Appendix L

The Loop Representation

introducing heavy mathematical tools, often unfamiliar to the average physicist

to achieve certainty is to work at a high level of mathematical precision.

we search for a mathematical precision is that in quantum gravity

in the absence of any experimental observation at least for the moment by having a consistent theory

Ashtekar and Isham the representation of the loop algebra by using $C^{\star}-$algebra representation theory: $\overline{\mathcal{A}/\mathcal{G}}$ is the Gelfand spectrum (complex valued, bounded functions on a compact Hausdorff space) on the abelian part of the loop algebra.

$\mathcal{H}$ can be constructed as the projective limit of the projective family of the Hilbert spaces $\mathcal{H}_{\gamma}$, associated to a graph γ in $\mathcal{M}$.

L.1 Loop Representation

quantizing field theories requires one to smear fields, i.e. to integrate them over regions in order to obtain a well-defined algebra without $\delta-$functions. Usually this is done by integrating both configuration and momentum variables over three-dimensional regions, which requires an integration measure.

There is now a different smearing available which does not require a background metric. Instead of using three-dimensional regions we integrate the connection along one-dimensional curves e and exponentiate in a path-ordered manner, resulting in holonomies.

densitized vector fields can naturally be integrated over 2-dimensional surfaces, resulting in fluxes

$$F_S(E) = \int_S \tau^i E_i^a n_a d^2y \tag{L.0}$$

with the co-normal n_a to the surface.

The Poisson algebra of holonomies and fluxes is now well-defined and one can look for representations on a Hilbert space. We also require diffeomorphism group on the representation by moving edges and surfaces in space.

Spatial geometry can be obtained from fluxes representing the densitized triad. Since these are now momenta, they are represented by derivative operators with respect to values of connections on the flux surface. States as constructed above depend on the connection only along edges of graphs such that the flux operator is non-zero only if there are intersection points between its surface and the graph in the state it acts on

L.2 Algebraic Quantization of Loop Representation

$$W_k(x) = \exp(ikx) \tag{L.0}$$

$$\Psi(k) := \int dx W_k^*(x)\Psi(x) \tag{L.0}$$

$$\begin{aligned}
\{\mathcal{T}^0(k_1), \{\mathcal{T}^0(k_2)\} &= 0, \\
\{\mathcal{T}^1(k_1), \{\mathcal{T}^0(k_2)\} &= -ik_1\mathcal{T}^0(k_1+k_2), \\
\{\mathcal{T}^1(k_1), \{\mathcal{T}^1(k_2)\} &= i(k_1-k_2)\mathcal{T}^0(k_1+k_2),
\end{aligned} \tag{L.-2}$$

Its action on a wavefunction is to affect a translation

$$\hat{\mathcal{T}}^0\Psi(k) = \int dx e^{-ikx} e^{ik_1\hat{x}}\Psi(x) = \Psi(k-k_1) \tag{L.-2}$$

Simple example of a free algebra

for any finite order polynomial can be generated by multiplication and addition of the elementary variables

$$a \;\text{ and }\; x \tag{L.-2}$$

the associative algebra generated by finite sums and products of these elementary operators

$$a_0 + a_1 x + \cdots + a_k x^k + \cdots + a_N x^N \tag{L.-2}$$

L.2.1 Loop Algebra for $U(1)$

Let $\mathcal{A}$ be the space of smooth $U(1)$ connections whose cartesian components are functions of rapid decrease at infinity.

L.2.2 Mathematical Description

Let $\mathcal{L}_{x_0}$ denote the collection (or space) of oriented loops on R^3 with basepoint x_0. Them being oriented means there is a certain sense of flow around the loop. We can form a composition between loops as illustrated in fig(L.2.2) We denote the composition between two loops α and β as $\alpha \circ \beta$.

$$(a \cdot b)(t) := \begin{cases} a(2t), & 0 \leq t \leq 1/2 \\ b(2t-1) & 1/2 \leq t \leq 1 \end{cases} \tag{L.-2}$$

For any path α, we denote by α^{-1} the *inverse loop* formed by transversing α in the opposite direction.

can be found in section C.16.4 and in the maths glossary.

The set of all x_0-based loops in X is a semi-group with identity (a monoid?)

The collection of all equivalence classes of paths in a topological space X is called the *fundamental groupoid*, denoted $\Gamma(X)$.

The retracing identity

$$\mathcal{T} = \mathcal{T}[\alpha \cdot l \cdot l^{-1}]. \tag{L.-2}$$

Here l is a curve with one end on α, and $\alpha \cdot l \cdot l^{-1}$ is the loop obtained by going around α, along the curve and back along the curve α. And

$$\lim_{\gamma \to 0} \mathcal{T} = 1, \tag{L.-2}$$

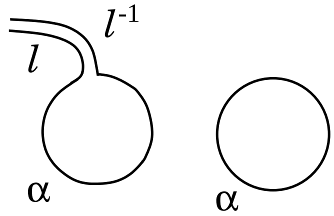


Figure L.1: Bridge.

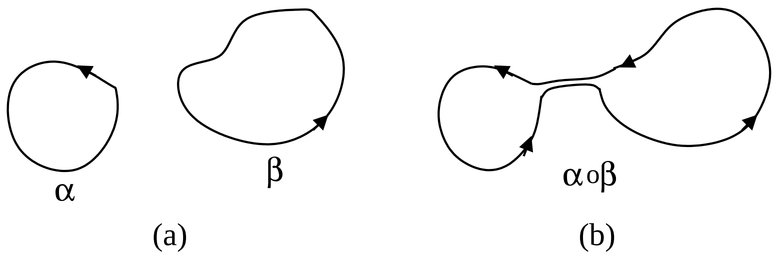


Figure L.2: Definition of the composition of two loop α and β (both in $\mathcal{L}_{x_0}$) - it is denoted as $\alpha \circ \beta$.

where $\gamma \to 0$ means γ shrinks to the loop to a point.

If we define multiplication of loops as the composition, the elements of $\mathcal{L}_{x_0}$ form a group under under this multiplication. The identity is just the loop contracted to a point at x_0. The inverse of the loop α is the same loop with opposite orientation, which we denote as α^{-1}. Associativity $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$ is demonstrated if fig().

i) The composition of parametrized curves is not associative, since the curves $(c_3 \circ c_2) \circ c_1$ and $c_3 \circ (c_2 \circ c_1)$ are related by a reparametrization:

$$c_3 \circ (c_2 \circ c_1) = c_3 \circ \left\{ \begin{array}{c} c_2(2t), \quad 0 \le t \le 1/2 \\ c_1(2t-1), \quad 1/2 \le t \le 1 \end{array} \right. = \left\{ \begin{array}{c} c_3(2t), \quad 0 \le t \le 1/2 \\ c_2(4t-2), \quad 1/2 \le t \le 3/4 \\ c_1(4t-3), \quad 3/4 \le t \le 1 \end{array} \right. \tag{L.-2}$$

$$(c_3 \circ c_2) \circ c_1 = \left\{ \begin{array}{c} c_3(2t), \quad 0 \le t \le 1/2 \\ c_2(2t-1), \quad 1/2 \le t \le 1 \end{array} \right. \circ c_1 = \left\{ \begin{array}{c} c_3(4t), \quad 0 \le t \le 1/4 \\ c_2(4t-1), \quad 1/4 \le t \le 1/2 \\ c_1(2t-1), \quad 1/2 \le t \le 1 \end{array} \right. \tag{L.-2}$$

Definition The set of equivalence classes of curves is denoted by $\mathcal{P}$. In order to distinguish the equivalence classes from their representative curves we will refer to them as **paths**.

