Appendix I

Loop Quantum Cosmology

I.1 Introduction

I.2 Quantum Cosmology

I.2.1 Classical theory

$$S[q_{ab}, N, N^a] = \int dt \int d^3xN({}^3q)^{1/2}(K_{ab}K^{ab} - K^2 + {}^3R - 2\Lambda) \tag{I.1}$$

$$G^{abcd} = \frac{1}{4}({}^3q)^{1/2}(q^{ac})q^{bd} + q^{ad})q^{bc} - 2q^{ab})q^{cd} \tag{I.2}$$

$$p^{ab} = -2G^{abcd}K_{cd} \tag{I.3}$$

we can write the action as

$$S[q_{ab}, p^{ab}N, N^a] = \int dt \int d^3 \left(p^{ab}\frac{dq_{ab}}{dt} - N\mathcal{C} - N^a\mathcal{C}\right) \tag{I.4}$$

where

$$\begin{aligned}
\mathcal{C} &= \frac{1}{2}G_{abcd}p^{ab}p^{cd} - (q)^{1/2}({}^3R - 2\Lambda), && \text{(I.5)}\\
\mathcal{C}_a &= -2g_{ac}\nabla_d p^{cd}, && \text{(I.6)}\\
G_{abcd} &= (q)^{-1/2}(g_{ac}g_{bd} + g_{ad}g_{ad} - 2g_{ad}g_{cd}). && \text{(I.7)}
\end{aligned}$$

I.2.2 Minsuperspace

The collection of all admissible three-metrics ia called superspace (nothing to do with supersymmetry!). This space can be given properties such as a "metric" to calculate. Is the DeWitt supermetric given by

$$ds^2 = N^2\, dt^2 - q_{ab}(\, dx^a + N^a\, dt)(\, dx^b + N^b\, dt) \tag{I.8}$$

$$ds^2 = N^2\, dt^2 - q_{ab}\, dx^a\, dx^b \tag{I.9}$$

Shift N^a is zero and is lapse function N is homogenious

$$S[q^i, N] = \int_{t_1}^{t_2} N \left(\frac{1}{2N^2} G_{ij} \frac{dq^i}{dt} \frac{dq^j}{dt} - V(q) \right) dt \tag{I.10}$$

The involving the Hamiltonian is

$$S[q^i, N] = \int_{t_1}^{t_2} \left(p_i \frac{dq^i}{dt} - N\mathcal{H} \right) dt \tag{I.11}$$

where

$$\mathcal{H} = G^{ij} p_i p_j + V(q). \tag{I.12}$$

momenta are proportional to the coordinates:

$$p_i = \frac{1}{N} G_{ij} \frac{dq^i}{dt}. \tag{I.13}$$

$$\frac{dq^i}{dt} = N[q^i, \mathcal{H}], \qquad \frac{dp_i}{dt} = N[p_i, \mathcal{H}] \tag{I.14}$$

$$\mathcal{H} \approx 0 \tag{I.15}$$

(weak equality means it is valid only if you restrict yourself to the constraint surface).

The Wheeler-DeWitt equation of cosed isostropic universe with scalar field ϕ and zero cosmological constant. With generic dependence of the potential on ϕ, namely $V(\phi)$.

$$\left(\frac{\partial^2}{\partial\Omega^2} - \frac{\partial^2}{\partial\phi^2}V(\phi)e^{6\Omega} - e^{4\Omega}\right)\Psi(\Omega,\phi) = 0. \tag{I.16}$$

WKB solutions

$$\mathcal{H} = -\frac{3aK}{8\pi G} - \frac{8\pi G p_a^2}{12a} + \frac{p_\phi^2}{2a^3} + a^3V(\phi) = 0. \tag{I.17}$$

Quantization $[\hat{a}, \hat{p}_a] = i$, $[\hat{\phi}, \hat{p}_\phi] = i$ we have the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\psi(a,\phi) = 0. \tag{I.18}$$

I.2.3 Mathematical Excurtion: Symmetry

One way to characterize the invariance of the metric under spacial transformations is to consider

$$e^i_m(x)\,dx^m = e^i_m(x')\,dx'^m \tag{I.19}$$

$$dl^2 = \eta_{ij}(e^i_m(x)\,dx^m)(e^j_n(x)\,dx^n) \tag{I.20}$$

that is, the three metric tensor is given by

$$q_{mn} = \eta_{ij}e^i_m e^j_n \tag{I.21}$$

$$e^m_i e^j_m = \delta^j_i, \quad e^m_i e^j_n = \delta^m_n. \tag{I.22}$$

$$e^m_i \frac{\partial e^n_j}{\partial x^m} - e^m_j \frac{\partial e^n_i}{\partial x^m} = C^k_{ij} e^n_k \tag{I.23}$$

These are the structure constants of the groups of transformations. Ifwe denote by X_i the following differential operator:

$$X_i = e^m_i \frac{\partial}{\partial x^m}, \tag{I.24}$$

then (N.-19) can be written as

$$[X_i, X_j] = C^k_{ij} X_k. \tag{I.25}$$

Some of the geometrical properties of a manifold $\mathcal{M}$ can most easily be examined by constructing a fibre bundle, which is locally a direct product of $\mathcal{M}$ and a suitable space.

The three dimensional rotation gorup $O(3)$ is the isometry group for of the ordinary round sphere S^2.

A group that is also a manifold and for which the group operations are continuous a Lie group.

Now any Lie group is a manifold and can be made into a group of transformations acting on itself as follows: the element g of G defines the transformation

$$L_g(h) = gh. \tag{I.26}$$

left-invariant vector field

$$A^i_a = c\Lambda^i_I \omega^I_a, \quad E^a_i = p\Lambda^I_i X^a_I \tag{I.27}$$

$$A = A_x(x)\Lambda_3 dr + (A_1(x)\Lambda_1 + A_2(x)\Lambda_2)d\theta + (A_1(x)\Lambda_2 - A_2(x)\Lambda_1)\sin\theta d\varphi + \Lambda_3 \cos\theta d\varphi \tag{I.28}$$

$$E = E^x(x)\Lambda_3 \sin\theta \frac{\partial}{\partial x} + (E^1(x)\Lambda 1 + E^2(x)\Lambda_2)\sin\theta \frac{\partial}{\partial\theta} + (E^1(x)\Lambda_2 E^2(x)\Lambda_1)\frac{\partial}{\partial\varphi} \tag{I.29}$$

I.2.4 Symmetries and Backgrounds

It is important to realize that the action of the symmetry group on a space manifold provides a partial background such that the situation is always slightly different from the full theory.

It is impossible to introduce symmetries in a completely background independent manner.

I.2.5 Loop Quantum Cosmology

We don't use metrics

Enough loops so that for any two different connections, one can find some holonomy on these two different connections. We need only one loop because of the symmetry.

There is now another gauge degree of freedom, upon which physical observables should not depend, and so under the action of the Euclidean group on the gauge field and electric field should be

$$A^i_\mu \mapsto g^{-1} A^i_\mu g + g^{-1} \frac{\partial g}{\partial x^\mu}, \qquad E^\mu_i \mapsto g^{-1} E^\mu_i g \tag{I.30}$$

the Poisson brackets of any two functions f and g on this phase space is given by:

$$\{f, g\} = \frac{\kappa\gamma}{3} \left(\frac{\partial f}{\partial c} \frac{\partial g}{\partial p} - \frac{\partial g}{\partial c} \frac{\partial f}{\partial p} \right) \tag{I.31}$$

because of homogeneity and isotropy, we do not need all edges e and surfaces S. Symmetric connections A in $\mathcal{A}$ can be recovered knowing holonomies $h(e)$ along straight lines in $\mathcal{M}$. Similary,

symmetric states exist as disributions supported on invariant connections only

isotropic connection / triad: $A^i_a = c\Lambda^i_I \omega^I_a$, $E^a_i = p\Lambda^I_i X^a_I$ with ω^I_a, X^a_I invariant 1-forms/vector fields.

Λ^i_I internal $su(2)$-triad (purely gauge)

$$h_e(A) = \mathcal{P} \exp \left(\int_e A^i_a(e(t)) \dot{e}^a \tau_i \, dt \right) \in SU(2) \tag{I.32}$$

given a surface $S : [0, 1] \to \Sigma$ we can form a flux as a function of the triads

$$E(S) = \int_S E^a_i(y) n_a(y) \tau^i \, d^2 y \tag{I.33}$$

where n_a is co-normla to the surface S. The co-normal is defined as

$$n_a = \frac{1}{2} \epsilon_{abc} \epsilon^{de} \frac{\partial x^b}{\partial y^d} \frac{\partial x^c}{\partial y^e} \tag{I.34}$$

without using a background metric, where x^a are coordinates of Σ and y^a coordinates of the surface S.

$$A_a^i(x)\,dx^a = c\omega^i, \qquad E_i^a \frac{\partial}{\partial x^a} = pX_i \tag{I.35}$$

where ω^i are invariant 1-forms and X_i invariant vector fields. For spacially flat configuration, $\omega^i = dx^i$ are just coordinate differentials, while X_i are the derivatives.

