. $\mathcal{M} = D \times R$.

$$\begin{aligned} \mathcal{CS} &= \frac{k}{4\pi}\int_{\mathcal{M}} (A_i dx^i) \wedge (\partial_i A_k dx^j \wedge dx^k) \\ &= \frac{k}{4\pi}\int_{\mathcal{M}} dv \epsilon^{ijk} A_i \partial_j A_k \end{aligned} \tag{H.12}$$

where $dv = dx^1 \wedge dx^2 \wedge dx^3$.

$$\mathcal{G}_\Lambda = \int_D \Lambda G(A) d^2x = \int_D \Lambda \wedge dA = \int_D dv \epsilon^{ij} \Lambda \partial_i A_j \approx 0. \tag{H.13}$$

Now we require that this functional generates gauge transformations, which in turn requires that $\left(\frac{\delta \mathcal{G}_\Lambda}{\delta A}\right)$ exists. However, we note from (H.26) that

$$\mathcal{G}_\Lambda = \int_D \epsilon^{ij} A_i \partial_j \Lambda + \int_{\partial D} \Lambda A \tag{H.14}$$

$$\mathcal{G}_\Lambda = \int_D A\, d\Lambda + \int_{\partial D} \Lambda A, \tag{H.15}$$

and therefore differentiability requires the boundary condition

$$\Lambda|_{\partial D} = 0. \tag{H.16}$$

Hence the Gauss law reduces to

$$\mathcal{G}_\Lambda = \int_D \epsilon^{ij} A_i \partial_j \Lambda \tag{H.17}$$

$$\mathcal{G}_\Lambda = \int_D dA\Lambda, \tag{H.18}$$

with the gauge parameter Λ subject to (H.16). Then, due to this boundary condition,

$$\begin{aligned} \{\mathcal{G}_\Lambda, \mathcal{G}_{\Lambda'}\} &= \left\{ \int_D dv \epsilon^{ij} \Lambda \partial_i A_j, \int_D dv' \epsilon^{i'j'} \Lambda' \partial_{i'} A_{j'} \right\} \\ &= \int_D dv \int_D dv' \epsilon^{ij} \epsilon^{i'j'} \partial_i \Lambda \partial_{i'} \Lambda' \{A_j(x), A_{j'}(x')\} \\ &= \frac{i}{\kappa} \int_D dv \epsilon^{ij} \epsilon^{i'j'} \partial_i \Lambda \partial_{i'} \Lambda' \epsilon_{jj'} \end{aligned} \tag{H.19}$$

$$\{A_j(x), A_{j'}(x')\} = \frac{i}{\kappa} \epsilon_{jj'} \delta(\vec{x} - \vec{x'}) \tag{H.20}$$

$$\{\mathcal{G}_\Lambda, \mathcal{G}_{\Lambda'}\} = \frac{2\pi}{k} \int_D \Lambda\, d\Lambda' = 0. \tag{H.21}$$

Finally we see that these observables generate a $U(1)$ Lie algebra at the edge,

$$\{\mathcal{Q}(\xi), \mathcal{Q}(\xi')\} = \frac{2\pi}{k} \int_{\partial D} \xi\, d\xi' \tag{H.22}$$

Translated:

$$A = A_i dx^i, \quad dA = \partial_i A_j dx^i \wedge dx^j, \quad dx^i \wedge dx^j = \epsilon^{ij} dx^1 \wedge dx^2,$$
$$dx^i \wedge dx^j = \epsilon^{ijk} dx^1 \wedge dx^2 \wedge dx^3. \tag{H.23}$$

$$\mathcal{CS} = \frac{k}{4\pi} \int_{\mathcal{M}} A \wedge \, dA \tag{H.24}$$

$$G(A) \equiv \epsilon^{ijk} \partial_i A_j \approx 0. \tag{H.25}$$

$$\mathcal{G}_\Lambda = \int_D \Lambda G(A) \, d^2x = \int_D \Lambda \, dA \approx 0. \tag{H.26}$$

$$\{A_j(x), A_{j'}(x')\} =? \tag{H.27}$$

$$\begin{aligned} \{\mathcal{G}_\Lambda, \mathcal{G}_{\Lambda'}\} &= \\ &= \frac{2\pi}{k} \int_D dx^i \Lambda \partial_j \Lambda' \end{aligned} \tag{H.28}$$

$$\int_{\partial D} \Lambda d\Lambda' = \int_{\partial D} dx^i \Lambda \partial_i \Lambda' = \int_{\partial D} ds \Lambda \frac{d\Lambda'}{ds} \tag{H.29}$$