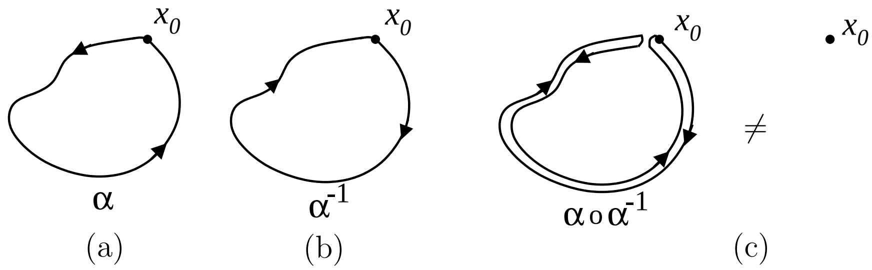


Figure L.3: inverse α^{-1} is a bit of a misnomer, $\alpha \circ \alpha^{-1} \neq o$.

Note that the definition of curves and paths is somewhat opposite to the defintions given in the appandix on the Hawking-Penrose singularity theorems.

The advantage of dealing with paths $\mathcal{P}$ rather than curves is that we now have almost a group structure since composition becomes associative and the path $p_c \circ p_c^{-1} = b(p_c)$ is trivial (stays at the beginning point). However, we still do not have a natural identity element in $\mathcal{P}$ and not all of its elements can be composed. The natural structure behind this is that of a **groupoid**.

L.2.3 Loops of Connections

Given a loop $\alpha \in \mathcal{L}_{x_0}$, the holonomy of $A_\mu(x)$ around α is $H_\alpha(A) := \exp(i \oint_\alpha A_\mu dx^\mu)$.

$$H_{\alpha\circ\beta}(A) = \tag{L.-2}$$

If two loops α and β have the same holonomy

$$H_\alpha(A) = H_\beta(A) \tag{L.-2}$$

for every $A_\mu(x)$ then we say they belong to the same holonomy loop class or just hoop. We denote such a class as $\tilde{\alpha}$.

$\mathcal{FA}$ free algebra

$$\sum_{i=1}^{N} a_i \alpha_i \in K \text{ if and only if } \sum_{i=1}^{N} a_i H_{\alpha_i}(A) = 0 \tag{L.-2}$$

The function $H_{\alpha_i}(A)$ is a homorphism. Forms an ideal

$$H_{\alpha_i}(A)\left(\left(\sum_{i=1}^{N} b_i\beta_i\right)\left(\sum_{i=1}^{N} a_i\alpha_i\right)\right) = H_{\alpha_i}(A)\left(\sum_{i=1}^{N} b_i\beta_i\right) \times 0 = 0 \ \in \qquad \text{(L.-2)}$$

α and β have a common point. Here # indicates joining of two loop at an intersection.

$$\mathcal{T}^0[\gamma] = U_\gamma(s) = \mathcal{P}e^{\oint_\gamma A}. \qquad \text{(L.-2)}$$

$$\{\mathcal{T}^0[\alpha], \mathcal{T}^0[\beta]\} = 0 \qquad \text{(L.-2)}$$

$$\mathcal{T}^a[\alpha](\alpha(s)) := \frac{1}{2}[U_\alpha(t,s)\tilde{E}^a(\alpha(s))U_\alpha(s,t)\tilde{E}^b(\alpha(t)) \qquad \text{(L.-2)}$$

$$\{\mathcal{T}^a[\alpha], \mathcal{T}^0[\beta]\} = \frac{1}{2i}\int dt\delta^3(\gamma s, \eta(t))\dot{\eta}(t)[\mathcal{T}^0[\gamma\#\eta] - [\mathcal{T}^0[\gamma\#\eta^{-1}]]. \qquad \text{(L.-2)}$$

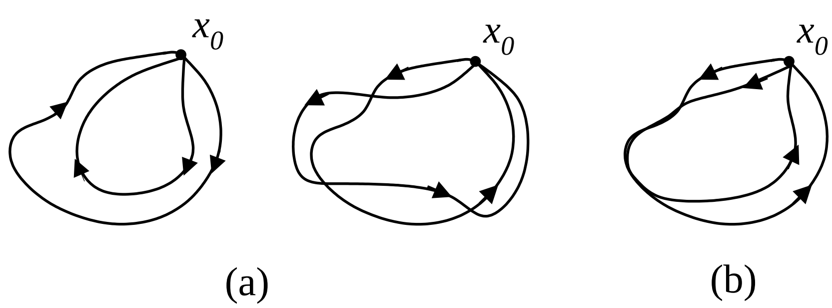


Figure L.4: (a) Two examples of strongly independent loops in $\mathcal{L}_{x_0}$. (b) An example of two strongly dependent loops in $\mathcal{L}_{x_0}$ - they have a segment in common.

$$\left(\hat{\mathcal{T}}^0[\alpha]\Psi\right)[\gamma] \equiv \Psi[\alpha \cup \gamma] \qquad \text{(L.-2)}$$

$\cup$ stands for union in the set of loops.

$$(\hat{\mathcal{T}}^a[\alpha](s))[\gamma] \equiv \hbar c \int dt\delta^3(\alpha(s), \gamma(t))\dot{\gamma}^a(t)[\Psi[\gamma \circ \alpha] - [\gamma \circ \alpha^{-1}] \qquad \text{(L.-2)}$$

can be found in section **??** and in the maths glossary.

The collection of all equivalence classes of paths in a topological space X is called the *fundamental groupoid*, denoted $\Gamma(X)$.

1. A finite set $\{e_1, \ldots, e_N\}$ of edges is said to be independent if the edges e_i can only intersect each other at their sources $s(e_i)$ or targets $t(e_i)$.

2. A finite graph is a collection of a finite set $\{e_1, \ldots, e_N\}$ of independent edges and their vertices. We denote by $E(\gamma)$ and $V(\gamma)$ respectively as sets of independent edges and vertices of a given finite graph γ. N_γ is the number of elements in $E(\gamma)$.

Details: loop algebra

$$T^a[\gamma](s) := \sqrt{2}\mathrm{tr}U_\gamma(s)^{AB}\tilde{\sigma}^a_{AB}(\gamma(s)) \tag{L.-2}$$

$$\begin{aligned} i\{T^a[\gamma](s), T^b[\eta](t)\} &= 2U_\gamma(s)^{AB}\{\tilde{\sigma}^a_{AB}(\gamma(s)), U_\eta(t)^{CD}\}\tilde{\sigma}^b_{CD}(\eta(t)) \\ &+ 2\tilde{\sigma}^a_{AB}(\gamma(s))\{U_\gamma(s)^{AB}, \tilde{\sigma}^b_{CD}(\eta(t))\}U_\eta(t)^{CD} \end{aligned} \tag{L.-2}$$

