Appendix K

Quantization Schemes

K.1 Introduction

In view of these new physical settings we must first re-examine the quantization problem from a somewhat broader perspective. Standard methods?

One constructs a suitable algebra of functions on the classical phase space, promotes it to an operator algebra and then seeks its representations by operators on a Hilbert space.

[arXiv: quant-ph/ 0412015]. Quantisation is the problem of deriving the mathematical framework of a quantum mechanical system from the mathematical framework of the corresponding classical mechanical system. A method of quantisation must contain a map $\mathcal{A}$ from the set of classical observables to the set of quantum observables with the following properties:

Linear functionals on an observable's operator algebra. quantization as functionals on the set of observables. in other words expectation values of observables - these are called states ω

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a collection 'observables' and is capable of being in certain 'states'. We can define the state of a system as knowledge of the expected values of the observables. that is , a state is an assignment of an expected value of each observable.

K.2 Algebraic Quantum Theory

In quantum mechanics we often start by taking classical observables an write down Poisson brackets. We then promote these to operator equations that express that they don't commute. The first to spring to mind might well be

$$\hat{p}\hat{q} - \hat{q}\hat{p} = i\hbar \tag{K.1}$$

which came from

$$\{p, q\} = -1 \tag{K.2}$$

but there are many others.

When we do this, we are doing is defining an **"algebra of observables"** (or what mathematicians refer to as an algebra).

One way to get a hold of states is to take your algebra of observables and represent it as an algebra of operators on a Hilbert space. Then the unit vectors in your Hilbert space represent states. However, the same algebra of observables can have different representations as operators on a Hilbert space.

K.2.1 The Harmonic Oscillator

with eigenvector equation

$$\left(\frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2\right)\psi = E\psi \tag{K.3}$$

where and are self-adjoint operators such that $[\hat{p}, \hat{x}] = -i\hbar I$.

The algebra $\mathcal{A}$ of observables for the one dimensional harmonic oscillator is generated by the operators the Hamiltonian $\mathcal{H}$, the operator P momentum, and Q. The defining algebraic relations are:

$$\mathcal{H} = \frac{1}{2m}\hat{p}^2 + \frac{1}{2}m\omega^2\hat{x}^2, \qquad [P, Q] = -i\hbar I. \tag{K.4}$$

The elements of $\mathcal{A}$ are assumed to be linear operators defined on a linear space Ψ. There is a scalar product $(\cdot, \cdot)$ defined on Ψ that provides probability amplitudes (but Ψ is not a Hilbert space).

There are many realisations of the vector space Ψ on which $\mathcal{A}$ is an algebra of operators. We have to make one further assumption to fully specify the realisation of $\mathcal{A}$

there exists at least one non-degenerate eigenvalue of $\mathcal{H}$ whose corresponding eigenvector

$$\Omega \text{ is an element of } \Psi. \tag{K.5}$$

the physics of the harmonic oscillator is described by an algebra of observables that satisfy the algebraic assumptions (K.4)-(K.5).

To construct the space Ψ, we make elements of $\mathcal{A}$ act on the eigenvector Ω.

$$\hat{a} = \hat{p} - im\omega\hat{x} \tag{K.6}$$

and its adjoint $\hat{a}^\dagger$

$$\hat{a}^\dagger = \hat{p} + im\omega\hat{x} \tag{K.7}$$

$$[\hat{a}, \hat{a}^\dagger] = \tag{K.8}$$

$$(\hat{p} \mp im\omega\hat{x})^*(\hat{p} - \pm im\omega\hat{x}) = \hat{p}^2 + m^2\omega^2\hat{x}^2 \mp m\omega\hbar I = 2m(\mathcal{H} \mp \frac{1}{2}\hbar\omega). \tag{K.9}$$

$$\begin{aligned}
(\varphi, a\psi) &= (a^\dagger\varphi, \psi), \qquad \text{for all } \varphi, \psi \in \Psi, && \text{(K.10)}\\
(\varphi, N\psi) &= (N\varphi, \psi), \qquad \text{for all } \varphi, \psi \in \Psi. && \text{(K.11)}
\end{aligned}$$

$$[a, a^\dagger] = I. \tag{K.12}$$

The assumption about the existence of Ω implies that there exists a $\varphi_\lambda \neq 0$ in Ψ such that

$$N\varphi_\lambda = \lambda\varphi_\lambda. \tag{K.13}$$

From (K.11) and (K.13) it follows that

$$\lambda(\varphi_\lambda, \varphi_\lambda) = (\varphi_\lambda, N\varphi_\lambda) = (N\varphi_\lambda, \varphi_\lambda) = \overline{\lambda}(\varphi_\lambda, \varphi_\lambda). \tag{K.14}$$

Therefore, $\lambda = \overline{\lambda}$, i.e., λ is real. From the commutation relation (K.12), it then follows that

$$\begin{aligned}
N(a\varphi_\lambda) &= a^\dagger a\varphi_\lambda = (aa^\dagger - I)a\varphi_\lambda\\
&= a(N - I)\varphi_\lambda = a(\lambda - 1)\varphi_\lambda\\
&= (\lambda - 1)a\varphi_\lambda && \text{(K.15)}
\end{aligned}$$

This implies either $a\varphi_\lambda$ is an eigenvector of N with eigenvalue $(\lambda - 1)$ or $a\varphi_\lambda = 0$. From (K.10) and from the commutation relation (K.12) it follows that

$$\|a^\dagger\varphi_\lambda\|^2 = (\varphi_\lambda, a^\dagger a\varphi_\lambda) + (\varphi_\lambda, I\varphi_\lambda) = \|a\varphi_\lambda\|^2 + \|\varphi_\lambda\|^2 \neq 0, \tag{K.16}$$

since φ_λ is different from the zero vector. This means that $a^\dagger\varphi_\lambda \neq 0$. equation (K.12) implies that

$$N(a^\dagger\varphi_\lambda) = (\lambda + 1)a^\dagger\varphi_\lambda, \tag{K.17}$$

i.e., $a^\dagger\varphi_\lambda$ is an eigenvector of N with eigenvalue $(\lambda + 1)$.

States as Expectation Values

In quantum mechanics one finding the expectation value of an operator. If the system is described by a pure state with wave function $\psi(\mathbf{r}, t)$ then the expectation value is given by

$$< \psi|\hat{A}|\psi >= \int \psi^*(\mathbf{r}, t)\hat{A}\psi(\mathbf{r}, t)d^3r \tag{K.18}$$

For every ψ in $\mathcal{H}$ the expectation value E_ψ has the following properties:

(i) E_ψ (I), where I denotes the identity operator on $\mathcal{H}$.

(ii) $E_\psi(A)$ is real for all self-adjoint operators A.

(iii) $E_\psi(A) \geq 0$ for all positive operators A.

(iv) $E_\psi(A)$ depends linearly on A, that is $E_\psi(\alpha A + \beta B) = \alpha E_\psi(A) + \beta E_\psi(B)$ for all complex numbers α and β and all linear operators A and B.

Proof:

(iii) When A is the identity operator

$$E_\psi(I) = \frac{< \psi|\psi >}{||\psi||^2} = 1. \tag{K.19}$$

(ii) Since A is self-adjoint we have

$$E_\psi(A) = \frac{< \psi|A\psi >}{||\psi||^2} = \frac{< A\psi|\psi >}{||\psi||^2} = \frac{\overline{< \psi|A\psi >}}{||\psi||^2} = \overline{E_\psi(A)}, \tag{K.20}$$

so that E_ψ is real.

(iii) A linear transformation is said to be positive if

$$< A\psi|\psi > \geq 0 \tag{K.21}$$

for all vectors ψ.

(iv)

$$\begin{aligned} E_\psi(\alpha A + \beta B) &= \frac{< \psi|\alpha A + \beta B\psi >}{||\psi||^2} \\ &= \alpha\frac{< \psi|A\psi >}{||\psi||^2} + \beta\frac{< \psi|B\psi >}{||\psi||^2} \\ &= \alpha E_\psi(A) + \beta E_\psi(B) \end{aligned} \tag{K.22}$$

Statistical Mechanical States

$$\mathrm{tr}(\mathcal{Q}_\psi A) = \frac{< \psi|\hat{A}\xi >}{\|\psi\|^2} = E_\psi \tag{K.23}$$

suppose we only know that there is a probability p_k that he state is described by the vector ψ_k. $< \psi|\hat{A}\ .\ > /\|\psi\|^2$ is sometimes referred to as a trace class operator. The expectation value should then be given by the weighted average

$$\sum_j p_j E_{\psi_j}(A) = \sum_j p_j \mathrm{tr}(\mathcal{Q}_\psi A) = \mathrm{tr}\left(\sum_j p_j \mathcal{Q}_\psi A)\right). \tag{K.24}$$

Definition In a quantum mechanical system whose state is described by a density operator ρ the expectation of the observable A is given by

$$E_\rho(A) = \mathrm{tr}(\rho A). \tag{K.25}$$

Definition definition of folium here??

A folium of a given state ω which may be defined to be the set of all states ω_σ which arise in the form $\mathrm{Tr}(\sigma\pi_\omega(\,\cdot\,))$ where σ ranges over the density operators (trace-class operators with unit trace) on $\mathcal{H}_\omega$.

K.2.2 Abstract Formalism

An algebraic structure called a $C\star$ algebra. A concrete C^*-algebra is a linear space $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, that is, a bunch of operators closed under addition, multiplication, scalar multiplication, and taking adjoints which is also complete with respect to the operator norm.

A $C\star$ algebra can be defined abstractly without any reference to linear operators acting on a Hilbert space. An abstract C^*-algebra is given by a set on which addition, multiplication, adjoint conjugation, and a norm are defined, satisfying the same algebraic relations as their concrete counterparts.

Can we solve quantum mechanics problems without resorting to differential equations

we start by giving the main properties of an abstract $\star-$algebra. We then show a how a measure, ω, called a weight is introduced, which plays the role of the state. ω is a functional, mapping elements of the algebra onto the real fields. It is not difficult to show that this is equivalent to introducing the density operator in the usual approach (exercise)

Definition: Field

Examples of field:

Rings are the "number systems" in mathematics.

Definition: If $\mathcal{A}$ has both the structure

(i) of a vector space

(ii) and of a ring with identity such that for all $A \in \mathcal{A}$ and α,

$$(\alpha I)A = \alpha A \tag{K.26}$$

then $\mathcal{A}$ is said to be an *algebra*.

Definition: If in addition to the properties in Definition there exists a map $* : \mathcal{A} \to \mathcal{A}$ (meaning the map, $*$, sends any element of $\mathcal{A}$ into another element of $\mathcal{A}$), such that for all

(i) $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$

(ii) $(AB)^* = B^*A^*$

(iii) $A^{**} = A$

then $\mathcal{A}$ is said to be a $\star$-*algebra*.

Definition: A state on a $\star$-algebra $\mathcal{A}$ is a linear functional $\omega : \mathcal{A} \to C$ that satisfies:

(i) $\omega(A^*A)$ is real and non-negative for all $\mathcal{A}$

(ii) $\omega(I) = 1$.

Definition: A $\star$-algebra that contains all its conjugates is known as a C$\star$-algebra.

Proposition: For any state, ω, on a $\mathcal{A}$, and for any $A, B \in \mathcal{A}$,

$$\omega(A^*B) = \overline{\omega(B^*A)}. \tag{K.27}$$

An immediate corollary is $\omega(A^*) = \overline{\omega(A)}$, and so if A is self-adjoint, $A^* = A$, then $\omega(A)$ is real.

there is a Cauchy-Schwartz inequality:

$$\omega(A^*A)\omega(B^*B) \geq |\omega(A^*B)|^2. \tag{K.28}$$

Proof:

$$\omega((A + \lambda B)^*A(A + \lambda B)) = \omega(A^*A) + |\lambda|^2\omega(B^*B) + \lambda\omega(A^*B) + \overline{\lambda}\omega(B^*A). \tag{K.29}$$

The left hand side and the first two terms of the right-hand side are all positive and so real. This forces the sum of the remaining two terms to be real and gives

$$\lambda\omega(A^*B) + \overline{\lambda}\omega(B^*A) = \overline{\lambda\omega(A^*B)} + \lambda\overline{\omega(B^*A)}. \tag{K.30}$$

Rearranging this we obtain

$$\lambda(\omega(A^*B) - \overline{\omega(B^*A)}) = \overline{\lambda}(\overline{\omega(A^*B)} - \omega(B^*A)). \tag{K.31}$$

This is true for any value of λ. Let us denote the quantity inside the brackets on the left-hand side α and that of the right-hand side β. The then condition reads

$$\lambda\alpha = \overline{\lambda}\beta. \tag{K.32}$$

By taking $\lambda = 1$ we have $\alpha = \beta$ and by taking $\lambda = i$ we have $\alpha = -\beta$ hence both α and β are zero and so

$$\omega(A^*B) = \overline{\omega(B^*A)}. \tag{K.33}$$

Taking $B = I$ gives $\omega(A^*) = \overline{\omega(A)}$ and so, in particular, when $A^* = A$ we deduce that $\omega(A)$ is real.

It is straightforward to get the Cauchy-Schwartz inequality (exercise).

K.2.3 GNS (Gel'fand, Naimark, Segal)

If $(\mathcal{H}, \pi, \Omega)$ is a cyclic representation of a C^*-algebra $\mathcal{A}$, then $A \to \omega(A) := < \Omega|\pi(A)\Omega >$ defines a state on $\mathcal{A}$. The converse is also true, and is known as the GNS construction.

formulating things so that the $*-$algebra is a C^*-algebra, then the GNS representation is as every-where defined on $\mathcal{H}$ bounded operators and is irreducible if and only if the state is pure.

Theorem K.2.1 *Let ω be a state on a $\star$-algebra $\mathcal{A}$. There exists an inner product space $\mathcal{H}_\omega$, a unit vector $\Omega_\omega \in \mathcal{H}_\omega$, and a homomorphism $\pi : \mathcal{A} \to (\mathcal{H}_\omega)$, such that for all $A \in \mathcal{A}$*

$$\omega(A) =< \Omega_\omega|\pi(A)\Omega_\omega > . \tag{K.34}$$

Proof

The space $\mathcal{H}_\omega$ will be a subspace of the dual space $\mathcal{A}'$ of linear functionals on $\mathcal{A}$. We can define a homomorphism $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{A}')$ by setting, for $f \in \mathcal{A}'$ and $X \in \mathcal{A}$,

$$(\pi(A)f) = f(XA). \tag{K.35}$$

To show that this defines a homomorphism we must establish $\pi(AB) = \pi(A)\pi(B)$ and $\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$.

$$(\pi(AB)f) = f(XAB) = (\pi(B)f)(XA) = (\pi(A)\pi(B)f)(X), \tag{K.36}$$

so that $\pi(AB) = \pi(A)\pi(B)$.

$$(\pi(\alpha A+\beta B)f) = f(X(\alpha A+\beta B)) = \alpha f(XA)+\beta f(XB) = \alpha(\pi(A)f)(X)+\beta(\pi(B)f)(X) \tag{K.37}$$

$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)$.

We have yet to define the inner product. That is a properties of an inner-product see (N.-19).

We now restrict attention to the subspace

$$\mathcal{H}_\omega = \{\pi(A)\omega : A \in \mathcal{A}\}, \tag{K.38}$$

which is invariant under the action of any $\pi(B)$, (that is $\pi(B)(\pi(A)\omega) \in \mathcal{H}_\omega$). To see that it is invariant take any member of $\mathcal{H}_\omega$, say $\pi(A)\omega$, and act on it with $\pi(B)$

$$\pi(B)(\pi(A)\omega) = (\pi(B)\pi(A))\omega = \pi(C)\omega \tag{K.39}$$

where $C \in \mathcal{A}$ therefore $\pi(B)(\pi(A)\omega) \in \mathcal{H}_\omega$. This is needed to ensure that the map $\pi : \mathcal{A} \to \mathcal{L}\mathcal{A}'$ is a homomorphism. On this subspace we define an inner product

$$< \pi(A)\omega|\pi(B)\omega >= \omega(A^*B). \tag{K.40}$$

This is well defined, since it can also be written as

$$\omega(A^*B) = (\pi(B)\omega)(A^*), \tag{K.41}$$

we need to check it has the properties of an inner:

(i) $< u|v >= \overline{< v|u >}$ for all $u, v \in V$,

(ii) $< u|\alpha v + \beta w >= \alpha < u|v > +\beta < u|w >$ for all $u, v, w \in V$, and for all $\alpha, \beta \in \mathbf{C}$,

(iii) $< u|u >> 0$ for all non-zero vectors u in V. $< 0|0 >= 0$

$$< \pi(A)\omega|\pi(B)\omega >= \omega(A^*B) = \overline{\omega(B^*A)} = \overline{< \pi(B)\omega|\pi(A)\omega >}, \tag{K.42}$$

giving the conjugate symmetry (condition (ii)). The linearity property follows from the linearity of ω

$$\begin{aligned} < \pi(A)\omega|\pi(\alpha B + \beta C)\omega > &= \omega(A^*(\alpha B + \beta C)) \\ &= \alpha\omega(A^*B) + \beta\omega(A^*C) \\ &= < \pi(A)\omega|\pi(B) > +\beta < \pi(A)\omega|\pi(C) > . \end{aligned} \tag{K.43}$$

It is clear form the definition of ω that

$$< \pi(A)\omega|\pi(A)\omega >= \omega(A^*A) \geq 0, \tag{K.44}$$

Hence to complete the proof we need to show that $<\pi(A)\omega|\pi(B)\omega>$ is strictly positive. To do this we use the Cauchy-Schwarz inequality; it tells us that

$$|(\pi(A)\omega)(B^*)|^2 = |\omega(B^*A)|^2 \leq \omega(B^*B)\omega(A^*A), \tag{K.45}$$

so that $\omega(A^*A)$ can only vanish only if $(\pi(A)\omega)(B^*) = 0$ for all B, which forces $\pi(A)\omega = 0$ that is $<0|0>= 0$.

$$<\omega|\pi(A)\omega>= \omega(IA^*) = \omega(A), \tag{K.46}$$

so Ω_ω.

Let us stop and sum up what has been proven here.

A reason for caution here is that a given physical system may lead to inequivalent spaces $\mathcal{H}_\omega$. It is the algebra that is associated with the physical system, not the inner product space. We give as an example a ferromagnetic material is described by different spaces, $\mathcal{H}_\omega$, according to whether the it is magnetized or not. The algebra, however, is the same.

This is the reason for concern whether we are working in an unphysical sector of the theory, that is, a sector that does not contain enough physical solutions for us to be able to recover the correct semi-classical limit.

If we have not only a unital $\star$-algebra but in fact a C$\star$-algebra one can show that the Hahn-Banach theorem that representations always exist, that ever non-degenerate representation is a direct sum of cyclic representations and that ever states is continuous so that the GNS representations are always bounded operators (see details).

Example: The Harmonic Oscillator

A second use of this term is *an expectation value functional.* This is to be chosen at some point in the quantization.

$$F_{\text{Sch}}(W(\zeta)) := e^{-\frac{1}{2}|\zeta|^2}. \tag{K.47}$$

The expectation values of $\hat{U}$ and $\hat{V}$ are given by:

$$F_{\text{Sch}}(U(\lambda)) = e^{-\frac{1}{2}\lambda^2 d^2} \quad \text{and} \quad F_{\text{Sch}}(V(\mu)) = e^{-\frac{1}{2}\frac{\mu^2}{d^2}} \tag{K.48}$$

The corresponding GNS "vacuum" - the cyclic state ψ_{Sch} is

$$\psi_{\text{Sch}} = \frac{1}{(\pi d^2)^{\frac{1}{4}}} e^{-\frac{x^2}{2d^2}}, \tag{K.49}$$

the ground state of the simple harmonic oscillator with fundamental length scale d.

observables form an algebra both in quantum and classical mechanics. Then states are defined as positive linear functionals on those algebras: in quantum mechanics pure states are labelled by certain vectors in a Hilbert spaces and classic pure states correspond to points of the phase space. Measurements are represented by evaluation of observables on particular states.

$$\hat{U}^{-1}\hat{q}\hat{U} = \hat{q} + a, \qquad \hat{U}^{-1}\hat{p}\hat{U} = \hat{p} \tag{K.50}$$

K.2.4 Null Ideals of the Algebra

A property you want the inner product to have is that it be positive definite, i.e. $(A, A) = 0$ implies that A is the null vector??. If an inner product $(A, A) = 0$ does not imply that A is null it is said to be only positive semidefinite.

$$(A, B)_\omega = \omega(A^* B) \tag{K.51}$$

$\omega(A^* A) = 0$ does not imply that A is null, means the algebra has a *null ideal*, usually denoted $\mathcal{N}$.

$$\mathcal{N} = \{A \in \mathcal{A} : \omega(A^* A) = 0\} \tag{K.52}$$

Recall the definition of a two-sided ideal, (or simply an ideal): say that $\mathcal{I}$ is a linear subspace of $\mathcal{A}$. If $\mathcal{I}$ satisfies the conditions $ai \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ and if $ia \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $a \in \mathcal{A}$ then $\mathcal{I}$ is a two-sided ideal.

The presence of a null ideal $\mathcal{N}$ requires to construct the Hilbert space by transferring our attention from the $\star-$algebra $\mathcal{A}$ and products of ... to consideration of equivalence classes ... and the induced multiplication between classes. This forces the positive semidefinite inner product to become positive definite.

System of semi-norms???

Let

$$\xi_\omega(\mathcal{A}) = \mathcal{A} = \mathcal{A}/\mathcal{N} \tag{K.53}$$

We write $\xi_\omega(A)$ for the projection of $A \in \mathcal{A}$ to $[A] \in \mathcal{A}/\mathcal{N}$.

K.2.5 Algebra Automorphisms: Time evolution and Symmetries

Symmetries and Time evolutions: Algebra Automorphisms

Given a state (i.e. a positive, normalized, linear functional) on $\mathcal{U}$ that is invariant under the classical symmetry automorphisms of $\mathcal{U}$.

Most important is that when a classical theory has symmetries that act on $\mathcal{U}$ by a group of automorphisms in the GNS construction these automorphisms are unitarily implemented. the state is is invariant under some automorphism of $\mathcal{U}$, its action is automatically unitarily implemented in the representation.

$$
\begin{aligned}
U(t) \cdot f &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\{f, T\}_n \\
&:= f + t\{f, T\} + \frac{t^2}{2!}\{\{f, T\}, T\} + \dots
\end{aligned}
\tag{K.54}
$$

the Poisson structure of phase space is preserved, including an automorphisms on the algebra of observables. The corresponding quantum operator $\hat{U}$ should be an automorphism in the algebra of quantum operators, and therefore, a unitary operator.

It is desirable to have a cyclic and invariant representation of $\mathcal{A}$ satisfying

$$\omega \circ \alpha_{\varphi} = \omega \tag{K.55}$$

for all φ in the symmetry group. Then the corresponding representation is then the GNS representation corresponding to ω. That the positive linear functional ω is invariant under a symmetry, general theorems from algebraic quantum mechanics tell us that we have a unitary representation of the symmetry on the GNS Hilbert space $\mathcal{H}_{\omega}$ defined by

$$U(\varphi)\pi_{\omega}(a)\Omega_{\omega} = \pi_{\omega}\Omega_{\omega}(\alpha_{\varphi}(a))\Omega_{\omega} \tag{K.56}$$

Properties of GNS construction

K.2.6 Summary

mostly from [hep-th/0601035]

The formulation of quantum mechanics in terms of the algebra of observables. The starting point of this formulation is a unital associative algebra $\mathcal{A}$ over $\mathbb{C}$ (the algebra of observables). One assumes that this algebra is equipped with antilinear involution $A \to A^*$. One says that a linear functional ω on A specifies a state if $\omega(1) = 1$ and $\omega \geq (AA^*)0$ (i.e. if the functional is normalized and positive). The probability distribution $\rho(\lambda)$ of real observable $A = A^*$ in the state ω is defined by the formula $\omega(A^n) = \int \lambda^n(\rho)d\lambda$.

In the textbooks on quantum mechanics the algebra of observables consists of operators acting on a (pre)Hilbert space. Every vector x having a unit norm specifies a state by the formula $\omega(A) = < Ax, x >$. (More generally, a density matrix K defines a state $\omega(A) = TrAK$). This situation is in some sense universal: for every state ω on $\mathcal{A}$ one construct a (pre)Hilbert space $\mathcal{H}$ and a representation of $\mathcal{A}$ by operators on this space in such a way that the state ω corresponds to a vector in this space. (To construct $\mathcal{H}$ one defines inner product on $\mathcal{A}$ by the formula $< A, B >= \omega(A^*B)$. The space $\mathcal{H}$ can be obtained from $\mathcal{A}$ by means of factorization with respect to zero vectors of this inner product (i.e. $< A, A >= 0$ when A is not the zero element of the vector space - the inner product is only positive semidefinite). The inner product on $\mathcal{A}$ descends to $\mathcal{H}$ providing it with a structure of preHilbert space. The state ω is represented by a vector of $\mathcal{H}$ that corresponds to the unit element of $\mathcal{A}$.)

It is important to notice that although every state of the algebra $\mathcal{A}$ can be represented by a vector in Hilbert space in general it is impossible to represent all states by vectors in the same Hilbert space.

Time evolution in algebraic formulation is specified by one-parameter group $\alpha(t)$ of automorphisms of the algebra $\mathcal{A}$ preserving the involution. This group acts in obvious way on the space of states. If ω is a stationary state (a state invariant with respect to time evolution) then the group $\alpha(t)$ descends to a group $U(t)$ of unitary transformations of corresponding space $\mathcal{H}$. The generator $\mathcal{H}$ of $U(t)$ plays the role of Hamiltonian. If the spectrum of $\mathcal{H}$ is non-negative one says that the stationary state ω is a ground state.

States

A state is an assignment of an expectation value to each member of a collection of 'observables' (the elements of a C^*-algebra). More precisely, a linear functional ω on a C^*-algebra $\mathcal{A}$ is a state if $\omega \geq 0$ (ie, for positive $A \in \mathcal{A}$, $\omega(A) \geq 0$), and $||\omega|| = 1$.

Representations

A representation of a C^*-algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and π is a morphism of $\mathcal{A}$ to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. The representation π is said to be faithful if, for $A \in \mathcal{A}$, $\pi(A) = 0 \Rightarrow A = 0$.

A cyclic representation of $\mathcal{A}$ is a triple $(\mathcal{H}, \pi, \Omega)$, where $\Omega \in \mathcal{H}$ such that $||\Omega|| = 1$ and $\pi(\mathcal{A})\Omega$ is dense in $\mathcal{H}$.

Automorphisms

A morphism g is a map from an algebra to itself that, that is $g : \mathcal{O} \to \mathcal{O}$. A morphism that has an inverse g^{-1} is called an automorphism. We say a state ω is invariant under a group of automorphisms G if

$$\omega(gA) = \omega(A)$$

for all $g \in G$ and $A \in \mathcal{O}$.

GNS construction

Given a state ω over an abstract C^*-algebra $\mathcal{A}$, the Gelfand-Naimark-Segal construction provides us with a Hilbert space $\mathcal{H}_\omega$ with a preferred state Ω_ω, and a representation π_ω of $\mathcal{A}$ as a concrete algebra of bounded operators on $\mathcal{H}_\omega$, such that

$$\omega(A) = < \Omega_\omega | \pi_\omega(A) | \Omega_\omega > . \tag{K.57}$$

Definition A representation $(\mathcal{H}, \pi)$ of the C^*-algebra $\mathcal{A}$ is said to be non-degenerate if .

Proposition K.2.2 *Let $(\mathcal{H}, \pi)$ be a non-degenerate representation of the C^*-algebra $\mathcal{A}$. Then π is the direct sum of a family of cyclic sub-representations.*

Let $\{\Omega_\alpha\}_{\alpha \in I}$ denote a maximal family (not a proper subset of any other such family) of nonzero vectors in $\mathcal{H}$ such that

$$(\pi(A)\Omega_\alpha, \pi(B)\Omega_\beta) = 0$$

for all $A, B \in \mathcal{A}$, whenever $\alpha \neq \beta$. To prove the existence of such a family we need to resort to the Zorn's lemma: The collection of all families of such vectors in $\mathcal{H}$ can be partially ordered by inclusion. Moreover, any linearly ordered chain (a subfamily totally order by inclusion) has an upper bound (the union of sets of the chain). Hence, Zorn's lemma implies the existence of a maximal element.

Next define $\mathcal{H}_\alpha$ as the Hilbert subspace formed by closing the linear subspace $\{\pi(A)\Omega_\alpha; A \in \mathcal{A}\}$. This is an invariant subspace so we can introduce π_α by $\pi_\alpha = P_{\mathcal{H}_\alpha} \pi_\alpha P_{\mathcal{H}_\alpha}$ on $\mathcal{H}_\alpha$ where

$P_{\mathcal{H}_\alpha} : \mathcal{H} \to \mathcal{H}_\alpha$ is the orthogonal projection. Then $\mathcal{H} = \oplus_\alpha \pi_\alpha$ and $(\pi_\alpha, \mathcal{H}_\alpha, \Omega_\alpha)$ is a cyclic representation of the $\mathcal{A}$.

K.2.7 Modifications for $^*-$algebras

not necessarily bounded operators on the Hilbert space $\mathcal{H}$.

A representation of a $^*-$algebra $\mathcal{A}$ is a pair $(\mathcal{H}, \pi)$ consisting of a Hilbert space $\mathcal{H}$ and a morphism $\pi : \mathbb{U} \to \mathcal{L}(\mathcal{H})$ into the algebra of linear (not necessarily bounded) operators on $\mathcal{H}$ with common and invariant dense domain.

A representation is said to be cyclic if there exists a normed vector $\Omega \in \mathcal{H}$ in the common domain of all the $a \in \mathbb{U}$ such that $\pi(\mathbb{U})\Omega$ is dense in $\mathcal{H}$. Notice that the existence of a cyclic vector implies that the states $\pi(b)\Omega$, $b \in \mathbb{U}$ lie in the common dense and invariant domain for all $\pi(a)$, $a \in \mathbb{U}$. A representation is said to be irreducible if every vector in a common dense and invariant (for $\mathbb{U}$) domain is cyclic.

K.3 Stone-Von Neumann Theorem

It assures us that there is nothing in the detailed theory of wave functions that cannot be, in principle, also be achieved by the algebraic approach, since they are isomorphic spaces.

$$\hat{Q}\hat{P} - \hat{P}\hat{Q} = i\hbar\mathbf{1} \tag{K.58}$$

Scrödinger found a representation of Eq(K.58) in the context of his wave mechanics:

$$(Q\Psi)(x) = x\Psi(x),\ x \in R, \tag{K.59}$$

on $L_2(R)$ and P is the differential operator

$$(P\Psi)(x) = i\hbar\frac{d\Psi}{dx}(x),\ x \in R, \tag{K.60}$$

on $L_2(R)$.

The relations in Eq(K.58) cannot be understood on the whole of the Hilbert space. To remedy this difficulty, Weyl introduced the *unitary* operators

$$U(a) = e^{-i2\pi aP/\hbar} \text{ and } V(a) = e^{-i2\pi aQ/\hbar}. \tag{K.61}$$

As is well known, unitary operators preserve the norm of a state and hence are well defined on the whole Hilbert space.

The algebraic relations between Q and P expressed in Eq(K.58) are replaced by

$$U(a)V(a) = e^{i2\pi ab/\hbar}V(b)U(a). \tag{K.62}$$

This is the Weyl form of the CCR for one degree of freedom. We can then ask *formally* what the algebra for "generated",

$$\begin{aligned} U(a)QU^{-1}(a) &= \exp(iaP)Q\exp(-iaP) \\ &= Q + ia[P,Q] + \frac{(ia)^2}{2!}[P,[P,Q]] + \ldots \\ &= Q + a\hbar \quad (\text{as } [P,Q] = i\hbar \text{ a scalar}). \end{aligned} \tag{K.63}$$

$$(U(a)\Psi)(x) = \Psi(xa) \ \text{ and } \ (V(b)\Psi)(x) = e^{-i2\pi bx/\hbar}\Psi(x), \tag{K.64}$$

The formula involving bounded operators will typically imply the one for unbounded operators but not vice versa.

von Neumann uniqueness theorem in QM:

Theorem If $\{\tilde{U}(a) : a \in R\}$ and $\{\tilde{V}(a) : a \in R\}$ are (**weakly continuous**footnote here) families of unitary operators acting irreducibly on a (separable) Hilbert space $\mathcal{H}$ such that

$$\begin{aligned} \tilde{U}(a)\tilde{U}(b) = \tilde{U}(a+b) \quad &, \quad \tilde{V}(a)\tilde{V}(b) = \tilde{V}(a+b) \\ \tilde{U}(a)\tilde{V}(b) &= e^{i2\pi ab/\hbar}\tilde{V}(b)\tilde{U}(a), \end{aligned} \tag{K.65}$$

then there exists a Hilbert space isomorphism $W : \mathcal{H} \to L^2(R)$ such that

$$W\tilde{U}(a)W^{-1} = U(a) \ \text{ and } \ W\tilde{V}(a)W^{-1} = V(a), \tag{K.66}$$

for all $a \in R$, where $U(a)$ and $V(a)$ are the Weyl unitaries in the Schödinger representation in Eq(K.64).

Each Weyl system with a finite number M of degrees on freedom is unitarily equivalent to the Schrödinger representation. Each reducible Weyl system is with finite number M of degrees of freedom is the direct sum of irreducible representations; hence it is a multiple of the Schrödinger representation.

The Stone-von Neumann theorem ensures that every irreducible 1-parameter representation of $\mathbf{W}$ *which is weakly continuous in the parameter* ξ is unitarily equivalent to the standard Schrödinger representation, where the Hilbert space is the space of $L^2(R, dx)$ of square integrable functions on R. $W(\xi)$ are represented via:

$$\hat{W}(\xi)\psi(x) = e^{\frac{1}{2}\alpha\beta} e^{i\alpha x}\psi(x+\beta) \tag{K.67}$$

$U(f+g) = U(f)U(g)$, $V(f+g) = V(f)V(g)$, and satisfy Weyl's form of the CCRs

$$V(f)U(g) = e^{i<f,g>}U(g)V(f). \tag{K.68}$$

unlike the case of finite dimensions in which we have the Stone-von Neumann theorem which asserts that there is only one irreducible representation of the CCRs (up to unitary equivalence)[1], in infinite dimensional case the Stone-von Neumann theorem does not hold. There are an infinite number of irreducible representations of the CCRs .

K.3.1 Proof of the Stone-von Neumann Theorem

The Schödiinger representation

Instead of $U(a)$ and $V(b)$ one can consider the two parameter family

$$S(a,b) = \exp\left(-\frac{1}{2}iab\right)U(a)V(b). \tag{K.69}$$

The Weyl form of the CCR entails commutation relation for $S(a,b)$:

$$S(a,b)S(c,d) = \exp\left(\frac{1}{2}i(ad-bc)\right)S(a+c,b+d). \tag{K.70}$$

$$\begin{aligned} S(a,b)S(c,d) &= \exp\left(-\frac{1}{2}iab\right)\exp\left(-\frac{1}{2}icd\right)U(a)V(b)U(c)V(d) \\ &= \exp\left(-\frac{1}{2}i(ab+cd+2cb)\right)U(a+c)V(b+d) \\ &= \exp\left(\frac{1}{2}i(ad-cb)\right)\exp\left(-\frac{1}{2}i(a+c)(b+d)\right)U(a+c)V(b+d) \end{aligned}$$

We now define the representation of CCR in terms of bounded operators

[1]The theorem that guarantees the equivalence between Schrodinger's and Heisenberg's approaches to quantum mechanics.

Definition

$$(a, b) \in \mathbb{R} \mapsto S(a, b) \in \mathcal{B}(\mathcal{H})$$

is a representation of (the Weyl form) of CCR if

$$\begin{aligned} S(-a, -b) &= S(a, b)^\dagger \\ S(a, b)S(c, d) &= \exp\left(-\frac{1}{2}i(ad - bc)\right) S(a + c, b + d). \end{aligned} \tag{K.71}$$

□

Two representations S and S' of the CCR on $\mathcal{H}$ are unitarily equivalent if there exists a unitary $U : \mathcal{H} \to \mathcal{H}$ such that

$$S(a, b) = US'(a, b)U^\dagger \quad \text{for all} \quad a, b. \tag{K.72}$$

A representation S of CCR on $\mathcal{H}$ is unique if S is unitarily equivalent to every representation S' of CCR on $\mathcal{H}$.

These are the conditions ensuring uniqueness of representation:

The closed linear subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ is called invariant if

$$S(a, b)\xi \in \mathcal{H}_0$$

for all $\xi \in \mathcal{H}_0$ and for all a, b. The representation $(a, b) \mapsto S(a, b)$ is

(i) irreducible if there are no non-trivial invariant subspaces

(ii) (strongly) continuous if

$$(a_n, b_n) \to (a, b) \quad \text{entails} \quad S(a_n, b_n)\xi \to S(a, b)\xi \quad \text{for all} \quad \xi \in \mathcal{H}$$

Theorem K.3.1 *Stone-von Neumann's theorem. The Schrodinger representation of the CCR on a Hilbert space $\mathcal{H}$ is the unique irreducible, (strongly) continuous representation of the CCR.*

In detail: The theorem says that if S is any irreducible, continuous representation of CCR on $\mathcal{H}$ and S^{Sch} is the Schrodinger representation on $L^2(\mathbb{R}, \mu)$, then there exists a unitary operator

$$U : L^2(\mathbb{R}, \mu) \to \mathcal{H}$$

such that

$$S(a,b) = US^{Sch}(a,b)U^\dagger \quad \textit{for all} \quad a, b.$$

Proof: Any proof of this theorem had to construct with aid of P, Q or

$$U(\alpha) = e^{i\alpha P} \qquad V(\beta) = e^{i\beta Q}$$

some operator, which has easily identifiable propeerties, determining it in a unique way - and which operator on the other hand can be used to determine some vectors in Hilbert space.

The general form of the operator determined by U and V is

$$A = \int\int \varphi(\alpha,\beta) U(\alpha) V(\beta) d\alpha d\beta \tag{K.73}$$

with an integrable function $\mathbb{R}^2 \mapsto a(\alpha, \beta)$.