H.3 Quantum Geometry and Black Hole Entropy

In the classical theory, the fields in the bulk - metric, triad, connection... determine the fields on the boundary by continuity. So there would appear to by no independent surface degrees of freedom! This is correct in the classical theory but it is not the case in the quantum theory. A feature of any quantum field theory, not just quatum GR, is that quantum fields are really distributional and so can be arbriarly discontinous. So just because you know the quantum field in the bulk, it doesn't tell you what the field at the boundary is.

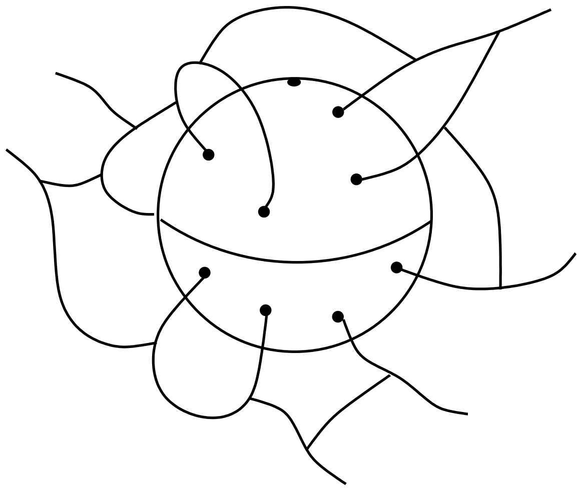

Figure H.2: sepbulksurface.

Basic Strategy:

Canonically quantize the vacuum Einstein equtions with boundary conditions describing horizon of a non-rotating black hole, using techniques of loop quantum gravity:

Seperate “bulk” and “surface” degrees of freedom and count surface states with an area $\mathcal{A}$.

Also works for gravity coupled to electromagonetism and dilaton field!

H.3.1 Classical Phase Space

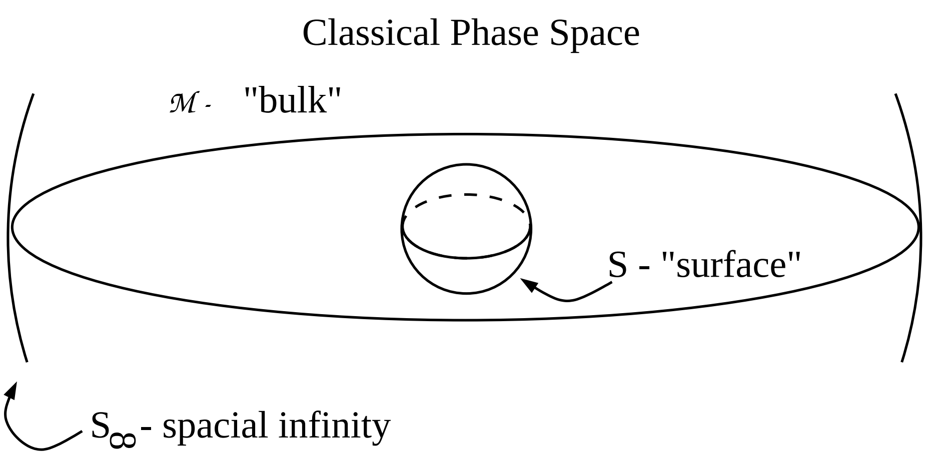

Figure H.3: classphase.

A point in the classical space consists of the fields:

$$A_a^i = \Gamma_a^i - \gamma K_a^i \tag{H.30}$$

γ real and $\gamma \neq 0$,

$$E_{ab}^i = \frac{1}{\gamma}\Sigma_{ab}^i \tag{H.31}$$

satisfying:

(i) Asymptotic flatness at S_∞

(ii) A_a^i reduces to $U(1)$ connection A_a on S.

(iii) $F_{ab} = -\frac{2\pi\gamma}{\mathcal{A}}$

There consists generating:

(i) $SU(2)$ gauge transformations on $\mathcal{M}$ reducing to $U(1)$ on S and identify on S_∞.