The symmetry condition can be implemented by using onlt invariant connections (I.35) in holonomies as creation operators, i.e.

$$\begin{aligned} h_i(c) &= \exp(c\tau_i) = \exp(-i\frac{c}{2}\sigma_i) \\ &= 1 + \left(\frac{-ic}{2}\right)\sigma_i + \frac{1}{2!}\left(\frac{-ic}{2}\right)^2\sigma_i^2 + \frac{1}{3!}\left(\frac{-ic}{2}\right)^3\sigma_i^3 + \ldots \\ &= I\left(1 - \frac{1}{2!}\left(\frac{c}{2}\right)^2 + \ldots\right) - i\sigma_i\left(\frac{c}{2} - \frac{1}{3!}\left(\frac{c}{2}\right)^3 + \ldots\right) \quad (\text{using } \sigma^2 = I) \\ &= \cos(c/2) + 2\tau_i\cos(c/2) \end{aligned} \tag{I.36}$$

physical components $c = \frac{1}{6}\dot{a}$ extrinsic curvature (flat model) $p = \epsilon a^2$ a: scale factor , ϵ : orientation

$$A_a = cV_0^{-\frac{1}{3}}\,{}^0\omega_a^i\tau_i, \qquad E^a = pV_0^{-\frac{2}{3}}\sqrt{q}e_i^a\tau^i. \tag{I.37}$$

Symmetric States

$$A(e) = \cos\frac{l_C}{2} + 2[\sin\frac{l_C}{2}](\dot{e}^{a0}\omega_a^i)\tau^i \tag{I.38}$$

$$F(A) = \sum_j \xi_j e^{il_jc} \tag{I.39}$$

These are precisely the **almost periodic functions**

$$\{F(A), p\} = \frac{8\pi\gamma G}{6}\sum_I (il_j\xi_j)e^{il_jc} \tag{I.40}$$

$$\mathcal{N}_l(\overline{A}) = e^{il_jc} \tag{I.41}$$

the space of square integrable fucntions on a suitable completion $\overline{\mathcal{A}}_S$ of the classical configuration space.

$\mathcal{A}_S = R$ and $\overline{\mathcal{A}}_S$ is the Gel'fand spectrum of the C* algebra of almost periodic functions on $\mathcal{A}_S$.

Say X is a measurable space and Y is a topological space. A function $f : X \to Y$ is a measuable function if the pre-image $f^{-1}(V)$ of every open set of Y is a measure subset of X.

orthonormal states $|m>$ in connection representation:

$$< c|m >= \frac{\exp\left(\frac{1}{2}imc\right)}{\sqrt{2}\sin\frac{c}{2}} \qquad n \in \mathbf{Z} \tag{I.42}$$

isotropic spin network states $|m| = 2j + 1$

geometric operators

$$\hat{p}|m >= \frac{1}{6}\gamma l_P^2 m|m > \tag{I.43}$$

$$\hat{V}|m >= \left(\frac{1}{6}\gamma l_P^2\right)^{3/2} \sqrt{(|m| - 1)|m|(|m| + 1)}|m > \tag{I.44}$$

sgn m: orientation

Poisson brackets.

$$\begin{aligned} \{F(A), p\} &= \{\sum_j \xi_j \exp(il_j c), p\} \\ &= \sum_j \xi_j \{\exp(il_j c), p\} \\ &= \frac{8\pi\gamma G}{6} \sum_I (il_j \xi_j) \exp(il_j c) \end{aligned} \tag{I.45}$$

Details Hamiltonian.

Quantization of $1/a$ (Test curvature singularity)

$\hat{p}$, $\hat{v}$ have eigenvalues zero implies no inverse

rewrite

$$\frac{1}{a}\delta_{IJ} = \frac{q_{IJ}}{\sqrt{\det q}} = \frac{e^i_I e^i_I}{\det e} =: m_{IJ} \tag{I.46}$$

with cotriad

$$e^i_I = \sqrt{\det q}(E^{-1})^i_I = \frac{2}{\gamma\kappa}\{A^i_I, V\} \tag{I.47}$$

$$m_{IJ} = \frac{16}{\gamma^2\kappa^2}\{A^i_I, \sqrt{V}\}\{A^i_J, \sqrt{V}\} \tag{I.48}$$

quantized:

$$\hat{m}_{IJ} = \frac{64}{\gamma^2 l_P^4}\left[(\sqrt{\hat{V}} - \cos\frac{c}{2}\sqrt{\hat{V}}\cos\frac{c}{2} - \sin\frac{c}{2})^2 - \delta_{IJ}(\sin\frac{c}{2}\sqrt{\hat{V}}\cos\frac{c}{2} - \cos\frac{c}{2}\sqrt{\hat{V}}\sin\frac{c}{2})^2\right] \tag{I.49}$$

eigenvalues bounded, finite even if $V = 0$ rapidly approach the classical behaviour

upper bound:

$$\hat{m}_{IJ,z} = \frac{32(2-\sqrt{2})}{3\sqrt{\gamma}l_P} \tag{I.50}$$

$$\hat{m}_{IJ}|0>= 0 \tag{I.51}$$

due to

$$m_{IJ} = \frac{sgn(a)^2}{|a|}\delta_{IJ} \tag{I.52}$$

evolution equations don't breakdown even though the volume of the universe goes to zero!

$$\mathcal{H} = -12\gamma^{-2}\kappa^{-1}(c(c-k) + (1+\gamma^2)k^2/4)\sqrt{|p|} \tag{I.53}$$

One has $|p| = a^2$ while $c = (k - \gamma\dot{a}$. If we insert this constraint equation $\mathcal{H} + \mathcal{H}_{\mathrm{matter}}$

• All except a finite number of degrees of freedom are "frozen". The shift function is zero and the lapse is homogeneous. N is still a function of τ, so that separation beween two succesive three-surfaces is still undetermined. Reparametrization invariance is what remains of general covariance of the full theory.

• Are there corresponding solutions in full theory?

• Only considering states that are symmetric at the microscopic level.

I.2.6 Continuum Limit

Pre-classicality

$$\frac{3}{\gamma^3 l_P^2}\left[\left(V_{\frac{|m+1|}{2}}V_{\frac{|m+1|}{2}-1}\right)s_{m+4}(\phi) - \left(V_{\frac{|m|}{2}}V_{\frac{|m|}{2}-1}\right)s_m(\phi) + \left(V_{\frac{|m-1|}{2}}V_{\frac{|m-1|}{2}-1}\right)s_{m-4}(\phi)\right] = -\hat{\mathcal{H}}_\phi(m)s_m(\phi) \tag{I.54}$$

at large volume ($m >> 1$) assume s_m to be only mildy varying at small scales (from m to $m+1$) conjtinuum approximation

$$\psi(p,\phi) := s_{s(p)}(\phi) \tag{I.55}$$

with

$$n(p) = \frac{6p}{\gamma l_P^2} \tag{I.56}$$

and interpolation

I.2.7 Inflation from Loop Quantum Cosmology

effective Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{16\pi}{3}Ga^{-3}\left(\frac{1}{2}a^{-3}p(3a^2/jl_P^2)^6 p_\phi^2 + a^3V(\phi)\right). \tag{I.57}$$

Since the right hand side now depends on a for small a the classical behaviour, the dynamics is clearly modified.