$A^\dagger = A$ implies by

$$\begin{aligned}
(\int\int \varphi(\alpha,\beta)U(\alpha)V(\beta)d\alpha d\beta)^\dagger &= \int\int \overline{\varphi(\alpha,\beta)} V(\beta)^\dagger U(\alpha)^\dagger d\alpha d\beta \\
&= \int\int \overline{\varphi(\alpha,\beta)} V(-\beta) U(-\alpha) d\alpha d\beta \\
&= \int\int \left(\overline{\varphi(-\alpha,-\beta)} e^{i\alpha\beta} \right) U(\alpha) V(\beta) d\alpha d\beta
\end{aligned}$$

that

$$\varphi(-\alpha, -\beta) = e^{-i\alpha\beta} \overline{\varphi(\alpha,\beta)}$$

and $A = A^2$ implies by

$$
\begin{aligned}
A^2 &= \int\int \varphi(\alpha,\beta)U(\alpha)V(\beta)\int\int \varphi(\gamma,\sigma)U(\gamma)V(\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \int\int\int\int \varphi(\alpha,\beta)\varphi(\gamma,\sigma)e^{-i\gamma\beta}U(\alpha)U(\gamma)V(\beta)V(\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \int\int\int\int \varphi(\alpha,\beta)\varphi(\gamma,\sigma)e^{-i\gamma\beta}U(\alpha+\gamma)V(\beta+\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \int\int \Big(\int\int \varphi(\alpha,\beta)\varphi(\gamma-\alpha,\sigma-\beta)e^{i\alpha-\gamma,\beta}d\alpha d\beta\Big)U(\gamma)V(\sigma)d\gamma d\sigma \\
&= \int\int \varphi(\gamma,\sigma)U(\gamma)V(\sigma)d\gamma d\sigma
\end{aligned}
$$

that

$$
\int\int \varphi(\alpha,\beta)\varphi(\gamma-\alpha,\sigma-\beta)e^{-i(\alpha-\gamma)\beta}d\alpha d\beta = \varphi(\gamma,\sigma).
$$

Furthermore, by Fourier analysis the operator A can vanish only if $\varphi \equiv 0$. As noted by von Neumann, if

$$
\varphi(\alpha,\beta) = \frac{1}{2\pi}e^{-i\alpha\beta}\exp\left(-\frac{1}{4}(|\alpha|^2+|\beta|^2)\right),
$$

then the conditions for $A = A^\dagger = A^2$ are satisfied. So A can be written in terms of $S(a,b)$ as

$$
A = \frac{1}{2\pi}\int\int \exp\left(-\frac{1}{4}(|\alpha|^2+|\beta|^2)\right)S(\alpha,\beta)d\alpha d\beta. \qquad \text{(K.74)}
$$

$$
\begin{aligned}
A^2 &= \frac{1}{(2\pi)^2}\int\int\int\int e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)-\frac{1}{4}(|\gamma|^2+|\sigma|^2)}S(\alpha,\beta)S(\gamma,\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \frac{1}{(2\pi)^2}\int\int\int\int e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)-\frac{1}{4}(|\gamma|^2+|\sigma|^2)}e^{\frac{1}{2}i(\alpha\sigma-\beta\gamma)}S(\alpha+\gamma,\beta+\sigma)d\alpha d\beta d\gamma d\sigma \\
&= \frac{1}{(2\pi)^2}\int\int\int\int e^{-\frac{1}{4}(|\alpha-\gamma|^2+|\beta-\sigma|^2+|\gamma|^2+|\sigma|^2)}e^{\frac{i}{2}[(\alpha-\gamma)\sigma-(\beta-\sigma)\gamma]}S(\alpha,\beta)d\alpha d\beta d\gamma d\sigma \\
&= \frac{1}{(2\pi)^2}\int\int\int e^{-\frac{1}{2}|\gamma|^2+\frac{1}{2}(\alpha+i\beta)\gamma}d\gamma\int e^{-\frac{1}{2}|\sigma|^2+\frac{1}{2}(\beta+i\alpha)\sigma}d\sigma\times e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)}S(\alpha,\beta)d\alpha d\beta \\
&= \frac{1}{2\pi}\int\int e^{-\frac{1}{4}(|\alpha|^2+|\beta|^2)}S(\alpha,\beta)d\alpha d\beta
\end{aligned}
$$

The crucial observation is that the operator

$$P = A$$

is a projection operator.

Crucial to what follow is the notion of an operator which relates two representations to one another

Definition Let U and W be two reperesentations of the same group G on spaces $\mathcal{H}$ and $\mathcal{K}$ respectively. An interwining operator for U and W is a linear transformation, T, from $\mathcal{H}$ to $\mathcal{K}$ which satisfies

$$TU(x) = W(x)T$$

for all x in G. If there exists an invertible interwining operator then U and W are said to be equivalent.

If the representation $(a, b) \mapsto S(a, b)$ is irreducible then P is one dimensional, spanned by a unit vector $\xi \in \mathcal{H}$; hence if S and S' are two irreducible representations of the CCR's and ξ' is the analogously defined vector determined by S', then the map $U : \mathcal{H} \to \mathcal{H}$ defined by

$$US(a, b)\xi = S'(a, b)\xi'$$

extends lineaily to a unit operator of $\mathcal{H}$ that interwines between the two representations S and S'.

FOLLAND:

A matrix representation of the Heisenberg Lie algebra is

$$m(p, q, t) = \begin{pmatrix} 0 & p_1 & \cdots & p_n & t \\ 0 & 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & 0 & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & q_n \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

It is easily verified that

$$m(p, q, t)m(p', q', t') = m(0, 0, pq')$$

and so

$$[m(p,q,t), m(p',q',t')] = m(0,0,pq'-qp').$$

Using

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}$$

we have

$$\exp m(p,q,t)\exp m(p',q',t') = \exp m\big(p+p', q+q', t+t'+\frac{1}{2}(pq'-qp')\big).$$

One identifies a point $X \in \mathbb{R}^{2n+1}$ with the matrix $e^{m(X)}$, and makes $\mathbb{R}^{2n+1}$ into a group with group law

$$(p,q,t)(p',q',t') = \big(p+p', q+q', t+t'+\frac{1}{2}(pq'-qp')\big).$$

This is called the Heisenberg group and is denoted $\mathbf{H_n}$. The element $(0,0,0)$ is the identity and the inverse of the element (p,q,t) is $(-p,-q,-t)$.

We regard the operators P_j, Q_j as continuous linear (bounded) operators on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. As such they satisfy the commutation relations

$$[P_j, P_k] = [Q_j, Q_k] = 0, \qquad [P_j, Q_k] = \frac{h\delta_{jk}}{2\pi i}1 \tag{K.74}$$

and it follows that the map

$$d\rho_h(p,q,t) = 2\pi i(hpD + qX + tI)$$

Lie algebra homomorphism.

If $f \in L^2$, let

$$g(x,t) = [e^{2\pi i t(pD+qX)}f](x)$$

$$\frac{\partial g}{\partial t} - \sum_j p_j \frac{\partial g}{\partial x_j} = 2\pi q x g, \qquad g(x,0) = f(x)$$

$$\frac{dt}{d\tau}\frac{\partial g}{\partial t} + \sum_j \frac{dx_j}{d\tau}\frac{\partial g}{\partial x_j} - 2\pi qxg = 0$$

$\vec{a}$ is perpendicular to the normal to the solution surface. We parameterize the lines using the variable τ, then they satify

$$\frac{dt}{d\tau} = 1, \quad \frac{dx_j}{d\tau} = -p_j, \quad \frac{dg}{d\tau} = 2\pi qxg$$

$$\frac{dt}{d\tau} = 1 \quad \text{implies} \quad t = \tau + k(x)$$

But $t = 0$ when $\tau = 0$ implies $k = 0$. So

$$t = \tau.$$

$$\frac{dx_j}{dt} = -p_j \quad \text{implies} \quad x_j(t) = x_j - p_j t$$

and

$$\frac{dg}{dt} = 2\pi q_j(x_j - p_j t)g$$

says

$$g(x,t) = g_0(x)e^{2\pi i t q_j x_j - \pi i t^2 p_j q_j}$$

or

$$g = f(x)e^{2\pi i t q_j x_j - \pi i t^2 p_j q_j}$$

Setting $t = 1$ and replacing x by $x + p$, we obtain

$$e^{2\pi i(pD+qX)}f(x) = e^{2\pi i qx + \pi i pq}f(x+p)$$

Obviously

$$\|e^{2\pi i(pD+qX)}f\|_2^2 = \|f\|_2^2$$

so $e^{2\pi i(pD+qX)}$ is a unitary operator on L^2, and it is easily checked that

$$\begin{aligned} e^{2\pi i(pD+qX)}e^{2\pi i(rD+sX)} &= e^{2\pi i[(p+r)D+(q+s)X]+\frac{1}{2}(2\pi i)^2[pD+qX,rD+sX]} \\ &= e^{2\pi i[(p+r)D+(q+s)X]+\frac{1}{2}(2\pi i)^2(ps-qr)[D,X]} \\ &= e^{2\pi i[(p+r)D+(q+s)X]+\pi i(ps-qr)I} \end{aligned}$$

Therefore the map ρ from $\mathbf{H_n}$ to the group of unitary operators on L^2 defined by

$$\rho(p,q,t) = e^{2\pi i(pD+qX+tI)} = e^{2\pi it}e^{2\pi i(pD+qX)}$$

that is,

$$\rho(p,q,t)f(x) = e^{2\pi it+2\pi iqx+\pi ipq}f(x+p) \tag{K.71}$$

is a unitary representation of $\mathbf{H_n}$.

□

$$\rho(F) = \int\int F(p,q)\rho(p,q)dpdq = \int\int F(p,q)e^{2\pi i(pD+qX)}dpdq \tag{K.71}$$

$$\rho(F)f(x) = \int\int F(p,q)e^{2\pi iqx+\pi ipq}f(x+p)dpdq$$

Proposition K.3.2

$$\rho(a,b)\rho(F) = \rho(G) \quad \textit{and} \quad \rho(F)\rho(a,b) = \rho(H)$$

where

$$G(p,q) = e^{\pi i(bp-aq)}F(p-a,q-b) \quad \textit{and} \quad H(p,q) = e^{\pi i(aq-bp)}F(p-a,q-b).$$

Proof:

$$\begin{aligned}
\rho(a,b)\rho(F) &= \int\int F(p,q)\rho(a,b)\rho(p,q)dpdq \\
&= \int\int F(p,q)e^{\pi i(aq-bp)}e^{2\pi i[(a+p)D+(b+q)X]}dpdq \\
&= \int\int F(p-a,q-b)e^{\pi i(a(q-b)-b(p-a))}e^{2\pi i(pD+qX)}dpdq \\
&= \int\int e^{\pi i(aq-bp)}F(p-a,q-b)\rho(p,q)dpdq
\end{aligned}$$

□

We return to $L^1(\mathbf{H_n^{red}})$. This space is a Banach algebra under convolution,

Definition Twisted convolution.

$$\begin{aligned}
F\natural G(p,q) &= \int\int F(p',q')G(p-p',q-q')e^{\pi i(p'q-q'p)}dp'dq' \\
&= \int\int F(p-p',q-q')G(p',q')e^{\pi i(pq'-qp')}dp'dq'
\end{aligned}$$

We call $F\natural G$ the twisted convolution of F and G.

□

Its definition is set up so that

$$\rho(F\natural G) = \rho(F)\rho(G).$$

$$\begin{aligned}
\rho(F\natural G) &= \int_0^1\int\int\int_0^1\int\int e^{2\pi i(t'+t)}F(p',q')G(p-p',q-q')e^{\pi i(p'q-q'p)}\rho(p,q)dp'dq'dt'dpdqdt \\
&= \int_0^1\int\int\int_0^1\int\int F(p,q)G(p',q')\rho(p'+p,q'+q,t'+t+\frac{1}{2}(p'q-q'p))dp'dq'dt'dpdqdt \\
&= \int_0^1\int\int\int_0^1\int\int F(p,q)G(p',q')\rho\big((p,q,t)(p',q',t')\big)dpdqdtdp'dq'dt' \\
&= \int\int F(X)G(Y)\rho(XY)dXdY \\
&= \rho(F)\rho(G)
\end{aligned}$$

$$
\begin{aligned}
V(f,g)(p,q) &= (\rho(p,q)f,g) = (e^{2\pi i(pD+qX)}f,g) \\
&= \int e^{2\pi iqx+\pi ipq} f(x+p)\overline{g(x)}dx \\
&= \int e^{2\pi iqy} f(y+\frac{1}{2}p)\overline{g(y-\frac{1}{2}p)}dy
\end{aligned}
$$

By the Schwartz inequality

$$
\begin{aligned}
|V(f,g)| &\leq \left(\int |f(x+\frac{1}{2}p)|^2|g(x-\frac{1}{2}p)|^2 dy\right)^{1/2} \\
&\leq \left(\int |f(y+\frac{1}{2}p)|^2 dy \int |g(y-\frac{1}{2}p)|^2 dy\right)^{1/2} \\
&= \left(\int |f(y)|^2 dy\right)^{1/2} \left(\int |g(y)|^2 dy\right)^{1/2} \\
&= \|f\|_2 \|g\|_2
\end{aligned}
$$

then as

$$
\|V(f,g)\|_\infty := \sup |V(f,g)|
$$

we have

$$
\|V(f,g)\|_\infty \leq \|f\|_2 \|g\|_2
$$

Proposition K.3.3 *The representation ρ_h is irreducible for any $h \in \mathbb{R}/\{0\}$.*

Proof: Suppose $M \subset L^2(\mathbb{R}^n)$ is a nonzero closed invariant subspace and $f \neq 0 \in M$. If $g \perp e^{2\pi i(hpD+qX)}f$ for all $p,q \in \mathbb{R}^n$, i.e. $V(f,g)=0$. But this implies that $\|f\|_2\|g\|_2 = 0$, whence $g = 0$ and $M = L^2(\mathbb{R}^n)$.

□

Proposition K.3.4 *Let*

$$
\phi(x) = 2^{n/4}e^{-\pi x^2}, \qquad \Phi = V(\phi,\phi), \qquad \Phi^{ab} = V(\phi,\rho(a,b)\phi).
$$

Then

(a) $\Phi(p,q) = e^{-(\pi/2)(p^2+q^2)}$.

(b) $\Phi^{ab}(p,q) = e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}$.

(c) $\rho(\Phi)\rho(a,b)\rho(\Phi) = e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)$.

(d) $\Phi\natural\Phi^{ab} = e^{-(\pi/2)(a^2+b^2)}\Phi$.

Proof:

$$\begin{aligned}
\Phi(p,q) &= 2^{n/2}\int e^{2\pi iqy}e^{-\pi[y+(p/2)]^2-\pi[y-(p/2)]^2}dy \\
&= 2^{n/2}e^{(-\pi/2)p^2}\int e^{2\pi iqy}e^{2\pi y^2}dy = e^{(-\pi/2)(p^2+q^2)}
\end{aligned}$$

$$(p,q,0)(p,q,0) = \big(p-c, q-d, \frac{1}{2}(dp-cq)\big).$$

using this

$$\begin{aligned}
V(f,\rho(c,d)g)(p,q) &= (\rho(p,q)f, \rho(c,d)g) \\
&= (\rho(-c,-d)\rho(p,q)f, g) \\
&= e^{\pi i(dp-cq)}(\rho(p-c,q-d)f,g) \\
&= e^{\pi i(dp-cq)}V(f,g)(p-c,q-d)
\end{aligned} \tag{K.49}$$

$$\begin{aligned}
\Phi^{ab}(p,q) &= V(\phi,\rho(a,b)\phi)(p,q) \\
&= e^{\pi i(bp-aq)}V(\phi,\phi)(p-a,q-b) \\
&= e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}
\end{aligned} \tag{K.48}$$

$$\begin{aligned}
\rho(\Phi)\rho(p,q)\rho(\Phi)f &= \rho(\Phi)\rho(a,b)[(f,\phi)\phi] \\
&= \rho(\Phi)[(f,\phi)\rho(a,b)\phi] \\
&= (f,\phi)(\rho(a,b)\phi,\phi)\phi \\
&= (f,\phi)\Phi^{ab}(a,b)\phi \\
&= e^{-(\pi/2)(a^2+b^2)}(f,\phi)\phi \\
&= e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)f
\end{aligned} \tag{K.44}$$

Consider $\rho(a,b)\rho(\Phi)$, it follows from proposition N.-19 that

$$G(p,q) = e^{\pi i(bp-aq)}\Phi(p-a,q-b) = \Phi^{ab}(p,q)$$

and so $\rho(a,b)\rho(\Phi) = \rho(\Phi^{ab})$. Now by (c)

$$\begin{aligned}\rho(\Phi\natural\Phi^{ab}) &= \rho(\Phi)\rho(\Phi^{ab})\\ &= \rho(\Phi)\rho(a,b)\rho(\Phi)\\ &= e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)\end{aligned}\tag{K.43}$$

Since ρ is faithful (i.e. $\mathrm{Ker}\rho = \{0\}$), $\Phi\natural\Phi^{ab} = e^{-(\pi/2)(a^2+b^2)}\rho(\Phi)$.

$\square$

$$\phi(x) = 2^{n/4}e^{-\pi x^2}$$

by (K.3.1)

$$\phi^{ab}(x) := \rho(a,b)\phi(x) = 2^{n/4}e^{2\pi ibx+\pi iab}e^{-\pi(x+a)^2}$$

$$\begin{aligned}(\phi^{pq},\phi^{ab}) &= (\rho(p,q)\phi,\rho(a,b)\phi)\\ &= V(\phi,\rho(a,b)\phi)(p,q)\\ &= \Phi^{ab}(p,q) = e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}\end{aligned}\tag{K.42}$$

$$\Phi\natural\Phi^{ab} = e^{-(\pi/2)(a^2+b^2)}\Phi \tag{K.42}$$

$$\pi(p,q)\pi(r,s) = \pi\big(p+r,q+s,\frac{1}{2}(ps-qr)\big) = e^{\pi i(ps-qr)}\pi(p+r,q+s)$$

We consider the integrated version of π,

$$\pi(F) = \int\int F(p,q)\pi(p,q)dpdq \qquad (F\in L^1(\mathbb{R}^{2n})),$$

and just as with ρ, we have

$$
\begin{aligned}
\pi(F)\pi(G) &= \pi(F\natural G) && (K.43)\\
\pi(F)\pi(a,b) &= \pi(G) \quad \text{where} \quad G(p,q)=e^{\pi i(aq-bp)}F(p-a,q-b) && (K.44)\\
\pi(a,b)\pi(F) &= \pi(H) \quad \text{where} \quad H(p,q)=e^{\pi i(bp-aq)}F(p-a,q-b). && (K.45)
\end{aligned}
$$

Proposition K.3.5 *We prove that π is faithful on $L^1(\mathbb{R}^{2n})$.*

Proof: If $\pi(F)=0$ then, by (K.44) and (K.45), for any $u,v\in\mathcal{H}$ and $a,b\in\mathbb{R}^n$,

$$
\begin{aligned}
0 &= (\pi(a,b)\pi(F)\pi(-a,-b)u,v)\\
&= (\pi(a,b)[\int\int e^{\pi i(bp-aq)}F(p+a,q+b)\pi(p,q)dpdq]\ u,v)\\
&= \int\int e^{2\pi i(bp-aq)}F(p,q)(\pi(p,q)u,v)dpdq
\end{aligned}
$$

Thus by the Fourier inverse theorem,

$$
F(p,q)(\pi(p,q)u,v)=0 \quad \text{for a.e. } (p,q),
$$

and sinse u and v are arbitary, $F=0$ a.e.

□

Proposition K.3.6 *If we take F to be given by $\Phi=V(\phi,\phi)$ where $\phi(x)=2^{n/2}e^{-\pi x^2}$, $\pi(\Phi)$ is a projection.*

Proof: By (K.45), (K.42), (K.43), and (K.3.1).

$$
\begin{aligned}
\pi(\Phi)\pi(a,b)\pi(\Phi) &= \pi(\Phi)\big(\pi(a,b)\pi(\Phi)\big)\\
&= \pi(\Phi)\int\int e^{\pi i(bp-aq)}e^{-(\pi/2)[(p-a)^2+(q-b)^2]}\pi(p,q)dpdq\\
&= \pi(\Phi)\pi(\Phi^{ab})\\
&= \pi(\Phi\natural\Phi^{ab})\\
&= e^{-(\pi/2)(a^2+b^2)}\pi(\Phi). && (K.39)
\end{aligned}
$$

taking $a=b=0$ we obtain

$$\pi(\Phi)^2 = \pi(\Phi),$$

and since Φ is even and real it is easy to see that $\pi(\Phi)$ is self-adjoint:

$$\pi(\Phi)^\dagger = \pi(\Phi).$$

Thus $\pi(\Phi)$ is an orthogonal projection which is non-zero since $\Phi \neq 0$ and π is faithful.

$\square$

Proposition K.3.7 *Let $\{v_\alpha\}$ be an orthogonal basis for Ran$(\pi(\Phi))$, and let*

$$\mathcal{H}_\alpha := \overline{span\{\pi(p,q)v_\alpha\}}\ , \qquad p,q \in \mathbb{R}^n.$$

$\mathcal{H}_\alpha \perp \mathcal{H}_\beta$ and $(\oplus \mathcal{H}_\alpha)^\perp = \{0\}$.

Proof: By

we have

$$\begin{aligned} (\pi(p,q)u, \pi(r,s)v) &= (\pi(-r,-s)\pi(p,q)\pi(\Phi)u, \pi(\Phi)v) \\ &= e^{\pi i(ps-qr)}(\pi(\Phi)\pi(p-r,q-s)\pi(\Phi)u, v) \\ &= e^{\pi i(ps-qr)}e^{-(\pi/2)[(p-r)^2+(q-s)^2]}(u,v) \end{aligned} \tag{K.38}$$

For $\alpha \neq \beta$

$$(\pi(p,q)u_\alpha, \pi(r,s)u_\beta) = e^{\pi i(ps-qr)}e^{-(\pi/2)[(p-r)^2+(q-s)^2]}\ (u_\alpha, u_\beta) = 0$$

for all $p,q,r,s \in \mathbb{R}^n$. That is

$$\mathcal{H}_\alpha \perp \mathcal{H}_\beta.$$

$\square$

We claim that $\pi\big|\mathcal{H}_\alpha$ is equivalent to ρ for all α. Indeed, fix an α and let $v^{pq} = \pi(p,q)v_\alpha$. Then by () and (),

$$(v^{pq}, v^{rs}) = (\phi^{pq}, \phi^{rs})$$

so the correspondence $v^{pq} \to \phi^{pq}$ is an isometric map V,

$$T : v^{pq} \mapsto \phi^{pq}$$

The first obvious extension of elements which can be written as linear combinations of the v^{pq}'s.

$$T\big[\sum a_{jk} v^{p_j q_k}\big] = \sum a_{jk} T v^{p_j q_k} = \sum a_{jk} \phi^{p_j q_k}$$

It follows that if $u = \sum a_{jk} v^{p_j q_k}$ and $f = \sum a_{jk} \phi^{p_j q_k}$ then

$$\|u\|_{\mathcal{H}} = \sum a_{jk} \overline{a_{lm}} (v^{p_j q_k}, v^{p_l q_m}) = \sum a_{jk} \overline{a_{lm}} (\phi^{p_j q_k}, \phi^{p_l q_m}) = \|f\|_2$$

Therefore, if the v_n converge to some element, $\tilde{v}$, of $\mathcal{H}_\alpha$, so $T v_n$ must converge to some element of $L^2(\mathbb{R}^n)$.

In particular $u = 0$ if and only if $f = 0$. Therefore T is unitary.

$$T(\pi \big| \mathcal{H}_\alpha)(p,q) = \rho(p,q) T$$

for all $p, q \in \mathbb{R}^n$.

i.e. the map extends by linearity and continuity to a unitary map from $\mathcal{H}_\alpha$ to $L^2(\mathbb{R}^n)$ that interwines $\pi \big| \mathcal{H}_\alpha$ and ρ.

K.3.2 Superselection Sectors

Recall that a subspace $\mathcal{S}$ of $\mathcal{H}$ is said to be invariant under a set of operators $\mathcal{A}$ if

$$\mathcal{A}\mathcal{S} \subseteq \mathcal{S}. \tag{K.38}$$

A set of operators is reducible if there is a subspace other than than the whole space or the zero vector, are invariant under the set of operators. If a set of operators is not reducible, we say it is irreducible. A representation $\mathcal{A}$ is irreducible if the Hilbert space does not split into orthogonal subspaces and so is preserved by the action of a set of operators $\mathcal{A}$.

$$\Psi = a\psi + b\phi$$

The existence of a restriction to the superposition principle for pure states, represented by vectors in the physical Hilbert space of quantum field theory, was discovered by Wightman and Wigner, showed that this Hilbert space is the direct sum of superselection sectors in such a way that the phase relations between vectors belonging to different sectors are unobservable. Haag and Kastler, this was recognized as an aspect of the representation problem in quantum field theory, i.e. the existence of several inequivalent irreducible representations of the algebra of observables for systems with an infinite number of degrees of freedom (in contrast to the situation for non-relativistic finite systems)

The representation should be irreducible on physical grounds, if there were superselection sectors this would imply that the physically relevant information is already contained in a closed subspace.

Irreducibility of a representation is equivalent to the condition that there must exist in the Hilbert space a cyclic vector whose image under the action of $\mathcal{A}$ is dense in the Hilbert space. We prove this below and also that superselection sectors can be identified with the irreducible sectors of a reducible representation.

Proposition K.3.8 *Let $\mathcal{A}$ be a self-adjoint set of bounded operators on the Hilbert space $\mathcal{H}$. The following conditions are equivalent:*

1) $\mathcal{A}$ is irreducible;

2) the commutant $\mathcal{A}'$ of $\mathcal{A}$, i.e., the set of all bounded operators on $\mathcal{H}$ which commute with each $A \in \mathcal{A}$, consists of multiples of the identity operator;

3) every nonzero vector $\psi \in \mathcal{H}$ is cyclic for $\mathcal{A}$ in $\mathcal{H}$, or $\mathcal{A} = 0$ and $\mathcal{H} = \mathbb{C}$.

Let $\mathcal{S}$ be a subspace which is invariant under $\mathcal{A}$, then its orthogonal complement $\mathcal{S}^{\perp}$ is also invariant. The proof is as follows: let P be the projection operator onto $\mathcal{S}$

$$P : \mathcal{H} \to \mathcal{S}.$$

Let A be in $\mathcal{A}$. Then $AP\psi \in \mathcal{S}$, so $AP\psi = PAP\psi$ for all $\psi \in \mathcal{H}$, or

$$AP = PAP$$

As the set $\mathcal{A}$ is self-adjoint, if A is in $\mathcal{A}$ then so is its adjoint $A^{\dagger}$,

$$A^{\dagger}P = PA^{\dagger}P$$

By taking the adjoint we get

$$PA = APA$$

and we see that

$$PA = AP.$$

So that if $\varphi \in \mathcal{S}^\perp$ then $P(A\varphi) = A(P\varphi) = 0$ for all $A \in \mathcal{A}$, i.e., $\mathcal{A}\mathcal{S}^\perp \subseteq \mathcal{S}^\perp$.

Proof of proposition K.3.8.

(2) $\Rightarrow$ (1)

Let P be the projection operator onto $\mathcal{S}$, by the above, P commutes with $\mathcal{A}$. If multiples of the identity operator are the only bounded operators which commute with $\mathcal{A}$ then P is either the identity operator or zero, and hence $\mathcal{S}$ is either the whole space or the subspace containing only the zero vector, so the set of operators is irreducible.

(1) $\Rightarrow$ (3)

Suppose that $\mathcal{A}$ is irreducible. Assume there is a nonzero vector ψ such that the set $\{A\psi : A \in \mathcal{A}\}$ is not dense in $\mathcal{H}$. The orthogonal complement contains at least one nonzero vector and is invariant under $\mathcal{A}$ (unless $\mathcal{A} = 0$ and $\mathcal{H} = \mathbb{C}$), and this contradicts that $\mathcal{A}$ is irreducible.

(3) $\Rightarrow$ (2)

Let B be a bounded non-self-adjoint operator which commutes with every operator in $\mathcal{A}$. If A is in $\mathcal{A}$ then so is A^*. Taking the adjoint of $BA^* = A^*B$ gives $B^*A = AB^*$. It follows that the Hermitian operators

$$\mathrm{Re}B = \frac{B + B^\dagger}{2}, \qquad \mathrm{Im}B = \frac{B - B^\dagger}{2i}$$

are also in $\mathcal{A}'$. Thus there is a self-adjoint operator $H \in \mathcal{A}'$. Each of the projection operators in the spectral decomposition of H commutes with every operator in $\mathcal{A}$. But if P is any such projector and ψ is a vector in the range of P then $\psi = P\psi$ cannot be cyclic:

$$A\psi = AP\psi = P(A\psi)$$

for all $A \in \mathcal{A}$ so that $\mathcal{A}\psi$ is not dense in $\mathcal{H}$.

[157]

Proposition K.3.9 *Let $(\mathcal{H}, \pi)$ be a nondegenerate*[2] *representation of the C^*-algebra $\mathcal{A}$. It follows that π is the direct sum of a family of cyclic sub-representaions.*

reduces the discussion of general representations down to cyclic representations. This is important because there is a canonical manner of constructing cyclic representations.

For infinite systems there are not unique representations, however, uniqueness may be restored by imposition of physical requirements.

Poincare invariance and the ground state being the state having the lowest energy - these suffice to selection a unique vacuum state and a representation of the observable operators.

K.3.3 Existence of Representations

States

K.3.4 Construction of Representations

K.4 Gel'fand−Neumark Theorem

Since we require the analogue of a Hermitian conjugate we will be interested those algebras which are equipped with an involution (or $\star$ operation). An algebra $\mathcal{A}$ written

$$\star : \mathcal{A} \rightarrow \mathcal{A} \tag{K.38}$$

A Banach algebra is a normed space which is *complete*.

A Banach $\star-$algebra is a complex Banach algebra endowed with an $\star-$ operation such that for all $a \in \mathcal{A}$, $\|a^*\| = \|a\|$. In particular for all $a \in \mathcal{A}$,

$$\|a^* a\| \leq \|a^*\| \, \|a\| = \|a\|^2. \tag{K.38}$$

Definition A C^*-algebra is a Banach $\star-$algebra $\mathcal{A}$ such that for all $a \in \mathcal{A}$,

$$\|a^* a\| = \|a\|^2 \tag{K.38}$$

[2]A representation $(\mathcal{H}, \pi)$ is said to be nondegenerate if $\{\psi; \psi \in \mathcal{H}, \pi(A)\psi = 0 \text{ for all } A \in \mathcal{A}\} = \{0\}$.

Example 1:

One of the most important Banach algebras is the set $\mathcal{C}(X)$ of all bounded continuous complex functions defined on a topological space X. The case in which X is a compact Hausdorff space will have particular significance for us in this report. If X has only one point, then $\mathcal{C}(X)$ can be identified with the simplest of all Banach algebras, the algebra of complex numbers.

Example 2:

(a)

If B is a non-trivial complex Banach space, then the set $\mathcal{B}(B)$ of all bounded operators on B is a Banach algebra.

$$\begin{aligned}
\|AB\| &= \sup_{\|x\|=1} \|(AB)x\| \\
&= \sup_{\|x\|=1} (\|Bx\| \, \|A\Big(\frac{Bx}{\|Bx\|}\Big)\|) \\
&\leq \sup_{\|x\|=1} \|Bx\| \sup_{\|y\|=1} \|Ay\| \\
&= \|A\| \, \|B\|
\end{aligned}$$

We assume that B is non-trivial in order to guarantee that the identity in the algebraic sense.

(b)

Example 3:

An L_p space essentially consists of all measurable functions f defined on a measure space X with measure μ which are such that $|f(x)|^p$ is integrable, with

$$\|f\|_p = (\int |f(x)|^p d\mu(x))^{1/p} \tag{K.34}$$

taken as the norm.

Gel'fand-Neumark theorem asserts that that *every commutative C⋆-algebra with identity is isomorphic to the C⋆-algebra of all continuous, bounded complex functions on a compact, Hausdorff space.*

like this

$\boxed{\text{like this}}$

$$\left(2\sqrt{\frac{a}{b}}\right)^m$$

$$\left(a+b\right)$$

K.5 Gel'fand Theory for Abelean C* Algebras

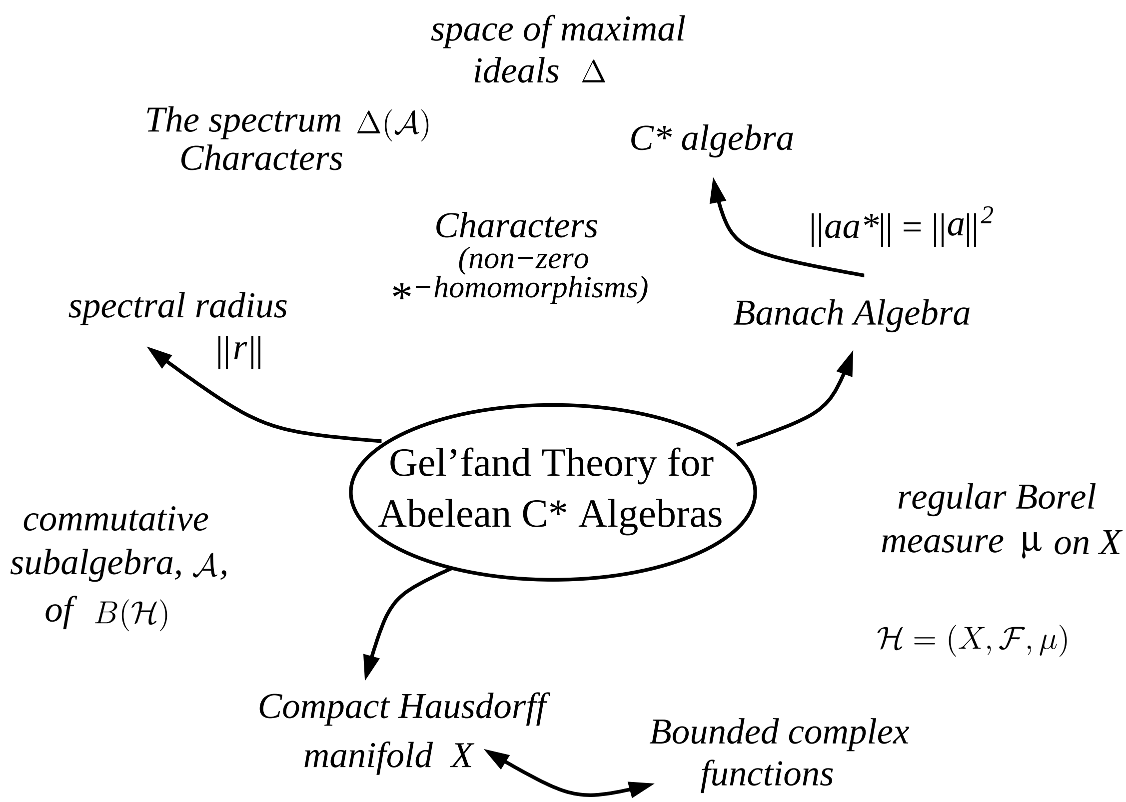

Figure K.1: Gelfand. Compact (and Hausdorff) in a natural topology. The algebra of the commutative C^*-algebra is isomorphic to $C(\Delta)$, the algebra of continuous functions on Δ.

K.6 Details of the Gel'fand−Neumark Theorem

If were only working with topologies of metric spaces (spaces that are equipped with a norm), we would need only consider sequence convergence. All the rest of the topological notions such as continuity, denseness, boundedness, closure, completeness, e.t.c. would be derived from the notion of sequence convergence. However, here we will also have to deal with topologies that are not defined by a norm. In these cases net convergence is sufficient to characterize closure of sets and that compactness can be characterized in terms of convergence of subnets.

K.6.1 Review of what we Wish to Establish

To help keep track of the spaces, maps and topologies that will be encountered in this section below we provide a list of definitions. Don't worry too much if you don't understand them straight away, their meaning will become more clear as you work through the section.

$\mathcal{A}$	an algebra
$\mathcal{A}'$	is the topological dual of $\mathcal{A}$
$\Delta(\mathcal{A})$	spectrum of the algebra $\mathrm{Hom}(\mathcal{A},\mathbb{C})$ or space of maximal ideals
$\sigma(A)$	spectrum of the operator A
$\chi(a)$	commutative continuous homomorphism
Gel'fand map	a map from the commutative C^*-algebra $\mathcal{A}$ onto the space of continuous bounded complex functions on the spectrum of the algebra, $C(\Delta(\mathcal{A}))$
$\overset{\vee}{a}$	function on the maximal ideals
weak topology	weakest topology (the one with the least number of open sets) making all the maps $x \mapsto < \phi, x >$ continuous,
weak* topology	weakest topology making all the maps $\phi \mapsto < \phi, x >$ continuous,
Gel'fand topology	topology on the spectrum of an Abelean Banach algebra with unit is the weak* topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$

a)

b) Functional calculus

c) Isometric... commutative C^*-algebra is isomorphic with complex continuous bounded functions on a compact Hausdorff space.

K.6.2 Properties of the Spectrum of Operators

Let $\mathcal{A}$ be a Banach algebra with unit $\mathcal{I}$. An element A of $\mathcal{A}$ is invertible if there exists an element A^{-1} of $\mathcal{A}$ such that

$$A^{-1}A = AA^{-1} = \mathbb{I}.$$

Definition If $\mathcal{A}$ is a Banach algebra, then let $G(A)$ denote the group of invertible elements.

One calls the resolvent set of A the set

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda\mathcal{I} - A \text{ is invertible}\}. \tag{K.34}$$

The partial sum which defines an element

$$S_n = \frac{1}{\lambda}\sum_{m=0}^{n}\left(\frac{A}{\lambda}\right)^m \tag{K.34}$$

From the inequality

$$\|S_n\| \leq \frac{1}{|\lambda|}\sum_{m=0}^{n}\left(\frac{\|A\|}{|\lambda|}\right)^m \tag{K.34}$$

we see that if $|\lambda| > \|A\|$ the partial sums $S_n := (1/\lambda)\sum_{m=1}^{n}(A/\lambda)^n$ form a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete, these partial sums converge to an element of $\mathcal{A}$, which is easily identified as the inverse of $\lambda\mathbb{I} - A$,

$$(\lambda\mathbb{I} - A)\frac{1}{\lambda}\Big[\mathbb{I} + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\Big] = \frac{1}{\lambda}\Big[\mathbb{I} + \frac{A}{\lambda} + \frac{A^2}{\lambda^2} + \cdots\Big](\lambda\mathbb{I} - A) = \mathbb{I}. \tag{K.34}$$

An immediate corollary of this is that an element A for which $\|A - \lambda\mathbb{I}\| < |\lambda|$ is invertible, and the inverse of such an element is given by the formula

$$A^{-1} = (\lambda\mathbb{I} - (\lambda\mathbb{I} - A))^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\mathbb{I} - A)^n}{\lambda^n} \tag{K.34}$$

From the above, we see that $\rho(A)$ is included in $B(0, \|A\|)$.