$$\begin{aligned} &= U_\gamma(s)^{AB} - i\sqrt{2}\int^t du\delta^3(\eta(u),\gamma(s))\dot{\eta}^a(u)U_\eta(0,u)^C{}_{(A}U_\eta(u,t)_{B)}^D\tilde{\sigma}^b_{CD}(\eta(t)) \\ &+ i\sqrt{2}\tilde{\sigma}^a_{AB}(\gamma(s))\int^s du\delta^3(\gamma(u),\eta(t)\dot{\gamma}^b(u)U_\gamma(0,u)^C{}_{(A}U_\gamma(u,s)_{B)}^D \\ &= \sqrt{2}\Delta^a[\gamma,\eta]U_\gamma(s)^{AB}U_\eta(0,u)^C{}_{(A}U_\eta(u,t)_{B)}^D\tilde{\sigma}^b_{CD}(\eta(t)) \\ &+ \sqrt{2}\Delta^b[\eta,\gamma]\tilde{\sigma}^a_{AB}(\gamma(s))U_\gamma(0,u)^A{}_{(C}U_\gamma(t,u)_{D)}^B U_\eta(t)^{CD} \end{aligned} \tag{L.-4}$$

$$\begin{aligned} \sqrt{2}\Delta[\gamma,\eta] \quad & \Big[U_\gamma(s)^{AB}U_\eta(u,t)_B{}^D\tilde{\sigma}^b_{CD}(\eta(t))U_\eta(t,u)^C{}_A \\ + \quad & U_\gamma(s)^{AB}U_\eta(u,t)_A{}^D\tilde{\sigma}^b_{CD}(\eta(t))U_\eta(t,u)^C{}_B\Big] \end{aligned} \tag{L.-4}$$

$$\{T^a[\gamma](s), T^b[\eta](t)\} = i\Delta^b[\eta,\gamma](t)\; T^a \tag{L.-4}$$

$$U_\gamma(s)_{AB} = -U_{\gamma^{-1}}(s)_{BA} \tag{L.-4}$$

$$U^A{}_B(0,u)U^B{}_C(u,s) = U^A{}_C(0,s) \tag{L.-4}$$

$$\Delta^a[\gamma,\eta](s) \equiv \frac{1}{2}\int dt\; \delta^3(\gamma(s),\eta(t))\; \dot{\eta}^a(t) \tag{L.-4}$$

$$\sqrt{2}\{\tilde{\sigma}^a_{AB}(x), U_\gamma^{CD}(0,s)\} = -i\int_0^s du\delta^3(\gamma(u),x)\;\dot{\gamma}^a(u)\; U_\gamma(0,u)^C{}_{(A}U_\gamma(u,s)_{B)}^D \tag{L.-4}$$

L.2.4 Differentiability Classes of Manifolds and Loops

See Chapter 3

L.2.5 Loop Space

$\mathcal{L}_\Sigma$ denotes the space of parametrize, differntiable, loops in Σ, which are the maps $\gamma : s \to \Sigma$. We also include in $\mathcal{L}_\Sigma$ those loops that with nowhere vanishing tangent vector, $\dot{\gamma}^a(s)$.

there is a subset of Diff(Σ) which leaves the curve γ invariant, and only reparamertizes it. The infinitesimal elements of this subset are the vector fields on Σ that are tangent to γ. Globally, one may show that the subset is thw diffeomorphism group of the complement of γ, that is, $\Sigma - \gamma$.

Exercise: Prove loop space is a differential manifold.

Proof:

L.2.6 Regularization of Holonomies

The Poison-brackets among the holonomies and the fluxes can be calculated by regularizing the edges and surfaces in three dimensions and then taking the limit of a family of functions, theat converge exactly to the holonomy along the particular edge and the flux through the particular surface.

$$H_\alpha(A) = \exp i \int_{R^3} X^\mu_\gamma(x) A_\mu(x) d^3x \tag{L.-4}$$

$$X^\mu_\gamma(x) := \oint_\gamma ds \delta^3(\vec{\gamma}(s), \vec{x}) \dot{\gamma}^\mu \tag{L.-4}$$

where s is a parametrization of the loop $\gamma, s \in [0, 2\pi]$. $X^\mu_\gamma(x)$ is called the form factor of γ. Its Fourier transform is

$$\begin{aligned} X^\mu_\gamma(k) &:= \frac{1}{(2\pi)^{3/2}} \int d^x X^\mu_\gamma(x) e^{-ik\cdot x} \\ &= \frac{1}{(2\pi)^{3/2}} \oint ds \dot{\gamma}^\mu e^{-i\vec{k}\cdot\vec{\gamma}(s)} \end{aligned} \tag{L.-4}$$

regularize by replacing the delta function $\delta^3(\vec{y}-\vec{x})$ with $f_r(\vec{y}-\vec{x})$ that approximates the delta function and such that $\lim_{r\to 0} f_r(\vec{y}-\vec{x}) \to \delta^3(\vec{y}-\vec{x})$

$$X^\mu_{\gamma(r)}(\vec{x}) := \int_{R^3} d^3y f_r(\vec{y}-\vec{x}) X^\mu_\gamma(\vec{y}) \tag{L.-4}$$

L.2.7 Classical Loop Algebra

L.2.8 Holonomy-Flux $*-$Algebra

The elementary classical observables in our representaion theory are the complex valued functions of holonomies $A(e)$ along paths e in Σ, and fluxes $E_i(S)$ of triad filed across 2-surfaces S, which are defined by

$$E_i(S) := \int_S \eta_{abc} \tilde{E}^c_i. \tag{L.-4}$$

$$\lim_{\epsilon\to 0} \int_S d^2y \; f_\epsilon(x^1, x^2; y^1, y^2) \; g(y^1, y^2) = g(x^1, x^2) \tag{L.-4}$$

$$[E_i]_f(x) := \int dy^a \wedge dy^b \; f_\epsilon(x,y) \eta_{abc} E^c_i(y) \tag{L.-4}$$

if the surface is given in local coordinates by $x^3 = const$, as ϵ tends to zero $[E_i]_f$ tends to $E^3_i(x)$.

$$\begin{aligned} \mathcal{T}[\alpha] \;&:=\; \frac{1}{2}\mathrm{trP}\exp\left[G \oint dt\dot{\alpha}^b A_b(\alpha(t))\right], & \text{(L.-3)} \\ \mathcal{T}^a[\alpha](\alpha(s)) \;&:=\; \frac{1}{2}\mathrm{trP}\left\{\exp\left[G \oint dt\dot{\alpha}^b A_b(\alpha(t))\right] \tilde{E}^a(\alpha(s))\right\} & \text{(L.-2)} \end{aligned}$$

where $A_a(x) = A^i_a(x)\tau_i$ and $E^a(x) = 4E^{ai}(x)\tau_i$ are the Ashtekar connection and its conjugate frame field. (τ_i is the Paulli matrix divided by $2i$)

Invariance under inversion of loop is expressed as

$$\boxed{\mathcal{T}[\alpha]^{-1} = \mathcal{T}[\alpha^{-1}].} \tag{L.-2}$$

The spinor identity

$$\boxed{\mathcal{T}[\alpha]\mathcal{T}[\beta] = \frac{1}{2}\left(\mathcal{T}[\alpha\#\beta] + \mathcal{T}[\alpha\#\beta^{-1}]\right),} \tag{L.-2}$$

if α and β have a common point. Here # indicates joing of two loop at an intersection.