I.3 Quantum Configuration Space of LQC

I.3.1 Review of the Schrodinger Representation

In the Schrodinger representation, the Weyl operators

$$W(\alpha,\beta) := e^{i(\alpha q+\beta p)} \tag{I.58}$$

with standard momentum and position operators p and q and $\alpha, \beta \in \mathbb{R}$, satisfy the Weyl relation

$$W(\alpha_1,\beta_1)W(\alpha_2,\beta_2) := e^{-\frac{i}{2}(\alpha_1\beta_2-\alpha_2+\beta_1)}W(\alpha_1+\alpha_2,\beta_1+\beta_2). \tag{I.59}$$

Together with the unitarity condition

$$W(\alpha,\beta)^* = W(-\alpha,-\beta) \tag{I.60}$$

these relations alone define a unique simple C^*-algebra, the Weyl algebra. The Schrodinger representation is (up to unitary equivalence) the only irreducible representation of the Weyl algebra in which the Weyl operators are continuous fucntions of α and β with respect to the weak operator topology:

$$\lim_{\alpha'\to\alpha} < W(\alpha',\beta)\phi|\psi >=< W(\alpha,\beta)\phi|\psi > \tag{I.61}$$

for all $\alpha \in \mathbb{R}$ and for all $\phi, \psi \in \mathcal{H}$ and similarly for β.

$$[q,p] = i. \tag{I.62}$$

$$\int_{-\infty}^{\infty} < p|\psi(p)dp \;\text{ with }\; \int_{-\infty}^{\infty} |\psi(p)|^2 dp < \infty. \tag{I.63}$$

The momentum operator acting to the left on a coordinate eigenstate has the realization as the derivative with reaspect to the eigenvalue q'

Let us define

$$U = 1 + i\delta q' p \tag{I.64}$$

To infinitesimal order, this is a unitary operator,

$$U^{\dagger}U = 1,$$

and hence when acts on a state vector it preserves the norm of the vector.

the canonical commutation relation

$$UqU^{-1} = q + i[p, q]\delta q', \tag{I.65}$$

and so U is the operator which performs infinitesimal tasnlations on the coordinate operator $\hat{q}$ (but leaves p unchanged).

$$\begin{aligned} < q'|Uq &= < q'|UqU^{-1}V \\ &= < q'|(q' + \delta q')U \\ &= (q' + \delta q') < q'|U, \end{aligned} \tag{I.66}$$

which implies

$$< q'|U =< q' + \delta q'|. \tag{I.67}$$

The action of U on $< q'|$ is to infintesimal trasnslate the eigenvalue q' to $q' + \delta q'$, with $< q'|$ and $< q' + \delta q'|$ having the same norm.

$$< p'|p >= \delta(p' - p) \tag{I.68}$$

Transformation function $< q'|p' >$

$$< q'|p' > p' =< q'|p|p' >= \frac{1}{i}\frac{\partial}{\partial q'} < q'|p' >, \tag{I.69}$$

integrating this differential equation giving

$$< q'|p' >= \frac{1}{\sqrt{2}\pi} e^{iq'p'}, \tag{I.70}$$

$$\begin{aligned}\int dp' < q'|p' >< p'|q'' > &= \int \frac{dp'}{2\pi} e^{ip'(q'-q'')} \\ &= \delta(q'-q'') \\ &= < q'|q' >\end{aligned} \tag{I.71}$$

the resolution of identity

$$\int dp'|p' >< p'| = 1. \tag{I.72}$$

Similarly

$$\int dq' < p'|q' >< q'|p'' > = < p'|p'' >, \tag{I.73}$$

and

$$\int dq'|q' >< q'| = 1. \tag{I.74}$$

I.3.2 Polymer Representation

Let $\mathcal{H}_{\mathcal{P}}$ be a non-separable Hilbert space spanned by mutually orthoganol vectors $|p >$, $p \in \mathbb{R}$, $< p'|p >= \delta_{p'p}$, where $\delta_{p'p}$ is the Koneckera delta. A general element of $\mathcal{H}_{\mathcal{P}}$ is of the form

$$\sum_{p\in\mathbb{R}} \psi(p)|p > \text{ with } \sum_{p\in\mathbb{R}} |\psi(p)|^2 < \infty. \tag{I.75}$$

Thus the polymer Hilbert spae $\mathcal{H}_{\mathcal{P}}$ can also be defined as the space of complex functions on $\mathbb{R}$ that are square integrable with respect to the discrete measure. Necessarily, any wave fucntion $\psi(p)$ can be non-zero only on a countable subset of $\mathbb{R}$, as the uncountable sum of finite terms is divergent (for example the sum of numbers between 0 and ϵ is greater than $\frac{\epsilon}{2} + \frac{\epsilon}{2} + \cdots = \frac{\epsilon}{2}\infty = \infty$).

The momentum operator $\hat{p}$ is defined in this representation by:

$$\hat{p}(k)|p >= p|p > \quad \text{or} \quad \hat{p}\psi(p) = p\psi(p). \tag{I.76}$$

Since the discrete measure is translation invariant, there are also well defined unitary operators $\mathcal{U}(k)$ implimenting translaions in momentum space:

$$\mathcal{U}(k)|p>=|p+k> \quad \text{or} \quad \mathcal{U}(k)\psi(p)=\psi(p-k), \; k\in\mathbb{R}. \tag{I.77}$$

The exponentiation of the operator $\hat{p}$ together with the operators $\mathcal{U}(k)$ provide a representation of the Weyl relations:

$$\begin{aligned} e^{iap}e^{ibp}\psi(p) &= e^{i(a+b)p}\psi(p) \\ \mathcal{U}(a)\mathcal{U}(b)\psi(p) &= \psi(p-a-b) \\ &= \mathcal{U}(a+b)\psi(p) \end{aligned} \tag{I.78}$$

$$\begin{aligned} \mathcal{U}(a)e^{ibp}\psi(p) &= e^{ib(p-a)}\psi(p-a) \\ &= e^{-iab}e^{ibp}\mathcal{U}(a)\psi(p) \end{aligned} \tag{I.79}$$

Definition of weakly continuous

We say that a sequence k_n converges to k weakly if for all $p\in\mathcal{H}$

$$\lim_{n\to\infty} <k_n|p> = <k|p>$$

We say that $\mathcal{U}(p)$ is weakly continuous if

$$\lim_{n\to\infty} <\mathcal{U}(k_n)p|p> = <\mathcal{U}(k)p|p>$$

for all $p\in\mathcal{H}$ as k_n converges weakly to k.

$k\mapsto\mathcal{U}(k)$ is weakly continuous if

$$\lim_{\tau\to t} <\mathcal{U}(\tau)p'|p> = <\mathcal{U}(t)p'|p>$$

for all $t\in\mathbb{R}$ and $p',p\in\mathcal{H}$

However, the representation of $\mathbb{R}$ given by $k\mapsto\mathcal{U}(k)$ is not continuous as the general element of the Hilbert space $\mathcal{H}_{\mathcal{P}}$ is non-zero only on a countable subset of $\mathbb{R}$. In fact, for arbitrary small k, a vector $|p>$ is mapped by $\mathcal{U}(k)$ to an orthogonal one $|p+k>$ so that

$$< p|\mathcal{U}(k_n)p >= 0 \quad \text{for all } n. \tag{I.80}$$

However $\mathcal{U}(k)$ is not weakly continuous, since $|p>$ and $|p+k>$ are orthogonal to each other no matter how small the parameter t is. So one always has

$$| < p|\mathcal{U}(k)|p > - < p|p > | = < p|p > \neq 0, \tag{I.81}$$

even in the limit that k goes to zero. Therefore, the infintesimal generator

Thus, the generator that would correspond to the configuration operator $\hat{x}$ is not defined on $\mathcal{H}_{\mathcal{P}}$. The best we can do is:

$$-i\frac{(\mathcal{U}(\delta k) - 1)}{\delta k}|p> \quad \text{or} \quad -i\frac{(1 - \mathcal{U}(\delta k))}{\delta k}\psi(p) \tag{I.82}$$

where δk coresponds to the nearest point above p for which $\psi(p)$ is non-zero. The operators $\mathcal{U}(k)$ can then nevertheless be seen as giving a quantization of the classical configuration functions e^{ikx}. Reality conditions for these "approximated position operators" are satified, since $\mathcal{U}^{\dagger}(k) = \mathcal{U}(-k)$ Thus, the polymer representation provides a quantization of the space of the Poisson algebra of phase space functions made out of finite linear combinations of the functions

$$p \quad \text{and} \quad e^{ikx}, \;\; k \in \mathbb{R}. \tag{I.83}$$

In particular, the configuration part of the Poison algebra of phase space algebra is the linear space of continuous and bounded complex functions in $\mathbb{R}$ of the form

$$f(x) = \sum_j c_j e^{ik_j x}, \tag{I.84}$$

where the sums are finite, k_j are arbitrary real numbers and c_j are complex coefficents. The set of functions (I.84) clearly separates points in $\mathbb{R}$, i.e. given $x, x' \in \mathbb{R}$, $x \neq x'$, one can find a function f such that $f(x) \neq f(x')$. Infact, two functions are sufficient to separate points, e.g. $e^{ik_1 x}$ and $e^{ik_2 x}$, with k_1/k_2, an irrational number. To see this consider the case when k_1/k_2 is a rational number fig.(I.3.2). Neither function separates the points x_0 and x_1 as both functions are periodic over $x_1 - x_0$. However, if k_1/k_2 is an irrational number there are no finite intervals over which both functions are periodic.