(ii) Diffeomorphisms of $\mathcal{M}$ mapping S to itself and reducing to identity at S_∞.

(iii) Time evolution with lapse goinig to 0 at S and constant at S_∞.

H.3.2 Sympletic Structure

The Lie algebra of vector fields on Γ induces a Lie algebra structure on the space of functions, given by (need to improve on this)

$$\{f, g\} := \Omega^{ab}\partial_a f \partial_b g \tag{H.32}$$

$$\Omega(y_1, y_2) = \sum_\mu (p_{1\mu}q_{2\mu} - p_{2\mu}q_{1\mu}) \tag{H.33}$$

$$\omega\left((\delta A, \delta E), (\delta A', \delta E')\right) = \frac{1}{8\pi}\left[\int_\mathcal{M} \mathrm{tr}(\delta E \wedge \delta A' - \delta E' \wedge \delta A) - \frac{\mathcal{A}}{\pi\gamma}\int_S \delta A \wedge \delta A'\right] \tag{H.34}$$

$U(1)$ Chern-Simons surface tern! Equals

$$-\frac{k}{2\pi}\int_S \delta A \wedge \delta A' \tag{H.35}$$

where the “level”

$$k = \frac{\mathcal{A}}{4\pi\gamma} \tag{H.36}$$

must be an *integer* to quantize the theory!

diads on a 2-sphere. Connection compatible to these diads. This connection is a $U(1)$ connection. On sphere we get first $SO(2)$ - because of double covering we get $U(1)$ connection W. $SU(2)$ is trival on an oreinetable 3-manifold - so don't have to do patches. Cannot have a global field of diads on a 2-sphere because every vector field on a sphere has to vanish somewhere. Spin bundle on 2-sphere is not rrival. Need 2 patches and in the overlap they are related to each other by a $U(1)$ transformation. W not globally defined 1-form.

H.3.3 Quantization Strategy

We separately quantize:

BULK: (A, E) on $\mathcal{M}$ with usual symplectic structure.

SURFACE: $U(1)$ connection A on S with Chern-Simons symplectic structure. $k = \frac{\mathcal{A}}{4\pi\gamma}$ must be integer!

Obtaining Hilbert space spaces H_{bulk} and H_{surface}.

$$H_{\text{bulk}} = \lim_{P} H^{P}_{\text{bulk}} \tag{H.37}$$

$$H_{\text{surface}} = \lim_{P} H^{P}_{\text{surface}} \tag{H.38}$$

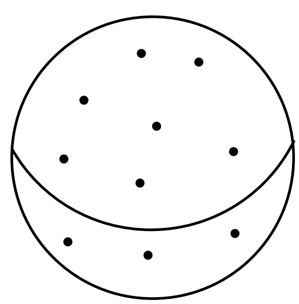

Figure H.4: Classical bourndary conditions for isolated horizons.

$$P = \{p_1, \ldots, p_n\} \tag{H.39}$$

Then we set

$$H_{\text{physical}} = \lim_{P} \frac{H^P_{\text{bulk}} \otimes H^P_{\text{surface}}}{Gauge} \tag{H.40}$$

Note that $U(1)$ gauge transformations on S are generated by

$$F_{\alpha\beta} + \frac{2\pi\gamma}{\mathcal{A}} E_{\alpha\beta} \tag{H.41}$$

so "Gauge" includes imposing constraint!

H.3.4 Bulk States

$$\mathcal{H}_{bulk} = L^2(\{\text{generalized } SU(2) \text{ cons on } \mathcal{M}\}) \tag{H.42}$$

as defned in the loop quantum gravity. But we can save time by modding out by $SU(2)$ gauge transforms that reduce to identity on S, S_∞.

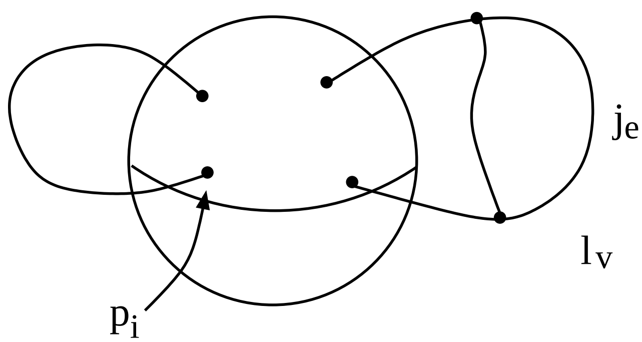

Figure H.5: Bulk states.