Furthermore, if λ_0 is in $\rho(A)$ and if $|\lambda_0 - \lambda| < \|(\lambda_0\mathbb{I} - A)^{-1}\|$, then the (von Neumann) series

$$\sum_{m=0}^{\infty}(\lambda_0 - \lambda)^m(\lambda_0\mathbb{I} - A)^{-m-1} \tag{K.34}$$

defines an element of $\mathcal{A}$ which is easily seen to be $(\lambda\mathbb{I} - A)^{-1}$:

$$
\begin{aligned}
&\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0\mathbb{I}-A)^{-m-1}(\lambda\mathbb{I}-A)\\
&=\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0\mathbb{I}-A)^{-m-1}[(\lambda_0\mathbb{I}-A)+(\lambda_0-\lambda)\mathbb{I}]\\
&=\sum_{m=0}^{\infty}(\lambda_0-\lambda)^m(\lambda_0\mathbb{I}-A)^{-m}-\sum_{m=0}^{\infty}(\lambda_0-\lambda)^{m+1}(\lambda_0\mathbb{I}-A)^{-(m+1)}\\
&=\mathbb{I}.
\end{aligned}
\tag{K.32}
$$

Lemma K.6.1 *$G(A)$ is open.*

□

Definition The resolvent $\rho(A)$ of an element $A \in \mathcal{A}$ is the set of scalars such that λ such that $A - \lambda\mathbb{I}$ is regular. The spectrum $\sigma(A)$ is the set of all scalars not in $\rho(\mathcal{A})$.

Let $\mathcal{A}$ be a unital Banach algebra with unit $\mathbb{I}$. If $A \in \mathcal{A}$, then the spectrum of x is

$$\sigma(x) = \{\lambda \in \mathbb{C} :\ \lambda \cdot \mathbb{I} - x \notin G(A)\}, \tag{K.32}$$

and the spectral radius of x is

$$r(x) = \sup\{|\lambda| \in \mathbb{C} :\ \lambda \in \sigma(x)\}, \tag{K.32}$$

Theorem K.6.2 *If $\mathcal{A}$ is a Banach algebra with unit, then for each $x \in A$,*

$$
\begin{array}{ll}
\textit{(a)} & \sigma(x) \textit{ is compact and nonempty.}\\
\textit{(b)} & \rho(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \inf_{n\geq 1} \|x^n\|^{1/n}
\end{array}
\tag{K.32}
$$

Now, if $\lambda \in \sigma(x)$, then

$$
\begin{aligned}
(\lambda^n\mathbb{I} - x^n) &= (\lambda\mathbb{I} - x)(\lambda^{n-1}\mathbb{I} + \lambda^{n-2}x + \cdots + x^{n-1})\\
&= (\lambda^{n-1}\mathbb{I} + \lambda^{n-2}x + \cdots + x^{n-1})(\lambda\mathbb{I} - x)
\end{aligned}
\tag{K.32}
$$

implies that $\lambda^n \in \sigma(x^n)$ otherwise, $(\lambda^{n-1}\mathbb{I} + \lambda^{n-2}x + \cdots + x^{n-1})(\lambda^n\mathbb{I} - x^n)^{-1}$ is an inverse for $(\lambda\mathbb{I} - x)$. Then $|\lambda^n| \leq \rho(x^n) \leq \|x^n\|$. In particular, $|\lambda| \leq \|x^n\|^{1/n}$. Combining with

$$\limsup_n \|x^n\|^{1/n} \;\leq\; \rho(x) \;\leq\; \inf_{n\geq 1} \|x^n\|^{1/n} \;\leq\; \liminf_n \|x^n\|^{1/n}. \tag{K.32}$$

This completes the proof.

□

Normal operators

Of particular interest will be operators that satisfy $x^*x = xx^*$. Such operators are called normal operators, self-adjoint and unitary operators are special cases. For normal operators we have $r(x) = \|x\|$. The proof uses the identity $\|y\|^2 = \|y^*y\|$.

$$\begin{aligned}
\|x^2\| &= (\|(x^2)\|^2)^{1/2} \\
&= \|(x^2)^*(x^2)\|^{1/2} \\
&= \|(x^*x)^*(x^*x)\|^{1/2} \qquad \text{from using } x^*x = xx^* \\
&= (\|x^*x\|^2)^{1/2} = (\|x^*x\|) \\
&= \|x\|^2
\end{aligned} \tag{K.29}$$

From this we have $\|x\|^4 = \|x^2\|^2 = \|x^4\|$. By induction we get $\|x\|^{2n} = \|x^{2n}\|$. Thus,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{1/n} = \|x\| \tag{K.29}$$

K.6.3 Functional Calculus

Theorem K.6.3 *Let A be a element of $\mathcal{A}$. The following are equivalent.*

i) A is positive

$t \cdot \mathbf{1} - A$ is a normal operator so $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\} = \|A\|$.

$$\begin{aligned}
\|t \cdot \mathbf{1} - A\| &= \sup\{|\lambda| : \lambda \in \sigma(t \cdot \mathbf{1} - A)\} \\
&= \sup\{|\lambda - t| : \lambda \in \sigma(A)\} \leq t \qquad \text{as } r(A) \leq \|A\| \leq t.
\end{aligned} \tag{K.29}$$

□

$$\| \, \|A\| + \|B\| - A - B\| \leq \| \, \|A\| - A\| + \| \, \|B\| - B\| \leq \|A\| + \|B\| \tag{K.29}$$

$\|t - (A + B)\| \leq t$ (with $t = \|A\| + \|B\|$), hence by Theorem N.-19 $A + B$ is positive.

Proposition K.6.4 *Let $(\mathcal{H}, \pi)$ be a representation of a C^*-algebra $\mathcal{A}$. Then the following are equivalent:*

(i) π is faithful (ker $\pi = \{0\}$).

(ii) $\|\pi(A)\| = \|A\|$ for all $A \in \mathcal{A}$

(iii) $\|\pi(A)\| > 0$ is $A > 0$.

Proposition K.6.5 *Let ω be a linear form on $\mathcal{A}$, a unital C^*-algebra. Then the following assertions are equivalent:*

(i) ω is positive;

(ii) ω is continuous with $\|\omega\| = \omega(\mathbf{1})$.

K.6.4 Null Ideals

Definition Say that $\mathcal{I}$ is a linear subspace of $\mathcal{A}$. If it satisfies the condition

(i) $i \in \mathcal{I} \Rightarrow xi \in \mathcal{I}$ for every element $x \in \mathcal{A}$, that is, $xi \in \mathcal{I}$ whenever $i \in \mathcal{I}$ and $x \in \mathcal{A}$

then $\mathcal{I}$ is what is called a **left-sided ideal**, and if it satisfies the condition

(ii) $i \in \mathcal{I} \Rightarrow ix \in \mathcal{I}$ for every element $x \in \mathcal{A}$

it is called a **right-sided ideal**. If $\mathcal{I}$ satisfies both conditions then it is called a **two-sided ideal**, or simply an **ideal**. (Note that any Abelean algebra with a left or right ideal is automatically a two sided ideal).

Definition A maximal left ideal, MLI, in $\mathcal{A}$ is a proper left ideal not properly contained in any other proper left ideal, (proper meaning $\neq \mathcal{A}$ but only a subset of $\mathcal{A}$). A maximal right ideal, MRI, is defined similarly.

$[x] = [x_1]$ means that there is an equivalence relation $x \sim x_1$, that is, that $x - x_1$ is in $\mathcal{I}$. The elements x and x_1 are called **representatives** of the coset that contains them.

When we say a Banach algebra is **unital** this means it includes an element,

$$\mathbf{1} \cdot a = a \cdot \mathbf{1} = a, \quad \text{for all } a \in \mathcal{A}. \tag{K.29}$$

Definition A **non-trivial** ideal is an ideal which does not contain invertible elements.

We explain the reason for the term non-trivial: say an ideal $\mathcal{I}$ contains an invertible element, denote it as a^{-1} with inverse a. The product $aa^{-1} = \mathbf{1}$ must also be in the ideal $\mathcal{I}$. Now, if $\mathbf{1}$ is in $\mathcal{I}$ the condition

$$x\mathbf{1} \in \mathcal{I} \text{ whenever } \mathbf{1} \in \mathcal{I} \text{ and } x \in \mathcal{A} \tag{K.29}$$

on x is satisfied by every element of $\mathcal{A}$, i.e., the algebra $\mathcal{A}$ **itself is the ideal which contains 1**.

Definition Define its kernal as $ker(\chi) := \{a \in \mathcal{A};\ \chi(a) = 0\}$

Lemma K.6.6 *The kernal $ker(\chi)$ is a two-sided ideal of $\mathcal{A}$.*

Proof: ab is in the kernal $ker(\chi)$ for all $a \in \mathcal{A}$ and $b \in ker(\chi)$, making it a left-sided ideal.

$$\chi(ab) = \chi(a)\chi(b) = \chi(b) \times 0 = 0 \tag{K.29}$$

It is obviously a right-sided ideal as well as the element ba is also in the kernel: $\chi(ba) = \chi(b)\chi(a) = 0$. The elements of $\mathcal{A}$ whose character vanishes forms an ideal of $\mathcal{A}$ for the multiplication. That is, the character of the product of an arbitrary element a with a element whose character vanishes b also vanishes.

$\square$

Lemma K.6.7 *The kernal of a character determines a maximal ideal in $\mathcal{A}$.*

Proof: Since χ is a linear functional on $\mathcal{A}$ considered as a vector space, that is, $\chi(\alpha a + \beta b) = \alpha\chi(a) + \beta\chi(b)$ for all $a, b \in \mathcal{A}$ and for all $\alpha, \beta \in \mathbb{C}$, it follows that $\ker(\chi)$ is a vector subspace of $\mathcal{A}$ (meaning for any $a, b \in \ker(\chi)$ we have $\alpha a + \beta b \in \ker(\chi)$ for all $\alpha, \beta \in \mathbb{C}$) of codimension one. To see this suppose the codimension is $N > 1$. Let $c_1, \ldots, c_N$ be a set of linearly independent elements not in $\ker(\chi)$, then

$$\chi(\alpha_1 c_1 + \cdots + \alpha_N c_N) = 0$$

can only be true if all the α_i's are zero. Now, χ being a linear functional this condition can be reexpressed

$$\alpha_1\chi(c_1) + \alpha_2\chi(c_2)\cdots + \alpha_N\chi(c_N) = 0. \tag{K.30}$$

But this can be satisfied for non-zero α_i's, for example choose $\alpha_1/\alpha_2 = -\chi(c_2)/\chi(c_1)$ with $\alpha_3 = \cdots = \alpha_N = 0$. There is only no contradiction when the codimension is at most one. However, $\chi(a) = 0$ for all $a \in \mathcal{A}$ implies χ is identically zero which is ruled out by the defintion of the character, hence the codimension is one.

After taking the closure in $\mathcal{A}$ it is either still of codimension one or zero, the latter being impossible again since then χ would be identically zero. It follows that there exist elements $a \in \mathcal{A}-\ker(\chi)$ and that $\mathcal{A}$ is the closure of the span of a,$\ker(\chi)$. Thus, if there is an ideal $\mathcal{I}$ of $\mathcal{A}$ properly containg $\ker(\chi)$ then $\mathcal{I} = \mathcal{A}$. We conclude that the kernal of a character determines a maximal ideal in $\mathcal{A}$.

$\square$

Definition The **quotient ring**, R with respect to the ideal $\mathcal{I}$:

Let $\mathcal{I}$ be an ideal in a ring, R, and let the coset of an element x in R be defined by $x + \mathcal{I} := \{x + i : i \in \mathcal{I}\}$. Then the distinct cosets form a partition of R. If we define addition and multiplication by

$$\begin{aligned}
(x+\mathcal{I}) + (y+\mathcal{I}) &= (x+y) + \mathcal{I} \quad \text{and} &\text{(K.31)}\\
(x+\mathcal{I})(y+\mathcal{I}) &= xy + \mathcal{I}, &\text{(K.32)}
\end{aligned}$$

then these cosets form a ring denoted by $R/\mathcal{I}$ and called the quotient ring of R with respect to $\mathcal{I}$

Lemma K.6.8 *If $\mathcal{I}$ is a unital Banach algebra $\mathcal{A}$ then its closure $\overline{\mathcal{I}}$ is still an ideal in $\mathcal{A}$. Every maximal ideal is already closed.*

Proof: Recall that the closure of a subset Y in a topological is Y together with the limit points of

Recall that the subset Y in a topological space X is Y together with the limit points of convergent nets in Y. Let $\mathcal{I}$ be an ideal in $\mathcal{A}$ and let (a^α) be a net in $\mathcal{I}$ converging to

$a \in \overline{\mathcal{I}}$. Then for any $b \in \mathcal{A}$ we have $ba^\alpha \in \mathcal{I}$ since $\mathcal{I}$ is an ideal and $ba^\alpha = ba$ in the limit $a^\alpha \to a$ since

$$\|b(a^\alpha - a)\| \leq \|b\| \, \|a^\alpha - a\| \to 0. \tag{K.32}$$

Thus (ba^α) is a net in $\mathcal{I}$ converging to $ba \in \mathcal{A}$ and since all limit points of a converging net lie in $\overline{\mathcal{I}}$ we have $ba \in \overline{\mathcal{I}}$. Thus, $\overline{\mathcal{I}}$ is an ideal.

In the next step we prove that the set of non-invertible elements of $\mathcal{A}$ is closed subset. To show this, first recall a couple of facts: 1) every $a \in \mathcal{A}$ such that $\|a - 1\| < 1$ is invertible; 2) that $b^\alpha \to b$ implies $\|b^\alpha\| \to \|b\|$ as follows from application of the triangle inequality Eq.(J.55):

$$\big| \|b^\alpha\| - \|b\| \big| \leq \|b^\alpha - b\|.$$

Now consider the set

$$\{c \in \mathcal{A};\ \|c - 1\| \geq 1\} \tag{K.32}$$

and any convergent net (a^α) in it, converging to some element $a \in \mathcal{A}$. The net of real numbers $(\|a^\alpha - 1\|)$ belongs to the set

$$\{x \in \mathbb{R};\ x \geq 1\}$$

and since

$$a^\alpha - 1 \to a - 1 \quad \Rightarrow \quad \|a^\alpha - 1\| \to \|a - 1\|$$

it follows that $\|a - 1\| \geq 1$ since $\{x \in \mathbb{R};\ x \geq 1\}$ is closed. Therefore every convergent net (a^α) in (K.6.4) converges to a point in (K.6.4) and we conclude it is a closed subset of $\mathcal{A}$.

We can now conclude that every non-trivial ideal $\mathcal{I}$, that is, those not containing invertible elements, must be contained in the closed set $\{c \in \mathcal{A};\ \|c-1\| \geq 1\}$ and so must its closure $\overline{\mathcal{I}}$. Obviously $1 \notin \{c \in \mathcal{A};\ \|c-1\| \geq 1\}$, hence, closures of non-trivial ideals are non-trivial.

Finally a maximal ideal must be closed as otherwise its closure would be a non-trivial ideal containing it.

□

K.6.5 The Gel'fand-Mazur Theorem

Theorem K.6.9 (Gel'fand)

If $\mathcal{A}$ is an Abelean, unital Banach algebra and $\mathcal{I}$ a two-sided, maximal ideal in $\mathcal{A}$ then the quotient algebra $\mathcal{A}/\mathcal{I}$ is isomorphic with $\mathbb{C}$.

$$[a] = \{a + \mathcal{I}\} \tag{K.32}$$

By lemma K.6.8 $\mathcal{I}$ is closed in $\mathcal{A}$. The proof is split into three parts.

(a):

> If $\mathcal{I}$ is a maximal ideal in a unital Banach algebra $\mathcal{A}$ then $\mathcal{A}/\mathcal{I}$ is a Banach algebra.

The norm on $\mathcal{A}/\mathcal{I}$ is given by

$$\|[a]\| := \inf_{b\in[a]} \|b\| = \inf_{i\in\mathcal{I}} \|a+i\| \tag{K.32}$$

(i)

$$\|[za]\| = \|z[a]\| = \inf_{b\in[a]} \|zb\| = |z|\ \|[a]\| \tag{K.32}$$

(ii)

$$\begin{aligned}\|[a+a']\| &= \|[a]+[a']\| = \inf_{b\in[a]+[a']} \|b\| = \inf_{b\in[a]+[a']} \|b+b'\| \\ &\leq \inf_{b\in[a],b'\in[a']} (\|b\| + \|b'\|) = \|[a]\| + \|[a']\|.\end{aligned} \tag{K.32}$$

(iii)

$$\|[a]\| = \inf_{b\in[a]} \|b\| = 0 \ \Rightarrow [a] = [0]. \tag{K.32}$$

Suppose that $([a'_k])$ is a Cauchy sequence in $\mathcal{A}/\mathcal{I}$. Then for each n, there is a number $N(n) \geq N(n-1)$ such that

$$\|[a'_l] - [a'_k]\| < 2^{-n}, \quad l, k \geq N(n).$$

Set $a_n = a'_{N(n)}$. Then $\|[a_{n+1}] - [a_n]\| = \|[a_{n+1} - a_n]\| < 2^{-n}$. Since

$$\|[a_{n+1}] - [a_n]\| = \inf_{b_{n+1} \in [a_{n+1}], b_n \in [a_n]} \|b_{n+1} - b_n\| < 2^{-n} \tag{K.32}$$

certainly find representatives $\|c_{n+k} - c_n\| < 2^{-n+1}$. Then

$$\|c_n - c_m\| = \| \sum_{k=m+1}^{n-1} c_{k+1} - c_k \| \leq \sum_{k=m+1}^{n-1} 2^{-k+1} = 2^{-m} \sum_{k=0}^{m-n-1} 2^k \leq 2^{-m+1} \tag{K.32}$$

which shows that (c_n) is a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{A}$ is complete this sequence converges to some $a \in \mathcal{A}$. But then

$$\|[a_n] - [a]\| = \inf_{b_n \in [a,], b \in [a]} \|b_n - b\| \leq \|c_n - a\| \tag{K.32}$$

so $([a_n])$ converges to $[a]$. It follows that $\mathcal{A}/\mathcal{I}$ is complete, that is, a Banach space with unit $[1]$.

(b):

> For an Abelean, unital algebra $\mathcal{A}$ an ideal $\mathcal{I}$ is maximal in $\mathcal{A}$ if and only if $\mathcal{A}/\mathcal{I} - [0]$ consists of invertible elements only.

First say the ideal $\mathcal{I}$ is maximal and suppose $\mathcal{A}/\mathcal{I} - [0]$ does not consist of invertible elements only, that is, we find $[0] \neq [a] \in \mathcal{A}/\mathcal{I}$ but that $[a]^{-1}$ does not exist. This means that a^{-1} does not exist since $[a]^{-1} = [a^{-1}]$ as follows from $[a][a^{-1}] = [1]$. Consider now the ideal

$$\mathcal{A} \cdot a = \{ba : b \in \mathcal{A}\}$$

(this is a two-sided ideal because $\mathcal{A}$ is Abelean). Since $\mathcal{I} \subset \mathcal{A}$ we certainly have $\mathcal{I} \cdot a \subset \mathcal{A} \cdot a$ and since $\mathcal{I} \cdot a = \mathcal{I}$ we have

$$\mathcal{I} \subset \mathcal{A} \cdot a.$$

Now $a \in \mathcal{A} \cdot a$ since $1 \in \mathcal{A}$ and $a \notin \mathcal{I}$ because otherwise $[a] = \{i : i \in \mathcal{I}\} = \{0 + i : i \in \mathcal{I}\} = [0]$, which we excluded. It follows that $\mathcal{I}$ is a proper subideal of $\mathcal{A} \cdot a$. Finally, since

$a^{-1} \notin \mathcal{A}$, the unit element 1 can not be an element of $\mathcal{A} \cdot a$ and so $\mathcal{A} \cdot a$ cannot be all of $\mathcal{A}$. It follows

$$\mathcal{I} \subset \mathcal{A} \cdot a \subset \mathcal{A}.$$

and so $\mathcal{I}$ is not maximal, which is a contradiction.

Now say $\mathcal{A}/\mathcal{I} - [0]$ consists of invertible elements only and suppose $\mathcal{I}$ is not a maximal ideal. Then we find a proper subideal $\mathcal{J}$ of $\mathcal{A}$ of which $\mathcal{I}$ is a proper subideal. Since every non-zero element of $\mathcal{A}/\mathcal{I}$ is invertible so is every element $[a]$ of $\mathcal{J}/\mathcal{I}$ since $\mathcal{J} \subset \mathcal{A}$. But then $\mathcal{J}$ contains the invertible element $a \in \mathcal{A}$ and thus $\mathcal{J}$ coincides with $\mathcal{A}$ which is a contradiction.

(c)

> A unital Banach algebra $\mathcal{B}$ in which every non-zero element is invertible is isomorphic with $\mathbb{C}$.

Consider $b \in \mathcal{B}$ then we claim that $\sigma(b) \neq \emptyset$. Suppose that $\rho(b)$ is the whole complex plane. Let ϕ be a continuous linear functional on $\mathcal{A}$ considered as a vector space with metric $d(a, b) = \|a - b\|$.

Using linearity of ϕ and the expansion of $r_z(b)$ into an absolutely geometric series

$$r_z(b) = -\frac{1}{z} \sum_{n=0}^{\infty} (\frac{b}{z})^n$$

we see that $z \mapsto \phi(r_z(b))$

$$\phi(r_z(b)) = -\frac{1}{z} \sum_{n=0}^{\infty} (\frac{\phi(b)}{z})^n$$

is an entire analytic function - see sections K.15.1 and K.15.2 for more details. An entire function is a complex function analytic an arbitary distance from the origin. Since ϕ is linear and continuous, it is bounded with bound $\|\phi\|$. Thus $|\phi(r_z(b))| \leq \|\phi\| \; \|r_z(b)\|$. Consider

$$
\begin{aligned}
\Big|\sum_{n=1}^{N} z^{-n}\phi(b^{n-1})\Big| &\leq \sum_{n=1}^{N}\frac{\|\phi\|\;\|b\|^{n-1}}{k^n\|b\|^n} \\
&\leq \frac{\|\phi\|}{\|b\|}\sum_{n=1}^{\infty}\frac{1}{k^n} \\
&= \frac{\|\phi\|}{(k-1)\|b\|}
\end{aligned} \tag{K.31}
$$

Thus

$$
\|r_z(b)\| \leq \frac{1}{(k-1)\|b\|}, \quad |z| \geq k\;\|b\|. \tag{K.31}
$$

This shows that $\|r_z(b)\| \to 0$ as $|z| \to \infty$. In particular, $z \mapsto \phi(r_z(b))$ is an entire bounded function which therefore, by Liouville's theorem, is a constant.

Proof of Liouville's theorem: by Cauchy's integral formula,

$$
f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz.
$$

If we take C to be a circle $|z-z_0| = r_0$, then

$$
\begin{aligned}
|f'(z_0)| &= \left|\frac{1}{2\pi i}\right|\oint_{C_0}\frac{|f(z)|}{|(z-z_0)^2|}|dz| \\
&< \frac{1}{2\pi r_0^2}M2\pi r_0 = \frac{M}{r_0},
\end{aligned} \tag{K.31}
$$

where $|f(z)| < M$ within and on C_0. As $f(z)$ is entire, we may take r_0 as large as we like. So given any $\epsilon > 0$ we may make $|f'(z_0)| < \epsilon$. That is, $|f'(z_0)| = 0$, which implies that $f'(z_0) = 0$ for all z_0, so $f(z_0) = \text{constant}$. QED

Since $\phi(r_z(b)) \to 0$ as $|z| \to \infty$, we must have $\phi(r_z(b)) = 0$. Since ϕ was arbitrary it follows that

$$
r_z(b) = (z_b 1 - b)^{-1} = 0
$$

implying that $b - z_b \cdot 1$ does not exist for otherwise it would imply

$$1 = (z_b 1 - b)(z_b 1 - b)^{-1} = 0.$$

Thus we find $z_b \in \sigma(b)$, that is, $b - z_b \cdot 1$ is not invertible. By assumption, only zero elements are not invertible, hence $b = z_b \cdot 1$ for some $z_b \in \mathbb{C}$ for any $b \in \mathcal{B}$. The map $b \mapsto z_b$ is then the searched for isomorphism $\mathcal{B} \to \mathbb{C}$. Notice that $b = 0$ if and only if $z_b = 0$.

Let then $\mathcal{I}$ be a maximal ideal in a unital, Abelean Banach algebra $\mathcal{A}$. Then by a) $\mathcal{B} := \mathcal{A}/\mathcal{I}$ is a unital Banach algebra and by b) each of its non-zero elements is invertible. Thus by c) it is isomorphic with $\mathbb{C}$.

□

Corollary K.6.10 *In an Abelean, unital Banach algebra $\mathcal{A}$ there is a one-to-one correspondence between its spectrum $\Delta(\mathcal{A})$ and the set $I(\mathcal{A})$ of maximal ideals $\mathcal{A}$ via*

$$\Delta(\mathcal{A}) \to I(\mathcal{A});\ \chi \mapsto ker(\chi). \tag{K.31}$$

Proof: We know by lemme K.6.7 that each character gives rise to a maximal ideal in $\mathcal{A}$.

Conversely, let $\mathcal{I}$ be a maximal ideal in a commutative unital Banach algebra then we can apply theorem K.6.9 and obtain a Banach algebra isomorphism

$$\chi : \mathcal{A}/\mathcal{I} \to \mathbb{C}; \quad [a] \mapsto \chi([a]).$$

We can extend this to a homomorphism

$$\chi : \mathcal{A} \to \mathbb{C}$$

by $\chi(a) := \chi([a])$ By construction $\chi(a) = 0$ if and only if $[a] = [0]$, that is, if and only if $a \in \mathcal{I}$.

□

Lemma K.6.11 *Let $\mathcal{A}$ be a unital, commutative Banach algebra and $a \in \mathcal{A}$. Then $z \in \sigma(a)$ if and only if there exists $\chi \in \Delta(\mathcal{A})$ such that $\chi(a) = z$.*

Proof:

The requirement $\chi(a) = z$ is equivalent to $\chi(a - z \cdot \mathbf{1}) = 0$ so that $a - z \cdot \mathbf{1} \in \ker(\chi)$. Since is a maximal ideal in $\mathcal{A}$ it cannot contain invertible elements, thus $(a - z \cdot \mathbf{1})^{-1}$ does not exist, hence $z \in \sigma(a)$.

$\square$

Definition For a character χ in an Abelian, unital Banach algebra we define its norm by

$$\|\chi\| := \sup_{a\in\mathcal{A}} |\chi(a)| \tag{K.31}$$

Lemma K.6.12 *The characters of an Abelian, unital Banach algebra form a subset of the unit sphere in $\mathcal{A}'$, the continuous linear functionals on $\mathcal{A}$ considered as a topological vector space.*

Proof:

$$\begin{aligned} \|\chi\| &= \sup_{a\in\mathcal{A}} \frac{|\chi(a)|}{\|a\|} \leq \sup_{a\in\mathcal{A}} \frac{\sup\{|\chi'(a)|;\ \chi' \in \Delta(\mathcal{A})\}}{\|a\|} \\ &= \sup_{a\in\mathcal{A}} \frac{\sup\{|z|;\ z \in \sigma(z)\}}{\|a\|} \qquad \text{by lemma K.6.11} \\ &= \sup_{a\in\mathcal{A}} \frac{r(a)}{\|a\|} \leq 1 \end{aligned} \tag{K.30}$$

since $r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} \leq \|a\|$. On the other hand $\chi(1) = 1$, hence $\|\chi\| = 1$ for every character χ. This shows that every character is a bounded linear functional $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$.

$\square$

Definition Recall that the topological dual X' of the topological vector space X is the set of continuous (bounded) linear functionals. The weak * topology on the topological dual X' is defined by pointwise convergence, that is a net (ϕ^α) in X' converges to ϕ if and only if for any $x \in X$ the net of complex numbers $(\phi^\alpha(x))$ converges to $\phi(x)$).

More details here

This is equivalent to:

it is the weakest topology such that all the functions $x:\ X' \to \mathbb{C}; \phi \to \phi(x)$ are continuous.

K.6.6 Gel'fand Topology

weak* convergence of a sequence of functionals. Then weak* convergence of (ϕ_n) means that there is a $\phi \in X'$ such that $\phi_n(x) \to \phi(x)$ for all $x \in X$.

Every character is a bounded linear functional on $\mathcal{A}$, that is, $\Delta(\mathcal{A}) \subset \mathcal{A}'$. The Gel'fand topology on the spectrum of a unital, Abelean Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

Proof

Product of topologies

Definition: Let $\{X_\alpha\}$ be a family of topological spaces, $\alpha \in A$. Define a subset

$$B(\beta, U_\beta) = \{f | f \in \times X_\alpha, f(\beta) \in U_\beta\} \tag{K.30}$$

The Gel'fand topology on the spectrum of a unital, Abelian Banach algebra is the weak * topology induced from $\mathcal{A}'$ on its subset $\Delta(\mathcal{A})$.

or put as (notes on the Spectral...):

the weak$-\star$ topology is the on A^* is the smallest topology making the maps $\phi \mapsto \phi(x)$ continuous from A^* to $\mathbb{C}$ for each $x \in A$. It follows that the Gel'fand topology is the restriction of the weak$-\star$ topology to $\Delta(A)$.

Lemma K.6.13 *Let X be a Banach space and X' its topological dual. Then the unit ball in X' is closed and compact in the weak topology.*

Proof: The unit ball B in X' is defined as the subset of elements ϕ with norm smaller than or equal to unity, that is,

$$\|\phi\| := \sup_{x \in X} \frac{|\phi(x)|}{\|x\|} \leq 1.$$

Let ϕ^α be a universal net in B and consider for any given $x \in X$ the net of complex numbers $(\phi^\alpha(x))$ which are bounded by $\|x\|$. Our x a function $X' \to \mathbb{C}; \phi \to \phi(x)$. If (x_α) is a universal net in a space X, and $f : X \to Y$ is a function, then $f(x_\alpha)$ is a universal net in Y, with no restriction on f. The net $(\phi^\alpha(x))$ is universal. It is contained in the set $\{z \in \mathbb{C} : |z| \leq \|x\|\}$ which is compact in $\mathbb{C}$ and therefore it converges. Define ϕ pointwise by the limit, that is, $\phi(x) := \lim_\alpha \phi^\alpha(x)$. Then

$$\|\phi\| = \sup_{x \in X} \lim_\alpha \frac{|\phi^\alpha(x)|}{\|x\|} \leq \|\phi^\alpha\| \leq 1. \tag{K.30}$$

Thus ϕ^α converges pointwise to $\phi \in B$. Since a topological space is compact if and only if every universal net converges we conclude that B is compact.

A subset Y of a topological space X is closed if for every convergent net $(x_\alpha) \in X$ with $x_\alpha \in Y$ for all α the limit lies in Y. Say there is a convergent net (ϕ_α) which converges to an element ϕ not in B. Consider any open set U containing ϕ. Then for any $\alpha \in I$ such that $\phi^\alpha \in U$, $(\phi_{F(\beta)})_{\beta \in J} \in U$ for all $\beta \succeq \beta'$ for some β' by definition of a subnet. Therefore any subnet is eventually in U. But every net has a universal subnet. This contradicts that every universal net of B converges to a limit point in B. So in particular we have shown that B is closed.

□

Theorem K.6.14 *In Gel'fand topology, the spectrum $\Delta(\mathcal{A})$ of a unital, Abelian algebra is compact.*

Proof of Theorem K.6.14

By lemma K.6.12 $\Delta(\mathcal{A})$ is a subset of the unit ball B in $\mathcal{A}'$ and by lemma K.6.13 B is compact in the weak * topology. By () we know that closed subspaces of compact spaces are compact in the subspace topology. As the Gel'fand topology is the subspace topology induced from B, to prove the theorem we need only show that $\Delta(\mathcal{A})$ is closed in B

Let then (χ^α) be a net in $\Delta(\mathcal{A})$ converging to $\chi \in B$. We have, e.g., $\chi(ab) = \lim_\alpha \chi^\alpha(ab) = \lim_\alpha \chi^\alpha(a)\chi^\alpha(b)$ and similar for pointwise addition, scalar multiplication and involution in $\mathcal{A}$. It follows that the limit χ is a character as well, that is, $\chi \in \Delta(\mathcal{A})$.

□

K.6.7 The Gel'fand Transformation

Definition The Gel'fand transform is defined by

$$\bigvee : \mathcal{A} \to \Delta(\mathcal{A})'; \quad a \mapsto \overset{\vee}{a} \quad \text{where} \quad \overset{\vee}{a}(\chi) := \chi(a). \tag{K.30}$$

Where $\Delta(\mathcal{A})'$ denotes the continuous linear functionals on $\Delta(\mathcal{A})$ considered as a topological vector space.

Theorem K.6.15 *The Gel'fand transform extends to a homomorphism*

$$\bigvee : \mathcal{A} \to C(\Delta(\mathcal{A})); \quad a \to \overset{\vee}{a} \tag{K.30}$$

with the following additional properties:

i) range$(\overset{\vee}{a}) = \sigma(a)$.

ii) $\| \overset{\vee}{a} \| := \sup_{\chi \in \Delta(\mathcal{A})} | \overset{\vee}{a} (\chi)| = r(a)$.

iii) The image $\bigvee(\mathcal{A})$ *separates the points of* $\Delta(\mathcal{A})$.

Proof:

i) We have

$$\mathrm{range}(\overset{\vee}{a}) = \{\overset{\vee}{a} (\chi) : \chi \in \Delta(\mathcal{A})\} = \sigma(a) \tag{K.30}$$

as follows from lemma K.6.11.

ii) We have

$$\| \overset{\vee}{a} \| = \sup_{\chi \in \Delta(\mathcal{A})} | \overset{\vee}{a} (\chi)| = \sup_{\chi \in \Delta(\mathcal{A})} |\chi(a)| = \sup(\{|\chi(a)| : \chi \in \Delta(\mathcal{A}\}) = r(a) \tag{K.30}$$

by definition of the spectral radius.

iii) Consider any $\chi_1, \chi_2 \in \Delta(\mathcal{A})$ with $\chi_1 \neq \chi_2$. By definition of $\Delta(\mathcal{A})$ there exists then $a \in \mathcal{A}$ such that $\chi_1(a) = \overset{\vee}{a} (\chi_1) \neq \chi_2(a) = \overset{\vee}{a} (\chi_2)$.

□

Lemma K.6.16 *The Gel'fand topology is on the spectrum of a unital Abelean Banach algebra is Hausdorff.*

Proof: The proof follows trivially from the fact that by theorem K.6.15 $\mathcal{C} := \{\overset{\vee}{a} \,; a \in \mathcal{A}\}$ is a system of continuous functions separating the points of $\Delta(\mathcal{A})$ and applying lemma J.9.7.

□

Theorem K.6.17 *Let $\mathcal{A}$ be a unital commutative C^*-algebra (not only Banach algebra). Then the Gel'fand transform is an isometric isomorphism between $\mathcal{A}$ and the space of continuous functions on its spectrum.*

Proof:

For Abelean C^*-algebras we have

$$r(a) = \|a\|. \tag{K.30}$$

By theorem K.6.15 we therefore have

$$\| \overset{\vee}{a} \| = \|a\| \tag{K.30}$$

that is, isometry.

Consider now the system of complex valued functions on the spectrum given by $\mathcal{C} := \{\overset{\vee}{a}: a \in \mathcal{A}\}$. We claim that it has the following properties:

i) $\mathcal{C} \subset C(\Delta(\mathcal{A}))$

ii) $\mathcal{C}$ separates points of $\Delta(\mathcal{A})$

iii) $\mathcal{C}$ is a closed (in the sup-norm topology) $*$ subalgebra of $C(\Delta(\mathcal{A}), \mathbb{C})$

iv) The constant functions belong to $\mathcal{C}$.

We show iii) $\mathcal{C}$ is a closed * algebra in $C(\Delta(\mathcal{A}))$. Suppose that $(\overset{\vee}{a^\alpha})$ is a net in $\mathcal{C}$ converging to some $f \in C(\Delta(\mathcal{A}))$. Thus, $(\overset{\vee}{a}{}^\alpha)$ is in particular a Cauchy sequence, meaning that $\| \overset{\vee}{a}{}^\alpha - \overset{\vee}{a}{}^\beta \| = \|a^\alpha - a^\beta\|$ becomes arbitrarily small as α, β grow, where we have used isometry. It follows that (a^α) is a Cauchy sequence and therefore converges to some $a \in \mathcal{A}$ since $\mathcal{A}$ ia a banach algebra and therefore complete.

Now we can establish isomorphism. From theorem K.6.14 and corollary K.6.16 that $\Delta(\mathcal{A})$ is a compact Hausdorff space. Recall the Stone-Weierstrass theorem J.11.4 which states that if Y be a compact Hausforff space, and $\mathcal{C}$ be a closed subalgebra of $C(Y, \mathbb{C})$ such that $\mathcal{C}$ is unital, closed under complex conjugation and separates points of $C(Y, \mathbb{C})$ then $\mathcal{C} = C(Y)$. Properties i), ii), iii), iv) of $\mathcal{C}$ enable us to apply the theorem, Y being $\Delta(\mathcal{A})$, and conclude $\mathcal{C} = C(\Delta(\mathcal{A}))$. In other words, the Gel'fand tranform is a surjection. Finallly it is an injection since $\overset{\vee}{a} = \overset{\vee}{a}{}'$ implies $\| \overset{\vee}{a} - \overset{\vee}{a}{}' \| = \|a - a'\| = 0$ by isometry, hence $a = a'$.

□

Corollary K.6.18 *Every compact Hausdorff space X arises as the spectrum of an Abelean unital C^*-algebra $\mathcal{A}$, specifically $\mathcal{A} = C(X)$, $\Delta(\mathcal{A}) = X$.*

Let X be a compact Hausdorff space and define $\mathcal{A} := C(X)$ equipped with the sup-norm.

Next let (x^α) be a net in X which converges in $\Delta(C(X))$ then by theorem J.10.5 $\overset{\vee}{f}(x^\alpha)$ converges in $\mathbb{C}$ for any $\overset{\vee}{f} \in C(\Delta(C(X)))$, i.e., $f(x^\alpha)$ converges in $\mathbb{C}$ for any $f \in C(X)$. But by theorem J.10.5 $f : X \to \mathbb{C}$ is continuous only if whenever $f(x^\alpha)$ is a net convergent in $\mathbb{C}$ then the net (x^α) converges to $x \in X$. From theorem J.10.3 it follows that X is closed in $\Delta(C(X))$.