$$\mathcal{T}^0[\gamma] = \mathrm{tr}U_\gamma(s) = \mathrm{tr}\mathcal{P}e^{\oint_\gamma A}. \tag{L.-2}$$

$$\{\mathcal{T}^0[\alpha], \mathcal{T}^0[\beta]\} = 0 \tag{L.-2}$$

$$\mathcal{T}^a[\alpha](\alpha(s)) := \frac{1}{2}\mathrm{tr}[U_\alpha(t,s)\tilde{E}^a(\alpha(s))U_\alpha(s,t)\tilde{E}^b(\alpha(t)) \tag{L.-2}$$

$$\{\mathcal{T}^a[\alpha], \mathcal{T}^0[\beta]\} = \frac{1}{2i}\int dt\delta^3(\gamma s, \eta(t))\dot{\eta}(t)[\mathcal{T}^0[\gamma\#\eta] - [\mathcal{T}^0[\gamma\#\eta^{-1}]]. \tag{L.-2}$$

L.2.9 Quantization of Loop Algebra

The holonomy (corresponding to the configuration variable) opertor acts by multiplication as does (x in one particle quantum mechanics):

$$\hat{\mathcal{T}}[\alpha]\Psi_S(A) = -\mathrm{Tr}\left(\mathcal{P}\int_\alpha A\right)\Psi_S(A) \tag{L.-2}$$

$$[\hat{\mathcal{T}}^0[\alpha], \hat{\mathcal{T}}^0[\beta]] = 0 \tag{L.-2}$$

$$[\hat{\mathcal{T}}^0[\alpha], \hat{\mathcal{T}}^a[\beta](s)] \tag{L.-2}$$

$$\boxed{||T_\gamma|| := \sup_{[A]\in\mathcal{A}}|\ T_\gamma[A]\ |} \tag{L.-2}$$

and complete $\mathcal{HA}$ with respect to this norm we obtain a commutative C^*-algebra $\overline{\mathcal{HA}}$.

first key result we will use is the

> *Gel'fand-Naimark theorem, that every C^*-algebra with identity is isomorphic to the C^*-algebra of all continuous bounded functions on a compact Hausdorff space called the* spectrum *of the algebra.*

Completetion w.r.t. this norm gives us a commutative C$*$-algebra with identity, $\overline{\mathcal{HA}}$. We will call the spectrum of $\overline{\mathcal{HA}}$ by:

$$\overline{\mathcal{A}/\mathcal{G}} \tag{L.-2}$$

The algebra structure allows us to construction of its representaions on Hilbert spaces. For every cyclic representation of $\overline{\mathcal{HA}}$ there is a Borel measure μ on $\overline{\mathcal{A}/\mathcal{G}}$ using which we get a Hilbert space:

$$\mathcal{H}_{aux} := L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu). \tag{L.-2}$$

(Exercise) the operator equation

$$e^{-\hat{B}}\hat{A}e^{\hat{B}} = \mathbb{I} + t\{\hat{A},\hat{B}\} + \frac{t^2}{2!}\{\hat{A},\{\hat{A},\hat{B}\}\} + \cdots \tag{L.-2}$$

L.3 Spinor Network States

The basic canonical degrees of freedom are holonomies of a distributional $SU(2)$ connection and fluxes of the densitized triad conjugate to this connection. The Gauss law (local $SU(2)$ invariance) and momentum (spatial diffeomorphism) constraints are realized as self-adjoint operators constructed out of these variables. States annihilated by these constraint operators span the kinematical Hilbert space. Particularly convenient bases for this kinematical Hilbert space are the spin network bases. In any of these bases, a state is described in terms of links $l_1, \ldots, l_n$ carrying spins ($SU(2)$ irreducible representations) $j_1, \ldots, j_n$ and vertices carrying invariant $SU(2)$ tensors (intertwiners).

$$[\rho_{j_e}(H_e(A))]^{\alpha}_{\beta}, \quad \alpha, \beta = 1, \ldots, d_{\rho_{je}} \tag{L.-2}$$

where $d_{\rho_{je}} = 2j_e + 1$ is the dimension of the representation.

$$[\rho_{j_1}(H_{e_1}(A))]^{\alpha_1}_{\beta_1} \cdots [\rho_{j_n}(H_{e_1}(A))]^{\alpha_n}_{\beta_n} \, v^{\beta_1 \ldots \beta_n} = v^{\alpha_1 \ldots \alpha_n} \tag{L.-2}$$

$$\overline{v_i^{\alpha_1 \ldots \alpha_n}} \, v_{i'\alpha_1 \ldots \alpha_n} = \delta_{ii'} \tag{L.-2}$$

L.3.1 Spinor Network Decomposition of Kinematic Hilbert Space

Spinor Network Decomposition on Single edge

The decomposition of $\mathcal{H}_e = L^2(SU(2), d\mu_H)$ is provided by the Peter-Weyl theorem.

$$\int_{\overline{\mathcal{A}}_e} \overline{\rho^{j'}_{\alpha'\beta'}} \rho^{j}_{\alpha\beta} \, d\mu_e = \frac{1}{2j+1} \delta^{j'j} \delta_{\alpha'\alpha} \delta_{\beta'\beta} \tag{L.-2}$$

Spinor Network Decomposition on Finite Graph

L.4 Cylindrical Measure Theory

A key ingredient for discussing quantum physics is to have at hand an inner product to compute expectation values. It is not easy to develop functional measures in infinite dimensional spaces.

One wishes to compute: $(\psi_1, \psi_2) = \int_{\mathcal{A}/\mathcal{G}} d\mu([A]) \overline{\psi_1([A])} \psi_2([A])$

To motivate the functional space we will consider let us start with a simpler example, that of a scalar field ϕ satisfying the Klein-gordon equation.

A configuration space C for such a theory would be given by the set of all smooth field configurations with appropriate fall off conditions at infinity, for instance C^2 functions. One therefore expects to have wavefunctions $\Psi(\phi)$, and wishes to compute,

$$(N_1; N_2) = N_1(p) N_2(p) \tag{L.-2}$$

And we therefore need a suitable measure and integration theory. To construct this, let us consider the set of test (or smearing) functions on R^3, that is, functions that fall off such that the integral,

$$F_f(\phi) = \langle f, \phi \rangle = \int_{\mathcal{R}^d} d^d x f(x) \phi(x). \tag{L.-2}$$

The functions f are called "Schwarz space" and define the simplest linear functionals on C.

A set of functions on C one can introduce are the "cylindrical" functions. Consider a finite dimensional subspace of the Schwarz space V_n, with a basis $(e_1, \ldots, e_n)$. We can define the projections, For any function $F : R^n \to C$

$$\pi_{e_1,\ldots,e_n}(\phi) = \{\langle e_1, \phi\rangle, \ldots, \langle e_n, \phi\rangle\} \tag{L.-2}$$

This represntation is not unique. In particular any function cylindrical with respect to V_n is cylindrical with respect to any V_m that contains V_n.