The smaler "$\mathcal{H}_{bulk}$ has a basis given by spin networks in $\mathcal{M}$ with" loose ends" at points $p_i \in S$:

• edges labeled by spins j_e

• vertices labelled by interwiners l_v

• punctures p_i: labelled by vectors $|m_i >$ in spin-j_i representation, where j_i is spin of incident edge:

$$m_i = -j_i, -j_i + 1, \ldots, j_i. \tag{H.43}$$

If $P = \{p_1, \ldots, p_n\}$ of bulks states with loose ends at points in P.

H.3.5 Geometric Interpretation of Bulk States

Consider $\psi \in \mathcal{H}^P_{\text{bulk}}$:

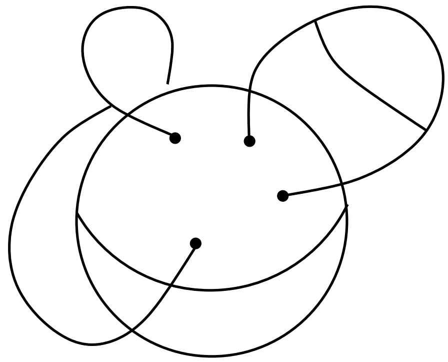

Figure H.6: geobulk1.

punctures p_i labelled by representations j_i and vectors $|m_i>$ in these representations.

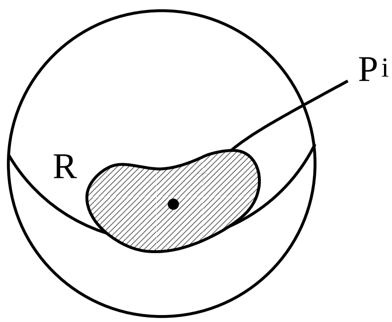

Figure H.7: geobulk2.

Then Area $(R)\gamma \int_R \sqrt{\vec{E} \cdot \vec{E}}$ has eigenvalue

$$8\pi\gamma\sqrt{j_i(j_i+1)} \tag{H.44}$$

and $\int_R \sqrt{\vec{E} \cdot \vec{r}}$ has eigenvalue

$$8\pi m_i \tag{H.45}$$

in the state ψ.

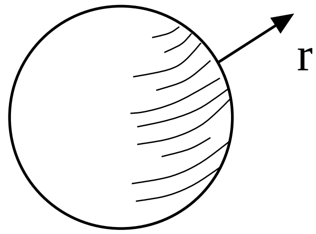

Figure H.8: geobulk3 $U(1)$ fixes internal vector r^i.

H.3.6 Surface States

Since we will impose constraint $F = -\frac{2\pi\gamma}{\mathcal{A}} E \cdot r$ on S, and $E \cdot r$ vanishes except at points $p_i \in P$ in states in $\mathcal{H}^P_{surface}$, we take as our phase space:

$X^P_{surface}$ = { generalized $U(1)$ conns on S, flat except at $p_i \in P$, mod gauge except at $p_i \in P$ }

A point in $X^P_{surface}$ is described by $2(n-1)$ holonomies:

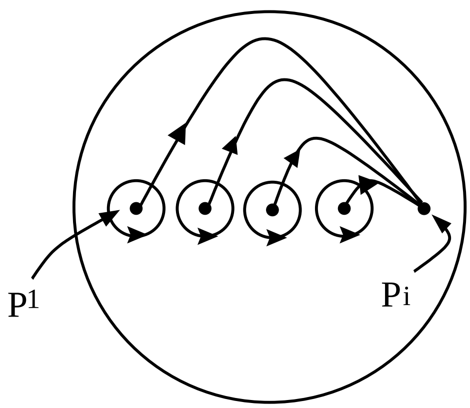

Figure H.9: surfacestate.

so $X^P_{surface} \sim (U(1) \times U(1))^{n-1}$ with sympletic structure equal to $\frac{k}{2\pi}$ times that comming from usual sympletic structure on

$$U(1) \times U(1) = \frac{R}{2\pi Z} \times \frac{R}{2\pi Z} \tag{H.46}$$

??The U(1) Chern-Simons theory action gives this phase space when we take spacetime to be R x M and M is a 2-sphere with two holes removed. If you put suitable boundary conditions at the holes, a flat U(1) connection on M is determined (up to gauge transformations) by its holonomies around one hole and along a path from one hole to the other. So we get two elements of U(1), i.e., a point on the torus. ??