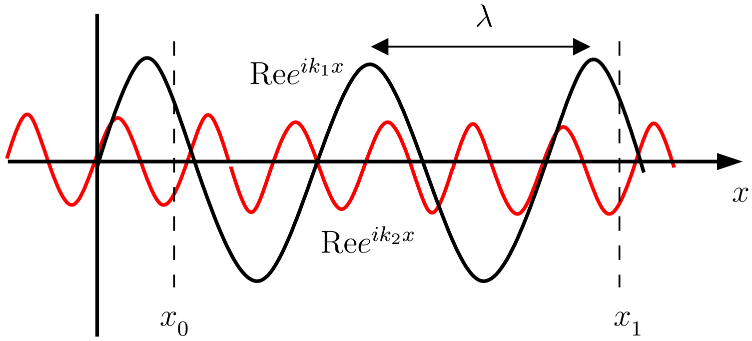

Figure I.1: separatenot. As $k_1/k_2 = 2.5$ both e^{ik_1x} and e^{ik_2x} are periodic over intervals of lengh 2λ, these two functions do not separate points an interger number of 2λ apart. However, if k_1/k_2 were irrational there would be no finite interval over which both functions were periodic.

I.3.3 Quantum Configuration Space as a Compact Group

The space $\overline{\mathbb{R}}$ is introduced as a set of homomorphisms, corresponding to a similar charachterization of $\overline{\mathcal{A}}$, section ??. The role of the group of hoops (or the groupoid of paths) is here played by the discrete group $\mathbb{R}$. The group $SU(2)$ is replaced by T, the unit circle in the complex plane $\mathbb{C}$.

The quantum configuration space of LQG includes all those connections which are discontinuous but all the same asign well defined holonomies (section ??), specifically:

$$\overline{A}(\gamma^{-1}) = (\overline{A}(\gamma))-1 \quad \text{and} \quad \overline{A}(\gamma_2 \cdot \gamma_1) = \overline{A}(\gamma_2)\overline{A}(\gamma_1) \tag{I.85}$$

We can understand these connections as the homomorphisms from $\mathcal{G}$ to the $SU(2)$ group.

$$\overline{\mathcal{A}} \equiv \mathrm{Hom}[\mathcal{G}??, SU(2)]. \tag{I.86}$$

This compactification can be imagined as being obtained from enlarging the classical configuration space $\mathbb{R}$ by adding points, and thus more continuity conditions, until only functions of the given algebra survive as continuous ones.

Let us consider the real line $\mathbb{R}$ equipped with the commutative group structure given by addition of real numbers. The Bohr compactification $\overline{\mathbb{R}}$ can be described as the set $[\mathbb{R}, T]$ of all, not necessary continuous, group homomorphisms from the group $\mathbb{R}$ to the multiplication group T of the unit circle of $\mathbb{C}$. we identify with the $\mathrm{Hom}[\mathbb{R}, T]$

$$\overline{\mathbb{R}} \equiv \mathrm{Hom}[\mathbb{R}, T]. \tag{I.87}$$

The generic element of $\bar{\mathbb{R}}$ will be denoted by $\bar{x}$. So, every $\overline{x} \in \overline{\mathbb{R}}$ is a map, $\overline{x} : \mathbb{R} \to T$ such that

$$\overline{x}(0) = 1 \quad \text{and} \quad \overline{x}(k_1 + k_2) = \overline{x}(k_1)\overline{x}(k_2), \text{ for all } k_1, k_2 \in \mathbb{R}. \tag{I.88}$$

Since T is a commutative group, it is clear that

$$\overline{x}\,\overline{x}'(k) := \overline{x}(k)\overline{x}'(k) \tag{I.89}$$

defines a group structure on $\overline{\mathbb{R}}$

From the fact that $\bar{\mathbb{R}}$ contains only homomorphisms, it is a closed subset of $\times_{k\in\mathbb{R}}T$, and it is therefore compact. Verify this.

PROOF HERE

$\square$

I.3.4 Projective Aspects

For arbitrary $n\ in \mathbb{N}$, a finite set of real numbers $\gamma = \{k_1, \ldots, k_n\}$ will said to be independent if $k_1, \ldots, k_n$ are algebraically independent if

$$\sum_{i=1}^{n} m_i k_i = 0, \quad m_i \in \mathbb{Z} \tag{I.90}$$

implies $m_i = 0$, for all i. The set of all such independent sets γ will be denoted by Γ.

Let G_γ denote the subgroup of $\mathbb{R}$ freely generated by the set $\gamma = \{k_1, \ldots, k_n\}$:

$$G_\gamma := \{\sum_{i=1}^{n} m_i k_i,\ m_i \in \mathbb{Z}\}. \tag{I.91}$$

By (I.90) we have an unique unit element corresponding to $m_i = 0$, for all i and a unique inverse of $\sum_i n_i k_i$ given by $\sum_i (-n_i) k_i$.

From this group structure we have a partial order relation making Γ a directed set: a set γ' is said to be greater than γ, and we write $\gamma' \geq \gamma$, if G_γ is a subgroup of $G_{\gamma'}$. It is clear that given γ and γ' one can always find γ'' such that $\gamma'' \geq \gamma$ and $\gamma'' \geq \gamma'$, and so Γ becomes a directed set. Also, Γ has no maximal element.

$$\overline{\mathbb{R}}_\gamma := \mathrm{Hom}[G_\gamma, T]. \tag{I.92}$$

For any pair γ, γ' such that $\gamma' \geq \gamma$ there are surjective projections

$$p_{\gamma\gamma'}; \overline{\mathbb{R}}_{\gamma'} \to \overline{\mathbb{R}}_\gamma \tag{I.93}$$

I.3.5 C^*-algebra Aspects

We will now see explicitely that $\overline{\mathbb{R}}$ is in fact the spectrum of the C^*-algebra of almost periodic functions in $\mathbb{R}$. This characterization of the quantum configuration space corresponds to the original introduction of the space of generalized connections $\overline{\mathcal{A}}$ as the spectrum of the holonomy algebra section(??).

Let us consider the configuration $*-$algebra $\mathcal{C}$ of functions given by finite sums of the form

$$f(x) = \sum_j c_j e^{ik_j x}, \tag{I.94}$$

with respect to the supremum norm, $\|f(x)\| = \sup_{x\in\mathbb{R}} |f(x)|$ this becomes a C^*-algebra (as $\|f^*(x)\| = \|f(x)\|$ and $\|f^*(x)f(x)\| = \|f(x)\|^2$). We form the C^*-completion $\overline{\mathcal{C}}$ with respect to this norm.

The spectrum $\Delta(\overline{\mathcal{C}})$ of the algebra $\overline{\mathcal{C}}$ is the set of all non-zero multipliactive linear functionals on $\overline{\mathcal{C}}$, i.e. non-zero linear functionals $\varphi : \overline{\mathcal{C}} \to \mathbb{C}$ such that

$$\varphi(fg) = \varphi(f)\varphi(g), \text{ for all } f, g \in \overline{\mathcal{C}}. \tag{I.95}$$

such functions are necessarily continuous as . One can check that $\varphi(e^{ikx})$ takes values in T, for all $\varphi \in \Delta(\overline{\mathcal{C}})$, for all k

$$|\varphi(e^{ikx})| =$$

$\{k_1, \ldots, k_n\}$, let $\mathcal{C}_\gamma \subset \mathcal{C}$ denote the $*-$subalgebra generated by the set of fucntions $\{e^{ik_1x}, \ldots, e^{ik_nx}\}$, whose elements are finite sums of the form

$$f(x) = \sum_k c_k e^{ikx} \quad \text{with} \quad k \in G_\gamma. \tag{I.96}$$

I.3.6 The Schrodinger Representation

We will now see how the usual Schrodinger representation is obtained in the present context. It is given by a different measure on $\overline{\mathbb{R}}$. The introduction of this measure corresponds to the so called $r-$Fock measwures in the loop quantization of $U(1)$ connections [??].

consider the Gausian measure $d\nu_G = e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}$ in $\mathbb{R}$, and the corresponding space of square integrable fucntions $L^2(\mathbb{R}, \nu_G)$. The Hilbert space $L^2(\mathbb{R}, \nu_G)$ carries (a representation unitarily equivalent to) the usual Schrodniger representation of the Weyl relations.