Suppose now that $\Delta(C(X)) - X \neq \emptyset$. Thus there exists $\chi_0 \in \Delta(C(X)) - X$. By lemma J.9.3 we know that one point sets in a Hausdorff space are closed. Therefore the sets $X, \{\chi_0\}$ are disjoint closed sets in the compact Hausdorff space $\Delta(C(X))$. Since compact Hausdorff spaces are normal spaces (theorem J.9.8) we may apply Urysohns's lemma (lemma J.9.10) to conclude that there is a continuous function $F : \Delta(C(X)) \to \mathbb{R}$ with range in $[0, 1]$ such that $F_{|X} = 0$ and $F|\{\chi_0\} = F(\chi_0) = 1$.

Consider then any $f \in C(X)$. Since $C(\Delta(C(X)))$ are all continuous functions on $\Delta(C(X))$, from the functions F just constructed we find different continuous extensions of f to $C(\Delta(C(X)))$: $\overset{\vee}{f} \to \overset{\vee}{f} + F$. However, this contradicts the fact that $\bigvee$ is an isomorphism since it would not be surjective.

$\square$

Corollary tells us that a compact Hausdorff space can be reconstructed from its Abelean, unital C^*-algebra of continuous fnctions by constructing its spectrum.

K.7 Spectral Theorem and GNS-Construction

[407]:

"...C^*-algebras occupy a a very special place among all Banach algebras. This should be compared with the role played by Hilbert spaces among Banach spaces. In fact, as we will see there is an intimate relationship between Hilbert spaces and C^*-algebras thanks to the GNS construction and the Gel'fand-Neumark embedding theorem."

To each state, ω, the GNS-construction associates a representation, π_ω, of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_\omega$ together with a cyclic vector $\Omega \in \mathcal{H}_\omega$ such that

$$\omega(A) = < \Omega | \pi_\omega(A) \Omega > \tag{K.30}$$

A folium of a given state ω which may be defined to be the set of all states ω_σ which arise in the form $\mathrm{Tr}(\sigma \pi_\omega(\,\cdot\,))$ where σ ranges over the density operators (trace-class operators with unit trace) on $\mathcal{H}_\omega$.

Given a state, ω, and an automorphism, α, which preserves the state (i.e. $\omega \circ \alpha = \omega$) then there will be a unitary operator, U, on $\mathcal{H}_\omega$ which implements α in the sense that $\pi_\omega(\alpha(A)) = U^{-1}\pi_\omega(A)U$ and U is chosen uniquely by the condition $U\Omega = \Omega$.

Normal operators form a C^*-algebra

Let $\mathcal{A}$ be a unital abelean C^*-algebra generated by $1, x, x^*$. For normal operators $x^*x = xx^*$ the algebra generated by $C(\{1, x, x^*\})$ is a C^*-algebra.

$$\|x\psi\|^2 =< \psi, x^*x\psi >= < \psi, xx^*\psi >= \|x^*\psi\|^2$$

so that $\|x\| = \|x^*\|$ for any $x \in \mathcal{A}$. By the Schwartz inequality

$$\|x\|^2 = \|x^*\|^2 = | < \psi, x^*x\psi > | \leq \|\psi\| \ \|x^*x\psi\|$$

implying $\|x\|^2 = \|x^*\|^2 \leq \|x\|^2$. On the other hand, we always have $\|x^*x\| \leq \|x\| \ \|x^*\|$. Hence,

$$\|x^*x\| = \|x\|^2. \tag{K.30}$$

Consider the spectrum $\Delta(\mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathbb{C})$ and the map

$$z : \Delta(\mathcal{A}) \to \mathbb{C} \quad ; \quad \chi \mapsto \chi(a)$$

which is continuous by the definition of the Gel'fand topology on the spectrum. We have seen already that the range of this map coincides with $\sigma(a)$. Moreover, z is injective because $\chi(a) = \chi'(a)$ implies that χ, χ' coincide on all polynomials of a, a^* since they are homomorphisms,

$$\begin{aligned}
\chi(p(a, a^*)) &= \chi(c_0 + c_1 a + d_1 a^* + c_1 a^2 + \dots) \\
&= c_0 + c_1\chi(a) + d_1\overline{\chi(a)} + c_2\chi(a)^2 + \dots \\
&= c_0 + c_1\chi'(a) + d_1\overline{\chi'(a)} + c_2\chi'(a)^2 + \dots \\
&= \chi'(c_0 + c_1 a + d_1 a^* + c_2 a^2 + \dots) \\
&= \chi'(p(a, a^*))
\end{aligned}$$

and thus on all of $\mathcal{A}$ by continuity whence $\chi = \chi'$. Thus, z is a continuous bijection between the spectra $\Delta(\mathcal{A})$ and $\sigma(a)$. Since a is bounded, both spectra are compact Hausdorff spaces. Now a continuous bijection between compact Hausdorff spaces is automatically a homeomorphism.[3] We conclude that we can identify $\Delta(A)$ topologically with $\sigma(a)$.

By defintion the polynomials p in a, a^* lie dense in $\mathcal{A}$ and we have for $\chi \in \Delta(\mathcal{A})$ that

$$\chi(p(a,a^*)) = p(\chi(a), \overline{\chi(a)}) = p(z(\chi), \overline{z(\chi)}) = [p \circ (z, \overline{z})](\chi) = p(a,a^*)^{\vee}(\chi) \qquad \text{(K.25)}$$

so that the Gel'fand isometric isomorphism can be thought of as a map

$$\bigvee : \mathcal{A} \to C(\sigma(a)) \quad ; \quad b \mapsto \overset{\vee}{b} \quad \text{with} \quad \overset{\vee}{b}(z) = \chi(b)_{z=\chi(a)}.$$

Now consider any state $\psi \in \mathcal{H}$ with $\|\psi\| = 1$. Then

$$\omega_\psi : \mathcal{A} \to \mathbb{C} \quad ; \quad b \mapsto < \psi, b\psi > \qquad \text{(K.25)}$$

is obviously a state on $\mathcal{A}$. Via the Gel'fand transform we obtain a positive linear functional on $C(\sigma(a))$ by

$$\Lambda_\psi : C(\sigma(a)) \to \mathbb{C} \quad ; \quad \overset{\vee}{b} \mapsto \omega_\psi(b) \qquad \text{(K.25)}$$

and since $\sigma(a)$ is a compact Hausdorff space we can apply the Riesz representation theorem in order to find a unique, regular Borel measure μ_ψ on $\sigma(a)$ such that

$$\omega_\psi(b) = \int_{\sigma(a)} d\mu_\psi \, b(z). \qquad \text{(K.25)}$$

The measure μ_ψ is caled a spectral measure. The meaning of this formula is explained by the following definition.

Extension to normal operators

Theorem K.7.1 *Let (a_I) be a self-adjoint collection of mutually commuting elements of a C^*-algebra $\mathcal{C}$. Then there exists a representation of the sub-C^*-algebra $\mathcal{A}$ generated by this collection on a Hilbert space $\mathcal{H}$ such that the $\pi(a_I)$ become multiplication operators.*

[3] Proof: Let $f : X \to Y$ be a continuous bijection and let X be compact and Y Hausdorff. We must show that $f(U)$ is open in Y for every open subset $U \subset X$, or by taking complements, that images of closed sets are closed. Now since X is compact, it follows that every closed set U is also compact. Since f is continuous, it follows that $f(U)$ is compact. Since Y is Hausdorff it follows that $f(U)$ is closed. See theorems N.-19 and N.-19.

K.7.1 Bits and Pieces

$$1 \in \mathcal{B} \subseteq \mathcal{A}.$$

Then clearly

$$\sigma_{\mathcal{A}}(x) \subseteq \sigma_{\mathcal{B}}(x). \tag{K.25}$$

Theorem K.7.2 *Suppose that $\mathcal{B}$ is a unital C^*-subalgebra of a C^*-algebra $\mathcal{A}$ (i.e., $1 \in \mathcal{B} \subseteq \mathcal{A}$ with $\star : \mathcal{B} \to \mathcal{B}$). Then for all $x \in \mathcal{B}$, $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$.*

Fix $x \in \mathcal{B}$. We need only show that if $x \notin \sigma_{\mathcal{A}}(x)$ then $x \notin \sigma_{\mathcal{B}}(x)$ (i.e., $x^{-1} \in \mathcal{B}$), then $\sigma_{\mathcal{B}}(x) = \sigma_{\mathcal{A}}(x)$ follows from Eq.(K.7.1).

$x^{-1} = (x^*x)^{-1}x^* \in \mathcal{B}$ if x is. As $x^* \in \mathcal{B}$ it suffices to show that $(x^*x)^{-1} \in \mathcal{B}$ if x is.

Let $\mathcal{C}$ be a C^*-subalgebra of $\mathcal{A}$ generated by x and x^{-1}, and let $\mathcal{D}$ be the $\star-$subalgebra generated by 1 and x.

Therefore, $\mathcal{D} = \mathcal{C}$ by the Stone-Weierstrass Theorem. In particular, $x^{-1} \in \mathcal{D} \subseteq \mathcal{B}$.

$\square$

Theorem K.7.3 *J is a maximal ideal of $\mathcal{A}$ if and only if J is the kernal of some $h \in \Delta$.*

We establish the **first part** of K.7.3:

> If J is a maximal ideal of $\mathcal{A}$ then J is the kernal of some $h \in \Delta$.

Let J be the a maximal ideal of $\mathcal{A}$. Since J is closed, Now choose $x \in \mathcal{A}$ so that $\pi(x) \neq 0$. Thus, $x \notin J$, and

$$M = \{ax + y \ : \ a \in \mathcal{A} \text{ and } y \in J\} \tag{K.25}$$

is an ideal in $\mathcal{A}$ so that $J \subset M$ but $J \neq M$. Therefore $M\mathcal{A}$; in particular, for some $a \in \mathcal{A}$ and $y \in JU$,

$$ax + y = e. \tag{K.25}$$

...that π defines a complex homomorphism with kernal J.

We now move onto the **second part** of K.7.3:

If J is the kernal of some $h \in \Delta$ then J is a maximal ideal of $\mathcal{A}$.

If $h \in \Delta$, then we must show that $J = h^{-1}(0)$ is a maximal ideal.

□

Notes on operator algebras (G. Jungman)

Theorem K.7.4 (Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative C^*-algebra, and equip Δ with the Gelfand topology as usual. Then the Gelfand transform is an isometric $*-$isomorphism of $\mathcal{A}$ onto the algebra of continuous complex-valued functions on Δ, $C(\Delta)$.*

The next theorem provides a continuous symbolic calculus for operators as long as they generate a commutative C^*-algebra. So, for example, if x is a normal operator then we apply the above theorem to the algebra generated by x and x^*, and we get the continuous functional calculus for normal operators.

Theorem K.7.5 (Inverse Gelfand-Naimark). *Let $\mathcal{A}$ be a commutative C^*-algebra. Let $x \in \mathcal{A}$ be such that the polynomials in x and x^* are dense in $\mathcal{A}$. Then we can define an isometric isomorphism $\Phi : C(Spec(x)) \to \mathcal{A}$ by*

$$\widehat{(\Phi f)} = f \circ \hat{x},$$

and we have

$$\Phi f^* = (\Phi f)^*.$$

Proof. Let

□

By the Gelfand-Naimark theorem, $f \circ \hat{x}$ is thus the Gelfand transform of a unique element in $\mathcal{A}$ which we denote Φf, and $\|\Phi f\| = \|f\|_\infty$. If $f(\lambda) = \lambda$, then $f \circ \hat{x} = \hat{x}$ and $\Phi f = x$.

Compare to section J.3.4

Theorem K.7.6 *Let (a_I) be a self-adjoint collection of mutually commuting elements of a C^*-algebra $\mathcal{C}$. Then there exists a representation of the sub–C^*-algebra $\mathcal{A}$ generated by this collection on a Hilbert space $\mathcal{H}$ such that the $\pi(a_I)$ become multiplication operators.*

K.7.2 Extension of Spectral Theorem to Unbounded Operators

The extension of the spectral theorem to unbounded self-adjoint operators on a Hilbert space can be traced back to the bounded case by using the following trick. (Recall that a densely defined operator a with domain $D(a)$ is called self-adjoint if $a^\dagger = a$ and $D(a^\dagger) = D(a)$ where

$$D(a^\dagger) = \{\psi \in \mathcal{H}; \sup_{0 \neq \psi' \in D(a)} | < \psi, a\psi' > | \; / \; \|\psi'\| < \infty\}$$

K.8 *Further Details on Algebraic Quantization

If the set is over complete, we impose anti-commutation relations. For example if $FG = H$ where H is also a member of the algebra $\mathcal{S}$ then we impose the operator equation

$$\hat{F} \cdot \hat{G} + \hat{G} \cdot \hat{F} - 2\hat{FG} = 0, \tag{K.25}$$

More generally, if $F_1, F_2, \cdots, F_n$, as well as thier product, $F_1 F_2 \cdots F_n$, belong to the space $\mathcal{S}$, we require that

$$F_{(1} F_2 \cdots F_{n)} - (F_1 \cdot \hat{\cdot} \cdot F_n) = 0. \tag{K.25}$$

K.9 Examples: Shrödinger and Weyl-Representations

$$W(\xi_1)W(\xi_2) = e^{\frac{i}{2}\Im \xi_1 \bar{\xi}_2} W(\xi_1 + \xi_2), \tag{K.25}$$

$$[W(\xi)]^* = W(-\xi) \tag{K.25}$$

this property is called involution $*$

This is the Weyl-Heisenberg $\star$-algebra

$$W(\xi) = e^{\frac{i}{2}\lambda\mu} U(\lambda)V(\mu) \tag{K.25}$$

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1 + \lambda_2), \quad V(\mu_1)V(\mu_2) = V(\mu_1 + \mu_2),$$
$$U(\lambda)V(\mu) = e^{\frac{-i}{2}\lambda\mu} V(\mu)U(\lambda) \tag{K.25}$$

K.10 Uniqueness Proof of the Ashtekar-Lewandowski-Isham Representation

K.10.1 Uniqueness

K.10.2 Irreducibility

Imcomplete

By the Stone-von Neumann theorem, every irreducible, weakly continuous representation of the Weyl algebra is unitarily equivalent to the Schrodinger representation.

.....

The proof was analogous to the original proof by von Neumann for the Schrodinger representation of the standard Weyl algebra.

K.11 Algebraic Quantum Field Theory

In AQFT one cleanly separates two parts of quantizing a field theory, namely first to define a suitable algebra $\mathcal{A}$ and then study its representations in a second step.

Reformulating QFT on an axiomatic basis: that is, starting from what seem to be physically necessary and mathematically precise principles which any QFT would have to satisfy, and then finding QFTs which actually satisfy them. This program is now generally referred to as algebraic quantum field theory, or AQFT; (see [114], and references therein, for extensive discussion of it).

Very few concrete theories have been found which satisfy the AQFT axioms. To be precise, the only known background dependent theories in four dimensions which do satisfy the axioms are interaction-free. The only known AQFT-compatible interacting field theories, and in particular the standard model are the background independent quantum field theories.

Operators of observables localized in an open region of spacetime $\mathcal{O}$ form an algebra $\mathcal{A}(\mathcal{O})$ of bounded operators. It is possible to encode all physically relevant properties in terms of these algebras and their transformation behaviour under Poincare group.

In AQFT one uses the mathematical framework of operators, which have been surveyed above and combines it with the physical concept of locality of nets of local algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$. for each open region $\mathcal{O}$ one assigns a C^*-algebra $\mathcal{A}(\mathcal{O})$.

(i) locality: operators localized in causally disjoint region commute,

(ii) covariance: the spacetime symetries act ....

(iii) stability??

hep-th/9901015 a QFT theory is cast into an inclusion preserving map

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}) \tag{K.25}$$

a unital C^*-algebra $\mathcal{A}(\mathcal{O})$. The Hermition elements of the abstract C^*-algebra $\mathcal{A}(\mathcal{O})$ are interpreted as the observables which can be measured at times and locations in $\mathcal{O}$. The physical states are described by positive, linear, and normalized functionals.

GNS-construction any state ω on $\mathcal{A}$ gives rise to a Hilbert space $\mathcal{H}_\omega$ and a representation π_ω together with a cyclic vector Ω_ω, such that

$$\omega(a) = (\Omega_\omega, \pi_\omega(a)\Omega_\omega) \tag{K.25}$$

A measure μ_ω on K is induced by ω via the Riesz-Markov representation theorem.

The net $\mathcal{O} \to \mathcal{A}(\mathcal{O})$

M. Rainer, *Is Loop Quantum Gravity a QFT?*, [gr-qc/9912011].

$$\mathcal{L}(\mathcal{H}) \supset \mathcal{A}(\mathcal{O}) := \{\phi(x)\ x\ \in \mathcal{O}\} \tag{K.25}$$

Vice versa, the entity of point-like localized fields can be reconstructed from localized algebras as

$$\{\phi(x)\} = \cap_{x\in\mathcal{O}} \overline{\mathcal{A}(\mathcal{O})} \tag{K.25}$$

K.11.1 Wightman Axioms

The Wightman axioms describe the action of the vacuum of the action of the group of inhomogeneous Lorentz transformations $\{a, \Lambda\}$, where Λ denotes a homogeneous Lorentz transformation and where a denotes a spacetime translation. they involve constructing the fields $\varphi(x, t)$, along with a unitary, positive-energy representation $U(a, \Lambda)$ of the inhomogeneous Lorentz group - they act on the Hilbert space $\mathcal{H}$ of the quantum theory. Furthermore, one requires that a zero-energy ground state $\Omega \in \mathcal{H}$ of the full group $U(a, \Lambda)$, and that is unique.

The unitary representation $U(a, \Lambda)$ of the Lorentz group of spacetime symmetries determines a $*-$automorphism group of transformations of fields,,

$$\alpha_{a,\Lambda} : \varphi \to U(a,\Lambda)\ \varphi\ U(a,\Lambda)^*. \tag{K.25}$$

K.11.2 Haag-Kastler Axioms

Operator Algebras Department of Mathematics

Postulate 1

To each region $\mathcal{O}$ in Minkowski space, $\mathcal{M}$, there corresponds a sub$-C^*-$algebra $\mathcal{B}(\mathcal{O})$ of $\mathcal{B}$. Moreover, $\mathcal{B}$ generated by the algebras $\mathcal{B}(\mathcal{O})$ as $\mathcal{O}$ runs over regions of $\mathcal{M}$.

The next axiom expresses the notion that if one region is contained inside another, the bigger region will have as many or more observables associated with it.

Postulate 2 (Isotony)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are regions in Minkowski space with $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then

$$\mathcal{B}(\mathcal{O}_1) \subseteq \mathcal{B}(\mathcal{O}_2). \tag{K.25}$$

Observables associated with space-like separated regions should not affect each other and so should be simultaneously measurable. This means that elements of the C^*-algebra, they must commute.

Postulate 3 (Causality/locality)

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated regions, then the associated local algebras of observables $B \in \mathcal{B}(\mathcal{O}_1)$ and $B \in \mathcal{B}(\mathcal{O}_2)$ commute, i.e., for any $A \in \mathcal{B}(\mathcal{O}_1)$ and $B \in \mathcal{B}(\mathcal{O}_2)$, we have $AB = BA$.

Poincaré covariance of the theory is expressed in the following axiom.

Postulate 4 (Poincaré covariance)

There is a representation α of $\mathcal{P}_+^\dagger$, the restricted Poincaré group, in Aut $\mathcal{B}$ automorphisms group of $\mathcal{B}$, such that

$$\alpha(L)(\mathcal{B}(\mathcal{O})) = \mathcal{B}(\Lambda\mathcal{O} + a) \tag{K.25}$$

for any region $\mathcal{O}$ and $L = (a,\Lambda) \in \mathcal{P}_+^\dagger$.

Postulate 5

The quasilocal algebra $\mathcal{B}$ is primitive, that is, $\mathcal{B}$ possesses a faithful, irreducible representation.

K.12 Constructive Quantum Field Theory

K.13 Algebraic Quantization

Algebraic Quantization

1. The set $\mathcal{S}$ should be a vector space large enough so that every function on Γ can be obtained by (possibly a limit of) sum of products of elements of $\mathcal{S}$. The purpose of this condition is that we have enough observables be unambiguously quantized.

2. The set $\mathcal{S}$ should be small enough so that it is closed under Poisson brackets from the vector space $\mathcal{S}$ as the free associative algebra generated by $\mathcal{S}$. It is this quantum algebra that we impose the Dirac quantization condition: Given A, B and $\{A, B\}$ in $\mathcal{S}$ we impose

$$[\hat{A}, \hat{B}] = i\hbar \widehat{\{A, B\}} \tag{K.25}$$

We must now find a vector space V and a representation of the elements of $\mathcal{A}$ as operators on V. Real observables must be represented by Hermition operators. One then completes V to get the Hilbert space $\mathcal{H}$ of the theory.

Refined Algebraic Quantization

RHS

K.14 von Neumann Algebras

K.14.1 Introduction

We tend to perceive time as 'flowing', as though it were in smooth and perpetual continuous motion,

'emergence of time'

has a thermodynamical origin.

observables living in a region R can be computed from observables living in region S if R is in the causal shadow of S. von Neumann algebras also enter here.

Also very important in non-commutative geometry.

K.14.2 The Emergence of Time

Generally covariant theories and The problem of time

see week 41 beaz

Rovelli wants to use thermodynamics to **define** what we call time as we usualy mean. does this as follows. Given a mixed state with density matrix D, find some operator H such that D is the Gibbs state $\exp(-H/kT)$. In lots of cases this isn't hard; it basically amounts to

$$H = -kT \ln D \tag{K.25}$$

Of course, H will depend on T, but this is really is just saying that fixing your temperature fixes your units of time!

Operator theorists have pondered this notoin very carefully for a long time and generalised it into the Tomita-Takesaki theorem. This gives a very general way of finding a Hamiltonian (hence a notion of time evolution) from a state of a quantum system! For example, one can use this trick to start with a Robertson-Walkerr universe full of blackbody radiation, and recover a notion of "time".

Gibb's distribution

The quantum state is then given by the Gibbs density matrix

$$\omega = Ne^{\beta H} \tag{K.25}$$

where H is the hamiltonian, defined on a Hilbert space $\mathcal{H}$, and $N = tr[e^{\beta H}]$.

The KMS condition

$$\omega() = \omega() \tag{K.25}$$

Mathematics preliminaries of von Neumann Algebras

Given a state ω over an abstract C^*-algebra $\mathcal{A}$, the well known Gelfand-Naimark-Segal construction provides us with a Hilbert space $\mathcal{H}$ with a preferred state $|\Psi_0>$, and a representation π of $\mathcal{A}$ as a concrete algebra of operators on $\mathcal{H}$, such that

$$\omega(A) =< \Psi_0|\pi(A)|\Psi_0 > . \tag{K.25}$$

Some definitions

The name trace class comes from its property that if A is trace class, then for any orthononormal basis $\{\varphi_n\}$

$$tr(A) = \sum_n < e_n, Ae_n > \tag{K.25}$$

is finite and independent of the orthononormal basis.

A state's **folium** is the set of states ω_ρ on $\mathcal{U}$ defined by

$$\omega_\rho := \frac{\mathrm{tr}_{\mathcal{H}_\omega}(\rho\pi_\omega(a))}{\mathrm{tr}_{\mathcal{H}_\omega}(\rho)} \tag{K.25}$$

where ρ is a positive trace class operator on the GNS Hilbert space $\mathcal{H}_\omega$.

The folium of states is the set of states determined by the density matrices on the Hilbert space of the given representation.

In the following, we denote $\pi(A)$ simply as A. Given ω and the corresponding GNS representation of $\mathcal{A}$ in $\mathcal{H}$, the set of all the states ρ over $\mathcal{A}$ that can be represented as

$$\rho(A) = Tr[A\rho] \tag{K.25}$$

where ρ is a positive trace-class operator in $\mathcal{H}$, is denoted as the folium determined by ω. In the following, we shall consider an abstract C^*-algebra $\mathcal{A}$, and a preferred state ω. A von Neumann algebra $\mathcal{R}$ is then determined, as the closure of $\mathcal{A}$ under the weak topology determined by the folium of ω. (expand - make easier)

The KMS or modular condition is a mathematically rigorous requirement for a state on a C^*-algebra to be a thermal equilibrium states, [??]. Roughly this may be understood that for finite systems a thermal state is given by a (normal) trace-class opeerator in the GNS-Hilbert space. It is here that one is able to calculate the KMS condition as it is given in the below defition below. The important property of this relation now is, that it survives the thermodynamical limit, and it is a relation which can be formulated purely in terms of C^*-algebras without taking reference to any representation space.

Modular automorphisms

The modular flow of ω is

$$\alpha_t A = e^{i\beta tH} A e^{i\beta tH}, \tag{K.25}$$

namely it is the time flow generated by the hamiltonian, with the time rescaled as $t \to \beta t$.

K.14.3 von Neumann Algebras

Let $\mathcal{R}$ be an algebra acting on a Hilbert space $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be a cyclic vector for $\mathcal{R}$ if the set $\{x\Omega : x \in \mathcal{R}\}$ is dense in $\mathcal{H}$. A vector $\Omega \in \mathcal{H}$ is said to be separating for $\mathcal{R}$ if $x\Omega = 0$ for $x \in \mathcal{R}$ implies that $x = 0$.

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded operators on a complex Hilbert space $\mathcal{H}$ and 1 the identity operator on $\mathcal{H}$. The commutant of a set $\mathcal{U} \subset \mathcal{B}(\mathcal{H})$ is

$$\mathcal{U}' = \{A \in \mathcal{B}(H) : [A, B] = 0 \text{ for all } B \in \mathcal{U}\}.$$

If the commutant $\mathcal{U}'$ consists of multiples of the identity, then $\mathcal{U}$ is irreducible. Let $\mathcal{U}'' := (\mathcal{U}')'$, the double commutant of $\mathcal{U}$.

Lemma K.14.1 *(or theorem??) Ω is cyclic for $\mathcal{R}$ if and only if it is separating for the commutant $\mathcal{R}'$.*

Proof: Suppose Ω is cyclic for $\mathcal{R}$, we wish to show that Ω is separating for $\mathcal{R}'$. We suppose $y\Omega = 0$ for some y in $\mathcal{R}'$, then for any $x \in \mathcal{R}$ $yx\Omega = xy\Omega = 0$. So we have $y(x\Omega) = 0$ for all $x \in \mathcal{R}$. As $\mathcal{R}\Omega$ is dense in $\mathcal{H}$, it must be that $y = 0$.

Suppose Ω is separating for $\mathcal{R}'$, we wish to show that Ω is cyclic for $\mathcal{R}$. let p be the projection onto the closure of the subspace $\mathcal{R}\Omega$ in $\mathcal{H}$,

$$p : \mathcal{H} \to \overline{\mathcal{R}\Omega}$$

To prove the result we will show that in fact $p = \mathbf{1}$ as this implies that $\mathcal{R}\Omega$ is dense in $\mathcal{H}$. For any $f \in \mathcal{H}$, $pf \in \overline{\mathcal{R}\Omega}$ so that

$$pf = \lim_{n\to\infty} a_n \Omega$$

for some sequence $\{a_n\} \in \mathcal{R}$ (if $f \in \mathcal{H}$ but $f \notin \overline{\mathcal{R}\Omega}$ then $a_n = 0$??). For any $b \in \mathcal{R}\Omega$ we can write

$$bpf = \lim_{n\to\infty} ba_n\Omega = \lim_{n\to\infty} pba_n\Omega = pbpf$$

and so $bp = pbp$. For any $x \in \mathcal{R}$, setting $b = x$ and $b = x^*$ gives $xp = pxp$ and $x^*p = px^*p$ respectively. Taking adjoints of this last equality, we find $px = pxp$ and hence

$$px = xp,$$

i.e., $p \in \mathcal{R}'$. By the definition of p, $p\Omega = \Omega$, that is, $(\mathbf{1} - p)\Omega = 0$. It is obvious that $(\mathbf{1} - p) \in \mathcal{R}'$ and so as Ω is separating for $\mathcal{R}'$, we deduce $p = \mathbf{1}$.

□

It follows from the above that if Ω is both cyclic and separating for $\mathcal{R}$, it is also cyclic and separating for $\mathcal{R}'$.

Note that a weaker topology has less open sets so that if a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

Definition Let $\mathcal{R} \subseteq \mathcal{B}(\mathcal{H})$ be a self-adjoint algebra of operators containing the unit $\mathbf{1}$. $\mathcal{R}$ is said to be a von Neumann algebra if it is weakly closed in $\mathcal{B}(\mathcal{H})$.

$$\omega(a) = Tr[a\omega] \tag{K.25}$$

Let $\mathcal{R}$ be a von Neumann algebra generated by $\pi_\omega(\mathcal{A})$, i.e., $\mathcal{R} = (\pi_\omega(\mathcal{A}))''$, where $\mathcal{A}'$ denotes the cummutant of $\mathcal{A}$

K.14.4 von Neumann's Density Theorem

for the other four topologies sequencies are not sufficient. the resulting topologies are not first countable, and so the closure of a subset N of $\mathcal{B}(\mathcal{H})$ is generally larger than the set of all limit points of sequencies in N. Rather, the closure of N is the set of all limit points of generalised sequencies (nets) in N.

$$
\begin{aligned}
\sum_{i=1}^{n} | < y_i, (C_\alpha - C)x_i > | &\leq \sum_{i=1} \|y_i\| \, \|(C_\alpha - C)\| \, \|x_i\| \\
&\leq \|(C_\alpha - C)\| \Big[\sum_{i=1}^{n} \|y_i\|^2 \Big]^{1/2} \Big[\sum_{i=1}^{n} \|x_i\|^2 \Big]^{1/2} \\
&< \infty
\end{aligned}
\tag{K.24}
$$

Note that a weaker topology has less open sets so that if a set is closed in the weak topology it is necessarily closed in the strong and norm topologies.

A self-adjoint algebra $\mathcal{R}$ of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if it satisfies one of the following equivalent conditions:

(1) $\mathbf{1} \in \mathcal{R}$ and $\mathcal{R}$ is closed in the strong operator topology.[4]

(2) $\mathbf{1} \in \mathcal{R}$ and $\mathcal{R}$ is closed in the weak operator topology.[5]

(3) $\mathcal{R} = \mathcal{R}''$.

Preliminary proposition

Proposition K.14.2 *For any subsets $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$, we have that*

(1) $\mathcal{N}' \subseteq \mathcal{M}'$ and

(2) $\mathcal{M} \subseteq \mathcal{M}''$.

(3) For any subset $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$, the set $\{\mathcal{M} \cup \mathcal{M}^*\}'$ is a von Neumann algebra.

Proof: **(1)** All the elements of $\mathcal{B}(\mathcal{H})$ which commute with $\mathcal{N}$ also commute with $\mathcal{M}$. As $\mathcal{N}$ is larger, it is a more demanding requirement that elements of $\mathcal{B}(\mathcal{H})$ commute with it. Hence, $\mathcal{N}' \subseteq \mathcal{M}'$.

(2) If B belongs to $\mathcal{M}'$ and A belongs to $\mathcal{M}$ then $AB = BA$, and thus $A \in (\mathcal{M}')' = \mathcal{M}''$. This proves the inclusion, but there may be elements of $\mathcal{B}(\mathcal{H})$ which commute with $\mathcal{M}'$ but are not in $\mathcal{M}$. Hence, $\mathcal{M} \subseteq \mathcal{M}''$.

(3) If $M \in \{\mathcal{M} \cup \mathcal{M}^*\}$ then the self-adjoint operators $(M + M^*)/2$ and $(M - M^*)/2i$ are in $\{\mathcal{M} \cup \mathcal{M}^*\}$. So we can take the elements of $\{\mathcal{M} \cup \mathcal{M}^*\}$ to all be self-adjoint. Now say $MD - DM = 0$ for all $M \in \{\mathcal{M} \cup \mathcal{M}^*\}$, by taking the adjoint of this, we see that if

[4]this requirement means that if $\{x_n\}$ is a sequence of operators in $\mathcal{R}$ such that for all $\Phi \in \mathcal{H}$ one has $x_n\Phi \to x\Phi$ for some $x \in \mathcal{B}(\mathcal{H})$.

[5]this requirement means that if $\{x_n\}$ is a sequence of operators in $\mathcal{M}$ such that for all $\Phi \in \mathcal{H}$ and $x' \in \mathcal{B}^*(\mathcal{H})$?? one has $(x', x_n\Phi) \to (x', x\Phi)$ for some $x \in \mathcal{B}(\mathcal{H})$.

$D \in \{\mathcal{M} \cup \mathcal{M}^*\}'$ then so is D^*. Again we can take the elements of $\{\mathcal{M} \cup \mathcal{M}^*\}'$ to all be self-adjoint.

It is straightforward show that, for every subset $\mathcal{U}$ of $\mathcal{B}(\mathcal{H})$, $\mathcal{U}'$ is weakly closed in $\mathcal{B}(\mathcal{H})$. From (2) we have $\mathcal{U} \subset \mathcal{U}''$, and applying (1) to this gives $\mathcal{U}''' \subset \mathcal{U}'$. On the other hand from (2) we also have $\mathcal{U}' \subset (\mathcal{U}')''$. So we have $\mathcal{U}''' \subseteq \mathcal{U}' \subseteq \mathcal{U}'''$, that is,

$$\mathcal{U}' = (\mathcal{U}')''$$

(This also implies $\mathcal{U}'' = (\mathcal{U}'')''$, $\mathcal{U}''' = (\mathcal{U}''')''$, and so on). That $\mathcal{U}'$ is weakly closed easily follows. If (A_n) is a sequence in $\mathcal{U}'$ which converges weakly to A in $\mathcal{B}(\mathcal{H})$ then for all $B \in \mathcal{U}$ and all $x, y \in \mathcal{H}$ we have

$$| < x, (AB - BA)y > | \leq | < x, (A - A_n)By > | + | < x, B(A - A_n)y > | \tag{K.24}$$

Thus A belongs to $\mathcal{U}'$. Putting $\mathcal{U}' = \{\mathcal{M} \cup \mathcal{M}^*\}'$ completes the proof of (3).

□

Theorem K.14.3 (von Neumann density theorem (Bicommutant theorem)). *Let $\mathcal{R}$ be a self-adjoint algebra in $\mathcal{B}(\mathcal{H})$ which contains the identity $\mathbf{1}$. Then the statements are equivalent:*

(1) $\mathcal{R}$ is weakly dense in $\mathcal{B}(\mathcal{H})$

(2) $\mathcal{R}$ is strongly dense in $\mathcal{B}(\mathcal{H})$

(3) $\mathcal{R}$ is ultra-strongly dense in $\mathcal{B}(\mathcal{H})$

(4) $\mathcal{R}$ is ultra-weakly dense in $\mathcal{B}(\mathcal{H})$

and their respective closures are all the same and equal to $\mathcal{R}''$.

(5) $\mathcal{R} = \mathcal{R}''$

Proof. of Theorem K.14.3.

$\mathcal{R}$ is a von Neumann algebra if and only if $\mathcal{R} = \mathcal{R}''$.

Firstly note that, by applying (ii) twice we see that $\mathcal{R}''$ is a von Neumann algebra and hence we have

$$\mathcal{R} \subseteq \mathcal{R}'' = \overline{\mathcal{R}''}^{w}.$$

We need to show that

$$\overline{\mathcal{R}}^{us} = \mathcal{R}''$$

that is, we need to show that if $B_\alpha \in \mathcal{R}$ and if A such that $\sum_{i=1}^{\infty} \|(B_\alpha - A)x_i\|^2 \to 0$ then A is in $\mathcal{R}''$.

for $\sum_{i=1}^{\infty} \|x_i\|^2 < \infty$

$$\sum_{i=1}^{\infty} \|(Ax_i - Bx_i)\|^2 < \epsilon \tag{K.24}$$

$$x_i \in \mathcal{H}_i, \qquad \{x_1, x_2, \ldots, x_n, \ldots\} = x \in \bigoplus_i \mathcal{H}_i \tag{K.24}$$

with $\tilde{A}$ defined by

$$\tilde{A}x = \{Ax_1, Ax_2, \ldots, Ax_n, \ldots\} \in \bigoplus_i \mathcal{H}_i \tag{K.24}$$

We will say $\tilde{A} \in \tilde{\mathcal{R}}$. define

$$\|x\|^2 = \sum_{i=1}^{\infty} \|x_i\|^2 \quad \text{and} \quad \|(\tilde{A}x - \tilde{B}x)\|^2 = \sum_{i=1}^{\infty} \|(Ax_i - Bx_i)\|^2 \tag{K.24}$$

Thus we wish to find for any given $\tilde{B} \in \tilde{\mathcal{R}}''$ any point $x \in \bigoplus_i \mathcal{H}_i$ such that $\|x\|^2 < \infty$ and any $\epsilon > 0$, there exists an $\tilde{A} \in \mathcal{R}$ such that $\|(\tilde{A}x - \tilde{B}x)\|^2 < \epsilon$.

It will be enough to show that there is some linear combinations of $\{Ax_1, Ax_2, \ldots, Ax_n, \ldots\}$, $y = (L_i A x_i)_1^{\infty}$ (L_i is a scalar), such that $\|(y - \tilde{B}x)\|^2 < \epsilon$. That is, it will be enough to show that $\tilde{B}x$ is in the closed linear span $\mathcal{V}$ of $\{\tilde{A}x : A \in \mathcal{R}\}$ in $\bigoplus_i \mathcal{H}_i$.

$$P : \bigoplus_i \mathcal{H}_i \to \mathcal{V} \tag{K.24}$$

$C \in (\tilde{\mathcal{R}})'$

$$\sum_j AC_{ij} y_j = \sum_j C_{ij} A y_j \tag{K.24}$$

for all $i \in \mathbb{N}$ and all $(y_j) \in \bigoplus_j \mathcal{H}_j$ for any $A \in \mathcal{R}$.

Take $y_1, y_2, \ldots, y_n \in \mathcal{H}$ and $y = y_1 \oplus + \cdots \oplus y_n \in \mathcal{H} \oplus \cdots \oplus \mathcal{H}$. $y = (y_i)$

$\tilde{A}(x_1 \oplus \cdots \oplus x_n) = Ax_1 \oplus \cdots \oplus Ax_n$ or $\tilde{A}x = (Ax_i)$

let

$$C \in \mathcal{B}(\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n)$$

$$E_i : \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \to \mathcal{H}_i$$

$$C_{ij} = E_i C E_j$$

$C_{ij} \in \mathcal{B}(\mathcal{H})$

$C = \sum_{i,j} E_i C_{ij} E_j$

$$(\tilde{A}C - C\tilde{A})y = 0$$

for all $\tilde{A} \in \tilde{\mathcal{R}}$ then $C \in (\tilde{\mathcal{R}})'$

$$\sum_{i,j} (\tilde{A}C_{ij} - C_{ij}\tilde{A})y = 0$$

$$\sum_j A(C_{ij} y_i) = \sum_j C_{ij}(Ay_j)$$

In other words, $C \in (\tilde{\mathcal{R}})'$ if and only if $C_{ij} \in \mathcal{R}'$.