Applying geometric quantization to $X^P_{surface}$ get $\mathcal{H}_{surface}$. Basis of states given by labelling punctures with numbers $m_i = -\frac{k}{2}, -\frac{k}{2}+1, \ldots, \frac{k}{2}$. Really m_i defined only mod k.

$$W = W_1 + \sum_{i=1}^{n} c_i \frac{(x-i)dy - ydx}{(x-i)^2 + y^2} \tag{H.47}$$

where W_1 is a bounded smooth 1-form on $U - \mathcal{P}$. Since W is flat except at the punctures, W_1 must be closed. Note that the constants are not idependent: they must sum to zero modulas 2π, because the holonomy W around a loop enclosing all the punctures must be trivial,

Details: Surface states

(a) X_i satisfies

$$\int_{\gamma_j} X_i = \delta_{ij}, \quad \oint_{\eta_j} X_i = 0. \tag{H.48}$$

(b) Y_i satisfies

$$\oint_{\eta_j} Y_i = \delta_{ij}, \quad \int_{\gamma_j} Y_i = 0. \tag{H.49}$$

(c) X_i and Y_i satisfy

$$\int_{S^2} X_i \wedge X_j = 0, \quad \int_{S^2} Y_i \wedge Y_j = 0, \quad \int_{S^2} X_i \wedge Y_j = \delta_{ij}. \tag{H.50}$$

$$\int_{S^2} d^2x \epsilon^{ij} X_i X_j = 0, \quad \int_{S^2} d^2x \epsilon^{ij} Y_i Y_j = 0, \quad \int_{S^2} d^2x \epsilon^{ij} X_i Y_j = \delta_{ij}. \tag{H.51}$$

(d)

$$W = \tilde{W} + \sum_{i=1}^{n-1} (x_i X_i + y_i Y_i) \tag{H.52}$$

Note property (d) implies the 1-forms X_i and Y_i are closed $X_i = dX$

We actually define X_i and Y_i on regions slightly larger than $S - \mathcal{P}$. Define X_i on all of S by

$$X_i = df_i = dx^a \frac{\partial f_i}{\partial x^a} \tag{H.53}$$

where f_i is any smooth real-valued function on S with $f(p_j) = i$ for $j \neq i$ and $f_i = 0$ in an open disc containing p_i and the loop η_i. To define Y_i, first set

$$Y_i = \frac{1}{2\pi}\frac{(x-i)dy - ydx}{(x-i)^2 y^2} \tag{H.54}$$

in an open disc containing p_i and the loop η_i, and

$$Y_i = \frac{1}{2\pi}\frac{(x-n)dy - ydx}{(x-n)^2 y^2} \tag{H.55}$$

in an open disc containing p_i. Then extend Y_i smoothly to a closed 1-form on all of $S - \{p_i, p_n\}$.

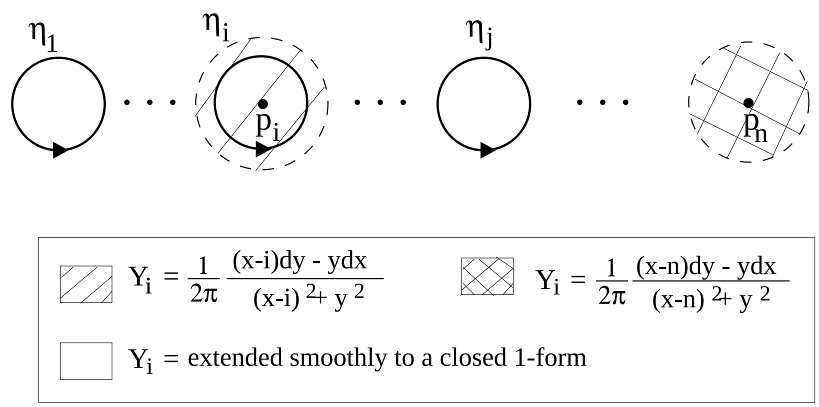

Figure H.10: surfaceY.