A cylindrical measure that allows to integrate cylindrical functions. Any measure in R^n would allow us to integrate cylindrical functions, but the tricky part is that there has to be consistency of these measures for different choices of V_n's.

$$\int_C d\mu(\phi) f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{\langle e_1,\phi\rangle,\ldots,\langle e_n,\phi\rangle} \tag{L.-2}$$

Suppose one has V_n and V_m which have non-vanishing intersection, and with $m > n$, and,

$$V_n^*(\eta_1, \ldots, \eta_n) \subset \tilde{V}_m^*(\tilde{\eta}_1, \ldots, \tilde{\eta}_n) \text{ with } e_i = \sum_{j=1}^{m} L_{ij}\tilde{e}_j \; ; \quad i = 1, \ldots, n \tag{L.-2}$$

Then for every cylindrical function f with respect to V_n defined by a function F on R^n one can make it cylindrical with respect to V_m via,

$$f(\phi) = F(\langle e_1, \phi\rangle, \ldots, \langle e_n, \phi\rangle) = F(\left\langle L_{1j}\tilde{e}_j, \phi \right\rangle, \ldots, \left\langle L_{nj}\tilde{e}_j, \phi \right\rangle = \tilde{F}(\left\langle \tilde{e}_j, \phi \right\rangle, \ldots, \left\langle \tilde{e}_j, \phi \right\rangle \tag{L.-2}$$

And therefore one has to have that,

$$\int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) = \int_{R^m} \tilde{F}(\tilde{\eta}_1, \ldots, \tilde{\eta}_m) d\mu_{\tilde{e}_1,\ldots,\tilde{e}_m}(\tilde{\eta}_1, \ldots, \tilde{\eta}_n) \tag{L.-2}$$

Any set of measures one finite dimensional spaces satisfying these conditions for any cylindrical function F, defines a cylindrical measure via,

$$\int_C d\mu(\phi) f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n) d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) \tag{L.-2}$$

And conversely, a cylindrical measure defines consistent sets of measures in finite dimensional settings.

$$f(\phi) = F\left(\langle e_1, \phi\rangle, \ldots, \langle e_n, \phi\rangle\right) \tag{L.-2}$$

A particularly simple example of this construction is to consider the normalized Gaussian measures in R^n. The resulting measure on C is the one used in textboks when quantizing the scalar field. The Fock space is obtained by completion of the sets of cylindrical measures with a certain weight.

However, in the Cauchy completion, we obtain states that which "genuinely" depend on an infinite number of degrees of freedom. these states can not be realized as functions on $\mathcal{C}$. appropriate measures are Gaussians and all quantun states can be realized on the space $\mathcal{S}'$ of tempered distributions, the topological dual of the space $\mathcal{S}$ of probes.

An obvious property of measures of integration involving a finite number of disjoint measure sets is that the measure of the union of the measure sets is equal to the sum of their measures, i.e.

$$\mu\left(\cup_{i=1}^N A_i\right) = \sum_{i=1}^N \mu(A_i), \quad \text{for } N < \infty, \quad \text{where } A_i \cap A_j = \emptyset \text{ for all } i \neq j. \tag{L.-2}$$

However, this is not in general true for *countable* unions.

More precise account:

The possibility of extending a measure μ on $\mathcal{F}$ to a $\sigma-$additive measure $\tilde{\mu}$ on $\mathcal{B}(\mathcal{F})$ is in particular relevant to physical applications in quantum mechanics. Recall that quantum mechanical systems are often defined by first giving a linear pre-Hilbert space and then completing this space with respect to an inner product. In general, if μ is cylindrical but not $\sigma-$additive, the space $\mathcal{H}$ of -square integrable cylindrical functions on X (denoted through $\mathcal{C}L^2(X, \mathcal{F}, \mu)$) is only a pre-Hilbert space. Such spaces will be discussed in section ??. However, if μ is extendible to a $\sigma-$additive measure $\tilde{\mu}$ on $(X, \mathcal{B}(\mathcal{F}))$ then the Cauchy completion of $\mathcal{H}$ leads to the space $\tilde{\mathcal{H}} = L^2(X, \mathcal{B}(\mathcal{F}), \tilde{\mu})$ (see section 5).

On the other hand if μ is not extendible then the Cauchy completion of $\mathcal{C}L^2(X, \mathcal{F}, \mu)$ leads in general to a space with state-vectors which cannot be expressed as functions on the initial space X. This is the case in scalar field theory if one considers $X = \mathcal{S}(\mathbb{R}3)$ (the Schwarz space of rapidly decreasing smooth C^∞ functions on $\mathbb{R}^3$) and μ is a cylindrical measure defined with the help of a positive definite function on $\mathcal{S}(\mathbb{R}^3)$, continuous in the nuclear space topology (see []).

$- - - - - - - - - - - - - - - - - - - -$

As we shall see in Sect. 5 this is also the case in Yang-Mills theory if we take $\mathcal{H} = \mathcal{C}L^2(\mathcal{A}/\mathcal{G}, \mathcal{F} = \mathcal{C}, \hat{\mu}_{AL})$, where $\hat{\mu}_{AL}$ is the Ashtekar-Lewandowski measure on $\mathcal{A}/\mathcal{G}$. In the scalar field case the Cauchy completion of $\mathcal{C}L^2(\mathcal{S}(\mathbb{R}^3), \mathcal{F}, \mu)$ gives the space of square integrable functions on $\mathcal{S}'(\mathbb{R}^3)$ (the space of tempered distributions), while in the Yang-Mills case the completion of $\mathcal{C}L^2(\mathcal{A}/\mathcal{G}, \mathcal{C}, \hat{\mu}_{AL})$ gives the space $L^2(\overline{\mathcal{A}/\mathcal{G}}, \mathcal{B}(\overline{\mathcal{C}}), \mu_{AL})$ of square integrable functions on the Ashtekar-Isham space $\overline{\mathcal{A}/\mathcal{G}}$ of generalized distributional connections modulo gauge transformations.

L.4.1 Probability Densities

$$C(\lambda) \equiv < 0|e^{i\lambda Q}|0 > = e^{-\frac{1}{2}\lambda^2} = \int \rho(x) e^{i\lambda x} dx.$$

Inverting the Fourier transform one finds a Gaussian ground state ensity

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}. \tag{L.-2}$$

$$\int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+(b,x)} d^n x = \prod_k \sqrt{\frac{(2\pi)^n}{det(A)}} e^{\frac{1}{2}(b,A^{-1}b)}. \tag{L.-2}$$

The "characteristic function"

$$C(\lambda) \;\equiv\; \sqrt{\frac{det(A)}{(2\pi)^n}} \int_{\mathbb{R}^n} e^{-\frac{1}{2}(x,Ax)+i(\lambda,x)} d^n x = e^{-\frac{1}{2}(\lambda,A^{-1}\lambda)}. \tag{L.-2}$$

It is the Fourier transform of a probability measure

$$d\mu(x) = \rho(x) dx, \tag{L.-2}$$

we determined the probability density

$$\rho(x) = \sqrt{\frac{det(A)}{(2\pi)^n}} e^{-\frac{1}{2}(x,Ax)}$$

by doing an inverse Fourier transform on C.

How do we know whether a given function C is the Fourier transform of a probability measure?

The function

$$C(\lambda) \equiv \int_{\mathbb{R}^n} \rho(x) e^{(\lambda,x)} d^n x \tag{L.-2}$$

has the properties

(i) C is normalized

$$C(0) = \int_{\mathbb{R}^n} \rho(x) d^n x = 1 \qquad \text{(L.-2)}$$

(ii) C is continuous at zero, since

$$\begin{aligned} C(\lambda) - 1 &= \int_{\mathbb{R}^n} \left(e^{i(\lambda,x)} - 1\right) \rho(x) d^n x \\ &= \int_{\mathbb{R}^n} (\cos((\lambda, x)) - 1) \rho(x) d^n x + i \int_{\mathbb{R}^n} (\sin((\lambda, x)) \rho(x) d^n x \end{aligned} \qquad \text{(L.-2)}$$

the principle of dominated convergence permits us to take the limit $\lambda \to 0$ inside the integrals.