Find B such that

such that $\|(A \oplus \cdots \oplus A - B \oplus \cdots \oplus B)y\| < \epsilon$

□

Factors

Classification of von Neumann algebras.

If $\mathcal{P}(\mathcal{R})$ stands for the set of projection operator of a von Neumann algebra $\mathcal{R}$, then $\mathcal{R}$ is the smallest von Neumann algerba containing $\mathcal{P}(\mathcal{R})$. Infact von Neumann algebras are completely determine by the collection of projection operators

Type I. $0 < p < s$

Type II.

Type III.

$$\mathcal{R} \cap \mathcal{R}' \tag{K.24}$$

K.14.5 Tomita-Takesaki Theorem

$(\mathcal{R}, \omega)$ a von Neumann algebra acting on on some Hilbert space $\mathcal{H}$, with normal faithful state ω on $\mathcal{R}$.

The GNS representation:

i) π is a morphism form $\mathcal{R}$ to $\mathcal{B}(\mathcal{H})$.

ii) $\omega(A) =< \Omega, \pi(A)\Omega >$.

iii) $\omega(\mathcal{R})\Omega$ is dense in $\mathcal{H}$.

Proposition K.14.4 *The vector Ω is cyclic and separating for $\mathcal{R}$ and $\mathcal{R}'$.*

Proof. Ω is cyclic for $\mathcal{R}$ by iii) above. Let us see it is separating for $\mathcal{R}$. If $A \in \mathcal{R}$ is such that $A\Omega = 0$ then $\omega(A^*A) = 0$, but as ω is faithful (i.e. $\omega(A^*A) = 0$ only if $A = 0$) we have $A = 0$. By the comment after lemma K.14.1, Ω is cyclic and separating for $\mathcal{R}'$.

□

Let ω be of the form

$$\omega(A) = tr(\rho A)$$

Let (σ_t) be the following group of automorphisms

$$\sigma_t(A) = e^{itH} A e^{-itH}$$

for some self-adjoint operator H in...

(1)

$$\omega(A\sigma_t(B)) = \omega(\sigma_{t-\beta i}(B)A) \tag{K.24}$$

(2)

$$\rho = \frac{1}{\mathcal{Z}} e^{-\beta H}, \tag{K.24}$$

where $\mathcal{Z} = tr(\exp(-\beta H))$.

Proof. (1) $\Rightarrow$ (2)

$$\begin{aligned}
\omega(A\sigma_t(B)) &= \frac{1}{\mathcal{Z}} tr(e^{-\beta H} A e^{itH} B e^{-itH}) \\
&= \frac{1}{\mathcal{Z}} tr(A e^{itH} B e^{(-it-\beta)H}) \\
&= \frac{1}{\mathcal{Z}} tr(A e^{-\beta H} e^{(it+\beta)H} B e^{(-it-\beta)H}) \\
&= \frac{1}{\mathcal{Z}} tr(e^{-\beta H} e^{(it+\beta)H} B e^{(-it-\beta)H} A) \\
&= \omega(\sigma_{t-\beta i}(B)A)
\end{aligned} \tag{K.21}$$

(2) $\Rightarrow$ (1)

$$\begin{aligned}
tr(AB\rho) &= tr(\rho AB) = \omega(AB) \\
&= \omega(\sigma_{-\beta i}(B)A) \\
&= tr(\rho e^{\beta H} B e^{-\beta H} A) \\
&= tr(A\rho e^{\beta H} B e^{-\beta H})
\end{aligned} \tag{K.19}$$

this is valid for any A so

$$B\rho = \rho e^{\beta H} B e^{-\beta H} \tag{K.19}$$

for all B.

$$B(\rho e^{\beta H}) = (\rho e^{\beta H})B.$$

we conclude that $\rho e^{\beta H}$ is a multiple of the identity.

The Tomita-Takesaki theorem asserts that the mappings $\alpha_t : \mathcal{R} \to \mathcal{R}$, $t \in \mathbb{R}$, given by

$$\alpha_t(b) = \Delta^{-it} b \Delta^{it} \tag{K.19}$$

Theorem K.14.5 *Let $\mathcal{R}$ be a von Neumann algebra with a cyclic and separating vector Ω. Then $J\Omega = \Omega = \Delta\Omega$ and the following hold:*

$$J\,\mathcal{R}\,J = \mathcal{R}' \quad \textit{and} \quad \Delta^{-it}\,\mathcal{R}\,\Delta^{it} = \mathcal{R}, \qquad \textit{for all } t \in \mathbb{R}. \tag{K.19}$$

two automorphisms of $\mathcal{R}$ are inner equivalent if the differ by an inner automorphism. are automorphisms which are inner equivalent

$$u\alpha''(a) = \alpha'(a)u \tag{K.19}$$

for every $a \in \mathcal{R}$ and some $u \in \mathcal{U}$.

The set of all equivalence classes of this relation is the outer automorphisms of $\mathcal{R}$, denoted $\mathrm{Out}(\mathcal{R})$.

K.14.6 The Statistical State of the Universe

Rovelli "The Statistical State of the Universe" [289].

K.15 Fock-Bargmann Representation

In the conventional representation the Hilbert space of vectors is formed by the space of complex valued, square intergrable, coordinate or momentum functions $\psi(q)$ and $\tilde{\psi}(p)$. No analyticity conditions are placed on these complex functions. However, there exists a representation in which any state vector is described by an entire analytic function (a function that is analytic in every open set of the complex plane $\mathbb{C}$ is called an entire analytic function). This is called the Fock-Bargmann representation.

Applications are.... Ashtekar variables quantization of simple model. Where explicitly one can get the inner production from the requirement that real observables should correspond to self-adjoint operators.

An arbitrary state $|\psi>$ of the Hilbert space can be expanded in the Harmonic oscillator basis states $\{|n>\}$,

$$|\psi> = \sum_{n=0}^{\infty} c_n |n>, \qquad <\psi|\psi> = \sum_{n=0}^{\infty} |c_n|^2 = 1 \tag{K.19}$$

For the coherent state

$$|\alpha> = \exp(-|\alpha|^2/2)\ \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\ |n> \tag{K.19}$$

The projection of the state $|\psi>$ onto the coherent state $|z>$ is

$$< \alpha|\psi> = \exp(-|\alpha|^2/2)\ \sum_{n=0}^{\infty} c_n \frac{\overline{\alpha}^n}{\sqrt{n!}} \tag{K.19}$$

then

$$\begin{aligned} < \alpha|\psi> &= \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} c_n u_n(\overline{\alpha}) \\ &= \exp(-|\alpha|^2/2)\psi(\overline{\alpha}) \end{aligned} \tag{K.19}$$

so we have

$$\psi(z) = \sum_{n=0}^{\infty} c_n u_n(z), \qquad u_n(z) = \frac{z^n}{\sqrt{n!}}. \tag{K.19}$$

The series (K.15) converges uniformally in any compact domain of the complex plane $\mathbb{C}$ because of the condition $\sum_{n=0}^{\infty} |c_n|^2 = 1$ (see next section on the Weierstrass M-test). Further as a consequence of this (see the section following the next section), $\psi(z)$ an entire analytic function.

normalized according to

$$\|\psi\|^2 = <\psi|\psi> = \int \exp(-|z|^2)|\psi(z)|^2 d\mu(z)\ < \infty \tag{K.19}$$

Tha scalar product of two entire functions, satifying (K.15), is defined by

$$< \psi_1|\psi_2 > = \int \exp(-|z|^2)\overline{\psi}_1(z)\psi_2(z)d\mu(z) \tag{K.19}$$

K.15.1 Weierstrass M-test

Theorem K.15.1 *Let $\sum_{k=1}^{\infty} f_k(z)$ be a series of functions, with each function defined on a subset U of $\mathbb{C}$. Suppose $\sum_{k=1}^{\infty} M_k$ is a series of real numbers such that:*

(i) $0 \leq |f_k(z)| \leq M_k$*;*

(ii) the series $\sum_{k=1}^{\infty} M_k$ converges

then $\sum_{k=1}^{\infty} f_k(z)$ converges uniformally.

Proof. For a series to be uniformally convergent, given any $\epsilon > 0$, there exists an integer N such that for all $n \geq N$, we have

$$|\sum_{k=1}^{\infty} f_k(z)| < \epsilon,$$

for all $z \in U$. Since $\sum_{k=1}^{\infty} M_k$ converges, we know that we can find an N so that for all $n \geq N$, we have

$$\sum_{k=1}^{\infty} M_k < \epsilon.$$

Since $0 \geq |f_k(z)| \leq M_k$, for all $z \in U$, we have

$$|\sum_{k=1}^{\infty} f_k(z)| \leq \sum_{k=1}^{\infty} |f_k(z)| \leq \sum_{k=1}^{\infty} M_k < \epsilon,$$

□

Example 1.

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

$$\left|\frac{z^{n+1}}{z^n}\right| = |z|, \tag{K.19}$$

the Taylor series converges for $|z| < 1.$, but fails to converge for $|z| > 1$. $|z| = 1$ is said to be the *radius of convergence*

Consider the geometric series

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k.$$

for $|z| < 1$.

we show that this series converges uniformally on any disc $|z| \leq a < 1$. Set

$$M_k = a^k.$$

For all z, we have $0 < |z|^n \leq a^n$. We know that the geometric series for $a < 1$, hence the geometric series converges uniformaly on any disc with radius less than 1 about the origin of the complex plane $\mathbb{C}$.

Example 2. Consider the series

$$\sum_{k=1}^{\infty} \frac{z^k}{k!}.$$

we show that this series converges uniformally on any disc $|z| \leq a$. Set

$$M_k = \frac{a^k}{k!}.$$

For all z , we have $0 < |z|^n/n! \leq a^n/n!$. Thus as the series $\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} a^k/k!$ converges, we will have uniform convergence. Convergence comes from the ratio test

$$\lim_{k\to\infty} \frac{M_{k+1}}{M_k} = \lim_{k\to\infty} \frac{\frac{a^{k+1}}{(k+1)!}}{\frac{a^k}{k!}} = \lim_{k\to\infty} \frac{a}{k+1} = 0.$$

We find the Taylor series for e^z converges uniformaly on any disc about the origin of the complex plane $\mathbb{C}$.

Example 3. Consider the function

$$\frac{1}{1-(1/z)} = \sum_{k=0}^{\infty} \frac{1}{z^k}.$$

for $|z| > 1$.

Set

$$M_k = \frac{1}{a^k}.$$

For all z, we have $0 < |1/z|^n \leq (1/a)^n$. Hence the function converges uniformaly outside any disc of radius more than 1 about the origin of the complex plane $\mathbb{C}$.

Coherent state expansion

The same proof slightly modified will give the required result.

Consider the series

$$f(z) = \sum_k \frac{c_n}{\sqrt{k!}} z^k \tag{K.19}$$

where $\sum_n |c_n|^2 = 1$.

Suppose, given ϵ, there exists an integer N such that for all $n \geq N$, we have

$$|\sum_{k=n}^{\infty} f_k(z)|^2 < \epsilon^2, \tag{K.19}$$

for all $z \in U$, then for all $n \geq N$, we have

$$|\sum_{k=n}^{\infty} f_k(z)| < \epsilon,$$

for all $z \in U$, i.e., the series $\sum_{k=1}^{\infty} f_k(z)$ is uniformally convergent.

We will prove the series (K.15.1) is uniformally convergent by proving it satifies the statement regarding (K.15.1). As $\sum_k |c_k|^2 = 1$ we have

$$\sum_{k=n} \frac{|c_k|^2}{k!} < \infty$$

from which follows that we can find an N so that for all $n \geq N$, we have $\sum_{k=n} |c_k|^2/k! < \epsilon^2$. Therefore

$$|\sum_{k=n}^{\infty} f_k(z)|^2 \leq \sum_{k=n}^{\infty} |f_k(z)|^2 \leq \sum_{k=n}^{\infty} |c_k|^2/k! < \epsilon^2,$$

proving the inequality (K.15.1).

$\square$

K.15.2 From the Space of Normalizable Functions to the Space of Entire Analytic Functions

Let U be an open set of the complex plane $\mathbb{C}$. A function $f(z)$ is analytic at z_0 if and only if in a neighbourhood of z_0, $f(z)$ is equal to a uniformaly convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{K.19}$$

Recall a squence of functions $f_k(z)$ converges uniformaly to a function $f(z)$ if eventually all the functions $f_k(z)$ fall within any $\epsilon-$tube about the limit function $f(z)$.

If case.

Here we take the sequence functions to be *partial sums*. Let $h_1(z), h_2(z), \ldots$ be a sequence of functions. The series of functions

$$f(z) = \sum_{k=1}^{\infty} h_k(z)$$

converges uniformaly to a function $f(z)$ if the sequence of partial sums $f_1(z) = h_1(z)$, $f_2(z) = h_1(z) + h_2(z)$, $f_3(z) = h_1(z) + h_2(z) + h_3(z) \ldots$ converges uniformaly to $f(z)$.

An example of particular interest is when $h_k(z) = a_k(z - z_0)^k$ and

$$f_k(z) = \sum_{n=0}^{k} a_n (z - z_0)^n. \tag{K.19}$$

This allows for a notion of uniform convergence for series. A series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ converges uniformaly in an open set U of the complex plane $\mathbb{C}$ if the sequence of polynomials $\{\sum_{n=0}^{N} a_n(z - z_0)^n\}$ converges uniformaly in U.

Theorem K.15.2 *Let the sequence $\{f_n(z)\}$ of analytic functions converge uniformaly on an open set U to a function $f : U \to \mathbb{C}$. Then the function $f(z)$ is also analytic and the sequence of derivatives $(f_n'(z))$ converge pointwise to the derivative $f'(z)$ on the set U.*

By this theorem, since polynomials are analytic, we conclude that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{K.19}$$

is a uniformaly convergent series, then the function $f(z)$ is analytic.

Only if case.

Suppose we have a function f which is analytic about a point z_0. Take a closed contour C around z_0. By the cauchy Integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw, \tag{K.19}$$

For any z inside C. Now we know for the geometric series

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}$$

for $|r| < 1$, it follows for all w and z with $|z - z_0| < |w - z_0|$.

$$\begin{aligned} \frac{1}{w - z} &= \frac{1}{w - z_0} \frac{1}{1 - \frac{z - z_0}{w - z_0}} \\ &= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n . \end{aligned} \tag{K.19}$$

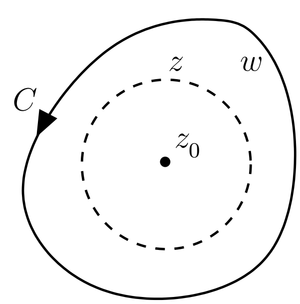

Figure K.2: contfornth. The contour used to evaluate the nth derivative, $|w - z_0| > |z - z_0|$.

$$
\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\frac{1}{1-\frac{z-z_0}{w-z_0}} \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n \\
&= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw \\
&= \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n \qquad \text{(K.16)}
\end{aligned}
$$

swaping the integral and summation follows from the fact that the series $\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$ converges uniformaly. The expansion (K.16) is finite wherever the series does.

Where we used the Cauchy Integral Formula

$$
f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw.
$$

Prove the first of these statements. From the triangle inequality

$$
\left|\int_C f(z)dz\right| \le \int_C |f(z)||dz| \le |\max f(z)|L \qquad \text{(K.16)}
$$

The series of partial sums

$$
s_k(z) = \sum_{n=0}^{k} f_n(z) \qquad \text{(K.16)}
$$

Uniform convergence implies that, given any $\epsilon > 0$, there exists some N such that $|f(z) - s_k(z)| < \epsilon$ everywhere on C. It follows

$$
\left|\int_C f(z)dz - \int_C s_n(z)dz\right| \le \epsilon L \qquad \text{(K.17)}
$$

or as $s_n(z)$ is a finite sum we can interchange integration and summation,

$$\left| \int_C f(z)dz - \sum_{k=0}^{n} \int_C f_k(z)dz \right| \leq \epsilon L \tag{K.18}$$

By choosing n large enough ϵL can be arbitary small, so

$$\lim_{n\to\infty} \sum_{k=0}^{n} \int_C f_k(z)dz = \int_C f(z)dz \tag{K.19}$$

or

$$\sum_{n=0}^{\infty} \int_C f_n(z)dz = \int_C \sum_{n=0}^{\infty} f_n(z)dz. \tag{K.20}$$

□

The proof

$$\begin{aligned}
\frac{f(\xi)-f(z)}{\xi - z} &= \frac{1}{2\pi i}\int_C \left[\frac{f(w)}{(w-\xi)} - \frac{f(w)}{(w-z)}\right]\frac{dw}{w-z} \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-\xi)(w-z)}dw \\
&= \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}\left(1+\frac{\xi - z}{w-\xi}\right)dw.
\end{aligned} \tag{K.19}$$

writing $\xi - z = \epsilon e^{i\theta}$

$$\left| \lim_{\xi\to z}\frac{f(\xi)-f(z)}{\xi - z} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw \right| \leq \frac{1}{2\pi}\lim_{\epsilon\to 0}\epsilon \oint_C \frac{|f(w)||dw|}{|(w-z)-\epsilon e^{i\theta}||w-z|^2} \tag{K.19}$$

We replace $|w-z|$ by its maximum value, m, and $|f(w)|$ by its maximum value, M, we obtain

$$\left| \lim_{\xi\to z}\frac{f(\xi)-f(z)}{\xi - z} - \frac{1}{2\pi i}\int_C \frac{f(w)}{(w-z)^2}dw \right| \leq \frac{1}{2\pi}\frac{ML}{m^2}\lim_{\epsilon\to 0}\frac{\epsilon}{m-\epsilon} = 0, \tag{K.20}$$

where L is the length of the contour. Repeating the process, we obtain for the nth derivative

$$f^{(n)} = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}dw. \tag{K.20}$$

Proof of theorem (K.15.2).

(a) We show term-by-term differentiation or integration of a power series yields a new power series with the same radius of curvature.

(b) The uniform-convergence property of a power series implies that term-by-term integration yields the integral of the sum function. We show that the integrated sum function is single-valued and analytic within the radius of convergence.

(c) We show that a power series converges to an analytic function within its circle of convergence.

□

(a) The convergence of a complex series is determined by the ratio test. The power series convereges for $|z| < R$, but fails to converge for $|z| > R$.

$$\begin{aligned}\frac{d}{dz}s(z) &= \frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-z_0)^n = \frac{d}{dz}\big(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\big)\\ &= \sum_{n=0}^{\infty} a_{n+1}(n+1)(z-z_0)^n \end{aligned} \tag{K.20}$$

The ratio test

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k} = \lim_{k\to\infty}\frac{(k+1)a_{k+1}}{ka_k} = \lim_{k\to\infty}\frac{a_{k+1}}{a_k} \tag{K.20}$$

Hence the differentiation of a power series yields a new power series with the same radius of curvature. Similarly for integration:

$$\begin{aligned}\int s(z)dz &= \int\sum_{n=0}^{\infty}\frac{a_n}{n}(z-z_0)^n dz = \int\big(a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots\big)dz\\ &= a_{-1} + \sum_{n=1}^{\infty}\frac{a_{n-1}}{n}(z-z_0)^n, \end{aligned} \tag{K.20}$$

the ratio test gives

$$\lim_{k\to\infty}\frac{M_{k+1}}{M_k}=\lim_{k\to\infty}\frac{ka_{k+1}}{(k+1)a_k}=\lim_{k\to\infty}\frac{a_{k+1}}{a_k} \tag{K.20}$$

(b) The integral sum fucntion is $F(z)=\sum_{n=0}^{\infty}\int_C f_n(z)dz$. $F(z)$ is also

$$F(z)=\int_{z_0}^{z}\sum_{n=0}^{\infty}f_n(w)dw=\int_a^z f(w)dw$$

a is a fixed point and z an arbitrary point of the region, $F(z)$ only depends on a and z.

$$\begin{aligned}\frac{F(z+\Delta z)-F(z)}{\Delta z} &= \frac{1}{\Delta z}\int_a^{z+\Delta z} f(z')dz' - \frac{1}{\Delta z}\int_a^{z} f(z')dz' \\ &= \frac{1}{\Delta z}\int_z^{z+\Delta z} f(z')dz' \\ &= \frac{f(z)}{\Delta z}\int_z^{z+\Delta z} dz' + \frac{1}{\Delta z}\int_z^{z+\Delta z}[f(z')-f(z)]dz' \qquad \text{(K.19)}\end{aligned}$$

Take the inequality

$$\left|\frac{1}{\Delta z}\int_a^{z+\Delta z}[f(z')-f(z)]dz'\right| \leq \max|[f(z')-f(z)]|. \tag{K.19}$$

The RHS tends to zero as $\Delta z\to 0$ because $f(z)$ is continuous and therefore from (K.19) we have

$$\frac{d}{dz}F(z)=f(z).$$

Hence the derivative exists and is single-valued, it is equal to $f(z)$. As its derivative is finite and unique, the integrated sum function $F(z)$ is analytic.

(c) If a fucntion $F(z)$ can be represented by

$$F(z)=\frac{1}{2\pi i}\int_C\frac{I(w)}{w-z}dw \tag{K.19}$$

and $I(z)$ is contiuous on C, then, $F(z)$ is analytic at any point z which doesn't lie on C.

In particular, a function $F(z)$ that is analytic in some region can be expressed in this region by Cauchy's integral formula - C will be an arbitrary closed contour encircling the point z and $I(z) = F(z)$.

Proof. Consider the expression

$$\Delta := \left| \frac{F(z+\Delta z) - F(z)}{\Delta z} - \frac{1}{2\pi i} \int_C \frac{I(w)}{(w-z)^2} dw \right| \tag{K.19}$$

Using the integral formula (K.15.2) for $F(z)$ and $F(z+\Delta z)$, one obtains

$$\Delta = \left| \frac{\Delta z}{2\pi} \int_C \frac{I(w)}{(w-z-\Delta z)(w-z)^2} dw \right| \tag{K.19}$$

Since z and $z+\Delta z$ dont lie on C and $I(w)$ is continuous on C, the integrand is bounded, therefore $\Delta \to 0$ as $\Delta z \to 0$. This proves the differentiability of $F(z)$. An analogous result holds for the nth derivavtive of $F(z)$.

□

Thus, $F(z)$ also has a second derivative and hence $f(z)$ is differentiable throughout the region. And so the power series converges to an analytic function within its circle of convergence.

□

K.15.3 The Segal-Bargmann-Hall Transformation

Hall generalized the Segal-Bargmann transformation to the phase space of aribitrary compact guage group. The role of $\mathbb{C}$ is replaced by the complexification $G^{\mathbb{C}}$ of G.

Construction of coherent states in LQG.

For each $f \in L^2(G; dx)$, where dx is the normalized Haar measure on G, the image of f by the CST, $C_t f$, is the analytic continuation to $G^{\mathbb{C}}$ of the solution of the heat equation,

$$\frac{1}{\pi} \frac{\partial u}{\partial t} = \Delta_G u, \tag{K.19}$$

in generalized coordinates the Laplacian

$$\triangle u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left(\sqrt{g} g^{ij} \frac{\partial u}{\partial q_j} \right) \tag{K.19}$$

with initial condition given by $u(0, x) = f(x)$.

K.15.4 The Ground-State Representation

K.16 Geometric Quantization

Not every sympletic manifold has the topology of a cotangent bundle over some configuration space.

Wave functions can be characterised as functions on phase space whic are annihilated by the Hamiltonian vector fields of maximally commuting set of observables. This way of looking at Schrodinger quantization is what will be generalised to other sympletic manifolds.

Geometric quantization provides a geometric, general framework for the quantization of a sympletic manifold M which is not neseccarily a cotangent bundle, for example when M it is compact.

Necessary background material in the quantization of black holes in LQG.

K.16.1 Pedestrian Overview

Let us begin by comparing [84]

Classical states are represented by on a symplectic manifold Γ, phase space. The space of observables consists of all the (smooth) real-valued functions on phase space. As a simple example: the classical state of a harmonic oscillator is represented by a point in a 2-d Euclidean manifold, coordinatized by the basic variables position, q and momentum k. The total energy, E, is an example of an observable. It corresponds to the function $E = \frac{1}{2}\omega^2 q^2 + \frac{1}{2m}k^2$ on phase space. Any other function of q and k is also an observable, although it wont in general have as direct physical meaning as the total energy. Let us return to the general case. The (ideal) measurement of an observable f in a state $p \in \Gamma$ yields the simple value $f(p)$ at the point p and the state is left undisturbed. These outcomes occur with complete certainty. Then the dynamics of an evolving observable f_t is given by the differential equation

$$\frac{\partial f_t}{\partial t} = \{\mathcal{H}, f_t\}. \tag{K.19}$$

where, as before, $\{,\}$ denotes the Poisson bracket. For the canonical choice of symplectic structure on T^*R^n, it is Hamilton's equations of motion as presented in Appendix E.

The arena for quantum mechanics, on the other hand, is a Hilbert space. States of the system correspond to normalized states, moduli a phase factor, (i.e. what are known as ***rays***), and observables are represented by self-adjoint linear operators on H. As in the classical description, the space of observables is a real vector space equip with

Taking the view in quantum mechanics that the observables evolve in time while the states remain fixed i.e. the Heisenberg picture. The fundamental equation describing the dynamical evolution of an observable $\mathcal{O}_t$ the equation,

$$\frac{d\mathcal{O}_t}{dt} = -\frac{i}{\hbar}[\hat{\mathcal{H}}, \mathcal{O}_t]. \tag{K.19}$$

This direct analogy with the classical theory was first realized by Dirac.

The measurement theory is strikingly different.In the textbook description based on the Copenhagen interpretation, the (ideal) measurement of an observable $\hat{A}$ in a state $\psi \in H$ yields an eigenvalue of $\hat{A}$ and, immediately after the measurement, the state is thrown into corresponding eigenstate.

This lineararity the basis of the superposition of quantum states,

$$\phi = a\psi_1 + b\psi_2 \tag{K.19}$$

however not ever linear combination of states is relevant because the coefficients must be chosen so that the new state is also normalized i.e. we must have $|a|^2 + |b|^2 = 1$.

Geometric quantization is a quantization scheme based on the above relation between Heisenberg's equation and Hamilton's equation.

fundamental objects

The fundamental objects are:

1) A set of observables $\mathcal{U}$

2) a set Ω of states

3) a probability interpretation map $\mathcal{U} \times \Omega \to \mathcal{P}$, where $\mathcal{P}$ denotes the set of all non-negative Lebesgue measurable functions $f : R \to R$ such that

$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \tag{K.19}$$

(i.e. probability distributions). For a state η and an observable $\mathcal{O}$, we write the associated probability distribution function as $\eta_{\mathcal{O}}(\lambda)$. Of course, there is a natural mean-value map from $\mathcal{P} \to R$, given by

$$\mapsto \int \lambda f(\lambda)\, d\lambda. \tag{K.19}$$

$\mathcal{U}$ and Ω are both form real vector spaces

The composition - observables $\times$ states

$$\mathcal{U} \times \Omega \to \mathcal{P} \to_{mean\ value} \to R, \quad \eta, \mathcal{O} \mapsto < \eta | \mathcal{O} > \tag{K.19}$$

there is a duality

It is evident that

In QM, the algebra is usually realized as an algebra of linear operators on a complex Hilbert space H, and the space $\mathcal{D}(H)$ of positive operators with unit trace (as we know from before *density matrices*) is taken as the space of states. In particular, this state space contains the *projective Hilbert space* (pure states)

$$P(\mathcal{H}) = \{\text{all projective operators onto } 1-\text{dimensional subspaces}\} \simeq \mathcal{H}/ \tag{K.19}$$

where . Elements of $P(\mathcal{H})$ are known as pure states, while elements of $\mathcal{D}(\mathcal{H})$ which cannot be represented as one-dimensional projectors are the mixed states.

$$\eta_{\mathcal{O}}(\lambda) = \mathrm{tr}_{\mathcal{H}}(\eta P_{\mathcal{O}}(\lambda)) \tag{K.19}$$

Pre-quantization

complexification - We can tensor a real vector space with the complex numbers and get a complex vector space; this process is called complexification. For example, we can complexify the tangent space at some point of a manifold, which amounts to forming the space of complex linear combinations of tangent vectors at that point. (from Geometric Quantization John Baez August 11, 2000)

Consider a 2n-dimensional symplectic manifold $(\Gamma, \Omega_{\alpha\beta})$ and the space of symplectic potentials ω_α (recall append E, $\Omega_{\alpha\beta} = 2\partial_{[\alpha}\omega_{\beta]}$).

$$\Psi(\omega_\alpha + \partial_\alpha f) = e^{if/\hbar}\Psi(\omega_\alpha). \tag{K.19}$$

More precisely, in the terminology of fibre bundles, we have the following. Pre-quantum wave functions are cross sections of a complex line bundle over Γ, associated with the principle $U(1)$ bundle which has $\Omega_{\alpha\beta}$ as the curvature tensor. The symplectic potentials are connection 1-forms whose curvature is given by $\Omega_{\alpha\beta}$. Equation (K.16.1) is just the cross sections under the $U(1)$ action.

$z = x + iy$ and $z^* = x - iy$

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial \overline{z}}\right), \quad \frac{\partial}{\partial z^*} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial z^*}\right). \tag{K.19}$$

$$\frac{\partial z}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z^*}\right)(x + iy) = 1, \quad \frac{\partial z^*}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial z^*}\right)(x - iy) = 0. \tag{K.19}$$

functions that satisfy the Cauchy-Riemann equations $f(z) = U + iV$ have the properties:

$$\frac{\partial f(z)}{\partial z^*} := \frac{1}{2}\left(\frac{\partial(U+iV)}{\partial x} + i\frac{\partial(U+iV)}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y}\right) + i\frac{1}{2}\left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}\right) = 0 \tag{K.19}$$

Consider the space of pre-quantum wave functions which have finite norm. The Cauchy completion of this space gives us the *pre-quantum Hilbert space*, $\mathcal{H}_p$.

Given the classical observable f, define an operator $\mathcal{O}_f$ on $\mathcal{H}_p$ by

$$\boxed{\mathcal{O}_f \circ \Psi := -i\hbar X_f^\alpha \nabla_\alpha \Psi + f\Psi \equiv -i\hbar X_f^\alpha \left(\partial_\alpha - \frac{i}{\hbar}\omega_\alpha\right)\Psi + f\Psi \qquad \text{(K.20)}}$$

(i) linear;
(ii) gauge-covariant;
(iii) symmetric (i.e. formally self-adjoint) with respect to the inner product.

Thus, at the pre-quantum level, the Dirac prescription of replacing Poisson brackets by $-i\hbar$ times the commutator is exact.

Moreover, this correspondence $f \mapsto \mathcal{O}_f$ is a linear, 1-1 mapping from the space of classical observables to the space of pre-quantum operators which preserves the natural Lie-algebra structure:

$$[\mathcal{O}_f, \mathcal{O}_g] = \frac{\hbar}{i}\mathcal{O}_{\{f,g\}}. \tag{K.20}$$

that is, for any classical observables f and g we require that

$$\mathcal{O}_{\{f,g\}} \circ \Psi = -i\hbar X^{\alpha}_{\{f,g\}} \nabla_{\alpha}\Psi + f\Psi, \tag{K.20}$$

where $X^{\alpha}_{\{f,g\}}$ is defined via,

$$X^{\alpha}_{\{f,g\}} \nabla_{\alpha}\Psi := [X^{\alpha}_{f}\nabla_{\alpha}, X^{\beta}_{g}\nabla_{\beta}]\Psi = [X_f, X_g]^{\alpha}\nabla_{\alpha}\Psi. \tag{K.20}$$

This is indeed the case for the pre-quantum operator (K.20), as is proven in box??.

Pre-quantum operators that preserve the natural Lie-algebra

We wish to show that the operator on $\mathcal{H}_p$ (K.20) satisfies (K.16.1), or equivalently:

$$\mathcal{O}_f \circ \mathcal{O}_g \circ \Psi - \mathcal{O}_g \circ \mathcal{O}_f \circ \Psi = \frac{\hbar}{i}\mathcal{O}_{\{f,g\}} \circ \Psi \tag{K.20}$$

First, we look at $\mathcal{O}_f \circ \mathcal{O}_g \circ \Psi$,

$$\begin{aligned} \mathcal{O}_f \circ \mathcal{O}_g \circ \Psi &= \mathcal{O}_f \circ (-i\hbar X^{\alpha}_{g}\nabla_{\alpha} + g)\Psi \\ &= (\mathcal{O}_f \circ g)\Psi - i\hbar\mathcal{O}_f \circ (X^{\alpha}_{g}\nabla_{\alpha}\Psi) + f\mathcal{O}_g \circ \Psi + g\mathcal{O}_f \circ \Psi \\ &= \left[(-i\hbar X^{\alpha}_{f}\nabla_{\alpha} g)\Psi - i\hbar\mathcal{O}_f \circ (X^{\alpha}_{g}\nabla_{\alpha}\Psi)\right] + \underbrace{fg\Psi + f\mathcal{O}_g \circ \Psi + g\mathcal{O}_f \circ \Psi}_{\textit{symmetric in } f \textit{ and } g} \end{aligned} \tag{K.18}$$

$$[\mathcal{O}_f, \mathcal{O}_g] \circ \Psi = \frac{\hbar}{i}\left[(X^{\alpha}_{f}\nabla_{\alpha} g - X^{\alpha}_{g}\nabla_{\alpha} f)\Psi + \mathcal{O}_g \circ (X^{\alpha}_{f}\nabla_{\alpha}\Psi) - \mathcal{O}_f \circ (X^{\alpha}_{g}\nabla_{\alpha}\Psi)\right] \tag{K.18}$$

Pre-quantum operators that preserve the natural Lie-algebra

$$\begin{aligned} [\mathcal{O}_f, \mathcal{O}_g] \circ \Psi &= (\mathcal{O}_f\mathcal{O}_g - \mathcal{O}_g\mathcal{O}_f)\Psi \\ &= (-i\hbar\nabla_{X_f} + f)(-i\hbar\nabla_{X_g} + g)\Psi - (-i\hbar\nabla_{X_g} + g)(-i\hbar\nabla_{X_f} + f)\Psi \\ &= -\hbar^2(\nabla_{X_f}\nabla_{X_g} - \nabla_{X_g}\nabla_{X_f})\Psi - i\hbar[\nabla_{X_f}(g\Psi) + f\nabla_{X_g}\Psi + fg\Psi \\ &\quad -\nabla_{X_g}(f\Psi) - g\nabla_{X_f}\Psi - gf\Psi] \\ &= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}]\Psi - i\hbar(\nabla_{X_f} g - \nabla_{X_g} f)\Psi \end{aligned} \tag{K.15}$$

Now

$$(\nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]})\Psi = 2\pi i \Omega(X_1, X_2)\Psi = 2\pi i\{f, g\}. \tag{K.15}$$

From section . Continuing where we left off,

$$\begin{aligned}
[\mathcal{O}_f, \mathcal{O}_g] \circ \Psi &= -\hbar^2[\nabla_{X_f}, \nabla_{X_g}]\Psi + 2\pi\hbar\{f, g\}\Psi \\
&= -\hbar^2(\nabla_{[X_f,X_g]} + \frac{i}{\hbar}\Omega(X_f, X_g))\Psi + 2\pi\hbar\{f, g\}\Psi \\
&= (\hbar^2\nabla_{[X_f,X_g]} - i\pi\hbar\{f, g\} + 2i\pi\hbar\{f, g\})\Psi \\
&= i\hbar(-i\hbar\nabla_{X_{\{f,g\}}} + \{f, g\})\Psi \\
&= i\hbar\mathcal{O}_{\{f,g\}}\Psi.
\end{aligned} \tag{K.12}$$

$$\delta^{\overset{\downarrow}{E}} \tag{K.12}$$

Quantization

To cut down the prequantum Hilbert space, we need to choose a POLARIZATION, say P. What's this? Well, for each point x in X, a polarization picks out a certain subspace Px of the complexified tangent space at x. We define the *quantum Hilbert space*, $\mathcal{H}$, to be the space of all square-integrable sections of L that give zero when we take their covariant derivative at any point x in the direction of any vector in Px. The quantum Hilbert space is a subspace of the prequantum Hilbert space.

To obtain quantum states, we need a new structure, called ***polarization.*** A polarization $\mathcal{P}$ is of the complexified tangent space such that:

(i) $v^\alpha \in \mathcal{P}|_\gamma$ for all $[v, w]^\alpha \in \mathcal{P}|_\gamma$ for all γ ;
(ii) Given any two vectors $v^\alpha, w^\alpha \in \mathcal{P}$, $\Omega_{\alpha\beta}v^\alpha w^\alpha = 0$ for all γ.

We devote this section to justify the main features related to the polarization, which are relevant to geometric quantization.

Given a polarization we define ***quantum wave functions*** as the cross-sections Ψ of (N.-19) satisfying:

$$v^\alpha\nabla_\alpha\Psi = 0, \quad \text{for all } v^\alpha \in \mathcal{P}. \tag{K.12}$$

This condition on the wave function is called the ***polarization condition*** and effectively eliminates n degrees of freedom from the wave functions.

$$\left[v^\alpha\nabla_\alpha, w^\beta\nabla_\beta\right]\Psi = [v,w]^\alpha\nabla_\alpha\Psi - v^\alpha w^\beta\Omega_{\alpha\beta}\Psi. \tag{K.12}$$

$$\mathcal{O}_{q^i}\Psi = q^i\Psi, \quad \mathcal{O}_{p_i}\Psi = -i\hbar\frac{\partial\Psi}{\partial q^i}, \tag{K.12}$$

Which agree with the usual representation of the configuration and momentum operators.

Let us return to the general quantizatoin procedure.

Pre-quantum operators that preserve the natural Lie-algebra

$$[v,w]^\alpha\Psi - vw\Omega_{\alpha\beta} \tag{K.12}$$

In usual "position representation" of a particle on the line, we start with the space of all (nice) functions on the phase space R^2, and then pick out the subspace of functions that depend only the position coordinate q. These are functions with

$$df/dp = 0 \tag{K.12}$$

In the "momentum representation" we pick out the functions that depend only on the momentum coordinate p:

$$df/dq = 0 \tag{K.12}$$

In the "Bargmann-Segal representation" we pick out the functions that depend only on the complex coordinate z:

$$df/dz* = 0z = q + ip, z* = q - ip \tag{K.12}$$

or in other words, functions that satisfy the Cauchy-Riemann equations.