$$\int_{\gamma_j} X_i = \int_{p_j}^{p_n} df_i = f_i(p_n) - f_i(p_j) = \begin{cases} 0 & j \neq i \\ 1 & j = i \end{cases} \tag{H.56}$$

Stoke's theorem and that $dX_i = 0$

$$\oint_{\eta_j} X_i = \int_{S^2} dX_i = 0. \tag{H.57}$$

Consider the phase space $\mathbb{R}^{2(N-1)}$ with canonical brackets $\{y_I, x_J\} = \delta_{IJ}\kappa'$, $\{x_I, x_J\} = \{y_I, y_J\} = 0$. In order to obtain the torus $U(1)^{2(N-1)}$ we devide $\mathbb{R}^{2(N-1)}$ by the action of the discrete translation group, that is, we identify points x_I, y_I up to translations by integer multiples of 2π.

H.3.7 Geometric Quantisation of Surface Degree's of Freedom

Before we proceed to quantise by means of geometric quantisation we must ensure that Weil's integrality criterion is satisfied...

Hence the number K, called the level of the Chern-Simos theorey, must be integral.

We choose a polarisation:

$$\Omega = \frac{\hbar K}{2\pi} dy^I \wedge dx^I \tag{H.58}$$

Let us set $z^I := x^I + iy^I$

We first define the quauntum theorey for $\mathbb{R}^{2(N-1)}$ and then pass to the torus.

$$\begin{aligned} \hat{x}^I &= i\hbar_{\chi_{x^I}} - \Theta(\chi_{x^I}) + x^I \\ \hat{y}^I &= i\hbar_{\chi_{y^I}} - \Theta(\chi_{y^I}) + y^I \end{aligned} \tag{H.59}$$

Moreover

$$\nabla_{\partial/\partial\overline{z}^I} = \partial/\partial\overline{z}^I - \frac{1}{i\hbar}\Theta(\partial/\partial\overline{z}^I) = \partial/\partial\overline{z}^I + \frac{K}{4\pi} z^I \tag{H.60}$$

Therefore polarised states satify

$$\left[\nabla_{\partial/\partial\overline{z}^I} \Psi\right](z, \overline{z}) = 0$$

whose solution is

$$\Psi(z, \overline{z}) = e^{-K\overline{z}^I z^I/(4\pi)} \psi(z) \tag{H.61}$$

H.3.8 Geometric Interpretation of Surface States

Consider $\psi \in \mathcal{H}_{surface}$:

punctures p_i labelled by $m_i = -\frac{k}{2}, \dots \frac{k}{2}$

Suppose only p_i lies in the region RS:

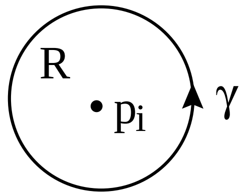

Figure H.11: surfaceR.

then the holonomy around γ has eigenvalue

$$e^{-4\pi i \frac{m_i}{k}} \tag{H.62}$$

in the state ψ. Classically

$$e^{i \oint_\gamma A} = e^{i \int_R F} \tag{H.63}$$

so we may say:

$\int_R F$ has eigenvalue $-4\pi i \frac{m_i}{k}$ mod 4π. Thus the geometry of S is flat except at p_i, where there are canonical singularities:

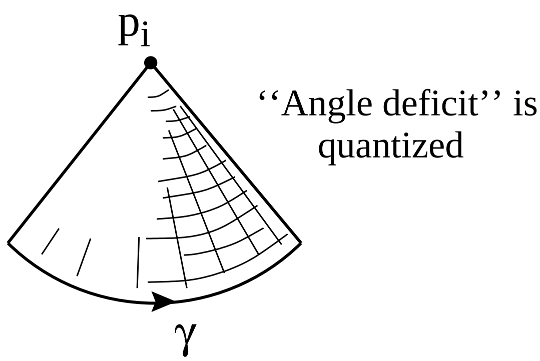

Figure H.12: cone.