(iii) For any complex $a_1, \ldots, a_n$, and real $\lambda_1, \ldots, \lambda_n$

$$\int_{\mathbb{R}^n} \left| \sum_l a_l e^{i(\lambda_l, x)} \right|^2 \rho(x) d^n x = \sum_{k,l} a_k^* a_l C(\lambda_k - \lambda_l) \geq 0 \qquad \text{(L.-2)}$$

as a consequence of the positivity of ρ (positive definiteness of C).

Infinite dimensions

problems come from trying to extend

$$\rho(x) d^n x = \sqrt{\frac{1}{(2\pi)^n}} e^{\frac{1}{2}(x,x)} d^n x \qquad \text{(L.-2)}$$

to $n = \infty$. d^∞ doesn't make sense, $(x, x) = \sum_1^\infty x_n^2$ would require this infinite sum to be convergent. The factor goes to zero as $n \to \infty$. However,

$$\begin{aligned} \lim_{n\to\infty} \left(\Omega, e^{i(\lambda, Q)} \Omega\right) &= \left(\Omega, e^{i\varphi(f)} \Omega\right) = \left(\Omega, e^{i \sum \lambda_n \varphi e_n} \Omega\right) \\ &= e^{-\frac{1}{2} \sum_{k=1}^\infty \lambda_k^2} = e^{-\frac{1}{2} \int f^2(x) dx} \end{aligned} \qquad \text{(L.-2)}$$

could be well defined. In particular we note that for

$$C(f) = \left(\Omega, e^{i\varphi(f)}\Omega\right)$$

(i)

$$C(0) = 1$$

(ii) C is continuous in the test functions f.

(iii) For any complex $a_1, \ldots, a_n$ and real test functions $f_1, \dot{s}, f_n$

$$\sum_{k,l} a_k^* a_l C(f_k - f_l) \geq 0. \tag{L.-2}$$

As we will come to later, we have the following generalization of Bochner's theorerm

(*Bochner-Minlos)* Any normalized continuous positive defininte complex function on test function space $\mathcal{S}(\mathbb{R}^n)$ is the Fourier transform of a probability measure μ on distribution space $\mathcal{S}(\mathbb{R}^n)$.

now we have

$$C(f) = \left(\Omega, e^{i\varphi(f)}\Omega\right) = \int_{\mathcal{S}^*} e^{i<\omega,f>} d\mu(\omega) \tag{L.-2}$$

where $<\omega, f>$ is the application of the generalized function $\omega \in \mathcal{S}^*$ to the test function f.

recall the expansion of test functions in terms of a basis, where

$$f(x) = \sum \lambda_n e_n(x).$$

only admit rapidly decreasing sequences of coefficients (λ_n). we have

$$f \in L^2(\mathbb{R}) \quad \leftrightarrow \quad \sum \lambda_n^2 < \infty$$

whereas

$$f \in \mathcal{S}(\mathbb{R}) \quad \leftrightarrow \quad \sum n^k \lambda_n^2 < \infty \text{ for all } k.$$

$$\omega(x) = \sum \omega_n e_n(x).$$

the coeficients ω_n are not square summable.

$$< \omega, f > = \sum \omega_n \lambda_n = \int f(x)\omega(x)dx.$$

The $\omega(x)$ on the right may fail to exist pointwise, but the sum is well defined and finite.

L.4.2 Bochner-Minlos Theorems

This is "algebraic" part of the problem.

we consider projective limits of infinite families of finite dimensional and measurable spaces.

The appropriate space of histories turns out to be the space $\mathcal{S}'$ of (tempered) distributions on the Euclidean space-time and regular measures $d\mu$ on this space are in one to one correspondence with the socalled generating functionals, which are functionals on the Schwarz space $\mathcal{S}$ of test functions satisfying certain rather simple conditions. (Recall that the tempered distributions are continuous linear maps from the Schwarz space to complex numbers.)

In the characterization of typical configurations of measures on functional spaces the so called Bochner-Minlos theorems play a very important role. These theorems are infinite dimensional generalizations of the Bochner theorem for probability measures on $\mathbf{R}^N$. Let us, for the convenience of the reader, recall the latter result. Consider any (Borel) probability measure on $\mathbb{R}^N$, i.e. a finite measure μ, normalized so that $\mu(\mathbb{R}^N) = 1$. The generating functional χ_μ of this measure is its Fourier transform, given by the following function on $\mathbb{R}^N(\mathbb{R}^N)'$, the prime denotes the topological dual, see below)

$$\chi_\mu(\lambda) = \int_{\mathbf{R}^N} N d\mu(x) e^{i(\lambda,x)}, \tag{L.-2}$$

where $(\lambda, x) = \sum_{j=1}^N \lambda^j x_j$. Generating functionals of measures satisfy the following three basic conditions,

(i) Normalization: $\chi(0) = 1$;

(ii) Continuity: is continuous on $\mathbb{R}^N$;

(iii) Positivity: $\sum_{k,l=1}^m c_k \overline{c_l}(\lambda_k - \lambda_l) \geq 0$, for all $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}^N$.

The last condition comes from the fact that $\|f\|_\mu \geq 0$, for $f(x) = \sum_k^m c_k e^{i(\lambda,x)}$, where $\|\cdot\|_\mu$ denotes the $L^2(\mathbf{R}^N, d\mu)$ norm. The finite dimensional Bochner theorem states that the converse is also true. Namely, for any function χ on $\mathbf{R}^N$ satisfying (i), (ii) and (iii) there exists a unique probability measure on $\mathbf{R}^N$ such that χ is its generating functional.

$$\chi(\lambda) = \int_{S'} e^{\lambda \cdot x} d\mu(\lambda) \tag{L.-2}$$

(i) $\chi(0) = 1$
(ii) χ is continuous in every finite dimensional subspace of $\mathcal{S}$
(iii) For every $e_1, \ldots, e_N \in \mathcal{S}$ and $c_1, \ldots, c_N \in C$ we have

$$\sum_{i,j=1}^{N} \bar{c}_i c_j \chi(-e_i + e_j) \geq 0. \tag{L.-2}$$

$$\begin{aligned} \chi(-e) &= \overline{\chi(e)} && \text{(L.-1)} \\ |\chi(e)| &\leq \overline{\chi(0)} && \text{(L.0)} \end{aligned}$$

i.e. $\chi(e)$ is bounded

Proof: take $N = 2$, $c_1 = 1$ and $c_2 = z$

$$\begin{aligned} \sum_{i,j=1}^{2} \bar{c}_i c_j \chi(-e_i + e_j) &= \chi(0) + \chi(e_1 - e_2)\overline{z} + \chi(e_2 - e_1)z + \chi(0)|z|^2 \\ &= \chi(0)(1 + |z|^2) + \chi(e)\overline{z} + \chi(-e)z && \text{(L.0)} \end{aligned}$$

where $e = e_1 - e_2$. Set $z = 1$, then we have $2\chi(0) + \chi(e) + \chi(-e) \geq 0$ so that $\chi(e) + \chi(-e)$ is real and for $z = i$ we have $-i\chi(e) + i\chi(-e) \geq 0$ so that $-i(\chi(e) - \chi(-e))$ is real. As such we have:

$$\begin{aligned} \chi(e) + \chi(-e) &= \overline{\chi(e)} + \overline{\chi(-e)} \\ \chi(e) - \chi(-e) &= \overline{\chi(-e)} - \overline{\chi(e)} && \text{(L.0)} \end{aligned}$$