Hilbert space from measure

The standard choice is to select the Hilbert space to be,

$$\mathcal{H} = L^2(\mathbb{R}, dq)$$

the space of square-integrable functions with respect to the Lebesgue measure dq (invariant under constant translations) on $\mathbb{R}$.

There is a representation of the Weyl algebra that can be called the 'Fock type'.

the measure in the Schrodinger representation becomes non trivial and thus the momentum operator acquires an extra term in order to render the operator self-adjoint.

$$(\hat{q} \cdot \psi)(q) = q\psi(q) \tag{I.97}$$

and

$$(\hat{p} \cdot \psi)(q) = -i\hbar\frac{d\psi}{dq} + \text{ multiplicative term} \tag{I.98}$$

where the second term in (I.98), depending on the configuration, is precisely there to render the operator self-adjoint when the measure is different from the "dx" measure, and depends on the details of the measure.

First we need to find the measure $d\mu$ on the quantum configuration space in order to get the Hilbert space $\mathcal{H}_s$ and second we need to find the multiplicative term of the basic operator ((I.98).

Quantum algebra and states

Let $\mathcal{A}$ be a C^*-algebra with unit and let $\omega : \mathcal{A} \to \mathbb{C}$ be a stste. Then there exists a Hilbert space $\mathcal{H}$, a representation $\pi : \mathcal{A} \to L(\mathcal{H})$ and a vector $|\Psi_0 >\in \mathcal{H}$ such that

$$\omega(A) = < \Psi_0, \pi(A)|\Psi_0 >_{\mathcal{H}} . \tag{I.99}$$

Furthermore, the vector $|\Psi_0 >$ is cyclic. The triplet $(\mathcal{H}, \pi(A)|\Psi_0 >)$ with these properties is unique (up to unitary equivalence).

One key aspect of the theorem is that one may have different, but unitarily equivalent, representations of the Weyl algebra, which yeild equivalent quantum theories. This is the precise sense in which the Fock and Schrodinger representations are related to each other.

Thus, by vertue of the GNS construction, the value of the state ω_{Fock} acting on the Weyl generators $\hat{W}()$ is interpreted as the expectation value of the corresponding operators $R_{Fock}(\hat{W}())$ on the vacuum state $\Omega_{\mathcal{F}}$

we can now compute the expectation values of the Weyl operators of the Fock vacuum and thus obtain a positive linear functional ω_{Fock} on the algebra $\mathcal{A}$. Now, the Schrodinger representation that will be equivalent to the Fock construction will be the one that the GNS construction provides for the *same* algebraic state ω_{Fock}. What we must do to complete the Schrodinger construction such that the expectation value of the corresponding Weyl operators coincide with those of the Fock representation.

$$\omega_{Fock}(\hat{W}(\lambda)) = e^{-\frac{1}{4}\mu(\lambda,\lambda)} \tag{I.100}$$

Construction of the Fock representation

The relations

$$\{q_\mu, q_\nu\} = \{p_\mu, p_\nu\} = 0 \tag{I.101}$$
$$\{q_\mu, q_\nu\} = \delta_{\mu\nu} \tag{I.102}$$

$$\Omega(y_1, y_2) = \sum_\mu (p_{1\mu} q_{2\mu} - p_{2\mu} q_{1\mu}) \tag{I.103}$$

$$W(y) = \exp[i\Omega(y, \cdot)] \tag{I.104}$$

every point of Γ uniquely determines a solution. We define $\mathcal{B}$ to be the space of solutions which arise from the initial data in Γ.

The fundamental Poisson brackets on Γ can be expressed as

$$\{\Omega([q_1, p_1], \cdot), \Omega([q_2, p_2], \cdot)\} = -\Omega([q_1, p_1], [q_2, p_2]). \tag{I.105}$$

the CCR read

$$\{\int f_1\phi, \int f_2\pi\} = \int f_1 f_2 \tag{I.106}$$

We specify a real inner product $\mu : \mathcal{B} \times \mathcal{B} \to \mathbb{R}$ satisfying, for all $\psi_1 \in \mathcal{B}$,

$$\mu(\psi_1, \psi_1) = \frac{1}{4} l.u.b._{\psi_2 \neq 0} \frac{[\Omega(\psi_1, \psi_1)]^2}{\mu(\psi_2, \psi_2)} \tag{I.107}$$

We define an operator $J : \mathcal{B}_\mu \to \mathcal{B}_\mu$ defined by

$$\mu(\psi_1, \psi_2) = 2\mu(\psi_1, J\psi_2) = (\psi_1, J\psi_2) \tag{I.108}$$

From the antisymmetry of Ω, it follows that

$J^\dagger = -J$. $J^2 = -I$. the specification of an inner product, μ, saticfying (??) gives rise to a complex structure, J, on $\mathcal{B}$.

we now complexify $\mathcal{B}_\mu$ and extend the actions of Ω, μ, and J from $\mathcal{B}_\mu$ to $\mathcal{B}_\mu^{\mathbb{C}}$. We define an inner product on $\mathcal{B}_\mu^{\mathbb{C}}$ by

$$(\psi_1, \psi_2) = 2\mu(\overline{\psi}_1, J\psi_2) \tag{I.109}$$

for $\psi_1, \psi_2 \in \mathcal{B}_\mu^{\mathbb{C}}$, thus making $\mathcal{B}_\mu^{\mathbb{C}}$ into a (complex) Hlbert space.

define the map $K : \mathcal{B}_\mu^{\mathbb{C}} \to \mathcal{H}$ to be the orthogonal projection onto the subspace, $\mathcal{H}$, of $\mathcal{B}_\mu^{\mathbb{C}}$.

$$(K\psi_1, K\psi_2) = -i\Omega(\overline{K\psi}_1, K\psi_2) = \mu(\psi_1, \psi_2) - \frac{i}{2}\Omega(\psi_1, \psi_2). \tag{I.110}$$

we have

$$\mathrm{Im}(K\psi_1, K\psi_2)_{\mathcal{H}} = -\frac{1}{2}\Omega(\psi_1, \psi_2). \tag{I.111}$$

Functional representation

the Schrodinger representation - find the measure $d\mu$ and the multiplicative term in (I.98), that corresponds to the given Fock representation.

$$R_{Sch}(\hat{W}(\lambda)) = e^{i\hat{p}[f]}. \tag{I.112}$$

Now, the equation (??) tells us that the state ω_{Sch} should be such that,

$$R_{Sch}(\hat{W}(\lambda)) = \exp\left[-\frac{1}{4}\mu(\lambda,\lambda)\right] = \exp\left[-\frac{1}{4}fBf\right] \tag{I.113}$$

where we have used (??) in the last step. On the other hand, the left hand side of (I.100) is the vacuum expectation value of the $\hat{W}(\lambda)$ operator. That is,

$$\omega_{Sch}(\hat{W}(\lambda)) = \int_{\overline{\mathcal{C}}} d\mu \overline{\Psi_0}(R_{Sch}(\hat{W}(\lambda)) \cdot \Psi_0) = \int_{\overline{\mathcal{C}}} d\mu e^{if\psi} \tag{I.114}$$

Let us compare (??) and (??),

$$\int_{\overline{\mathcal{C}}} d\mu e^{i\psi} = \exp\left[-\frac{1}{4}fBf\right] \tag{I.115}$$

The meaning of this is - the fourier tgransform of the measure $\tilde{\mu}$ is defined as

$$\chi_{\tilde{\mu}}(f) := \int d\tilde{\mu} e^{if\varphi}, \tag{I.116}$$

where f is an arbitrary continuous function(al) on $\mathcal{V}$,...Then the measure of the Gausian

I.4 Path Integral

$$\begin{aligned} I(x_M,\ldots,x_0,\Delta\tau) &= \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \cdots \int_0^{\tau_2} d\tau_1 \; (i)^M e^{i(\Delta\tau-\tau_M)x_M} e^{i(\tau_M-\tau_{M-1})x_{M-1}} \\ &\qquad \ldots e^{i(\tau_2-\tau_1)x_1} e^{i\tau_1 x_0} \end{aligned} \tag{I.117}$$

We prove by induction that the integral is

$$I(x_M,\ldots,x_0,\Delta\tau) = \sum_{i=0}^{M} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M}(x_i - x_j)} \tag{I.118}$$

This is true by inspection for $M = 0$.