H.3.9 Surface and Bulk

Having constructed the volume and surface Hilbert spaces, we now wish to impose the quantum bourndary condition, in order to pick out the kinemetical Hilbert space $\mathcal{H}$ as a subspace of $\mathcal{H}_V \otimes \mathcal{H}_S$.

$$\left(I \otimes e^{\int_R F}\right) \psi_{\text{bulk}} \otimes \psi_{\text{surface}} = \left(e^{-\frac{2\pi i \gamma}{2} \int_R E \cdot r} \otimes I\right) \psi_{\text{bulk}} \otimes \psi_{\text{surface}} \tag{H.64}$$

eigenvalue $e^{-4\pi i \frac{m_i}{k}}$ and $e^{-\frac{2\pi i\gamma}{\mathcal{A}}\cdot 8\pi m_i}$

These eignvalues match if the puncture labels m_i for surface and bulk states agree - since

$$k = \frac{\mathcal{A}}{4\pi\gamma}!! \tag{H.65}$$

$m_i = -j_i, \ldots, j_i$

H.3.10 Entropy Calculation

Recall that in classical general relativity in the Hamiltonian formulation, the bulk Hamiltonian is a first class constraint, so that the entire Hamiltonian consists of the boundary contribution HS on the constraint surface. In the quantum domain, the Hamiltonian operator can be written as

Other stuff

$$\frac{A \times B}{G} \sim \frac{A}{G} \times B \tag{H.66}$$

In $H_{phys} = \lim_p \frac{H^p_{bulk} \otimes H^p_{surface}}{Gauge}$ we form a density matrix from projection onto subspace of states where the horizon has area $\mathcal{A} \pm i_p^2$. Then we trace out ti get a density matrix ρ on $H_{surface}/Gauge$. If for every at least one solution of hte Hamiltonian constraint for any p_i, j_i, then

$$S = \mathrm{tr}(\rho \ln \rho) = \frac{\ln 2}{4\pi\sqrt{3}\gamma} \frac{\mathcal{A}}{l_P^2} + \mathcal{O}(\sqrt{\mathcal{A}}) \tag{H.67}$$

Get agreement with $S = \mathcal{A}/4l_P^2$ if:

$$\boxed{\gamma = \frac{\ln 2}{\pi\sqrt{3}}} \tag{H.68}$$

Heuristic estimate: $j = 1/2$ punctures dominate. These give an area of

$$8\pi\gamma\sqrt{j(j+1)}l_P^2 = 4\pi\sqrt{3}\gamma l_P^2 \tag{H.69}$$

and entropy $\ln 2$, so

$$\boxed{S \sim \frac{ln2}{4\pi\sqrt{3}\gamma}\,\mathcal{A}} \tag{H.70}$$

$$S \approx \frac{1}{4}\mathcal{A} \tag{H.71}$$

It was found, with some surprise, that the same calculation with Maxwell/dilaton fields gives $S = \mathcal{A}/4$ with same γ!

It is only those microscopic degrees of freedom that affect the energy exchanged between the blackhole and exteriorthat contribute to the entropy.

The interior degrees of freedom don't come into it.

Examples of normed spaces.

H.4 Maths

$$W(c)\psi = \psi$$

$$V(b)\sum \psi_l e^{il\cdot z} = \sum \psi_l e^{il\cdot z}$$

$$\begin{aligned} V(b)\sum \psi_l e^{il\cdot z} &= \exp\left(\frac{K}{2\pi}[ib\cdot z - b\cdot b/2]\right)\psi(z+ib) \\ &= \exp\left(\frac{K}{2\pi}[ib\cdot z - b\cdot b/2]\right)\sum \psi_l e^{il\cdot z} \\ &= \sum \left[\psi_l e^{-l\cdot b} e^{-\frac{1}{2}\frac{K}{2\pi}}\right] e^{i(l-\frac{Kb}{2\pi})\cdot z} \end{aligned} \tag{H.72}$$