$\implies \chi(e) = \overline{\chi(-e)}$. Now choose z such that:

$$\overline{z}\chi(e) + |\chi(e)| = 0 \tag{L.0}$$

So that $z\overline{\chi(e)} + |\chi(e)| = 0$. Substituting this into (L.0)

$$\chi(0)(1+\left|\frac{|\chi(e)|}{\chi(e)}\right|^2)+2|\chi(e)|\geq 0,$$

so that we have

$$2\chi(0)-2|\chi(e)|\geq 0 \tag{L.0}$$

Putting $z=0$ in (L.0) we find

$$\chi(0)\geq 0 \tag{L.0}$$

$$\begin{vmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{vmatrix} \geq 0 \tag{L.0}$$

because

$$\chi(0)|c_1|^2+\chi(e)c_1\overline{c}_2+\chi(-e)\overline{c}_1c_2+\chi(0)|c_2|^2\geq 0,$$

by the assumption of (L.4.2). This can be rexpressed as

$$(\overline{c}_1,\overline{c}_2)\begin{pmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}\geq 0, \tag{L.0}$$

As the matrix is hermitian, there exists eigenvectors $\tilde{c}_1$

$$\begin{pmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{pmatrix}\begin{pmatrix} \overline{c}_1 \\ \overline{c}_2 \end{pmatrix}=\lambda\begin{pmatrix} \overline{c}_1 \\ \overline{c}_2 \end{pmatrix} \tag{L.0}$$

$$\lambda_i\,(|\tilde{c}_1|^2+|\tilde{c}_2|^2)\geq 0 \quad i=1,2 \tag{L.0}$$

Hence both eigenvalues are real non-negative numbers

$$\lambda_1,\lambda_2\geq 0 \tag{L.0}$$

and so the determinte satisfies

$$\begin{vmatrix} \chi(0) & \chi(e) \\ \chi(-e) & \chi(0) \end{vmatrix}=\lambda_1\lambda_2\geq 0. \tag{L.0}$$

Conversley???

Every positive-definite continuous function defines a generalized function on $\mathcal{S}$.

$$M_{jk} = \chi(x_k - x_j) \tag{L.0}$$

The conditions for a matrix to be positive-definiteness is that it be Hermitian $M^*_{jk} = M_{kj}$ and its eigenvalues to be non-negative. The condition for arbitary x_k and x_j

$$f(x_k - x_j) = f(x_j - x_k)^* \quad \text{setting} \;\; x = x_k - x_j \quad \Longrightarrow f(-x) = \overline{f(x)}, \tag{L.0}$$

$$(f, \varphi) = \int \overline{f(x)} \varphi dx \tag{L.0}$$

on $\mathcal{S}$

$$\begin{vmatrix} f(0) & f(x) \\ f(x) & f(0) \end{vmatrix} \leq 0, \tag{L.0}$$

$$|f(x)| \leq f(0), \tag{L.0}$$

i.e., $f(x)$ is bounded.

this generalized function is positive-definite:

$$\int \overline{f(x-y)} \varphi(y) \overline{\varphi(x)} dx dy. \tag{L.0}$$

$$\int_{-T}^{T} \int_{-T}^{T} \overline{f(x-y)} \varphi(y) \overline{\varphi(x)} dx dy \tag{L.0}$$

$\varphi(x)$ is summable ($\int \varphi(x) dx < \infty$?) and $f(x)$ is bounded ??? For each T the integral () is the limit of sums

$$\sum_{j,k=1}^{m} \overline{f(x_k - y_j)} \varphi(x_k) \overline{\varphi(x_j)} \Delta x_k \Delta x_j \tag{L.0}$$

the generalized function (f, φ) is positive-definite

Every continuous postive-definite function $\chi(\phi)$ is the Fourier transform of a finite positive measure $d\mu$.

$$(f, \varphi) = (2\pi)^{-n} \int \tilde{\varphi}(\lambda) d\mu(\lambda) \tag{L.0}$$

is $d\mu(\lambda) < \infty$ $\varphi_m(x) = \alpha_m \star \alpha_m^*(x)$, where $\{\alpha_m(x)\}$ is a $\delta-$sequence in S. we obtain

$$(f, \varphi_m) = (2\pi)^{-n} \int \tilde{\varphi}_m(\lambda) d\mu(\lambda) \tag{L.0}$$

$$\tilde{\varphi}_m(\lambda) = |\tilde{\alpha}_m(\lambda)|^2 \tag{L.0}$$

$$\begin{pmatrix} 1 & \chi(t-s) & \chi(t) \\ \overline{\chi(t-s)} & 1 & \chi(s) \\ \overline{\chi(t)} & \overline{\chi(s)} & 1 \end{pmatrix} \tag{L.0}$$

which is $\{\phi(t_i - t_j)\}$ with $t_1 = t, t_2 = s$ and $t_3 = 0$. In particular the determinant has to be non-negative.

$$\begin{aligned} 0 &\leq 1 + \chi(s)\chi(t-s)\overline{\chi(t)} + \overline{\chi(s)\chi(t-s)}\chi(t) - |\chi(s)|^2 \\ &\quad -|\chi(t)|^2 - |\chi(t-s)|^2 \\ &= 1 - |\chi(s) - \chi(t)|^2 - |\chi(t-s)|^2 - \chi(t)\overline{\chi(s)}(1 - \chi(t-s)) \\ &\quad -\overline{\chi(t)}\chi(s)(1 - \chi(t-s)) \\ &\leq 1 - |\chi(s) - \chi(t)|^2 - |\chi(t-s)|^2 + 2|1 - \chi(t-s)| \end{aligned} \tag{L.-3}$$

or

$$\begin{aligned} |\chi(s) - \chi(t)|^2 &\leq 1 - |\chi(s) - \chi(t)|^2 + 2|1 - \chi(t-s)| \\ &\leq 4|1 - \chi(t-s)| \end{aligned} \tag{L.-3}$$

L.4.3 Proof of Bochner's Theorem

We use the dominated convergence theorem

$$\lim_{n\to\infty} \int f_n(t) d\mu = \int \lim_{n\to\infty} f_n(t) d\mu,$$

to prove

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} (1 - \frac{|t|}{T}) e^{-itx} \phi(t) dt,$$

$$= \lim_{T\to\infty} \frac{1}{2\pi} \int_{0}^{T} \int_{0}^{T} e^{-i(t-s)x} \phi(t-s) dt ds \qquad \text{(L.-3)}$$

a change of variables to show the second line

of χ to show the last

$$f(x) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{0}^{T} \int_{0}^{T} e^{-itx} e^{isx} \phi(t-s) dt ds \qquad \text{(L.-3)}$$

and finally a Riemann sum approximation to the integral and the positive definitness of ϕ to show that (L.4.3) is non-negative,

$$\begin{aligned} f(x) &= \lim_{T\to\infty} \frac{1}{2\pi} \int_{0}^{T} \int_{0}^{T} e^{-itx} e^{isx} \phi(t-s) dt ds \\ &= \lim_{A\to 0} \frac{A^2}{4^n} \sum_{j=1}^{2^n} \sum_{k=1}^{2^n} \phi(\frac{A(j-k)}{2^n}) e(\frac{a_j}{2^n}) e(\frac{a_k}{2^n})^* \\ &\geq 0. \end{aligned} \qquad \text{(L.-4)}$$

L.4.4 Generalization to Infinite Dimensional Spaces: Bochner-Minlos Theorem

$\lambda(i) := \lambda^i$ is replaced by $f(x)$.