The Kahler polarization

An almost complex structure on Γ is a tensor field $J^{\alpha}{}_{\beta}$ satisfying $J^{\alpha}{}_{\beta}J^{\beta}{}_{\mu} = -\delta^{\alpha}{}_{\mu}$.

$$g_{\alpha\beta} := \Omega_{\alpha\gamma}J^{\gamma}{}_{\beta} \tag{K.12}$$

$$\Omega = i\, dz \wedge\, d\overline{z}, \tag{K.12}$$

and the polarization condition (N.-19) we obtain:

$$\mathcal{O}_{z}\Psi = z\Psi, \quad \mathcal{O}_{\overline{z}}\Psi = \hbar\frac{\partial\Psi}{\partial z} + \frac{1}{2}\overline{z}\Psi \tag{K.12}$$

Then, the expressions of basic operators reduce to:

$$\mathcal{O}_{z}f(z) = zf(z), \quad \mathcal{O}_{\overline{z}}f(z) = \hbar\frac{\partial f(z)}{\partial z}. \tag{K.12}$$

(i) Construct the $\star$-algebra of quantum operators by incorporating the Poisson bracket relations and the reality conditions on the classical phase space.

(ii) Find an explicit represenation of the algebra by choosing a suitable polarization. At this stage, the $\star$-relations are ignored since one does not have accss to an inner product.

where this Hilbert space describes states of the black hole horizon.

This a field theory in 3 dimensions, and the reason it's called "topological" is that you don't need any metric or other geometrical structure on your 3d spacetime manifold for this theory to make sense.

Complex Manifolds

In two dimensions it is often very useful to combine the coordinates into complex coordinates $z = x_1 + ix_2$ and $\overline{z} = x_1 - ix_2$

A complex manifold is formally defined an a manner entirly similar to a real manifold. Instead of real coordinate neighborhoods $(U; x^a),\ x^a \in R$, we have complex coordinates neighborhoods $(U; z^a),\ z^a \in C,\ a = 1, \ldots, n.$

a complex manifold in which the vector space R^m is replaced with C^m and the overlap functions are required to be holomorphic. This latter requirement introduces profound changes in the manifold idea

K.16.2 Complex, Hermitian and Kähler Manifolds

To begin with, we define a holomorphic (or analytic) map on $\mathbb{C}^m$. This requires a simple higher-dimensional version of the Cauchy-Riemann relations.

Definition A complex-valued function $f : \mathbb{C}^m \to \mathbb{C}$ is holomorphic if $f(z) = u(x,y) + iv(x,y)$ satifies the Cauchy-Riemann relations for each $z^\mu = x^\mu + iy^\mu$,

$$\frac{\partial u}{\partial x^\mu} = \frac{\partial v}{\partial y^\mu}, \qquad \frac{\partial v}{\partial x^\mu} = -\frac{\partial u}{\partial y^\mu}. \tag{K.12}$$

A map $f \equiv (f^1, \ldots, f^n) : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic if each function f^ν $(\nu = 1, \ldots, n)$ is holomorphic, that is

$$\begin{aligned} f^\nu(z) &= u^\nu(x,y) + iv^\nu(x,y), \qquad z^\mu = x^\mu + iy^\mu \\ &\Rightarrow \frac{\partial u^\nu}{\partial x^\mu} = \frac{\partial v^\nu}{\partial y^\mu}, \qquad \frac{\partial v^\nu}{\partial x^\mu} = -\frac{\partial u^\nu}{\partial y^\mu}, \quad \nu = 1, \ldots, n,\ \mu = 1, \ldots, m. \end{aligned} \tag{K.12}$$

□

With an even dimensional manifold we can think of it being glued together by a number of coordinate patches, where each patch is an open region of the coordinate space $\mathbb{C}^m$ - the space of points are the $n-$tuples $(z^1, \ldots, z^m)$ of complex numbers. Given such a description, we then define a complex manifold to be one where the transition functions are given entirely by holomorphic functions. The formal definition is the following.

Definition $\mathcal{M}$ is a complex manifold if

(i) $\mathcal{M}$ is a topological space;

(ii) $\mathcal{M}$ is provided with a family of pairs $\{(U_I, z_I)\}$;

(iii) $\{U_I\}$ is a family of open sets which covers $\mathcal{M}$, z_I is a homomorphism from U_I to an open subset U'_I of $\mathbb{C}^m$;

(iv) Given U_I and U_J such that $U_I \cap U_J \neq \emptyset$, the map $\varphi_{IJ} = z_J \circ z_I^{-1}$ is holomorphic.

□

The number m is called the complex dimension of $\mathcal{M}$. Each complex manifold is also a smooth real manifold of dimension $2m$. Any chart U of a complex manifold has coordinates $(z^1, \ldots, z^m)$ which may be regarded as real coordinates $(x^1, y^1, \ldots, x^m, y^m)$. The analytic

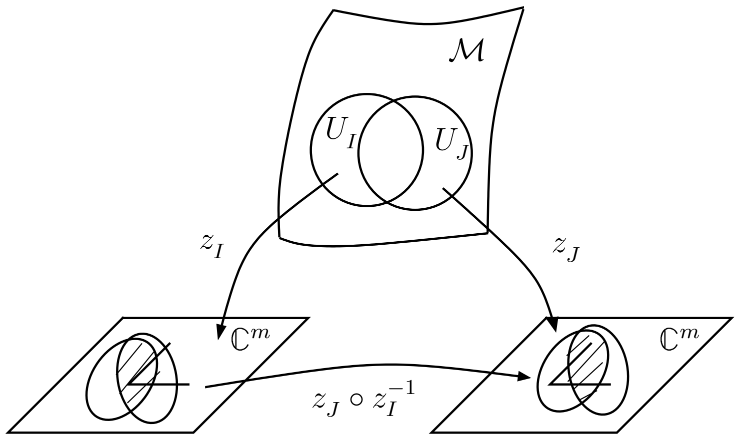

Figure K.3: Complex manifold.

property of the coordinate transformation functions ensures that they are differentiable when the manifold is regarded as a $2m$-dimensional differentiable manifold. However, holomorphic functions are much more rigid than smooth functions, so a complex structure (holomorphicity) is much stronger than a differentiable structure (smoothness) and hence most manifolds are not complex manifolds.

Complex forms

We define holomorphic vector fields, covectors, p-forms, tensors, etc., in just the same way as we did in the case of a real n-manifold. A complex manifold with local complex coordinates

$$z_I^\mu = x_I^\mu + i y_I^\mu$$

over U_I has the local coordinate vector fields

$$\frac{\partial}{\partial x_I^\mu}, \quad \frac{\partial}{\partial y_I^\mu}$$

and local coordinate one-forms

$$dx_I^\mu, \quad dy_I^\mu$$

as local real basis and co-basis respectively. Since we are allowed to take complex linear combinations in a complex manifold we may alternatively use complex basis and co-basis

$$
\begin{aligned}
\frac{\partial}{\partial z_I^\mu} &= \frac{1}{2}\left(\frac{\partial}{\partial x_I^\mu} - i\frac{\partial}{\partial y_I^\mu}\right), \qquad \frac{\partial}{\partial \overline{z}_I^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial x_I^\mu} + i\frac{\partial}{\partial y_I^\mu}\right) \\
dz_I^\mu &= dx_I^\mu + i dy_I^\mu, \qquad d\overline{z}_I^\mu = dx_I^\mu - i dy_I^\mu
\end{aligned} \tag{K.12}
$$

where $\overline{z} = x - iy$.

There is more than one way of formulating the notion of a complex structure. Form a complex vector field

$$
\zeta_I^\mu = \partial/\partial x_I^\mu + i\partial/\partial y_I^\mu .
$$

where $\partial/\partial x_I^\mu$ and $\partial/\partial y_I^\mu$ are ordinary real vector fields on the $2m$-manifold. Now consider the new complex vector field that arises when the complex vector field ζ_I^μ is multiplied by i:

$$
i\zeta_I^\mu = -\partial/\partial y_I^\mu + i\partial/\partial x_I^\mu
$$

The real vector field $\partial/\partial x_I^\mu$ is now replaced by $-\partial/\partial y_I^\mu$ and in turn $\partial/\partial y_I^\mu$ must be replaced by $\partial/\partial x_I^\mu$. The operation J which effects these replacements is a tensor field defined locally by

$$
\begin{aligned}
\left(J_I[\partial/\partial x_I^\mu]\right)(x_I(p)) &= \partial/\partial y_I^\mu \\
\left(J_I[\partial/\partial y_I^\mu]\right)(x_I(p)) &= -\partial/\partial x_I^\mu .
\end{aligned} \tag{K.12}
$$

Actually J is globally defined. Assume that $U_I \cap U_K \neq \emptyset$ and denote $z_K = \varphi_{IK}(z_I)$, we need to show that $J_I = J_K$ on their overlap. The Cauchy-Riemann equations then read

$$
\frac{\partial x_K^\mu}{\partial x_I^\nu} = \frac{\partial y_K^\mu}{\partial y_I^\nu}, \qquad \frac{\partial y_K^\mu}{\partial x_I^\nu} = -\frac{\partial x_K^\mu}{\partial y_I^\nu}
$$

Then

$$
\begin{aligned}
J_K[\partial/\partial x_I^\nu] &= J_K\left[\frac{\partial x_K^\mu}{\partial x_I^\nu}\frac{\partial}{\partial x_K^\mu}+\frac{\partial y_K^\mu}{\partial x_I^\nu}\frac{\partial}{\partial y_K^\mu}\right] \\
&= \frac{\partial x_K^\mu}{\partial x_I^\nu}J_K[\partial/\partial x_K^\mu]+\frac{\partial y_K^\mu}{\partial x_I^\nu}J_K[\partial/\partial y_K^\mu] \\
&= \frac{\partial x_K^\mu}{\partial x_I^\nu}\partial/\partial y_K^\mu-\frac{\partial y_K^\mu}{\partial x_I^\nu}\partial/\partial x_K^\mu \\
&= \frac{\partial y_K^\mu}{\partial y_I^\nu}\partial/\partial y_K^\mu+\frac{\partial x_K^\mu}{\partial y_I^\nu}\partial/\partial x_K^\mu \\
&= \partial/\partial y_I^\nu = J_I[\partial/\partial x_I^\nu]
\end{aligned}
\tag{K.9}
$$

We also find that $J_K[\partial/\partial y_I^\nu] = J_I[\partial/\partial y_I^\nu]$. J_I takes the form

$$
J_I = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{K.9}
$$

with respect to the basis

$$
\{\partial/\partial x_I^1, \ldots, \partial/\partial x_I^m; \partial/\partial y_I^1, \ldots, \partial/\partial y_I^m\}.
$$

Conversely, if $\mathcal{M}$ is a real $(2m)$-dimensional manifold which admits the globally defined tensor field J_0 then the diffeomorphisms between overlapping charts obey the Cauchy-Riemann equations: The requirement $J_K[\partial/\partial x_I^\nu] = J_I[\partial/\partial x_I^\nu]$ implies

$$
\frac{\partial x_K^\mu}{\partial x_I^\nu}\partial/\partial y_K^\mu-\frac{\partial y_K^\mu}{\partial x_I^\nu}\partial/\partial x_K^\mu = \partial/\partial y_I^\nu
$$

which becomes

$$
\frac{\partial x_K^\mu}{\partial x_I^\nu}\partial/\partial y_K^\mu-\frac{\partial y_K^\mu}{\partial x_I^\nu}\partial/\partial x_K^\mu = \frac{\partial y_K^\mu}{\partial y_I^\nu}\partial/\partial y_K^\mu+\frac{\partial x_K^\mu}{\partial y_I^\nu}\partial/\partial x_K^\mu
$$

upon writing $\partial/\partial y_I^\nu$ in terms of the basis $\{\partial/\partial x_K^\mu; \partial/\partial y_K^\mu\}$. We can simply read off the Cauchy-Riemann relations for the transition functions. The requirement $J_K[\partial/\partial x_I^\nu] = J_I[\partial/\partial x_I^\nu]$ leads to the same condition. Thus the existence of J_0 is equivalent to the existence of a complex structure.

Definition A $(2m)$-dimensional real manifold $\mathcal{M}$ admits a complex structure if and only if it admits a smooth tensor field $J_0 \in T_1^1(\mathcal{M})$ with $J_0^2(p) = -id_{T_p(\mathcal{M})}$ which in suitable coordinates has canonical component matrix $\epsilon \otimes 1_m$. We then call $\mathcal{M}$ a complex m-dimensional manifold with complex structure J.

□

The condition

$$\mathbf{J}^2 = -id_{T_p(\mathcal{M})}$$

alone defines what is referred to as an almost complex structure.

Definition An m-dimensional real manifold $\mathcal{M}$ with smooth tensor field such that $J^2(p) = -id_{T_p(\mathcal{M})}$ is called an almost complex manifold with almost complex structure J.

□

Notice that $\det(J^2(p)) = (-1)^m = [\det(J(p))]^2 > 0$, hence almost complex manifolds have even-dimension. Not every almost complex manifold admits a complex structure. To have an actual complex structure, so that a consistent notion of holomorphic can arise, a certain differential equation in the quantity J must be satisfied. There is a deep theorem, the Newlander-Nirenberg theorem, which gives us a necessary and sufficient (the difficult part - Newlander A and Nirenberg L Ann. Math. **65** 391) condition for a $2m$-dimensional real manifold, with this J-structure, to qualify as a complex m-manifold.

Theorem K.16.1 (Newlander and Nirenberg). *Let the Nijenhuis tensor field $N \in T_2^1(\mathcal{M})$ on an almost complex manifold $\mathcal{M}$ be defined by*

$$N[u,v] := [u,v] + J[[J[u],v]] + J[[u,J[v]]] - [J[u],J[v]]. \tag{K.9}$$

Then $\mathcal{M}$ admits a complex structure if and only if $N = 0$.

Proof omitted.

□

Definition Let $(\mathcal{M}, J)$ be a manifold of complex dimension m which at the same time is a Riemannian (2m)-dimensional real manifold with Riemannain structure g. Then $(\mathcal{M}, J, g)$ is called a **Hermitian manifold** provided that

$$g[J[u], J[v]] = g[u,v] \quad \text{for all } u, v \in T^1(\mathcal{M}). \tag{K.9}$$

Then g is called a Hermitian structure and is said to be J-compatible.

□

Definition Let $(\mathcal{M}, J, g)$ be a Hermitian manifold. The so-called **Kähler two-form** is defined by

$$\omega[u,v] := g[J[u],v] \quad \text{for all } u,v \in T^1(\mathcal{M}). \tag{K.9}$$

Notice that $\omega[u,v] = -\omega[v,u]$ due to $J^2(p) = -id_{T_p(\mathcal{M})}$. It also implies $\omega[J[u],v] = -g[u,v]$. So $g[J[u],J[v]] = \omega[u,J[v]]$ and $g[J[u],J[v]] = g[u,v]$ then becomes

$$\omega[u,J[v]] = -\omega[J[u],v]$$

or

$$\omega[J[u],J[v]] = \omega[u,v]. \tag{K.9}$$

Thus ω is also J-compatible.

□

Definition A **Kähler manifold** is a Hermitian manifold $(\mathcal{M}, J, g)$ such that the corresponding Kahler two-form is closed. Equivalently, a Kähler manifold $(\mathcal{M}, J, g)$ is a complex manifold which also carries a J-compatible sympletic structure ω.

□

Lemma K.16.2 *With the notation $z^{\overline{\mu}} = \overline{z}^{\mu}$ and similarly for $dz^{\overline{\mu}}$, $\partial_{\overline{\mu}}$*

$$\omega = i d \wedge \overline{d} K \tag{K.9}$$

where K is called the local Kähler potential for ω. K is a real-valued function which is uniquely determined by ω up to $K(z,\overline{z}) \to K(z,\overline{z}) + f(z) + g(\overline{z})$ where f, g are holomorphic and antiholomorphic functions respectively.

Proof:

In local coordinates we have

$$\omega = \omega_{\mu\nu} dz^{\mu} \wedge dz^{\nu} + \omega_{\overline{\mu}\nu} d\overline{z}^{\mu} \wedge dz^{\nu} + \omega_{\mu\overline{\nu}} dz^{\mu} \wedge d\overline{z}^{\nu} + \omega_{\overline{\mu}\overline{\nu}} d\overline{z}^{\mu} \wedge d\overline{z}^{\nu} \tag{K.9}$$

We have $J[\partial_{\mu}] = i\partial_{\mu}$, $J[\partial_{\overline{\mu}}] = -i\partial_{\overline{\mu}}$ and

$$\begin{aligned}\omega_{\mu\nu} &= \omega[\partial/\partial_\mu, \partial/\partial_\nu] \\ \omega_{\mu\overline{\nu}} &= \omega[\partial/\partial_\mu, \partial/\partial_{\overline{\nu}}] \\ \omega_{\overline{\mu}\nu} &= \omega[\partial/\partial_{\overline{\mu}}, \partial/\partial_\nu] \\ \omega_{\overline{\mu}\overline{\nu}} &= \omega[\partial/\partial_{\overline{\mu}}, \partial/\partial_{\overline{\nu}}].\end{aligned} \tag{K.7}$$

From the compatibility condition

$$\omega[\partial/\partial_\mu, \partial/\partial_\nu] = \omega[J[\partial/\partial_\mu], J[\partial/\partial_\nu]] = \omega[\partial/\partial_\nu, \partial/\partial_\mu]$$

implying $\omega_{\mu\nu} = 0$. Similarly we find $\omega_{\overline{\mu}\overline{\nu}} = 0$. Hence (K.16.2) simplifies to

$$\omega = [\omega_{\mu\overline{\nu}} - \omega_{\overline{\nu}\mu}]dz^\mu \wedge d\overline{z}^\nu =: \Omega_{\mu\overline{\nu}} dz^\mu \wedge d\overline{z}^\nu \tag{K.7}$$

Clearly $\Omega_{\overline{\nu}\mu} = -\Omega_{\mu\overline{\nu}}$. Reality $\omega = \overline{\omega}$ reads

$$\Omega_{\mu\overline{\nu}}\, dz^\mu \wedge d\overline{z}^\nu \equiv -\Omega_{\nu\overline{\mu}}\, d\overline{z}^\mu \wedge dz^\nu = \overline{\Omega_{\mu\overline{\nu}}\, dz^\mu \wedge d\overline{z}^\nu} = \overline{\Omega_{\mu\overline{\nu}}}\, d\overline{z}^\mu \wedge dz^\nu$$

implying

$$\overline{\Omega_{\mu\overline{\nu}}} = -\Omega_{\nu\overline{\mu}}. \tag{K.7}$$

Closure $\partial_{[\alpha}\omega_{\beta\gamma]} = 0$ for $\alpha, \beta \in \{\mu, \nu, \overline{\mu}, \overline{\nu}\}$ implies

$$\partial_{[\mu}\Omega_{\nu]\overline{\rho}} = \partial_{[\overline{\mu}}\Omega_{\overline{\nu}]\rho} = 0. \tag{K.7}$$

By Poincare's lemma this implies locally $\Omega_{\mu\overline{\nu}} = \partial_\mu f_{\overline{\nu}} = -\partial_{\overline{\nu}} g_\mu$.

$$= \tag{K.7}$$

The reality condition reads $\overline{i\partial_\mu \partial_{\overline{\nu}} K(z,\overline{z})} = -i\partial_\nu \partial_{\overline{\mu}} K(z,\overline{z})$ or

$$\partial_\nu \partial_{\overline{\mu}} \overline{K(z,\overline{z})} = \partial_\nu \partial_{\overline{\mu}} K(z,\overline{z})$$

□

Notice that by definition

$$g_{\mu\overline{\nu}} = \tag{K.7}$$

Any metric space defines a conformal or complex structure on S because one can always find isothermal coordinates, and the isothermal coordinates naturally define a complex structure.

n-dimensional Torus

The two-dimensional torus the points in R^2

$$(x, y) \;\; (x+1, y) \;\; (x, y+1) \tag{K.7}$$

are identified as a single point in T^2.

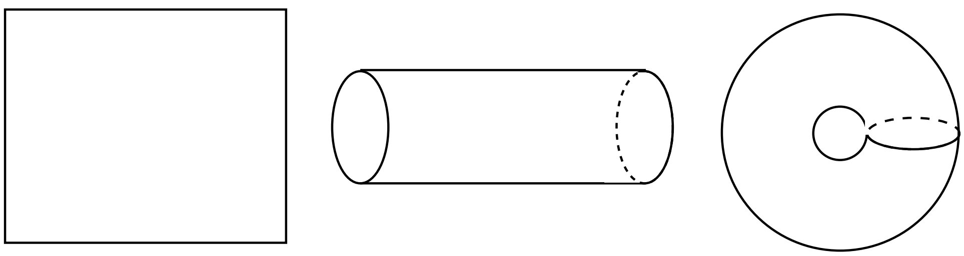

Figure K.4: Torus.

construct a holomorphic line bundle L over $\chi^{\mathcal{P}}$ with a connection whose curvature is $i\omega$.

$$\nabla_{x_i} = \partial_{x_i}, \quad \nabla_{y_i} = \partial_{y_i} + \frac{ik}{2\pi} z_i. \tag{K.7}$$

Note that

$$[\nabla_{x_i}, \nabla_{x_j}] = 0, \quad [\nabla_{y_i}, \nabla_{y_j}] = 0, \quad [\nabla_{x_i}, \nabla_{y_j}] = \frac{ik}{2\pi}\delta_{ij}, \tag{K.7}$$

using this curvature, we define parallel translation operators

$$U_i(t) = \exp(t\nabla_{x_i}), \quad V_i(t) = \exp(t\nabla_{y_i}) \tag{K.7}$$

for all t in R.

and the above commutation relations imply

$$U_i(s)U_j(t) = U_j(t)U_i(s), \quad V_i(s)V_j(t) = V_j(t)V_i(s), \quad U_i(s)V_j(t) = e^{\frac{ik}{2\pi}st}V_j(t)U_i(s). \quad \text{(K.7)}$$

the magnetic translation group quantum-Hall effect.

we obtain theta functions by the technique of group averaging. suppose we start with a holomorphic function on C^{n-1} that is invariant under $R(u)$ for real lattice vectors u, that is for $u \in (2\pi Z)^{n-1}$. Then try to average f with respect to the imaginary lattice directions, forming the function.

$$\psi = \sum_{v\in(2\pi Z)^{n-1}} R(iv)f. \quad \text{(K.7)}$$

A basis of such functions is given by

$$f_a(z) = \exp(ia \cdot z) \quad \text{(K.7)}$$

where $a \in (2\pi Z)^{n-1}$. If we apply group averaging to such a function f_a, we obtain a theta function

$$\psi_a(z) = \sum_{v\in(2\pi Z)^{n-1}} e^{\frac{k}{2\pi}(i\nu\cdot z - \frac{1}{2}v\cdot v)} e^{ia\cdot(z+iv)}, \quad \text{(K.7)}$$

$$\psi = \sum_{v\in(2\pi Z)} R(iv)\exp(ia \cdot z). \quad \text{(K.7)}$$

We define their inner product by

$$< f, g >= \int_{[0,2\pi]^{2(n-1)}} e^{-\frac{k}{2\pi}y\cdot y}\overline{f}(z)g(z)\, d^{n-1}x\, d^{n-1}y. \quad \text{(K.7)}$$

arXiv:quant-ph/0006037 v1 7 Jun 2000 HARMONIC ANALYSIS WITH RESPECT TO HEAT KERNEL

Roughly a polarization is a choice of d coordinates on the $2d-$phase space $\mathcal{M}$, with the idea that the functions in our quantum Hilbert space will be independent of these d variables.

For example if $\mathcal{M} = \mathbb{R}^{2d}$, then we may take the usual position and momentum variables $x_1, \ldots, x_d, p_1, \ldots, p_d$ and then consider functions that depend only on $x_1, \ldots, x_d$ and are independent of $p_1, \ldots, p_d$. (This is called the vertical polarization.)

Alternatively, one may consider complex variables $z_1, \dots z_d, \overline{z}_1, \dots, \overline{z}_d$ and then consider the functions that are independent of z_k, that is, holomorphic. (This is a complex polarization.) In that case our Hilbert space is the Segal-Bargmann space. To be more precise, in geometric quantization the elements of the quantum Hilbert space are not functions, but rather sections of a certain complex line bundle with connection. The sections are required to be covariantly constant in the directions corresponding to the polarization.

Theta Functions

Riemann θ-function

$$\theta(u_1, u_2, \dots, u_g; \tau_{ij}) := \sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} e^{2\pi i \sum_{j=1}^{g} u_j m_j} e^{\pi i \sum_{j,k=1}^{g} m_i \tau_{ij} m_j} \tag{K.7}$$

K.16.3 Mathematics Details of Geometric Quantization

The problem is that θ is neither necessarily globally defined nor unique. We can invoke fibre bundle theory to address this problem.

The integrality condition is closely related to the quantisation rule in old quantum theory.

1. A principal $(\mathbb{C} - \{0\})$ bundle B over M with globally defined connection A whose local sections θ have $\omega = d\theta$ as globally defined curvature.

2. A vector bundle E over M, associated with P under the defining representation of $\mathbb{C} - \{0\}$ with typical fibre $\mathbb{C}$ and local section ψ.

3. A $\nabla-$compatible fibre metric.

What is a necessary and sufficient criterion for the existence of these structures? To answer this we enter into the subject of Cech cohomology.

Introduction to Cech cohomology

We discussed de Rham cohomology previously. The point of cohomology came from the nilpotency of the exterior derivative: $d^2 = 0$.

the transition functions $g_{IJ} : U_I \cap U_J \to G$ satify

$$\begin{aligned} g_{IJ} g_{JI} &= 1, \\ g_{IJ} g_{JK} g_{KI} &= 1 \quad \text{(no summation)} \end{aligned}$$

or

$$\begin{aligned} g_{IJ} &= (g_{JI})^{-1}, & \text{(K.6)} \\ g_{JK}(g_{IK})^{-1}g_{IJ} &= (\delta g)_{IJK} = 1 \quad \text{(no summation)} & \text{(K.7)} \end{aligned}$$

Such relations fall under the subject of Cech cohomology. There is an isomorphism between Cech cohomology and de Rham cohomology. This isomorphism allows us to establish the condition for the existence of a globally defined connection.

Definition Let M be a manifold with open cover $\mathcal{U} = (U_I)_{I\in\mathcal{I}}$ subordinate to an atlas of M.

(i) An $n-$**cochain** $\{g\} \in C^n(\mathcal{U})$ is a system of functions defined on $U_{I_1} \cap \cdots \cap U_{I_n}$:

$$g_{I_1 \ldots I_n} : U_{I_1} \cap \cdots \cap U_{I_n} \to \mathbb{C} - \{0\}$$

such that

$$g_{I_{\pi(1)} \ldots I_{\pi(n+1)}} = (g_{I_1 \ldots I_{n+1}})^{sgn(\pi)}. \tag{K.7}$$

The n-cochains form an abelian group under pointwise multiplication for each muti-index.

(ii) We define a **cobounadry operator** (also called codifferential)

$$\delta_n : C^n(\mathcal{U}) \to C^{n+1}(\mathcal{U}),$$

$$(\delta g)(I_1, \ldots, I_{n+1}) := \prod_{k=1}^{n+1} [g(I_1, \ldots \hat{I}_k, \ldots, i_{n+1})]^{(-1)^{k-1}}$$

where the variable below the ˆ is omitted. The function δg is defined on $U_{I_1} \cap \cdots \cap U_{i_n} \cap U_{I_{n+1}}$ when it is non-empty. Some examples are

$$\begin{aligned} (\delta g)(I_1, I_2) &= g(I_2)g(I_1)^{-1} \\ (\delta g)(I_1, I_2, I_3) &= g(I_2, I_3)g(I_1, I_3)^{-1}g(I_1, I_2) & \text{(K.7)} \end{aligned}$$

Consider what happens when we take the operator δ again in these examples:

$$\begin{aligned}(\delta^2 g)(I_1, I_2, I_3) &= \delta(g(I_1)^{-1} g(I_2)) \\ &= (g(I_2)^{-1} g(I_3))\ (g(I_1)^{-1} g(I_3))^{-1}\ (g(I_1)^{-1} g(I_2)) \\ &= 1.\end{aligned} \tag{K.6}$$

Also

$$\begin{aligned}(\delta^2 g)(I_1, I_2, I_3, I_4) &= \delta(g(I_2, I_3) g(I_1, I_3)^{-1} g(I_1, I_2)) \\ &= (g(I_3, I_4) g(I_2, I_4)^{-1} g(I_2, I_3))\ (g(I_3, I_4) g(I_1, I_4)^{-1} g(I_1, I_3))^{-1} \\ &\times\ (g(I_2, I_4) g(I_1, I_4)^{-1} g(I_1, I_2))\ (g(I_2, I_3) g(I_1, I_3)^{-1} g(I_1, I_2))^{-1} \\ &= 1.\end{aligned} \tag{K.4}$$

We verify that $\delta^2(g) = \{1\}$ in general:

$$\begin{aligned}\delta^2(g) &= \delta(\prod_{k=1}^{n+1} [g(I_1, \ldots, \hat{I}_k, \ldots, I_{n+1})]^{(-1)^{k-1}}) \\ &= \prod_{l<k} g(I_1, \ldots, \hat{I}_l, \ldots, \hat{I}_k, \ldots, I_{n+2})^{(-1)^{k+l}} \\ &\times \prod_{l>k} g(I_1, \ldots, \hat{I}_k, \ldots, \hat{I}_l, \ldots, I_{n+2})^{(-1)^{k+l-1}} \\ &= 1.\end{aligned} \tag{K.2}$$

We have the power $k+l-1$ in the second product as we have passed through an ommitted index.

(iii) Instead of a multiplicative notation we can use an additive one by writing

$$g_{I_1 \ldots I_{n+1}} = \exp(f_{I_1 \ldots I_{n+1}}), \qquad f_{I_1 \ldots I_{n+1}} = sgn(\pi) f_{I_1 \ldots I_{n+1}}$$

This is obviously in accord with (K.16.3). The coboundray operation can be expressed

$$(\delta f)_{I_1 \ldots I_{n+2}} = (n+2) \chi_{[I_1} f_{I_2 \ldots I_{n+2}]} \tag{K.2}$$

where $\chi_I = \chi_{U_I}$ is the characteristic function of U_I. We verify the equivalence with the original definition:

$$
\begin{aligned}
(\delta g)_{I_1 \dots I_{n+1}} &= (\delta \exp(f))_{I_1 \dots I_{n+1}} \\
&= \exp(\delta(f))_{I_1 \dots I_{n+1}} \\
&= \exp\Big((n+1)\chi_{[I_1} f_{I_2 \dots I_{n+1}]}\Big) \\
&= \exp(\chi_{I_1} f_{[I_2 \dots I_{n+1}]} - \chi_{I_2} f_{[I_1 I_3 \dots I_{n+1}]} + \dots (-1)^n \chi_{I_{n+1}} f_{[I_1 \dots I_n]}) \\
&= \exp(\chi_{I_1} f_{I_2 \dots I_{n+1}} - \chi_{I_2} f_{I_1 I_3 \dots I_{n+1}} + \dots (-1)^{n+1} \chi_{I_{n+1}} f_{I_1 \dots I_n}) \\
&= (g_{I_2 \dots I_{n+1}})(g_{I_1 I_3 \dots I_{n+1}})^{-1} \dots (g_{I_1 \dots I_n})^{(-1)^{n+1}} \qquad \text{(K.-2)}
\end{aligned}
$$

It easy to verify that $(\delta^2 g) = \{1\}$. First we have

$$
(\delta^2 f)_{I_1 \dots I_{n+2}} = (n+2)(n+1)\chi_{[I_1}\chi_{I_2} f_{I_3 \dots i_{n+2}]} = 0
$$

So that

$$
\begin{aligned}
(\delta^2 g)_{I_1 \dots I_{n+3}} &= (\delta^2 \exp(f))_{I_1 \dots I_{n+3}} \\
&= \exp(\delta^2 f))_{I_1 \dots I_{n+3}} \\
&= 1. \qquad \text{(K.-3)}
\end{aligned}
$$

(iv) A $n-$cochain is called an $n-$**cocycle** if it is in the kernel of δ_n. The cocycle group is

$$
Z^n(\mathcal{U}) := \{g \in C^n(\mathcal{U}) : \delta_n g = \{1\}\}.
$$

(v) An $n-$cochain is called an $n-$**coboundary** if it is in the image of δ_{n-1}. The coboundary group is

$$
B^n(\mathcal{U}) := \{g \in C^n(\mathcal{U}) : g = \delta_{n-1} g', g' \in C^{n-1}(\mathcal{U})\}.
$$

As $\delta^2 = \{1\}$ every coboundary is also a cocycle.

(vi) Two cococyles g, g' that differ by a coboundary are said to be **cohomologous**. Rexpressed in terms of the additive notation

$$
f'_{I_1 \dots I_n} = f_{I_1 \dots I_n} + (\delta \tilde{f})_{I_1 \dots I_n}.
$$

(vii) The group

$$H^n(\mathcal{U}) = Z^n(\mathcal{U})/B^n(\mathcal{U})$$

is called the $n-th$**Cech cohomology group**. The equivalence class of f in $H^n(\mathcal{U})$ is denoted by $[f]$.

$\square$

The Cech cohomology seems to depend exlpicitly on the atlas $\mathcal{U}$. The dependence can be removed by taking an infinite refinement limit. However for our purposes this process is not required. Here we interested in the cases where M paracompact in which case one can choose a locally finite, contractible cover. We automatically have the so-called Leray cover for which the cohomology is already independent of the cover; the answer comes out the same whichever such covering is used. We have used the notation $H^n(\mathcal{U})$ in order to distinguish it from the de Rham cohomology $H^n(M)$ of forms.

In what follows we will only consider the Cech cohomology defined by locally constant functions. A function is locally constant, taking values for example in $\mathbb{R}, \mathbb{C}, \mathbb{Z}$, if it is constant in some neighbourhood of each point; it need not be globally constant if its domain is not connected. The resulting cohomology groups are said to have coefficients in $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and are denoted by $H^{\cdot}(\mathcal{U}, \mathbb{R}), H^{\cdot}(\mathcal{U}, \mathbb{C}), H^{\cdot}(\mathcal{U}, \mathbb{Z})$.

Definition A locally finite open cover of M is a cover such that any $p \in M$ is only in finitely many U_I. It is a contractible open cover of M if every U_I and every nonempty finite intersection $U_I \cap U_J \cap \dots$ is contractible to a point

$\square$

Definition Let M be paracompact. $\{f\}$ is a locally constant n-cochain if each $f_{I_1 \dots I_{n+1}}$ takes a constant value on each connected component of $U_{I_1} \cap \dots \cap U_{I_n}$, possibly a different one on each component.

$\square$

What is the relation between de Rham cohomology and Cech cohomology?

Let us define a map α from $C^n(\mathcal{U})$ to $C^n(M)$

$$\alpha : C^n(\mathcal{U}) \to C^n(M)$$

defined by

$$\alpha_{\{f\}}(p) := f_{II_1 \dots I_n}(p) e_I(p) \, de_{I_1}(p) \wedge \dots \wedge de_{I_n}(p) \tag{K.-3}$$

Theorem K.16.3 (de Rham isomorphism). *We have* $d\alpha_{\{f\}} = \alpha_{\{\delta f\}}$ *and* α *defines an isomorphism* $H^n(\mathcal{U}) \to H^n(M)$.

Proof: Notice that $df_{I_1 \ldots I_{n+1}} = 0$ in the compact support of $e_{I_1}\, de_{I_2} \wedge \cdots \wedge de_{I_{n+1}}$ due to local constancy. While it is not in general true that $df_{I_1 \ldots I_{n+1}} = 0$ on $\partial U_{I_1} \cap \ldots U_{I_n}$ this surface is not in the support of $e_{I_1}\, de_{I_2} \wedge \cdots \wedge de_{I_{n+1}}$. Hence

$$d\alpha_{\{f\}} = f_{I_1 \ldots I_{n+1}}\, de_{I_1} \wedge \cdots \wedge de_{I_{n+1}}. \tag{K.-3}$$

Now next notice the relation $\chi_I e_I = \sum_I e_I = 1$ because $\mathrm{supp}(e_I) \subset U_I$. Hence $0 = de_I \chi_I + e_I d\chi_I = de_I \chi_I$ where we have used the fact that $d\chi_I$ is non-vanishing on ∂U_I only, which however is outside the support of e_I.

$$\begin{aligned}
\alpha_{\delta\{f\}} &= (n+2)\chi_{[I} f_{I_1 \ldots I_{n+1}]} \\
&\quad + \sum_{k=1}^{n+1} (-1)^k \chi_{I_k}\, f_{I_1 \ldots \hat{I}_k I \ldots I_{n+1}}\, e_I\, de_{I_1} \wedge \cdots \wedge de_{I_{n+1}} \\
&= [\chi_I e_I] f_{I_1 \ldots I_{n+1}} de_{I_1} \wedge \cdots \wedge de_{I_{n+1}} \\
&= d\alpha_{\{f\}} - \sum_{k=1}^{n+1} \chi_{I_k}\, f_{I I_1 \ldots \hat{I}_k \ldots I_{n+1}}\, e_I\, de_{I_1} \wedge \cdots \wedge de_{I_{n+1}} \\
&= d\alpha_{\{f\}} + \sum_{k=1}^{n+1} (-1)^k \chi_{I_k}\, f_{I I_1 \ldots \hat{I}_k \ldots I_{n+1}}\, e_I\, de_{I_k} \wedge de_{I_1} \wedge \cdots \wedge d\hat{e}_{I_k} \wedge \cdots \wedge de_{I_{n+1}} \\
&= d\alpha_{\{f\}} + \left[\sum_{k=1}^{n+1} (-1)^k \right] [\chi_J de_J] \wedge \alpha_{\{f\}} = d\alpha_{\{f\}}
\end{aligned} \tag{K.-7}$$

Hence $\alpha_{\delta\{f\}} = d\alpha_{\{f\}}$ and so α maps coboundaries to closed forms. And if f and f' are cohomologous, then α_f and $\alpha_{f'}$ differ by an exact form.

We now define $\alpha : H^n(\mathcal{U}) \to H^n(M)$ by

$$\alpha_{[\{f\}]} := [\alpha_{\{f\}}]$$

where the brackets denote the respective cohomology classes.

Injectivity:

The map is one-to-one if for $f \neq f'$, $\alpha_{[\{f\}]} = \alpha_{[\{f'\}]}$ implies f and f' are Cech comologous. Now $\alpha_{[\{f\}]} = \alpha_{[\{f'\}]}$ means by definition $[\alpha_{\{f\}}] = [\alpha_{\{f'\}}]$. So $\alpha_{\{f'\}} = \alpha_{\{f\}} + d\alpha_{\{\tilde{f}\}}$

Surjectivity:

We now prove that the map is onto. We prove this only for $n = 2$, the general case is similar. Assume that $\tau \in Z^2(M)$, that is, τ is an arbitrary closed 2-form. We show that there exists an f_{IJK} such that $\tau - \alpha_{\{f\}}$ is exact, that is, $[\tau] = [\alpha_{\{f\}}] = \alpha_{[\{f\}]}$ proving surjectivity.