$$
\begin{aligned}
I(x_{M+1}, x_M, \ldots, x_0, \Delta\tau) &= \int_0^{\Delta\tau} d\tau_{M+1}\; ie^{i(\Delta\tau - \tau_{M+1})x_{M+1}} I(x_M, \ldots, x_0, \tau_{M+1}) \\
&= \int_0^{\Delta\tau} d\tau_{M+1}\; ie^{i(\Delta\tau - \tau_{M+1})x_{M+1}} \sum_{i=0}^{M} \frac{e^{ix_i\tau_{M+1}}}{\prod_{j\neq i}^{M}(x_i - x_j)} \\
&= ie^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j\neq i}^{M}(x_i - x_j)} \int_0^{\Delta\tau} d\tau_{M+1}\; e^{i(x_i - x_{M+1})\tau_{M+1}} \\
&= \sum_{i=0}^{M} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M+1}(x_i - x_j)} - e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j\neq i}^{M+1}(x_i - x_j)} \\
&= \sum_{i=0}^{M+1} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M+1}(x_i - x_j)} - e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M+1} \frac{1}{\prod_{j\neq i}^{M+1}(x_i - x_j)}
\end{aligned}
\tag{I.119}
$$

We use the identity

$$
\sum_{i=0}^{M+1} \frac{1}{\prod_{j\neq i}^{M+1}(x_i - x_j)} = 0 \tag{I.120}
$$

which we see from considering the simple case

$$
\begin{aligned}
\sum_{i=0}^{2} \frac{1}{\prod_{j\neq i}^{2}(x_i - x_j)} &= \frac{1}{(x_0 - x_1)(x_0 - x_2)} + \frac{1}{(x_1 - x_0)(x_1 - x_2)} \\
&\quad + \frac{1}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{(x_2 - x_1) - (x_2 - x_0) + (x_1 - x_0)}{(x_2 - x_1)(x_2 - x_0)(x_1 - x_0)} \\
&= 0
\end{aligned}
\tag{I.121}
$$

The integral can be written

$$
I(x_{M+1}, x_M, \ldots, x_0, \Delta\tau) = \sum_{i=0}^{M+1} \frac{e^{ix_i\Delta\tau}}{\prod_{j\neq i}^{M+1}(x_i - x_j)} \tag{I.122}
$$

$$
\begin{aligned}
I(y_1, n_1+1) &= \int_0^{\Delta\tau} d\tau_{n_1} \dots \int_0^{\tau_2} d\tau_1 \; (i)^{n_1} e^{iy_1 \Delta\tau} \\
&= (i)^{n_1} e^{iy_1\Delta\tau} \int_0^{\Delta\tau} d\tau_{n_1} \frac{(\tau_{n_1-1})^{n_1-1}}{(n_1-1)!} \\
&= \frac{(i\tau_{n_1})^{n_1}}{(n_1)!} e^{iy_1\Delta\tau}
\end{aligned} \tag{I.123}
$$

$$
I(y_1, n_1) = \left(\frac{\partial}{\partial y_1}\right)^{n_1-1} \frac{1}{(n_1-1)!} e^{iy_1\Delta\tau} \tag{I.124}
$$

Plugging in the assumed result for p distint values

$$
\begin{aligned}
I(y_{p+1}, n_{p+1}, \dots, y_1, n_1, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \dots \int_0^{\tau_{M-n_{p+1}+2}} d\tau_{M-n_{p+1}+1} \\
(i)^{n_{p+1}-1} e^{i(\Delta\tau - \tau_{M-n_{p+1}+1}) y_{p+1}} I(y_p, n_p, \dots, y_1, n_1, \tau_{M-n_{p+1}+1}) \\
= \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p} (y_i - y_j)} \\
(i)^{n_{p+1}-1} e^{i\Delta\tau y_{p+1}} \int_0^{\Delta\tau} d\tau_M \dots \int_0^{\tau_{M-n_{p+1}+2}} d\tau_{M-n_{p+1}+1} e^{i(y_i - y_{p+1})\tau_{M-n_{p+1}+1}}
\end{aligned} \tag{I.125}
$$

We are interested in evaluating the integral

$$
I(n,t) = (i)^n e^{ity} \int_0^t dt_n \dots \int_0^{t_2} dt_1 e^{i(y_i - y)t_1}
$$

we prove by induction that the answer is

$$
\frac{e^{iy_i t}}{(y_i - y)^n} - \sum_{m=1}^{n} \frac{e^{iyt}}{(y_i - y)^m} \frac{(it)^{n-m}}{(n-m)!} \tag{I.126}
$$

$$
\begin{aligned}
I(n+1,t) &= (i)^{n+1}e^{ity}\int_0^t dt_{n+1}\int_0^{t_{n+1}} dt_n \dots \int_0^{t_2} dt_1 e^{i(y_i-y)t_1} \\
&= (i)^{n+1}e^{ity}\int_0^t dt_{n+1}(i)^{-n}e^{-it_{n+1}y}\left[(i)^n e^{it_{n+1}y}\int_0^{t_{n+1}} dt_n \dots \int_0^{t_2} dt_1 e^{i(y_i-y)t_1}\right] \\
&= ie^{ity}\int_0^t dt_{n+1}e^{-it_{n+1}y}I(n,t_{n+1}) \\
&= ie^{ity}\int_0^t dt_{n+1}e^{-it_{n+1}y}\left[\frac{e^{iy_it_{n+1}}}{(y_i-y)^n}-\sum_{m=1}^{n}\frac{e^{it_{n+1}y}}{(y_i-y)^m}\frac{(it_{n+1})^{n-m}}{(n-m)!}\right] \\
&= ie^{ity}\left[\frac{e^{i(y_i-y)t}}{i(y_i-y)^{n+1}}-\frac{1}{i(y_i-y)^{n+1}}-\sum_{m=1}^{n}\frac{1}{(y_i-y)^m}\frac{(i)^{n-m}(t)^{n-m+1}}{(n-m+1)!}\right] \\
&= \frac{e^{iy_it}}{(y_i-y)^{n+1}}-\sum_{m=1}^{n+1}\frac{e^{ity}}{(y_i-y)^m}\frac{(it)^{n-m+1}}{(n-m+1)!}
\end{aligned}
$$

As

$$
\begin{aligned}
&\frac{1}{n-1}\frac{\partial}{\partial y}I(n-1,t;y) \\
&= \frac{1}{n-1}\frac{\partial}{\partial y}\left[\frac{e^{iy_it}}{(y_i-y)^{n-1}}-\sum_{m=1}^{n-1}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m-1}}{(n-m-1)!}\right] \\
&= \frac{e^{iy_it}}{(y_i-y)^n}-\frac{1}{n-1}\sum_{m=1}^{n-1}\left[\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m-1)!}+\frac{e^{iyt}}{(y_i-y)^{m+1}}\frac{(it)^{n-m-1}m}{(n-m-1)!}\right] \\
&= \frac{e^{iy_it}}{(y_i-y)^n}-\frac{1}{n-1}\sum_{m=1}^{n-1}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m-1)!}+\sum_{m=1}^{n}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}(m-1)}{(n-m)!} \\
&= \frac{e^{iy_it}}{(y_i-y)^n}-\frac{1}{n-1}\sum_{m=1}^{n-1}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m)!}[(n-m+(m-1)]-\frac{1}{n-1}\frac{e^{iyt}}{(y_i-y)^n}\frac{n-1}{0!} \\
&= \frac{e^{iy_it}}{(y_i-y)^n}-\sum_{m=1}^{n}\frac{e^{iyt}}{(y_i-y)^m}\frac{(it)^{n-m}}{(n-m)!} \\
&= I(n,t;y)
\end{aligned}
\tag{I.127}
$$

so

$$
\begin{aligned}
I(n) &= \frac{1}{(n-1)!}\left(\frac{\partial}{\partial y}\right)^{n-1}I(1) \\
&= \frac{1}{(n-1)!}\left(\frac{\partial}{\partial y}\right)^{n-1}\left(\frac{e^{iy_it}}{y_i-y}-\frac{e^{iyt}}{y_i-y}\right)
\end{aligned}
\tag{I.128}
$$

Using this in ()

$$
\begin{aligned}
I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \\
\left[\frac{1}{(n_{p+1}-1)!}\left(\frac{\partial}{\partial y_{p+1}}\right)^{n_{p+1}-1}\left(\frac{e^{iy_i\Delta\tau}}{y_i-y_{p+1}}-\frac{e^{iy_{p+1}\Delta\tau}}{y_i-y_{p+1}}\right)\right] \\
= \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1}\left[\sum_{i=1}^{p}\frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)}\left(\frac{e^{iy_i\Delta\tau}}{y_i-y_{p+1}}-\frac{e^{iy_{p+1}\Delta\tau}}{y_i-y_{p+1}}\right)\right] \\
= \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1}\left[\left(\sum_{i=1}^{p}\frac{e^{iy_i\Delta\tau}}{\prod_{j\neq i}^{p+1}(y_i-y_j)} - e^{iy_{p+1}\Delta\tau}\sum_{i=1}^{p}\frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)}\frac{1}{y_i-y_{p+1}}\right)\right]
\end{aligned}
\tag{I.129}
$$