So

$$\psi_{l+\frac{Kb}{2\pi}} = \psi_l e^{-l\cdot b} e^{-\frac{1}{2}\frac{K}{2\pi}b\cdot b}$$

or

$$\psi_l = \psi_{l-\frac{Kb}{2\pi}} e^{-l\cdot b} e^{\frac{1}{2}\frac{K}{2\pi}b\cdot b}$$

with solution

$$\psi_{l+nK} = \psi_l e^{-2\pi l\cdot n} e^{-2\pi\frac{K}{2}n\cdot n}$$

$$
\begin{aligned}
\psi(z) &= \sum_l \psi_l e^{il\cdot z} \\
&= \sum_{n\in\mathbb{Z}} \sum_{\prod_I l_I=1}^{K} \psi_{l+nK} e^{i(l+nK)\cdot z} \\
&= \sum_{l\in D_K}^{K} \sum_{n\in\mathbb{Z}} \psi_l e^{-2\pi l\cdot n} e^{-2\pi \frac{K}{2} n\cdot n} \exp(i(l+nK)\cdot z) \\
&= \sum_{l\in D_K} \psi_l \vartheta_l^{K,\mathcal{P}}(z)
\end{aligned}
\tag{H.73}
$$

where

$$
\vartheta_l^{K,\mathcal{P}}(z) = \sum_{n\in\mathbb{Z}^{N-1}} e^{-2\pi l\cdot n} e^{-2\pi \frac{K}{2} n\cdot n} \exp\left(i\left(l+nK\right)\cdot z\right) \tag{H.74}
$$

H.5 Entropy of Rotating and Axisymmetric Distorted Black Holes

With the notion employed, given a type II isolated horizon, its image under any diffeomorphism on Δ is again a type II isolated horizon. The diffeomorphism invariance on Δ remains intact.

In terms of these fields ψ, the surface part of the symplectic strucure is given by:

$$
\Omega_S(\delta_1,\delta_2) = \frac{1}{8\pi\gamma G} \oint_S \left[\delta_1\psi\delta_2(\Sigma^i r_i) - \delta_2\psi\delta_1(\Sigma^i r_i)\right] \tag{H.75}
$$

new connection W as

$$
dW = -\frac{2\pi\gamma}{a_0} \underline{\Sigma}^i r_i \epsilon \tag{H.76}
$$

the sympletic structure (H.75) reduces to the Chern-Simons form:

$$
\Omega(\delta_1,\delta_2) = \frac{1}{8\pi G}\frac{a_0}{\gamma\pi} \oint_S \delta_1 W \wedge \delta_2 W \tag{H.77}
$$

H.5.1 Quasi-normal Modes of Black Hole

H.6 Quantum Black Holes

horizon operators

$$\hat{\theta}_{\pm} = \frac{1}{2i\lambda}\left(\hat{U}_{\lambda} - \hat{U}_{\lambda}^{\dagger}\right) \pm \frac{2}{\epsilon l_P^2}\left(\hat{R}_{\epsilon} - \hat{R}_0\right). \tag{H.78}$$

H.7 Biblioliographical notes

In this chapter I have relied on the following references: Rovelli's paper *Loop Quantum Gravity and Black Hole Physics*.

H.7.1 Review of Chomology Group of Spherical Horizon

In the case of spherical horizons, one has a sphere with N-punctures due to the gravitational spin-network. The first cohomology group of the N-punctured sphere, denoted as $H^1(S - P_N)$, is $(N-1)$-dimensional which is one less than the number of punctures. $(N-1)$ pairs of forms are defined on the punctured sphere to yield the required symplectic structure (see figure ??, which is similar to the figure originally produced in [??]). These forms are constructed via their duality with chains on a punctured sphere as depicted in figure ??.

FIGURE HERE

There exist $N-1$ η paths and $N-1$ conjugate γ paths on this sphere. A basis for all the paths based at p_N

$$\{\gamma_1^{-1}\eta_1\gamma_1,\ \gamma_2^{-1}\eta_2\gamma_2,\ \cdots,\ \gamma_{N-1}^{-1}\eta_{N-1}\gamma_{N-1}\}. \tag{H.79}$$

there exists a fundamental relation

$$\eta_1 \cdot \eta_2 \cdot\ \ldots\ \cdot \eta_N = 1, \tag{H.80}$$

which is a mathematical relation indicating that a loop around all punctures can be shrunk to a point on the sphere. Another way to look at this relation is that a loop around all the $N-1$ punctures is equivalent to a loop around the N-th puncture but in reverse. In other words, η_N is expressible in terms of the other η paths.