Since U_I is contractible, by Poincare's lemma we find $\beta_I \in C^1(U_I)$

$$\tau = d\beta_I$$

on U_I. If $U_I \cap U_J \neq \emptyset$ we have

$$d(\beta_I - \beta_J) = 0$$

on $U_I \cap U_J$. Since $U_I \cap U_J$ is conractible, again by Poincare's lemmaa, we find $\gamma_{IJ} \in C^0(U_I \cap U_J)$ (the space of smooth functions on $U_I \cap U_J$) such that

$$\beta_I - \beta_J = d\gamma_{IJ}$$

on $U_I \cap U_J$ and where γ_{IJ} satisfies $\gamma_{IJ} = -\gamma_{JI}$. If $U_I \cap U_J \cap U_K \neq \emptyset$ then

$$d(\gamma_{IJ} + \gamma_{JK} + \gamma_{KI}) = (\beta_I - \beta_J) + (\beta_J - \beta_K) + (\beta_K - \beta_I) = 0$$

on $U_I \cap U_J \cap U_K$, hence

$$f_{IJK} := \gamma_{IJ} + \gamma_{JK} + \gamma_{KI} = (\delta\gamma)_{IJK}$$

is locally constant. Notice that f_{IJK} is locally constant but not necessarily the γ_{IJ}. As we are only considering Cech cohomology of locally constant functions, f_{IJK} does not generally qualify as exact. However, the identity $\delta^2 = 0$ proved in () holds for any type of function not just locally constant ones, hence we have that $\{f\}$ is closed and on $U_I \cap U_J \cap U_K \cap U_L$

$$(\delta f)_{IJKL} = f_{JKL} - f_{IKL} + f_{IJL} - f_{IJK} = 0. \tag{K.-7}$$

Contracting with $e_I \, de_J \wedge de_K$ and using $\sum e_I = 1$ and $\sum de_J = 0$ we get

$$
\begin{aligned}
\alpha_{\{f\}} &= f_{IJK} e_I de_J \wedge de_K \\
&= \left(f_{JKL} - f_{IKL} + f_{IJL}\right) e_I de_J \wedge de_K \\
&= f_{LJK} de_J \wedge de_K \\
&= d(f_{LJK} e_J \wedge de_K) =: d\omega_I
\end{aligned}
\tag{K.-9}
$$

on the interior of U_I. Similarly, contracting with $e_J\, de_K$ we find

$$
\begin{aligned}
f_{IJK} e_J\, de_K - f_{LJK} e_J\, de_K &= -f_{IKL} e_J\, de_K + f_{IJL} e_J\, de_K \\
&= f_{ILK} de_K \\
&= d(f_{ILK} e_K)
\end{aligned}
$$

on the interior of $U_I \cap U_J$. This expressed in terms of the just defined ω_I reads

$$
\omega_I - \omega_J = f_{IJK} de_K = d(f_{IJK} e_K) =: d\sigma_{IJ} \tag{K.-12}
$$

on the interior of $U_I \cap U_J$. Noticing that

$$
\sigma_{IJ} = f_{IJ} + f_{JK} e_K + f_{KI} e_K \tag{K.-12}
$$

and defining on the interior of U_I

$$
\lambda_I = \beta_I - \omega_I - d(f_{IJ} e_J) \tag{K.-12}
$$

we have on the interior of $U_I \cap U_J$

$$
\begin{aligned}
\lambda_I - \lambda_J &= \gamma_{IJ} - \omega_I + \omega_J - d(f_{IK} e_K - f_{JK} e_K) \\
&= -\omega_I + \omega_J + d(f_{IJ} + f_{JK} e_K + f_{KI} e_K) \\
&= -\omega_I + \omega_J + d\sigma_{IJ} = 0
\end{aligned}
\tag{K.-13}
$$

hence $\lambda = \lambda_I$ is globally defined. Hence on U_I

$$
d\lambda = d\beta_I - d\omega_I = \tau - \alpha_{\{f\}} \tag{K.-13}
$$

it follows that $\tau - \alpha_{\{f\}}$ is exact.

□

Theorem K.16.4 (Weil's integrality criterion). *A prequantisation of (M, ω), that is, a principal $\mathbb{C} - \{0\}$ bundle B with global connection ∇ and $\nabla-$compatible fibre metric ρ on an associated complex line bundle exists if and only if Weil's criterion holds: the Cech cohomology class of $\alpha^{-1}(\omega/(2\pi\hbar)$ is integral, that is, $[\alpha^{-1}(\omega/(2\pi\hbar)] \in \mathbb{Z}$ where $\alpha : H^2(\mathcal{U}) \to H^2(M)$ is the de Rham isomorphism*

Moreover, the inequivalent choices of (P, ∇, ρ) are parametrised by $H^1(\mathcal{U})$ with values in $U(1)$.

Proof:

Suppose first that Weil's criterion is satisfied and let $[\omega] = [\alpha_{\{f\}}]$. From the proof of the previous theorem we know that $f_{IJK} = \gamma_{IJ} + \gamma_{JK} + \gamma_{KI} = (\delta\gamma)_{IJK}$ on $U_I \cap U_J \cap U_K$ is locally constant with smooth functions $\gamma_{IJ} = -\gamma_{JI}$ on $U_I \cap U_J$. Moreover, by assumption

$$f_{IJK} = 2\pi\hbar n_{IJK}$$

where n_{IJK} takes locally constant integer values on $U_I \cap U_J \cap U_K$. Define

$$g_{IJ} = \exp(i\gamma_{IJ}/\hbar),$$

then

$$g_{IJ} g_{JI} = 1$$

on $U_I \cap U_J$ and because n_{IJK} is an integer

$$\begin{aligned} g_{IJ} g_{JK} g_{KI} &= \exp i(\gamma_{IJ} + \gamma_{JK} + \gamma_{KI})/\hbar \\ &= \exp(2\pi i n_{IJK}) = 1 \end{aligned} \tag{K.-13}$$

on $U_I \cap U_J \cap U_K$, hence g_{IJ} is a cocycle with values in $\mathbb{C} - \{0\}$ and therefore qualifies as the transition function of a principal $(\mathbb{C} - \{0\})$ bundle. If θ_I are the local potentials of ω then by definition

$$d\gamma_{IJ} = -i\hbar dg_{IJ} g_{IJ}^{-1} = \theta_I - \theta_J$$

or with $A_I = i\theta_I/\hbar$ we find

$$A_J = A_I - dg_{IJ} g_{IJ}^{-1} \tag{K.-13}$$

Here we use the results of section . We construct local one-forms $\nabla_U, \nabla_V, \ldots,$ on P from the local one-forms $A_U, A_V, \ldots,$ on $\mathcal{M}$. From the transformation (K.16.3) it follows that $\nabla_U = \nabla_V$ on $\pi^{-1}(U \cap V)$, thus the local one-forms $\nabla_U, \nabla_V, \ldots,$ collectively define a global connection one-form ∇ on P. The A_I qualify as the pull-backs by local sections of a globally defined $\mathbb{C} - \{0\}$ connection ∇.

Now we prove it the other way. Suppose that (P, ∇, ρ) exist and let g_{IJ} be the transition functions of the bundle P with values in $\mathbb{C} - \{0\}$. We wish to define a f_{IJK} in terms of the logarithm of g_{IJ}. However there is a subtlety in the definition of the logarithms. The complex logarithmic function has infinitely many branches. The function ln is called the fundamental or principal branch of the logarith.

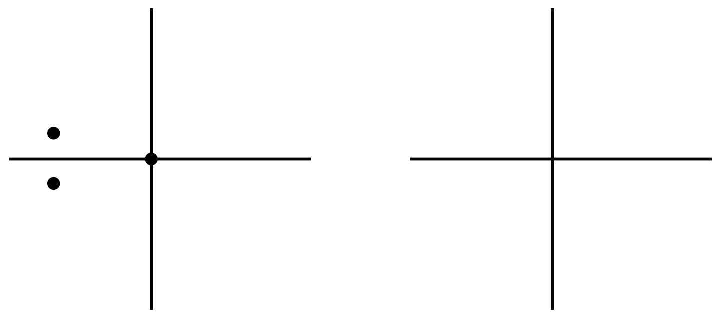

Figure K.5: .

We define

$$\frac{f_{IJK}}{2\pi\hbar} := \frac{1}{2\pi i}[\ln(g_{IJ}) + \ln(g_{JK}) + \ln(g_{KI})] \tag{K.-13}$$

where we choose the fundamental branch of the logarithm over $U_I \cap U_J$ with cut at $\varphi = \pi$ so that $\ln(g_{IJ}) = -\ln(g_{IJ})$. Hence we have

$$\ln g_{IJ} = \ln |g_{IJ}| + i(\theta + 2\pi n_{IJ})$$

where $n_{IJ} \in \mathbb{Z}$. Since g_{IJ} satisfy the cocycle condition

$$\begin{aligned} \frac{1}{2\pi i}[\ln(g_{IJ}) + \ln(g_{JK}) + \ln(g_{KL})] &= \frac{1}{2\pi i}\ln(g_{IJ}g_{JK}g_{KL}) + n_{IJ} + n_{JK} + n_{KL} \\ &= n_{IJ} + n_{JK} + n_{KL}. \end{aligned}$$

Hence the right-hand side of (K.16.3) is integral $n_{IJK} \in \mathbb{Z}$ and as (relying on antisymmetry $\ln(g_{IJ}) = -\ln(g_{JI})$)

$$\frac{f_{IJK}}{2\pi\hbar} = \frac{1}{2\pi i}(\delta \ln g)_{IJK}$$

$\delta\{f\} = 0$. Since...

$$\begin{aligned} \alpha_{\{f\}} &= -i \sum_{IJK} [\ln(g_{IJ}) + \ln(g_{JK}) + \ln(g_{KI})] \; e_I de_J \wedge de_K \\ &= \end{aligned} \tag{K.-15}$$

Finally we prove the last statement of the theorem. There is a freedom in the construction of P and ∇ from ω. If Weil's criterion is satisfied then $[\{f\}]$ is determined by γ_{IJ} only up to a coboundary $\delta\{x\}$

$\square$

Recall that two bundles P, P' are equivalent if... . Now if M is simply connected (intuitively a connected space without any holes) then $H^1(M) = \{0\}$, hence by the de Rham isomorphism also $H^1(\mathcal{U}) = \{0\}$.

This is sinificant because if M is simply connected, then $H^1(M)$ is trivial and there is a unique choice of B and ∇.

Corollary K.16.5 *Weil's criterion is equivalent to the requirement that for any closed two-surface S in phase space*

$$\int_S \frac{\omega}{2\pi\hbar} = integer \tag{K.-15}$$

Proof: Suppose first that Weil's criterion holds. We assume for simplicity that the contractible open cover $\mathcal{U}$ is such that the sets $D_I := S \cap U_I$ are open discs covering S such that no point of S lies in more than three different M_I.

$$\begin{aligned} M_I &:= D_I - \cup_{J \neq I}(D_I \cap D_J) && \text{(K.-14)} \\ M_{IJ} &:= D_I \cap D_J - \cup_{K \neq I,J}(D_I \cap D_J \cap D_K) && \text{(K.-13)} \\ M_{IJK} &:= D_I \cap D_J \cap D_K && \text{(K.-12)} \end{aligned}$$

$$\begin{aligned} S_1 &= \cup_I M_I \\ S_2 &= \cup_{I<J} M_{IJ} \\ S_3 &= \cup_{I<J<K} M_{IJK} \end{aligned} \tag{K.-13}$$

$$S = S_1 \cup S_2 \cup S_3$$

$$M_I = D_I - \cup_{J \neq I} M_{IJ} - \cup_{J<K;J,K\neq I} M_{IJK} \tag{K.-13}$$

$$\begin{aligned}
\int_S \Omega &= \sum_I \int_{M_I} \Omega + \sum_{I<J} \int_{M_{IJ}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega \\
&= \left(\sum_I \int_{D_I} \Omega - \sum_{J\neq I} \int_{M_{IJ}} \Omega - \sum_{J<K,\ J,K\neq I} \int_{M_{IJK}} \Omega \right) + \sum_{I<J} \int_{M_{IJ}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega
\end{aligned} \tag{K.-14}$$

As $\sum_{J\neq I} = \sum_{I<J} + \sum_{I>J}$:

$$\int_S \Omega = \sum_I \int_{D_I} \Omega - \sum_{I>J} \int_{M_{IJ}} \Omega - \sum_{J<K,\ J,K\neq I} \int_{M_{IJK}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega$$

Rewritting the second term

$$\int_S \Omega = \sum_I \int_{D_I} \Omega - \left(\sum_{I>J} \int_{D_I \cap D_J} \Omega - \sum_{I>J,\ K\neq I,J} \int_{M_{IJK}} \Omega \right) - \sum_{J<K,\ J,K\neq I} \int_{M_{IJK}} \Omega + \sum_{I<J<K} \int_{M_{IJK}} \Omega$$

Note

$$\begin{aligned}
\sum_{J<K,\ J,K\neq I} &= \sum_{I<J<K} + \sum_{J<I<K} + \sum_{J<K<I} \\
\sum_{I>J,\ J,K\neq I} &= \sum_{K<J<I} + \sum_{J<K<I} + \sum_{J<I<K} .
\end{aligned} \tag{K.-15}$$

Note the last two terms in the first and second lines are equal. We see

$$\begin{aligned}
\sum_{I<J<K} - \sum_{J<K,\ J,K\neq I} + \sum_{I>J,\ J,K\neq I} &= \sum_{I<J<K} - \sum_{I<J<K} + \sum_{K<J<I} \\
&= \sum_{I>J>K}
\end{aligned} \tag{K.-16}$$

So we finally have

$$\begin{aligned}\int_S \Omega &= \sum_I \int_{D_I} \Omega - \sum_{I>J} \int_{D_I \cap D_J} \Omega + \sum_{I>J>K} \int_{D_I \cap D_J \cap D_K} \Omega \\ &= \sum_I \int_{\partial D_I} \sigma_I - \sum_{I>J} \int_{\partial(D_I \cap D_J)} \sigma_J + \sum_{I>J>K} \int_{\partial(D_I \cap D_J \cap D_K)} \sigma_K \end{aligned} \qquad \text{(K.-16)}$$

where we used $\Omega = d\sigma_I$ on (any subset of) D_I.

We take all D_I with orientation such that the loop ∂D_I is counterclockwise for definiteness.

$$\begin{aligned}\partial(D_I \cap D_J) &= [(\partial D_I) \cap D_J] \cup [D_I \cap (\partial D_J)] \\ \partial(D_I \cap D_J \cap D_K) &= [(\partial D_I) \cap D_J \cap D_K] \cup [D_I \cap (\partial D_J) \cap D_K]] \cup [D_I \cap D_J \cap (\partial D_K)]\end{aligned} \qquad \text{(K.-17)}$$

$$\partial^2(D_I \cap D_J) = \partial[(\partial D_I) \cap D_J] \cup [D_I \cap (\partial D_J)] = 0 \qquad \text{(K.-17)}$$

$$\partial D_I = [\cup_{J \neq I} (\partial D_I) \cap M_{IJ}] \cup [\cup_{J<K;J,K \neq I} (\partial D_I) \cap M_{IJK}]. \qquad \text{(K.-17)}$$

$$\begin{aligned}\int_S \Omega &= \sum_{I>J} \int_{(\partial D_I) \cap D_J} (\sigma_I - \sigma_J) \\ &+ \sum_{I>J>K} \left\{ \int_{(\partial D_I) \cap D_J \cap D_K} (\sigma_K - \sigma_I) + \int_{D_I \cap (\partial D_J) \cap D_K} (\sigma_J - \sigma_K) \right\}\end{aligned}$$

$$\begin{aligned}\int_S \Omega &= \sum_{I>J} \int_{\partial[(\partial D_I) \cap D_J]} \gamma_{IJ} \\ &+ \sum_{I>J>K} \left\{ \int_{\partial[(\partial D_I) \cap D_J \cap D_K]} \gamma_{KI} + \int_{\partial[D_I \cap (\partial D_J) \cap D_K]} \gamma_{KJ} \right\}\end{aligned} \qquad \text{(K.-19)}$$

where $\sigma_I - \sigma_J = d(\gamma_{IJ})$ on (any subset of) $D_I \cap D_J$ was used.

$$\partial[(\partial D_I) \cap D_J] = \cup_{K \neq I,J} (\partial[(\partial D_I) \cap D_J)] \cap D_K \qquad \text{(K.-19)}$$

Next

$$\partial[D_I \cap (\partial D_J) \cap D_K] \;=\; (\partial[(\partial D_I) \cap D_J]) \cap D_K \cup (\partial[(\partial D_I) \cap D_K]) \cap D_J \tag{K.-19}$$

$$\begin{aligned}\sum_{I>J} \int_{\partial[(\partial D_I)\cap D_J]} \gamma_{IJ} &= \sum_{I>J} \sum_{K\neq I,J} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{IJ} \\ &= (\sum_{I>J>K} + \sum_{I>K>J} + \sum_{K>I>J}) \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{IJ}\end{aligned} \tag{K.-19}$$

$$\sum_{I>J>K} \int_{\partial[(\partial D_I)\cap D_J \cap D_K]} \gamma_{KI} \;=\; \sum_{I>J>K} \left\{ \int_{\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{KI} + \int_{(\partial[(\partial D_I)\cap D_K])\cap D_J} \gamma_{KI} \right\} \tag{K.-19}$$

$$\sum_{I>J>K} \int_{\partial[D_I \cap (\partial D_J)\cap D_K]} \gamma_{KJ} \;=\; \sum_{I>J>K} \left\{ \int_{\partial[(\partial D_J)\cap D_I])\cap D_K} \gamma_{KJ} + \int_{(\partial[(\partial D_J)\cap D_K])\cap D_I} \gamma_{KJ} \right\} \tag{K.-19}$$

Using $\gamma_{IJ} + \gamma_{JI} = 0$

$$\begin{aligned}\int_S \Omega &= \sum_{I>J>K} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} [\gamma_{IJ} + \gamma_{JK} + \gamma_{KI}] \\ &\quad +(\sum_{I>K>J} + \sum_{K>I>J}) \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{IJ} + 2\sum_{I>J>K} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} \gamma_{KJ} \\ &\quad + \sum_{I>J>K} \left\{ \int_{(\partial[(\partial D_I)\cap D_K])\cap D_J} \gamma_{KI} + \int_{(\partial[(\partial D_J)\cap D_K])\cap D_I} \gamma_{KJ} \right\}\end{aligned} \tag{K.-20}$$

We arrive at

$$\begin{aligned}\int_S \Omega &= \sum_{I>J>K} \int_{(\partial[(\partial D_I)\cap D_J])\cap D_K} [\gamma_{IJ} + \gamma_{JK} + \gamma_{KI}] \\ &= \sum_{I>J>K} n_{IJK} \in \mathbb{Z}\end{aligned} \tag{K.-20}$$

where we have used the fact that $\gamma_{IJ} + \gamma_{JK} + \gamma_{KI} = n_{IJK}$ is integral and constant on (any subset of) $D_I \cap D_J \cap D_K$.

We prove it the other way, that is, if $\int_S \omega/2\pi\hbar =$ integer then $[\alpha^{-1}(\omega/(2\pi\hbar))] \in \mathbb{Z}$. Fix some labels $I_0 > J_0 > K_0$ with $D_{I_0} \cap D_{J_0} \cap D_{K_0}$

□

Cotangent bundles T^*Q, equipped with the canonical sympletic strcuture $\omega = d\theta$, are always prequantisable as ω is exact.

Definition As sympletic manifold $(\mathcal{M}, \omega)$ is said to be prequantisable if and only if Weil's integrality criterion is satisfied. The associated structure (P, ∇, ρ) is called a prequantum bundle. The prequantum Hilbert space is $\mathcal{H}' = L_2(\mathcal{M}, \Omega)$ with inner product between smooth sections of compact support of the associated line bundle E given by

$$< \psi, \psi' > = \int_{\mathcal{M}} \Omega\rho[\psi, \psi'], \qquad \rho[\psi, \psi'] = \rho\psi\psi' \tag{K.-20}$$

where $d\ln(\rho) = \frac{2}{\hbar} Im(\theta)$ and symmetric operators associated with real-valued functions $f \in C^\infty(\mathcal{M})$ are densely defined on E by

$$\hat{f} = i\hbar\nabla_{\chi_f} + f, \qquad \nabla = d + \frac{1}{i\hbar}\theta, \qquad \omega = d\theta. \tag{K.-20}$$

□

K.16.4 Integrability Condition

Consider the system of partial differential equations

$$\frac{\partial f}{\partial x} = g(x, y), \qquad \frac{\partial f}{\partial y} = h(x, y) \tag{K.-20}$$

This can be written as

$$f_{,i} = a_i,$$

where $a_x = g$ and $a_y = h$. This equation has the coordinate independent form

$$df = a, \tag{K.-20}$$

where a is a one-form with components g and h. If f is a solution to this equation then we must have

$$d(df) = da.$$

We have the necessary condition for the solution to exist is that

$$da = 0.$$

In components this is

$$a_{[i,j]} = 0,$$

which is

$$\frac{\partial g}{\partial y} - \frac{\partial h}{\partial x} = 0. \qquad \text{(K.-20)}$$

These are sufficient conditions for the existence of a solution is guaranteed by Frobenius theorem in the differential form version.

K.16.5 Polarisation

The wave functions depend only on position. The wavefunctions of the associated prequantum Hilbert space would depend on momentum as well.

The prequantum Hilbert space H consists of functions ψ which depend on all the $2n$ coordinates of the sympletic manifold $(\mathcal{M}, \omega)$. Demand that the wavefunctions are constant along n vector fields in $\mathcal{M}$.

We choose some n-dimensional sub-bundle P of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$

As we see we proceed by choosing some k-dimensional subbundle P of the tangent bundle $T\mathcal{M}$ of $\mathcal{M}$ and consider only those wave functions that satisfy

$$D(X)\psi = 0 \qquad \text{for all } X \in P. \qquad \text{(K.-20)}$$

Now there could be non-trivial integrability conditions for those equations which would form an obstruction to finding (or a sufficient number of) solutions to (K.16.5). From (K.16.5) it follows that $[D(X), D(Y)]\psi = 0$ for all $X, Y \in P$. Combined with $\Omega(X, Y) = i([[D(X), D(Y)]] - D([X, Y]))$ this leads to the integrability condition

$$D([X,Y])\psi - (i/\hbar)\omega(X,Y)\psi = 0 \qquad \text{for all } X, Y \in P. \tag{K.-20}$$

We see that this condition is automatically satisfied provided that

$$X, Y \in P \;\Rightarrow\; [X,Y] \in P \tag{K.-20}$$

and

$$X, Y \in P \;\Rightarrow\; \omega[X,Y] = 0. \tag{K.-20}$$

The first condition means that P is integrable. The second condition means that the integral manifolds are Lagrangian, or if you like, maximally isotropic:

$$P = P^{\perp} = \{X \in V : \omega[X,Y] = 0 \text{ for all } Y \in P\}.$$

Integrability condition

Given a smooth nonvanishing vector field, by solvong a system of ordinary differential equations one can always locally find a smooth family of integral curves, that is, nonintersecting curves that fill up a region and are always tangent to the vector field.

System of Mayer-Lie

Frobenius' theorem arose in the study of partial differential equations.

This is important in the section on geometric quantisation.

Complex structures

It turns out that one needs to complexify the tangent bundle of $\mathcal{M}$, $T\mathcal{M} \to T\mathcal{M}^{\mathbb{C}}$, and to then consider integrable Lagrangian sub-bundles of $T\mathcal{M}^{\mathbb{C}}$.

For example say the symplectic manifold were the sphere, it is well known that any vector field on this space must vanish at at least one point. In this case how you cannot define a polarisation. A resolution to this problem is to take two vector fields that vanish at different points and consider their complex linear combination.

Definition Let V be a vector space involving only real coefficients. The complexification $V_{\mathbb{C}}$ of V is a vector space consisting of vectors of the form $w = u + iv$, $u, v \in V$.

The linear operations on $V_{\mathbb{C}}$ are defined as follows:

$$w_1 + w_2 := (u_1 + u_2) + i(v_1 + v_2),$$

for $z = x + iy \in \mathbb{C}$ we have

$$zw := (xu - yv) + i(xv + yu)$$

and we define

$$\overline{w} := u - iv.$$

$\square$

Note we connot have $zw = \overline{w}$ as $(xu - yv) + i(yu + xv)$ does not have a solution (except for $u = 0$, $y = 0$, $x = -1$), i.e. the equations

$$\begin{aligned}(xu - yv) &= u \\ (yu + xv) &= -v\end{aligned}$$

do not have a solution (except for $u = 0$, $y = 0$, $x = -1$).

Definition Let $(\mathcal{M}, \omega)$ be a sympletic manifold. A polarisation P of $(\mathcal{M}, \omega)$ is an integrable maximally isotropic (Lagrangian) sub-bundle of the complexified tangent bundle $T\mathcal{M}^{\mathbb{C}}$ of $\mathcal{M}$.

$\square$

A real distribution P on a manifold M is a sub-bundle of the tangent bundle.

Definition Let $(\mathcal{M}, \omega, J)$ be a sympletic vector space with ω-compatible complex structure J. The subspace $V^{\pm}$ of $V_{\mathbb{C}}$ consisting of vectors of the form

$$u^{\pm} := \frac{1}{2}(u \mp iJ[u])$$

is called the subspace of holomorphic (antiholomorphic) vectors since $J\left[u^{\pm}\right] = \pm iu^{\pm}$. We set $P_J := V^{+}$. Obviously u^{-} is the complex conjugate, $\overline{u^{+}}$, of u^{+} in the sense of complex vectors and so $\overline{P_J} = \{\overline{w},\ w \in P_J\} = V^{-}$.

$\square$

Lemma K.16.6 *P_J is a Lagrangian subspace of $V_{\mathbb{C}}$ with the additional property that $P_J \cap \overline{P_J} = \{0\}$. Conversely, every Lagrangian subspace with this property determines a compatible complex structure on V.*

Proof: Recall that a Lagrangian subspace F of a sympletic vector space V is defined by the property $F = F^{\perp} := \{v \in V : \omega(u,v) = 0 \text{ for all } u \in F\}$. We have for any $u, v \in V$

$$\begin{aligned} 4\omega[u^+, v^{\pm}] &= 4\omega[\frac{1}{2}(u - iJ[u]), \frac{1}{2}(v \mp iJ[v])] \\ &= (\omega[u,v] \mp \omega[J[u], J[v]]) - i(\omega[J[u], v] \pm \omega[u, J[v]]) \\ &= [\omega[u,v] - i\omega[J[u], v](1 \mp 1) \end{aligned} \tag{K.-23}$$

by compatibility ($\omega[J[u], J[v]] = \omega[u,v]$, equivalently $\omega[J[u], v] = -\omega[u, J[v]]$). Hence, because ω nondegenerate (if $\omega[u,v] = 0$ for all $v \in V$ then $u = 0$), for $u^+, v^+ \in P_J$ we have $\omega[u^+, v^+] = 0$ so $P_J \subset (P_J)^{\perp}$ and when $u^+ \in P_J$, $v^- \in (P_J)^{\perp}$ we have $\omega[u^+, v^-] \neq 0$ so $\overline{P_J}$ is not in $(P_J)^{\perp}$, that is, $\overline{P_J} \cap (P_J)^{\perp} = \{0\}$. Since $V^+ = P_J$, $V^- = \overline{P_J}$ span $V_{\mathbb{C}}$ and satisfy $P_J \cap \overline{P_J} = \{0\}$ by $P_J \subset (P_J)^{\perp}$ for $w \in (P_J)^{\perp}$, $w \notin P_J$ we must have $w \in \overline{P_J}$ but this contradicts $\overline{P_J} \cap (P_J)^{\perp} = \{0\}$ so there is no such w, i.e. $P_J = (P_J)^{\perp}$. Thus we have proved that P_J is a Lagrangian subspace of $V_{\mathbb{C}}$.

Conversely, given a Lagrangian subspace $P \subset V_{\mathbb{C}}$ with $P \cap \overline{P} = \{0\}$ we know that $V_{\mathbb{C}} = P \oplus \overline{P}$ and can decompose any $w \in V_{\mathbb{C}}$ uniquely as $w = w^+ + w^-$ with $w^+ \in P$, $w \in \overline{P}$. Notice that since ω is real and P is Lagrangian, $\overline{P}$ is also Lagrangian since $\omega[\overline{w}, \overline{w}'] = \overline{\omega[w, w']} = 0$. We now define

$$J[w] := i(w^+ - w^-)$$

which determines J uniquely. Then

$$\begin{aligned} \omega[J[w_1], J[w_2]] &= \omega[i(w_1^+ - w_1^-), i(w_2^+ - w_2^-)] \\ &= -\omega[w_1^+, w_2^+] + \omega[w_1^+, w_2^-] + \omega[w_1^-, w_2^+] - \omega[w_1^-, w_2^-] \\ &= \omega[w_1^+, w_2^-] + \omega[w_1^-, w_2^+] \end{aligned} \tag{K.-24}$$

where we have used the Lagrangian subspace property.

$\square$

Complex Lagrangian subspaces

Definition Let V be a vector space over the reals, or complex numbers, and let W be a subset of V. Then W is said to be a subspace of V. If W is itself a vector space, then W is said to be a vector subspace of V.

From the previous lemma we see that given a complex structure it is easy to construct Lagrangian subspaces of $V_{\mathbb{C}}$ with $P_J \cap \overline{P_J} = \{0\}$. The other extreme are Lagrangian subspaces of $V_{\mathbb{C}}$ with $P \cap \overline{P} = P$. They have the following property.

Lemma K.16.7 *Lagrangian subspaces P of $V_{\mathbb{C}}$ with the property $P \cap \overline{P} = P$ are the complexifications of Lagrangian subspaces of V.*

Proof:

$\square$

A polarisation picks out for each point $x \in \mathcal{M}$ out a certain subspace P_x of the complexified tangent space at x.

The easiest kind of polarisation is a real polarisation. This is when the polarisation comes from the complexification of subspaces of the tangent spaces.

We now systematically study the intermediate cases of complex Lagrangian subspaces, in which the dimension of $P \cap \overline{P}$ is between zero and n, utilising the following results.

Lemma K.16.8 *Let*

$$F \subset V$$

be a subspace of the real vector space V and let

$$G \subset V_{\mathbb{C}}$$

be a subspace of its complexification. We define the real subspace of G by

$$G_{\mathbb{R}} := G \cap V.$$

The annihilator subspaces $F^{\perp}, G^{\perp}$ of V, $V_{\mathbb{C}}$ respectively are defined by

$$F^{\perp} = \{v \in V : \omega[u, v] = 0 \text{ for all } u \in F\}$$

and

$$G^{\perp} = \{v \in V_{\mathbb{C}} : \omega[u,v] = 0 \textit{ for all } u \in G\}$$

respectively. Then the following results hold:

(i) $(F^{\perp})_{\mathbb{C}} = (F_{\mathbb{C}})^{\perp}$.

(ii) If $G = \overline{G}$ *then* $(G^{\perp})_{\mathbb{R}} = (G_{\mathbb{R}})^{\perp}$.

(iii) If $G = \overline{G}$ *then* $(G_{\mathbb{R}})_{\mathbb{C}} = G$.

(iv) Define $\tilde{F} = F/(F \cap F^{\perp}) = \{(u) : u \in F\}$ *where the rest of classes are defined by* $[u] = \{u + v : v \in F \cap F^{\perp}\}$. *Then* $\tilde{\omega}[(u),(u')] := \omega[u,u']$ *is well defined and* $(\tilde{F}, \tilde{\omega})$ *is a sympletic vector space.*

(v) if F *is co-isotropic* ($F^{\perp} \subset F$), $\pi : F \to \tilde{F} = F/F^{\perp}$ *the canonical projection,* P *a Lagrangian subspace in* V *then* $\tilde{P} := \pi(P \cap F)$ *is a Lagrangian subspace in* $(\tilde{F}, \tilde{\omega})$.

Proof:

(i):

$$p \in F^{\perp}$$

$$p_{\mathbb{C}} \in (F^{\perp})_{\mathbb{C}}, \quad p_{\mathbb{C}} = p_1 + ip_2, \quad p_1, p_2 \in F^{\perp}$$

now take

$$f_{\mathbb{C}} \in F_{\mathbb{C}}, \quad f_{\mathbb{C}} = f_1 + if_2, \quad f_1, f_2 \in F$$

$$\omega[p_{\mathbb{C}}, f_{\mathbb{C}}] = \omega[f_1 + if_2, p_1 + ip_2] = 0$$

so

$$p_{\mathbb{C}} \in (F_{\mathbb{C}})^{\perp}.$$

which implies

$$(F^{\perp})_{\mathbb{C}} \subset (F_{\mathbb{C}})^{\perp}$$

The other way around

$$f_{\mathbb{C}} \in F_{\mathbb{C}}, \quad f_{\mathbb{C}} = f_1 + if_2, \quad f_1, f_2 \in F$$

$$q \in (F_{\mathbb{C}})^{\perp}$$

means

$$\omega[f_{\mathbb{C}}, q] = 0 \quad \text{for all } f_{\mathbb{C}} \in F_{\mathbb{C}}$$

$$q \in V_{\mathbb{C}}, \quad q_{\mathbb{C}} = q_1 + iq_2, \quad q_1, q_2 \in V$$

$$\omega[q, f_{\mathbb{C}}] = 0$$

implies

$$\omega[q_1 + iq_2, f_1 + if_2] = \omega[q_1, f_1] - \omega[q_2, f_2] + i\left(\omega[q_2, f_1] + \omega[q_1, f_2]\right) \tag{K.-23}$$

set $f_1 = f_2$, then the real and imaginary parts are

$$\begin{aligned} \omega[q_1 - q_2, f_1] &= 0 \\ \omega[q_1 + q_2, f_1] &= 0 \end{aligned} \tag{K.-24}$$

Adding gives

$$\omega[q_1, f_1] = 0$$

which implies

$$q_1 \in F^{\perp}.$$

Thus

$$(F^\perp)_\mathbb{C} \subset (F_\mathbb{C})^\perp.$$

(ii): $G = \overline{G}$ implies if $u_1 + iu_2$ is in G then so is $u_1 - iu_2$.

$(G^\perp)_\mathbb{R}$ means

$$\{v \in V_\mathbb{C} : \omega[v, w] = 0, \quad \text{for all } w \in G\} \cap V$$

and $(G_\mathbb{R})^\perp$ is defined by

$$\{v \in V : \omega[v, w] = 0, \quad \text{for all } w \in G \cap V\}.$$

We first prove $(G_\mathbb{R})^\perp \subset (G^\perp)_\mathbb{R}$:

Let

$$g \in G$$

then

$$\overline{g} \in G$$

Let

$$f_\mathbb{C} \in F_\mathbb{C}, \quad f_\mathbb{C} = f_1 + if_2, \quad f_1, f_2 \in F$$

be in $G^\perp$, i.e.

$$f_\mathbb{C} \in G^\perp$$

then

$$f_1 \in (G^\perp)_\mathbb{R}.$$

Now as $G = \overline{G}$

$$\omega[f, g] = \omega[f, \overline{g}] = 0$$

or

$$\begin{aligned} \omega[f, g_1 + ig_2] &= 0 \\ \omega[f, g_1 - ig_2] &= 0 \end{aligned} \tag{K.-25}$$

Adding gives

$$\omega[f, g_1] = 0$$

which implies

$$\omega[f_1, g_1] = 0$$

therefore

$$f_1 \in (G_{\mathbb{R}})^{\perp} \tag{K.-25}$$

Which implies

$$(G_{\mathbb{R}})^{\perp} \subset (G^{\perp})_{\mathbb{R}}. \tag{K.-25}$$

The other way around.

$$f_1 \in (G^{\perp})_{\mathbb{R}}$$

(iii):

$$(G_{\mathbb{R}})_{\mathbb{C}} = (G \cap V)_{\mathbb{C}} = \{v_1 + iv_2 : v_1, v_2 \in G \cap V\}$$

Let

$$g_1, g_2 \in G_{\mathbb{R}}$$

(iv): It is well defined if $\omega[u_1, u_1'] = \omega[u_2, u_2']$ where u_1 and u_2 are anny two representatives of the same class and similarly for u_1' and u_2'. It follows that we can put $u_1 = u_2 + v$ and $u_1' = u_2' + v'$.

$$\begin{aligned}
\omega[u_1, u_1'] &= \omega[u_2 + v, u_2' + v'] \\
&= \omega[u_2, u_2'] + \omega[u_2, v'] + \omega[v, u_2'] + \omega[v, v'] \\
&= \omega[u_2, u_2'] + \omega[v, v'] \\
&= \omega[u_2, u_2']
\end{aligned}$$

where $\omega[u_2, v'] = \omega[v, u_2'] = 0$ as $u_2, u_2' \in F$ and $v, v' \in F \cap F^\perp \subset F^\perp$. We have $\omega[v, v'] = 0$ as $v, v' \in F \cap F^\perp$. We now prove that $(\tilde{F}, \tilde{\omega})$ is a symplectic vector space. Must show that $\tilde{\omega}$ is non-degenerate, i.e.

$$\tilde{\omega}[(u), (v)] = 0 \ \text{ for all } \ (v) \in \tilde{F} \ \text{ then } \ \in [u] = 0,$$

and closed.

(v): Note that in this case $F \cap F^\perp = F^\perp$. If $P \subset V$ is a Lagrangian subspace, then $\tilde{P} = \pi(P \cap F)$ is an isotropic subspace of $\tilde{F}$ i.e. $\tilde{P} \subset \tilde{P}^\perp$. We have to show $\tilde{P}^\perp \subset \tilde{P}$. Let $\tilde{u} \in \tilde{P}^\perp$. Then $\tilde{u} = \pi(u)$ for some $u \in F$ such that

$$\omega[u, v] = 0 \text{ for all } \ v \in F \cap P.$$

That is,

$$u \in (F \cap P)^\perp = F^\perp + P^\perp = F^\perp + P.$$

As $\pi(F^\perp + P) = \pi(P) = \tilde{P}$, we have $\pi(u) \in \tilde{P}$.

□

The following theorem is key to the classification of polarisations.

Theorem K.16.9 *Let $P \subset V_{\mathbb{C}}$ be a Lagrangian subspace and*

$$E := (P + \overline{P})_{\mathbb{R}}, \qquad D := (P \cap \overline{P})_{\mathbb{R}} \tag{K.-29}$$

(i) D is an isotropic ($D \subset D^\perp$) subspace of V.

(ii) E is a co-isotropic subspace of V and $E^\perp = D$.