Now we use the identity

$$
\sum_{i=1}^{p+1}\frac{1}{\prod_{j\neq i,j=1}^{p+1}(y_i-y_j)} = 0
$$

which implies

$$
\sum_{i=1}^{p}\frac{1}{\prod_{j\neq i,j=1}^{p+1}(y_i-y_j)} \equiv \sum_{i=1}^{p}\frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)}\frac{1}{y_i-y_{p+1}} = -\frac{1}{\prod_{j=1}^{p}(y_i-y_{p+1})}
$$

Using this in ()

$$
\begin{aligned}
I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \\
\left[\sum_{i=1}^{p}\frac{e^{iy_i\Delta\tau}}{\prod_{j\neq i}^{p+1}(y_i-y_j)} + \frac{e^{iy_{p+1}\Delta\tau}}{\prod_{j=1}^{p}(y_i-y_{p+1})}\right] \\
= \prod_{k=1}^{p+1} \frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p+1}\frac{e^{iy_i\Delta\tau}}{\prod_{j\neq i}^{p+1}(y_i-y_j)}
\end{aligned}
\tag{I.130}
$$

I.4.1 Eigenstates of Θ

Recall

$$
\begin{aligned}
(\Theta\Psi)(\nu) \;:=\; & -\frac{3\pi G}{4\ell_0^2}\left[\sqrt{|\nu(\nu+4\ell_0)|}(\nu+2\ell_0)\Psi(\nu+4\ell_0)-2\nu^2\Psi(\nu)+\right.\\
& \left.\sqrt{|\nu(\nu-4\ell_0)|}(\nu-2\ell_0)\Psi(\nu-4\ell_0)\right]
\end{aligned}
$$

$$
\chi(b) := \sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0} e^{\frac{i}{2}\nu b}\frac{\Psi(\nu)}{\sqrt{|\nu|}} \tag{I.131}
$$

$$
\begin{aligned}
(\Theta\chi)(b) &= \sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0} e^{\frac{i}{2}\nu b}\frac{(\Theta\Psi)(\nu)}{\sqrt{|\nu|}}\\
&= -\frac{3\pi G}{4\ell_0^2}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0}\frac{e^{\frac{i}{2}\nu b}}{\sqrt{|\nu|}}\left[\sqrt{|\nu(\nu+4\ell_0)|}(\nu+2\ell_0)\Psi(\nu+4\ell_0)-2\nu^2\Psi(\nu)+\right.\\
&\quad\left.\sqrt{|\nu(\nu-4\ell_0)|}(\nu-2\ell_0)\Psi(\nu-4\ell_0)\right]\\
&= -\frac{3\pi G}{4\ell_0^2}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0}\left[e^{\frac{i}{2}(\nu-4\ell_0)b}\sqrt{|\nu|}(\nu-2\ell_0)-2\frac{e^{\frac{i}{2}\nu b}}{\sqrt{|\nu|}}\nu^2+e^{\frac{i}{2}(\nu+4\ell_0)b}\sqrt{|\nu|}(\nu+2\ell_0)\right]\Psi(\nu)\\
&= -\frac{3\pi G}{4}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0} e^{\frac{i}{2}\nu b}\left[\sqrt{|\nu|}\nu(e^{2i\ell_0 b}+e^{-2i\ell_0 b})-\frac{2\nu^2}{\sqrt{|\nu|}}+\sqrt{|\nu|}2\ell_0(e^{2i\ell_0 b}-e^{-2i\ell_0 b})\right]\Psi(\nu)\\
&= -\frac{3\pi G}{4\ell_0^2}\sqrt{\frac{\ell_0}{\pi}}\sum_{\nu=4n\ell_0}\frac{e^{\frac{i}{2}\nu b}}{\sqrt{|\nu|}}\left[2\nu^2\left(\frac{\cos\ell_0 b}{\ell_0}\right)^2+4i\nu\frac{\sin\ell_0 b}{\ell_0}\cos\ell_0 b\right]\Psi(\nu)\\
&=
\end{aligned} \tag{I.132}
$$

This can then be wriitten as a simple differntial equation

$$
(\Theta\chi_k)(b) = -12\pi G\left(\frac{\sin\ell_0 b}{\ell_0}\partial_b\right)^2\chi_k(b) = \omega_k^2\chi_k(b) \tag{I.133}
$$

which have the solutions

$$
\chi_k(b) = A(k)\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}. \tag{I.134}
$$

Let us check this:

$$\begin{aligned}
-12\pi G\left(\frac{\sin\ell_0 b}{\ell_0}\partial_b\right)^2\chi_k(b) &= -A(k)\frac{12\pi G}{\ell_0^2}\sin\ell_0 b\,\partial_b\left[\frac{ik\sin\ell_0 b\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}}{\tan\frac{\ell_0 b}{2}}\frac{\ell_0}{2}\frac{1}{\cos^2\frac{\ell_0 b}{2}}\right]\\
&= -A(k)\frac{12\pi G}{\ell_0^2}\sin\ell_0 b\,\partial_b[ik\ell_0\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}]\\
&= A(k)12\pi G k^2\; e^{ik\;\ln(\tan\frac{\ell_0 b}{2})}\\
&= 12\pi G k^2\chi_k(b).
\end{aligned}$$

We see that the eigenvalues are

$$\omega_k^2 = 12\pi G k^2 \tag{I.135}$$

To express these eigenvectors in the ν representation we need the inverse transformation of (I.131). To this end let us write (I.131) as

$$\begin{aligned}
\chi(b) &= \sum_{-\infty}^{\infty} e^{in(2\ell_0 b)}\left(\sqrt{\frac{\ell_0}{\pi}}\frac{\Psi(4\ell_0 n)}{\sqrt{4\ell_0|n|}}\right)\\
&=: \sum_{-\infty}^{\infty} e^{inb'}\psi(n)
\end{aligned}$$

Using the Fourier inverse formula

$$\begin{aligned}
\psi(n) &= \frac{1}{2\pi}\int_0^{2\pi} db'\; e^{-inb'}\chi(b'/2\ell_0)\\
&= \frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\; e^{-in2\ell_0 b}\chi(b)
\end{aligned}$$

we obtain

$$\sqrt{\frac{\ell_0}{\pi}}\frac{\Psi(\nu)}{\sqrt{|\nu|}} = \frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db e^{-i\frac{\nu}{2}b}\chi(b)$$

or

$$\Psi(\nu) = \sqrt{\frac{\ell_0|\nu|}{\pi}} \int_0^{\pi/\ell_0} dbe^{-i\frac{\nu}{2}b}\chi(b) \tag{I.136}$$

Eigenvectors with non-zero eigenvalues can also be expressed in the ν representation by applying the inverse transformation

$$e_k(\nu) = A(k)\sqrt{\frac{\ell_0|\nu|}{\pi}} \int_0^{\pi/\ell_0} db\; e^{-\frac{i}{2}\nu b} e^{ik\ln(\tan\frac{\ell_0 b}{2})} \tag{I.137}$$

Put $\theta = b\ell_0$ then

$$\frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\; e^{-2ib\ell_0 n} e^{ik\;\ln(\tan\frac{\ell_0 b}{2})} = \frac{1}{\pi}\int_0^{\pi} d\theta\; e^{-2i\theta n} e^{ik\;\ln(\tan\frac{\theta}{2})}$$

now

$$\begin{aligned}
\tan\frac{x}{2} = \frac{e^{i\theta/2} - e^{-i\theta/2}}{i(e^{i\theta/2} + e^{-i\theta/2})} &= \frac{i(1-e^{i\theta})}{1+e^{i\theta}} \\
&= \exp[\ln i + \ln\frac{1-e^{i\theta}}{1+e^{i\theta}}] \\
&= \exp[e^{i\pi/2} + \ln\frac{1-e^{i\theta}}{1+e^{i\theta}}]
\end{aligned}$$

$$\frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\; e^{-2ib\ell_0 n} e^{ik\;\ln(\tan\frac{\ell_0 b}{2})} = \frac{e^{-\pi k/2}}{\pi}\int_0^{\pi} (e^{i\theta})^{-2n}\left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right)^{ik} d\theta =: J(k,n) \tag{I.138}$$

$$d\theta = \frac{dz}{iz}$$

$$J(k,n) = \frac{e^{-\pi k/2}}{\pi i}\int_{\mathcal{C}} (z)^{-2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz \tag{I.139}$$

where $\mathcal{C}$ is the unit semicircle in counterwise direction (note that $\mathcal{C}$ not a closed contour) in the upper half plane, $\mathrm{Im}z > 0$, of the complex plane.