(iii) $E_{\mathbb{C}} = P + \tilde{P}$, $D_{\mathbb{C}} = P \cap \tilde{P}$.

(iv) Let $\tilde{F} = E/D$. Then $\tilde{F}_{\mathbb{C}} = E_{\mathbb{C}}/D_{\mathbb{C}}$.

(v) If $\pi : E \to \tilde{F}$ *is the canonical projection then* $\tilde{P} := \pi(E_{\mathbb{C}} \cap P)$ *is Lagrangian in* $(\tilde{F}_{\mathbb{C}}, \omega)$.

(vi) $\tilde{P} \cap \overline{\tilde{P}} = \{0\}$.

Proof: We make use of the previous lemma, which is applicable since obviously $\overline{E} = E$, $\overline{D} = D$.

(i): $D^{\perp}$ is defined by

$$\{u \in V : \omega[u, v] = 0 \text{ for all } v \in D\}.$$

Consider

$$\omega[P \cap V, v] + \omega[\overline{P} \cap V, v] \text{ for } v \in D$$

As $D = (P \cap \overline{P}) \cap V$, $D \subset P$ and $D \subset \overline{P}$. It is easy to see that $D \subset D^{\perp}$ as $P = P^{\perp}$ and $\overline{P} = \overline{P}^{\perp}$.

(ii) We first prove $E^{\perp} = D$:

$$\begin{aligned}
E &= (P + \overline{P})_{\mathbb{R}} \\
&= (P + \overline{P}) \cap V \\
&= P \cap V + \overline{P} \cap V \\
&= P^{\perp} \cap V + \overline{P}^{\perp} \cap V \\
&= P_{\mathbb{R}}^{\perp} + \overline{P}_{\mathbb{R}}^{\perp} \\
&= (P_{\mathbb{R}} \cap \overline{P}_{\mathbb{R}})^{\perp} \\
&= ((P \cap \overline{P})_{\mathbb{R}})^{\perp} = D^{\perp}.
\end{aligned}$$

As $D \subset D^{\perp}$, the just proved equality implies $E^{\perp} \subset E$, or in other words, E is a co-isotropic subspace of V.

(iii) By (iii) of the previous lemma $E_{\mathbb{C}} = ((P + \tilde{P})_{\mathbb{R}})_{\mathbb{C}} = P + \tilde{P}$ and $D_{\mathbb{C}} = ((P \cap \tilde{P})_{\mathbb{R}})_{\mathbb{C}} = P \cap \tilde{P}$.

(iv) $\tilde{F}$ is defined by $F/F^{\perp} = \{[u] : u \in F\}$ where the equivalence classes are defined by $[u] = \{u + v : v \in F^{\perp}\}$. Consider an arbitrary element of $\tilde{F}_{\mathbb{C}}$: $[u_1] + i[u_2]$. Now $E_{\mathbb{C}}$ is $\{e_1 + ie_2 : e_1, e_2 \in E\}$ and $F_{\mathbb{C}}$ is $\{f_1 + if_2 : f_1, f_2 \in F\}$. It is fairly obvious that $\tilde{F}_{\mathbb{C}} = E_{\mathbb{C}}/F_{\mathbb{C}}$.

(v) E is a coisotropic subspace of V and

$$\pi : E \to \tilde{F} = E/D = E/E^{\perp}.$$

By (v) of the previous lemma, $\tilde{P}' = \pi(E \cap P)$ is Lagrangian subspace of $\tilde{F}$. Set $\tilde{P} = \pi(E_{\mathbb{C}} \cap P)$. Then $\tilde{P}$ is obviously a Lagrangian subspace of $\tilde{F}_{\mathbb{C}}$.

(vi) $\overline{\tilde{P}} = \overline{\pi(E_{\mathbb{C}} \cap P)} = \pi(\overline{E_{\mathbb{C}}} \cap \overline{P}) = \pi(E_{\mathbb{C}} \cap \overline{P})$.

$$\begin{aligned} \pi(E_{\mathbb{C}} \cap P) &= \pi((P + \overline{P}) \cap P) \\ &= \pi(P + P \cap \overline{P}) \\ &= \pi(P). \end{aligned}$$

So $\pi(E_{\mathbb{C}} \cap P) \cap \pi(E_{\mathbb{C}} \cap \overline{P}) = \pi(P) \cap \pi(\overline{P})$. As $\pi(D_{\mathbb{C}}) = \{0\}$, $\pi(P) \cap \pi(\overline{P}) = \{0\}$.

□

As $\tilde{P} \cap \overline{\tilde{P}} = \{0\}$, so $\tilde{P}$ determines a complex structure

Kahler polarisations are characterised by the condition $P \cap \overline{P} = \{0\}$. A Kahler manifold is a complex manifold with a compatible symplectic structure.

Definition The type of the Lagrangian subspace $P \subset V_{\mathbb{C}}$ is the pair of integers (r, s) just defined. Special types are:

(i) Kahler: $m = r + s$, that is $\dim(D) = 0$ so $P \cap \overline{P} = \{0\}$.

(ii) Positive: $m = r$, that is $s = 0$ and Kahler, hence $\omega = \tilde{\omega}$ and the associated Kahler metric $g[.,.] = \omega[.,.]$ is positive definite.

(iii) Non-negative: $s = 0$, that is, the Kahler metric $\tilde{g}$ on $\tilde{F}$ is positive definite, however, $\dim(D) > 0$ is possible in which case P contains a real subspace.

(iv) Real: $r = s = 0$, that is, $\dim(E) = \dim(D) = m$, hence $E = D$ and $P = \overline{P}$, so $P = L_{\mathbb{C}}$ where $L \subset V$ is Lagrangian.

□

After this preparation we can now generalise from sympletic vector spaces to sympletic manifolds $(\mathcal{M}, \omega)$. This will involve applying the previuos resluts to the indivdiual tangent spaces $T_p(\mathcal{M})$ of each $p \in \mathcal{M}$. In other words applying the results fibrewise.

Complex polarisations

We start by introducing a number of definitions.

Definition A complex distribution P on a real manifold $\mathcal{M}$ is an assignment of subspaces $p \mapsto P_p \subset (T_p(\mathcal{M}))_{\mathbb{C}}$ whose complex dimension k is constant and which is spanned by k complex vector fields in a neighbourhood of each point of $\mathcal{M}$.

$\square$

Definition A **complex polarisation** of a symplectic manifold $(\mathcal{M}, \omega)$ is a compex distribution P such that

(1) P_p is a Lagrangian subspace of $(T_p(\mathcal{M}))_{\mathbb{C}}$

and such that

(2) the type of P_p is constant (equivalently, the real dimension of $D_p = (P_p \cap \overline{P}_p) \cap T_p(\mathcal{M})$ is constant).

$\square$

Definition A complex distribution is called **integrable** provided that in a neighbourhood U of each point p of $\mathcal{M}$ there are smooth complex-valued functions $f_{k+1}, \ldots, f_{2m}$ with linearly independent differentials df_j, $j = k+1, \ldots, 2m$ such that $\overline{u}[f_j]$ for any vector field u tangential to P in U.

$\square$

Definition The complex distribution in the previous definition is said to be **strongly integrable** if in addition the real distribution $E_p = D_p^{\perp} = (P_p + \overline{P}_p) \cap T_p(\mathcal{M})$ is integrable.

$\square$

Definition A symplectic potential θ, $d\theta = \omega$ is said to be P-adapted provided that $i_{\overline{u}}\theta = 0$ for all u tangential to P. A polarisation is said to be **admissible** provided that local P-adapted symplectic potentials exist everywhere.

$\square$

Definition A polarisation is called **strongly admissible** if E is integrable and the spaces $\mathcal{M}/D$ and $\mathcal{M}/E$ are smooth Hausdorff manifolds.

$\square$

Naively, one would like to construct the quantum Hilbert space from sections of the prequantum line bundle covariantly constant (parellel) along P. This leads to proposing the following definition.

Definition A complex-valued function $\psi \in C^\infty(\mathcal{M})$ is said to be P-polarised provided that $\overline{u}[\psi] = 0$ for all u tangental to P. We use the notation $C^\infty_P(\mathcal{M})$ for such functions.

$\square$

Definition A polarisation P of a symplectic manifold $\mathcal{M}$ which satisfies $\overline{P} = P$ is the complexification of an integrable Lagrangian sub-bundle of $T(\mathcal{M})$ and is called a **real polarisation** of $\mathcal{M}$.

$\square$

The importance of the notation of a strongly integrable polarisation is demostrated by the following theorem:

Theorem K.16.10 *Let P be a strongly integrable polarisation of a symplectic manifold $(\mathcal{M}, \omega)$ of dimension $dim(\mathcal{M}) = 2(m' + \tilde{m}) = 2m$. Then in the neighbourhood of each point we find a system of coordinates $\{q^a, p_a, z^\alpha\}_{a=1}^{m'}{}_{\alpha=1}^{\tilde{m}}$ with q, p real z complex such that $P = span_{\mathbb{C}}\{\partial/\partial p_a, \partial/\partial z^\alpha\}$ and their is a real valued function $K(q, z, \overline{z}$ such that $\omega = d\theta$ where*

$$\theta = p_a dq^a - \frac{i}{2}(\partial_\alpha K)dz^\alpha + \frac{i}{2}(\partial_{\overline{\alpha}} K)dz^{\overline{\alpha}}. \tag{K.-39}$$

Proof:

$$\tilde{\omega}_q = i\partial_\alpha \partial_{\overline{\beta}} K_q(z, \overline{z}) dz^\alpha \wedge dz^{\overline{\beta}} \tag{K.-39}$$

for some Kahler scalar K_q.

The general form of ω is now given by

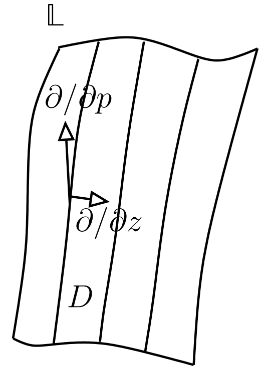

Figure K.6: Eleaf.

$$\begin{aligned}
\omega &= dp_a \wedge dq^a + \sigma_{ab} dq^a \wedge dq^b + \sigma_{a\alpha} dq^a \wedge dz^\alpha + \overline{\sigma_{a\alpha}} dq^a \wedge d\overline{z}^\alpha \\
&\quad + i(\partial_\alpha \partial_\beta K) dz^\alpha \wedge d\overline{z}^\beta
\end{aligned} \tag{K.-39}$$

We now compute

$$\begin{aligned}
d\omega &= d\left(\sigma_{ab} dq^a \wedge dq^b + \sigma_{a\alpha} dq^a \wedge dz^\alpha + \overline{\sigma_{a\alpha}} dq^a \wedge d\overline{z}^\alpha + \ i(\partial_\alpha \partial_\beta K) dz^\alpha \wedge d\overline{z}^\beta\right) \\
&=
\end{aligned} \tag{K.-39}$$

$$d(\sigma_{ab} dq^a \wedge dq^b) = \frac{1}{3!}(\partial_{q_c} \sigma_{ab} dq^c \wedge dq^a \wedge dq^b + \partial_{p_c} \sigma_{ab} dp^c \wedge dq^a \wedge dq^b + (\partial + \overline{\partial})\sigma_{ab} dq^a \wedge dq^b)$$

$$(\partial + \overline{\partial}) i(\partial_\alpha \partial_\beta K) dz^\alpha \wedge d\overline{z}^\beta = 0$$

$\partial_{[}$

$\square$

K.16.6 Quantisation

Let P be a strongly integrable polarisation of a sympletic manifold $(\mathcal{M}, \omega)$ and let B be a prequantum bundle over $\mathcal{M}$.

Definition A smooth section $s : \mathcal{M} \to B$ is said to be polarised if $\nabla_{\overline{X}} s = 0$ for every $X \in V_P(\mathcal{M})$. The space of polarised sections is denoted by S_P.

$\square$

We would like to qunatise $\mathcal{M}$ by replacing the prequantisation Hilbert space $\mathcal{H}'$ by the subspace of square integrable polarised sections by restricting the prequantum Hilbert space $\mathcal{H}'$ to (the completion of) S_P, i.e. $\mathcal{H}_P := \overline{\mathcal{H}' \cap S_P}$. This is problematic for the following reasons:

(A) Normalisation

For most polarisations it is not true that S_P contains any square integrable element.

(B) Operators

Obviously the operator $\hat{f}$ corresponding to $f \in C^\infty(\mathcal{M})$ is only admissible if it maps local polarised sections to polarised sections. Hence we must have $\nabla_{\overline{X}}(\hat{f}\psi)$ for every polarised section ψ. It is easy to see from

$$\nabla_{\overline{u}}(\hat{f}\psi) = \hat{f}(\nabla_{\overline{u}}\psi) - i\hbar\nabla_{[\overline{u},\chi_f]}\psi$$

that this is the case if and only if $\nabla_{[\overline{u},\chi_f]}\psi = 0$. Hence not every function can be realised as a prequantum operator on the Hilbert space $\mathcal{H}_P$. For example in the real case f must be of the form $v^a(q)p_a + u(q)$, in the Kahler case, χ_f must be a Killing vector.

K.17 Non-Commutative Field Theories and Thier Relation to Quantum Geometry

K.18 Summary

K.18.1 Algebraic quantum mechanics

representation theory

GNS

K.18.2 Algebraic quantum field theory

Can we reconstruct in detail the hamiltonian Hilbert space, as well as kinematic and dynamical operator of loop theory, starting from the covariant spin foam definition of the theory? [20]: This amounts to an extension to the diffeomorphism invariant context of the Wightmann and Osterwalder-Schrader reconstruction theorems.

K.18.3 Geometric quantization

Geometric quantisation provides a beautiful, geometric, general framework for the quantisation of a given sympletic manifold which is not necessarily of cotangent bundle type.

K.19 Biblioliographical notes

In this chapter I have relied on the following references:

Dana P. Williams notes on the SpectralTheorem for bounded normal operators

Introduction to topology and modern analysis G.F Simons.

K.20 Worked Exercises and Details

Cauchy-Schwartz inequality

To obtain the Cauchy-Schwartz inequality we first note that from (K.29) we have

$$0 \le E(A^*A) + |\lambda|^2 E(B^*B) + \lambda E(A^*B) + \overline{\lambda} E(B^*A) \tag{K.-39}$$

and then set $\lambda = tE(B^*A) = t\overline{E(A^*B)}$ for real t. So we see that

$$0 \le E(A^*A) + t^2 |E(A^*B)|^2 E(B^*B) + 2t|E(A^*B)|^2. \tag{K.-39}$$

If $0 < 2bt + at^2 + c$ for t real then the equation $at^2 + 2bt + c = 0$ can only be satisfied if the roots $t_\pm$ satisfying $t_\pm = a - b \pm \sqrt{b^2 - ca}$ are complex, that is $0 \le b^2 - ac$. By setting

$$\begin{aligned} a &= E(A^*A) \\ b &= |E(A^*B)|^2 \\ c &= |E(A^*B)|^2 E(B^*B) \end{aligned} \tag{K.-40}$$

we obtain the Cauchy-Schwarz inequality.

Adjoining a unit to a C^*- algebra.

If a C^*-algebra does not contain a unit one can be added as follows. Consider the vector space $\mathcal{A}' = \mathcal{A} \oplus \mathbb{C}$ with addition $(\lambda, A) + (\mu, B) = (\lambda + \mu, A + B)$, product defined by

$$(\lambda, A)(\mu, B) = (\lambda\mu, AB + \lambda B + \mu A),$$

with the involution $(\lambda, A)^* = (\lambda, A^*)$, and norm

$$\|(\lambda, A)\| = \sup_{\|B\|=1} \|AB + \lambda B\|.$$

Equipped this way $\mathcal{A}'$ is a C^*-algebra with a unit $(1, 0)$. The algebra $\mathcal{A}$ defines an ideal of $\mathcal{A}'$:

$$(0, A)(0, B) = (0, AB).$$

The triangle and product inequalities are easily verified. The triangle inequality:

$$\begin{aligned}
\|(\lambda, A) + (\mu, B)\| &= \|(\lambda + \mu, A + B)\| \\
&= \sup_{\|C\|=1} \|(A + B)C + (\lambda + \mu)C\| \\
&= \sup_{\|C\|=1} \|(AC + \lambda C) + (BC + \mu C)\| \\
&\leq \sup_{\|C\|=1} \|AC + \lambda C\| + \sup_{\|C\|=1} \|BC + \mu C\| \\
&= \|(\lambda, A)\| + \|(\mu, B)\| \qquad \text{(K.-43)}
\end{aligned}$$

To prove the product inequality we note that for any C^*-algebra we need $\|A\| = \sup_{\|B\|=1} \|AB\|$.

$$\begin{aligned}
\|(\lambda, A)(\mu, B)\| &= \|(\lambda\mu, AB + \lambda B + \mu A)\| \\
&= \sup_{\|C\|=1} \|(AB + \lambda B + \mu A)C + \lambda\mu C\| \\
&= \sup_{\|C\|=1} \|(A + \lambda)(B + \mu)C\| \\
&\leq \sup_{\|C\|=1} \|A + \lambda\| \sup_{\|C\|=1} \|BC + \mu C\| \\
&= \sup_{\|C\|=1} \|AC + \lambda C\| \sup_{\|C\|=1} \|BC + \mu C\| \\
&= \|(\lambda, A)\| \, \|(\mu, B)\| \qquad \text{(K.-47)}
\end{aligned}$$

We now check that $\|(\lambda, A)\| = 0$ implies $A = 0$ and $\lambda = 0$.

$$\|B - AB\| \leq \|B\| \, \|(-A, 1)\|$$

Thus if $\|(-A,1)\| = 0$ then $B = AB$ for all $B \in \mathcal{A}$. Considering the involution gives $B = BA^*$ for all $B \in \mathcal{A}$. In particular $A^* = AA^* = A$ and thus

$$B = AB = BA,$$

This means that A is an identity, which contradics the assumption.

Poincare invariance and the unique representation

The requirement of Poincare invariance and stability of theory unquiley picks out a preferred ground state.

C^*-algebras.

For self-adjoint operator x we know $r(x) = |x\|$ (Eq (K.6.2)). By definition of $r(x)$, $\|x\| = r(x) = \sup\{|\alpha + \beta| : \alpha + i\beta \in \sigma(x)\}$. So that $\|x\|^2 \geq \alpha^2 + \beta^2$ for $\alpha + i\beta \in \sigma(x)$. From this we have $\|x + i\lambda\|^2 \geq \alpha^2 + (\beta + \lambda)^2$. We can write

$$\alpha^2 + (\beta + \lambda)^2 \leq \|x + i\lambda\|^2 = \|(x + i\lambda)(x - i\lambda)\| = \|x^2 + \lambda^2\| \leq \|x^2\| + \lambda^2$$

$\alpha^2 + \beta^2 + 2\beta\lambda \leq \|x\|$

Existence of representations of C^*-algebras.

If we have not only a unital $\star$-algebra but in fact a C$\star$-algebra one can show that by the Hahn-Banach theorem that representations always exist, that every non-degenerate representation is a direct sum of cyclic representations and that every state is continuous so that the GNS representations are always bounded operators

a subspace $\mathcal{B}$ defined by

$$\mathcal{B} = \{\alpha \mathbf{1} + \beta A^* A \; ; \; \alpha, \beta \in \mathbb{C}\} \tag{K.-47}$$

A linear functional defined on $\mathcal{B}$

$$f(\alpha \mathbf{1} + \beta A^* A) = \alpha + \beta \|A\|^2 \tag{K.-47}$$

That $\lambda \cdot \mathbf{1} - A^* A$ non-invertable implies $(\alpha + \beta\lambda) \cdot \mathbf{1} - (\alpha \cdot \mathbf{1} + \beta A^* A)$ non-invertable. Now if T is normal ($TT^* = T^* T$) then by (K.6.2) the spectral radius of T is equal to $\|T\|$. One has by the spectral radius formula, (K.6.2), applied to the normal element $\alpha \mathbb{I} + \beta A^* A$,

$$\begin{aligned} |\alpha + \beta \|A\|^2| &\leq \sup\{|\alpha + \beta\lambda| : \lambda \in \sigma(A^* A)\} \\ &= \sup\{|\mu| : \mu \in \sigma(\alpha \mathbf{1} + \beta A^* A)\} \\ &= \|\alpha \mathbf{1} + \beta A^* A\| \end{aligned} \tag{K.-48}$$

Thus $\|f\| \leq 1$. but $f(\mathbf{1}) = 1$ and hence $\|f\| = 1 = f(\mathbf{1})$

□

Proof of proposition K.6.5.

a self-adjoint operator element A of $\mathcal{A}$, with $\|A\| = 1$, positive if and only if $\|(t\mathbf{1} - A)\| \leq t$ with $\|A\| \leq t$. In particular, we have for any $A \in \mathcal{A}$, we have that $\|A^* A\|\mathbf{1} - A^* A$ is positive. We see this is so as the condition

$$\|(t\mathbf{1} - \|A^* A\|\mathbf{1} + A^* A)\| \leq t$$

and

$$\| \|A^* A\|\mathbf{1} - A^* A\| \leq t$$

are met for the choice $t = \|A^* A\|$.

(i) $\Rightarrow$ (ii). If (i) holds then ω applied to the element $\|A^* A\|\mathbf{1} - A^* A$ gives

$$\omega(A^* A) \leq \|A^* A\| \, \omega(\mathbf{1})$$

By the Cauchy-Schwarz (K.28) we have

$$\begin{aligned} |\omega(A)| &= |\omega(A^* \mathbf{1})| \\ &\leq |\omega(A^* A)|^{1/2} \, \omega(\mathbf{1})^{1/2} \\ &\leq \|A^* A\|^{1/2} \, \omega(\mathbf{1}) \\ &= \|A\| \, \omega(\mathbf{1}) \end{aligned} \tag{K.-50}$$

So $|\omega(A)|$ is bounded and hence, because it is linear, is continuous. Also we have

$$\|\omega\| := \sup_{\|A\|=1} |\omega(A)| \leq \omega(\mathbf{1}),$$

from which this we easily see that $\|\omega\| = \omega(\mathbf{1})$.

(ii) $\Rightarrow$ (i). Supose ii) is satisfied. Assume $\omega(\mathbf{1}) = 1$. Let A be a self-adjoint element of $\mathcal{A}$. put $\omega(A) = \alpha + i\beta$ where α, β are real. For every $\lambda \in \mathbb{R}$ we have

$$\|A + i\lambda\mathbf{1}\|^2 = \|A^2 + \lambda^2\mathbb{I}\| = \|A\|^2 + \lambda^2.$$

Thus we have

$$\begin{aligned} \alpha^2 + \beta^2 + 2\lambda\beta + \lambda^2 &\leq |\alpha^2 + i(\beta + \lambda)|^2 \\ &= |\omega(A + i\lambda\mathbb{I})| \\ &\leq \|A\|^2 + \lambda^2 \end{aligned} \tag{K.-52}$$

If $|\lambda|$ is made large enough, we would have $\geq \|A\|$, so that that $\beta = 0$ and $\omega(A)$ is real. Consider now A positive. We have

$$\begin{aligned} |\ \|A\| - \omega(A)| &= |\omega(\|A\|\mathbf{1} - A)| \\ &\leq \|\ \|A\|\mathbf{1} - A\| \leq 1. \end{aligned} \tag{K.-53}$$

Thus $\omega(A)$ is positive.

□

Representations of C^*-algebras.

Let A, B, C be elements of a C^*-algebra $\mathcal{A}$. The following is true:

(a) if $A \geq B \geq 0$ then $\|A\| \geq \|B\|$;

(b) if $A \geq 0$ then $A\|A\| \geq A^2$;

We first need the following lemma.

Lemma K.20.1 *If $-M\mathbb{I} \leq A \leq M\mathbb{I}$ then*

$$\|A\| \leq M. \tag{K.-53}$$

Proof. First we prove that if $|a(x)| \leq Mb(x)$ then

$$|a(x,y)| \leq Mb(x)b(y). \tag{K.-53}$$

If $a(x,y)$ is complex then $a(x,y) = e^{i\theta}|a(x,y)|$, which can be expressed as $|a(x,y)| = a(e^{-i\theta}x, y)$. Hence, it suffices to prove (K.20) for $a(x,y)$ real.

$$\begin{aligned} 4|a(x,y)| &\leq |a(x+y)| + |a(x-y)| \\ &\leq M(b(x+y) + b(x-y)) \\ &= 2M(b(x) + b(y)) \end{aligned} \tag{K.-54}$$

as $a(\alpha x, y/\alpha) = a(x,y)$ for real α we have

$$2|a(x,y)| \leq M\Big(\alpha^2 b(x) + \frac{b(y)}{\alpha^2}\Big) \tag{K.-54}$$

Assuming $b(x) \neq 0$ and $b(y) \neq 0$ then putting $\alpha^2 = b(y)/b(x)$ we get

$$|a(x,y)| \leq Mb(x)^{1/2}b(y)^{1/2} \tag{K.-54}$$

$$| < Ax, y > | \leq M\|x\|^2 \tag{K.-54}$$

Hence, by the above,

$$| < Ax, y > | \leq M\|x\| \, \|y\|$$

takeing $y = Ax$ gives $\|Ax\|^2 \leq M\|x\| \, \|Ax\|$ implying (K.20.1).

□

Proof of (a). (K.20.1) gives $A \leq \|A\|\mathbb{I}$ and hence $0 \geq B \geq \|A\|\mathbb{I}$. A second application of (K.20.1) gives $\|B\| \leq \|A\|$.

Proof of (b). As $\sigma(A) \subseteq [0, \|A\|]$ for a positive element, $\sigma(A - \|A\|\mathbb{I}/2) \subseteq [-\|A\|/2, \|A\|/2]$ and hence $\sigma((A - \|A\|\mathbb{I}/2)^2) \subseteq [0, \|A\|/4]$. Thus

$$0 \leq \left(A - \frac{\|A\|\mathbb{1}}{2}\right)^2 \leq \frac{\|A\|^2\mathbb{1}}{4},$$

multiplying out the bracket we see this is equivalent to $0 \leq A^2 \leq \|A\|A$.

Ireduciability of representations of C^*-algebras.

$$A \geq 0 \quad \Rightarrow \quad \pi(A) \geq 0 \tag{K.-54}$$

π is continuous

$$\|\pi(A)\| \leq \|A\| \tag{K.-54}$$

for all $A \in \mathcal{A}$.

Proof of proposition K.6.4

(i) $\Rightarrow$ (ii).

For a general A we have

$$\|\pi(A)\|^2 = \|\pi(A^*A)\| \leq \|(A^*A)\| = \|A\|^2.$$

(i) $\Rightarrow$ (ii): If ker$= \{0\}$ then the map π is invertible and so we can define a morphism π^{-1} from the range of π into $\mathcal{A}$ by $\pi^{-1}(\pi(A)) = A$,

$$\begin{gathered}
\pi^{-1}(\alpha\pi(A) + \beta\pi(B)) = \pi^{-1}(\pi(\alpha A + \beta B)) = \alpha A + \beta B, \\
\pi^{-1}(\pi(A)\pi(B)) = \pi^{-1}(\pi(AB)) = AB \\
\pi^{-1}(\pi(A)^*) = \pi^{-1}(\pi(A^*)) = A^*
\end{gathered}$$

For a morphism π, $\|\pi(A)\| \leq \|A\|$. This result also applies to π^{-1} now,

$$\|A\| = \|\pi^{-1}(\pi(A))\| \leq \|\pi(A)\|$$

hence $\|\pi(A)\| = \|A\|$.

(ii) $\Rightarrow$ (iii):

(iii) $\Rightarrow$ (i): We prove this by presuming (i) does not hold and then show this contradicts (iii). If (i) is false then there is a $B \in ker\pi$ with $B \neq 0$ and $\pi(B^*B) = 0$. But $\|B^*B\| \geq 0$ and as $\|B^*B\| = \|B\|^2$ one has $B^*B > 0$.

Decomposition into Sub-Representations

invariant subspace if

$\pi(\mathcal{A})\mathcal{H}_1 \subseteq \mathcal{H}_1$

$$P_{\mathcal{H}_1}\pi(\mathcal{A})P_{\mathcal{H}_1} = \pi(\mathcal{A})P_{\mathcal{H}_1}$$

$$\begin{aligned}(P_{\mathcal{H}_1}\pi(\mathcal{A})P_{\mathcal{H}_1})^* &= P^*_{\mathcal{H}_1}\pi(\mathcal{A})^*P^*_{\mathcal{H}_1} = P_{\mathcal{H}_1}\pi(\mathcal{A})P_{\mathcal{H}_1} \\ &= (\pi(\mathcal{A})P_{\mathcal{H}_1})^* \\ &= P_{\mathcal{H}_1}\pi(\mathcal{A})\end{aligned} \qquad \text{(K.-55)}$$

$$\pi(\mathcal{A})P_{\mathcal{H}_1} = P_{\mathcal{H}_1}\pi(\mathcal{A})$$

$$\pi_1(A) = P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1} \qquad \text{(K.-55)}$$

$$\begin{aligned}\pi_1(A)\pi_1(B) &= (P_{\mathcal{H}_1}\pi(A)P_{\mathcal{H}_1})(P_{\mathcal{H}_1}\pi(B)P_{\mathcal{H}_1}) \\ &= (P_{\mathcal{H}_1}\pi(A))(\pi(B)P_{\mathcal{H}_1}) \\ &= P_{\mathcal{H}_1}\pi(AB)P_{\mathcal{H}_1} \\ &= \pi_1(AB)\end{aligned} \qquad \text{(K.-57)}$$

Every element of a C^*-algebra can be decomposed as a linear combination of four unitary elements.

Every element A of a C^*-algebra $\mathcal{A}$ can be written as a linear combination of unitaries in $\mathcal{A}$. As the conjugate of each element is also in the algebra, we can construct self-adjoint elements $A_+ = (A + A^*)/2$ and $A_- = (A - A^*)/2i$. But any self-adjoint element B can be written as $(U_+ + U_-)/2$ where

$$U_\pm = B \pm i\sqrt{\mathbb{I} - B^2}$$

Thus, for any $A \in \mathcal{A}$ we can write

$$
\begin{aligned}
A &= \frac{(A+A^*)}{2} + i\frac{(A-A^*)}{2i} = A^{(1)} + iA^{(2)} \\
&= \frac{(U_+^{(1)} + U_-^{(1)})}{2} + i\frac{(U_+^{(2)} + U_-^{(2)})}{2}. \qquad \text{(K.-57)}
\end{aligned}
$$

von Neumann algebras.

Check that $\tilde{A} = (Ax_i) \in \mathcal{H}(\mathcal{K})$ if $A \in \mathcal{H}(\mathcal{B})$

$$
\begin{aligned}
\|\tilde{A}\| &:= \sup_{\sum_i^\infty \|x_i\|=1} \sum_j^\infty \|A_j x_j\| \\
&\le \sup_{\sum_i^\infty \|x_i\|=1} \|A\| \sum_j^\infty \|x_j\| \qquad \text{as } A = A_j \\
&= \|A\| < \infty. \qquad \text{(K.-58)}
\end{aligned}
$$

Disjoint representations of C^*-algebras.

$(\pi_1 \oplus \pi_2)(\mathcal{A})'$ is the commutant of $(\pi_1 \oplus \pi_2)(\mathcal{A})$ taken in $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$

We may write B as

$$
B = \begin{pmatrix} X & S \\ T & Y \end{pmatrix} \in (\pi_1 \oplus \pi_2)(\mathcal{A})' \qquad \text{(K.-58)}
$$

for suitable $X \in \mathcal{B}(\mathcal{H}_1)$, $Y \in \mathcal{B}(\mathcal{H}_2)$ and bounded linear operators $S : \mathcal{H}_2 \to \mathcal{H}_1$ and $T : \mathcal{H}_1 \to \mathcal{H}_2$.

As $\pi_1(\mathcal{A}) \oplus \pi_2(\mathcal{A}) \subseteq (\pi_1 \oplus \pi_2)(\mathcal{A})$ we must have

$$
\begin{pmatrix} X & S \\ T & Y \end{pmatrix} \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} = \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} \begin{pmatrix} X & S \\ T & Y \end{pmatrix} \qquad \text{(K.-58)}
$$

so that

$$
\begin{pmatrix} X\pi_1(A) & S\pi_2(A) \\ T\pi_1(A) & Y\pi_2(A) \end{pmatrix} = \begin{pmatrix} \pi_1(A)X & \pi_2(A)S \\ \pi_1(A)T & \pi_2(A)Y \end{pmatrix} \qquad \text{(K.-58)}
$$

We must have $X \in \pi_1(\mathcal{A})'$, $Y \in \pi_2(\mathcal{A})'$, $T\pi_1(A) = \pi_2(A)T$, and $S\pi_2(A) = \pi_1(A)S$ for all $A \in \mathcal{A}$.

Next.

Let $\tilde{S}\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H})$ and write π for $\pi_1 \oplus \pi_2$, we see

$$\begin{aligned} \tilde{S}\pi(A) &= \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix} = \begin{pmatrix} 0 & S\pi_2(A) \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \pi_1(A)S \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \pi_1(A) & 0 \\ 0 & \pi_2(A) \end{pmatrix}\begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \\ &= \pi(A)\tilde{S} \end{aligned}$$

for all $A \in \mathcal{A}$. In particular, for any unitary element $U \in \mathcal{A}$, $\pi(U)$ is unitary and commutes with $\tilde{S}$. It follows that $\pi(U)$ also commutes with the polar isometry W, where $W|\tilde{S}|$ is the polar decomposition of $\tilde{S}$.

Operators and complex analysis.

(1)

From the "Taylor expansion" for $(z - A)^{-1}$ we have

$$\oint_C z^n (z-A)^{-1} dz = \sum_{k=1}^{\infty} A^{k-1} \oint_C z^n dz = A^n \oint_C \frac{dz}{z} = 2\pi i A^n. \tag{K.-61}$$

Thus

$$A^n = \frac{1}{2\pi i} \oint_C z^n (z-A)^{-1} dz. \tag{K.-61}$$

The partial fraction formula for constants $\mu, \lambda, \in \mathbb{C}$

$$\frac{1}{\mu - y} - \frac{1}{\nu - y} = (\mu - \nu)\frac{1}{\mu - y}\frac{1}{\nu - y}, \tag{K.-61}$$

where the variable $y \in \mathbb{C}$, has an operator (or matrix if you like) version. Let $x \in X$ and set

$$u = (\lambda - A)^{-1} x.$$

Thus $x = (\lambda - A)u$ and $(\mu - A)u = x + (\mu - \lambda)u$. Hence

$$u = (\mu - A)^{-1}x + (\mu - \lambda)(\mu - A)^{-1}u.$$

Substituting for u, we get

$$(\lambda - A)^{-1} - (\mu - A)^{-1} = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}. \tag{K.-61}$$

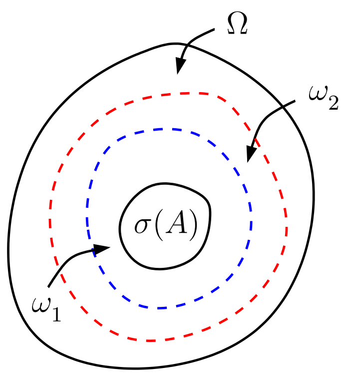

Figure K.7: OperProdsComp.

$$\begin{aligned} f(A)g(A) &= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)(z-A)^{-1}\oint_{\partial\omega_2} g(\zeta)(\zeta - A)^{-1}dzd\zeta \\ &= -\frac{1}{4\pi}\oint_{\partial\omega_1} f(z)\oint_{\partial\omega_2} g(\zeta)\frac{(z-A)^{-1} - (\zeta - A)^{-1}}{\zeta - z}dzd\zeta \end{aligned} \tag{K.-61}$$

From fig (K.20) we easily see

$$\oint_{\partial\omega_2} \frac{g(\zeta)}{\zeta - z}d\zeta = 2\pi i g(z) \tag{K.-61}$$

for any $z \in \partial\omega_1$.

$$\oint_{\partial\omega_1} \frac{f(z)}{z - \zeta}dz = 0 \tag{K.-61}$$

for any $\zeta \in \partial\omega_2$. Hence from (K.-61)

$$f(A)g(A) = \frac{1}{2\pi i}\oint_{\partial\omega_1} f(z)g(z)(z-A)^{-1}dz = h(A). \tag{K.-61}$$

□

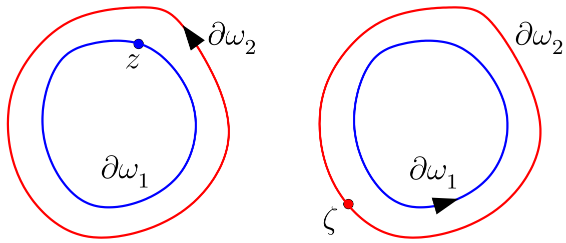

Figure K.8: OperProd2

(2)

the Taylor expansion around the point λ rather than μ,

$$\frac{1}{\lambda - x} = \frac{1}{\mu - x}\frac{1}{1 - \frac{\mu-\lambda}{\mu-x}} = \sum_{n=1}^{\infty}\frac{(\mu-\lambda)^{n-1}}{(\mu-x)^n} \tag{K.-60}$$

where $y \in \mathbb{C}$ (for $|\mu-\lambda|/|\mu-x| < 1$), has an operator (or matrix if you like) version: if $|\lambda-\mu|\cdot\|(\mu-A)^{-1}\| < 1$, then

$$(\lambda - A)^{-1} = \sum_{k=1}^{\infty}(\lambda-\mu)(\mu-A)^{-1} \tag{K.-60}$$

Note from (K.20) that $(\lambda-A)^{-1}$ and $(\mu-A)^{-1}$ commute. Substituting (K.20) it into itself one gets

$$\begin{aligned}(\lambda - A)^{-1} &= (\mu-A)^{-1} + (\mu-\lambda)(\lambda-A)^{-1}(\mu-A)^{-1} \\ &= (\mu-A)^{-1} + (\mu-\lambda)(\mu-A)^{-2} + (\mu-\lambda)^2(\lambda-A)^{-1}(\mu-A)^{-2}\end{aligned} \tag{K.-60}$$

Continuing to substitute (K.20) in this way one gets

$$(\lambda - A)^{-1} = \sum_{k=1}^{n}(\mu-\lambda)^{k-1}(\mu-A)^{-k} + (\mu-\lambda)^n(\lambda-A)^{-1}(\mu-A)^{-n} \tag{K.-60}$$

Since

$$\lim_{n\to\infty}\|(\mu-\lambda)^n(\lambda-A)^{-1}(\mu-A)^{-n}\| \le \lim_{n\to\infty}|\mu-\lambda|^n\|(\lambda-A)^{-1}\|\|(\mu-A)^{-1}\|^n = 0 \tag{K.-60}$$

completes the proof.