The second independent eigenfunction $e_{-k}(\nu)$ with the same eigenvalue ω_k^2 can be represented in a similar fashion by replacing $e^{i\theta} \to -e^{i\theta}$:

$$
\begin{aligned}
&J(-k,n)\\
= \quad &\frac{\ell_0}{\pi}\int_0^{\pi/\ell_0} db\, e^{-2ib\ell_0 n} e^{-ik\ \ln(\tan\frac{\ell_0 b}{2})} = -\frac{e^{\pi k/2}}{\pi}\int_{-\pi}^{0}(-e^{i\theta})^{-2n}\left(\frac{1+(-e^{i\theta})}{1-(-e^{i\theta})}\right)^{-ik} d\theta
\end{aligned}
$$

Set $z = -e^{i\theta}$

$$
d\theta = -\frac{dz}{iz}
$$

The result is a contour integral along the unit semicircle in the counterclockwise direction in the lower half, Im$z < 0$, of the complexplane.

$$
J(-k,n) = \frac{e^{\pi k/2}}{\pi i}\int_{-\mathcal{C}}(z)^{-2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz \tag{I.140}
$$

Combining the integrals

$$
\frac{1}{2}\left(e^{\pi k/2}J(k,n) + e^{-\pi k/2}J(-k,n)\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{ix})^{-2n}\left(\frac{1-e^{ix}}{1+e^{ix}}\right)^{ik} dx =: I(k,n) \tag{I.141}
$$

Being a linear combination of $e_k(\nu)$ and $e_{-k}(\nu)$, this $I(k,n)$ gives also an eigenfunction of Θ with eigenvalue ω_k^2. Using complex analysis we can evaluate $I(k,n)$.

$$
\frac{1}{2\pi i}\oint z^{-2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz \tag{I.142}
$$

Recall from basic complex anaylsis that

$$
\oint_C f(z)dz = 2\pi i\sum Res[f(z_0)]
$$

where, if a function has an m-th pole at z_0,

$$
Res[f(z_0)] = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^{m-1}f(z)]\right)
$$

Therefore

$$I(k,n) = \begin{cases} \frac{1}{(2n)!}\frac{d^{2n}}{z^{2n}}\left(\frac{1-s}{1+s}\right)^{ik}\Big|_{s=0} & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{I.143}$$

We repeat the same argument but taking $z = e^{-b\ell_0}$ and $z = -e^{-b\ell_0}$

Setting $z = e^{-ix}$ we get

$$J(-k,n) = \frac{e^{\pi k/2}}{\pi}\int_{-\pi}^{0}(e^{-ix})^{2n}\left(\frac{1-e^{ix}}{1+e^{ix}}\right)^{ik} dx \tag{I.144}$$

$$\frac{1}{2}(e^{-\pi k/2}J(k,n) + e^{\pi k/2}J(-k,n)) = \frac{1}{2\pi i}\oint z^{2n-1}\left(\frac{1-z}{1+z}\right)^{ik} dz = I(k,-n) \tag{I.145}$$

the basis is

$$\begin{aligned} e_k^{\pm}(\nu) &:= \frac{1}{2}(e^{\pm\pi k/2}e_k(\nu) + e^{\mp\pi k/2}e_k(\nu)) \\ &= A(k)\sqrt{\frac{\pi|\nu|}{\ell_0}}\, I(k, \pm\frac{\nu}{4\ell_0}) \end{aligned} \tag{I.146}$$

Normalisation of the vectors

There the functions describing the states are

$$\chi_k^{\pm}(b) = \frac{A(k)}{2}\left(e^{\pm\pi k/2}e^{ik\ln(\tan\frac{\ell_0 b}{2})} + e^{\mp\pi k/2}e^{-ik\ln(\tan\frac{\ell_0 b}{2})}\right) \tag{I.147}$$

$$
\begin{aligned}
< k'+|k+> &= \int_0^{\pi/\ell_0} db\; 2i\; \overline{\chi^+_{k'}}(b)\partial_b\chi^+_k(b) \\
&= -\frac{1}{2}A(k)\overline{A(k')}\; k\ell_0 \int_0^{\pi/\ell_0} db\; \left(e^{+\pi k'/2}e^{-ik'\ln(\tan\frac{\ell_0 b}{2})}+e^{-\pi k'/2}e^{ik'\ln(\tan\frac{\ell_0 b}{2})}\right) \\
&\quad \frac{1}{\sin\ell_0 b}\left(-e^{+\pi k/2}e^{ik\ln(\tan\frac{\ell_0 b}{2})}+e^{-\pi k/2}e^{-ik\ln(\tan\frac{\ell_0 b}{2})}\right) \\
&= -\frac{1}{2}A(k)\overline{A(k')}\; k\ell_0 \int_0^{\pi/\ell_0} \frac{\ell_0 db}{\sin\ell_0 b} \\
&\quad \left(-e^{+\pi(k+k')/2}e^{i(k-k')\ln(\tan\frac{\ell_0 b}{2})}+e^{-\pi(k+k')/2}e^{-i(k-k')\ln(\tan\frac{\ell_0 b}{2})}\right. \\
&\quad \left.+e^{+\pi(-k+k')/2}e^{-i(k+k')\ln(\tan\frac{\ell_0 b}{2})}-e^{+\pi(k-k')/2}e^{i(k+k')\ln(\tan\frac{\ell_0 b}{2})}\right)
\end{aligned}
$$

Make the change of variables $y=\ln(\tan\frac{\ell_0 b}{2})$

$$
dy=\frac{\ell_0 db}{4\sin\ell_0 b}
$$

then

$$
\begin{aligned}
< k'+|k+> &= -\frac{1}{2}A(k)\overline{A(k')}\; k\int_{-\infty}^{\infty} dy \\
&\quad \left(-e^{+\pi(k+k')/2}e^{i(k-k')y}+e^{-\pi(k+k')/2}e^{-i(k-k')y}\right. \\
&\quad \left.+e^{+\pi(-k+k')/2}e^{-i(k+k')y}-e^{+\pi(k-k')/2}e^{i(k+k')y}\right) \\
&= A(k)\overline{A(k')}\; k2\pi\; \frac{e^{k\pi}-e^{-k\pi}}{2}[\delta(k',k)+\delta(k',-k)] \\
&= \\
&= |A(k)|^2\; 2\pi k\sinh(\pi k)\;\delta(k',k)
\end{aligned}
\qquad (I.148)
$$

Using this we can find the normalisation of the eigenvectors $e_k(\nu)$:

From

$$
|k\pm>=\frac{1}{2}(e^{\pm\pi k/2}|k>+e^{\mp\pi k/2}|-k>)
$$

we get

$$|k> = \frac{2(e^{\pi k/2}|k+> - e^{-\pi k/2}|k->)}{e^{\pi k} - e^{-\pi k}}$$

and so

$$\begin{aligned}
<k'|k> &= \frac{2(e^{\pi k/2} < k+| - e^{-\pi k/2} < k-|)}{e^{\pi k} - e^{-\pi k}} \frac{2(e^{\pi k/2}|k+> - e^{-\pi k/2}|k->)}{e^{\pi k} - e^{-\pi k}} \\
&= 4\frac{e^{\pi k} < k+|k+> + e^{-\pi k} < k-|k->}{(e^{\pi k} - e^{-\pi k})^2} \\
&= |A(k)|^2 \, 4\pi k \sinh(\pi k) \frac{\cosh(\pi k)}{\sinh^2(\pi k)} \, \delta(k', k) \\
&= |A(k)|^2 \, 4\pi k \coth(\pi k) \, \delta(k', k) \qquad \text{(I.149)}
\end{aligned}$$

I.4.2 Matrix Elements for $f(\Theta)$

I.5 Summary

• Microscopic source for blackhole entropy. Once this parameter is fixed the correct formula for any no extremum blackhole (except rotating ones maybe)

• Removal of cosmological singurlarity. Evolutional equations don't break down at the place where the classical singularity is.

• Initial conditions are derived rather than guessed at. Not suprising as the constraint equations are admissible conditions on the initial data.

• First direct derivation of inflation from a candidate for quantum gravity. Due to a quatnum geometry effect in the early kinematic dominated universe.

• Some features in the minisuperspace are shared with the full theory. Allows proper investigation of the dynamics of minisuperspace that could shed some light on the dynamics of the full theory.