Then the simplest generalization of the Bochner theorem states that a function on $\mathcal{S}(\mathbb{R}^{d+1})$ satisfies the following conditions,

(i') Normalisation: $\chi(0) = 1$
(ii') Continuity: χ is continuous in every finite dimensional subspace of $\mathcal{S}(\mathbb{R}^{d+1})$
(iii') Positivity: $\sum_{k,l=1}^{m} c_k \overline{c}_l \chi(f_k - f_l) \geq 0$, for all $m \in \mathbb{N}$, $c_1, \ldots, c_m \in \mathbb{C}$ and $f_1, \ldots, f_m \in \mathcal{S}(\mathbb{R}^{d+1})$,

if and only of the Fourier transform of a probability measure μ on $\mathcal{S}(\mathbb{R}^{d+1})$, i.e.

$$\chi(f) = \int_{\mathcal{S}(\mathbb{R}^{d+1})} d\mu(\phi) e^{i\phi(f)}. \qquad \text{(L.-4)}$$

L.4.5 Background Independent Quantization of Linear Scalar Field Theory

[78]

$$\mathcal{L} = \frac{1}{2}\phi(x)\nabla_a\nabla^a\phi(x) + \frac{m}{2}\phi^2(x) \qquad \text{(L.-4)}$$

The background independent quantization of the real scalar field - polymer representation of the scalar field [?] The classical configuration space, $\mathcal{Q}$, consists of all real-valued smooth functions ϕ on Σ. Instead of loops, given a set of finite number of points $X = \{x_1, \ldots, x_N\}$ in Σ, denote Cly_X the vector space generated by finite linear combinations of the following functions of ϕ:

$$\Pi_{X,\lambda}(\phi) := \prod_{x_j \in X} \exp[i\lambda_j \phi(x_j)],$$

where $\lambda = (\lambda_1, \ldots, \lambda_N)$ are arbitrary real numbers, which play a role of labelling of loops. It is obvious that Cly of all cylindrical functions on $\mathcal{Q}$ is defined by

$$Cyl := \cup_X Cly_X, \qquad \text{(L.-4)}$$

(compare to (N.-19)). Completing Cyl with respect to the sup norm

$$\|\Pi_X\| := \sup_{\phi \in \mathcal{Q}} |\Pi_{X,\lambda}(\phi)|, \qquad \text{(L.-4)}$$

(compare to (L.2.9)) one obtains an Abliean C^*-algebra with unit $\overline{Cyl}$. Thus one can use the GNS structure to construct its cyclic representations. A preferred positive linear functional ω_0 on $\overline{Cyl}$ is defined by

$$\omega_0(\Pi_{X,\lambda}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \text{ for all } j \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(L.-4)}$$

which defines a diffeomorphism invariant faithful Borel measure on $\mathcal{Q}$ as

$$\int_{\mathcal{Q}} d\mu(\Pi_{X,\lambda}) = \begin{cases} 1 & \text{if } \lambda_j = 0 \text{ for all } j \\ 0 & \text{otherwise,} \end{cases} \qquad \text{(L.-4)}$$

Thus one obtains the Hilbert space, $\mathcal{H}^{KG}_{Kin} \equiv L^2(\overline{\mathcal{Q}}, d\mu)$, of square integrable functions on a compact topological space $\overline{\mathcal{Q}}$ with respect to μ, where $\overline{Cly}$ acts by multiplication. The quantum configuration space $\overline{\mathcal{Q}}$ is the Gel'fand spectrum of $\overline{Cyl}$.

for a single point set $X \equiv \{x_0\}$, Cyl_{x_0} is the space of all almost periodic functions on the real line $\mathbb{R}$. The Gel'fand spectrum of the corresponding C^*-algebra $\overline{Cyl}_{x_0}$ is the Bohr completion $\overline{\mathbb{R}}_{x_0}$ of $\mathbb{R}$ (see section N.-19), which is a compact topological space topological space such that $\overline{Cyl}_{x_0}$ is the C^*-algebra of all continuous functions on $\overline{R}_{x_0}$. Since $\mathbb{R}$ is densely embedded in in $\overline{R}_{x_0}$, $\overline{R}_{x_0}$ can be regarded as a completion of $\mathbb{R}$.

Given a pair (x_0, λ_0), there is an elementary configuration for the scalar field, the so-called point holonomy,

$$U(x_0, \lambda_0) := \exp[i\lambda_0\phi(x_0)]. \tag{L.-4}$$

It corresponds to a configuration operator $\hat{U}(x_0, \lambda_0)$, which acts on any cylindrical function $\psi(\phi) \in \mathcal{H}^{KG}_{Kin}$ by

$$\hat{U}(x_0, \lambda_0)\psi(\phi) = U(x_0, \lambda_0)\psi(\phi). \tag{L.-4}$$

All these operators are unitary. But since the family of operators $\hat{U}(x_0, \lambda_0)$ fails to be weakly continuous in λ, there is no operator $\hat{\phi}(x)$ on $\mathcal{H}^{KG}_{Kin}$ (in LQC this means the Stone-von Neumann theorem and so this quantization is not unitary equivalent to the usual Schrodinger representation). The momentum functional smeared on 3-dimensional region $R \subset \Sigma$ is expressed by

$$\pi(R) := \int_R d^3x\ \tilde{\pi}(x). \tag{L.-4}$$

The Poisson bracket between the momentum functional and a point holonomy can be easily calculated to be

$$\{\pi(R), U(x, \lambda)\} = -i\lambda\ \chi_R(x)U(x, \lambda), \tag{L.-4}$$

where $\chi_R(x)$ is the characteristic function for the region R. Recall from ordinary quantum mechanics: $\hat{p}\psi(q) := i\hbar\{p, \psi(q)\} = -i\hbar d\psi(q)/dq$. So the momentum operator is defined by the action on scalar network functions $\Pi_{c=(X,\lambda)}$ as

$$\hat{\pi}(R)\Pi_c(\phi) := i\hbar\{\pi(R), \Pi_c(\phi)\} = \hbar[\sum_{x_j \in X} \lambda_j\ \chi_R(x_j)]\Pi_c(\phi). \tag{L.-4}$$

so-called scalar network functions $\Pi_c(\phi)$ that are a orthonormal basis in $\mathcal{H}^{KG}_{Kin}$, where c denotes $(X(c), \lambda)$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$ are non-zero real numbers.

Tychonov Theorem

Then the direct product space $X_\infty = \prod_{l \in \mathcal{L}} X_l$ is a compact topological space in the Tychonov topology.

The Homotopy Group

We consider the set of paths with the same start and end points. Two such paths γ_1 and γ_2 are *homotopic* if one path may be continuously deformed into the other while holding the end points fixed. $\gamma_1 \sim \gamma_2$.

We introduce the group multiplication operator: we define the product of two such paths $\alpha \cdot \beta$ as a path that goes along α, then along β. That is, if $\gamma = \alpha \cdot \beta$, then

$$\begin{aligned} \gamma(t) &= \alpha(2t) \quad 0 \leq t \leq 1/2 \\ & \quad \beta(2t-1) \quad 1/2 \leq t \leq 1. \end{aligned} \tag{L.-4}$$

Notice that this product is compatible with the equivalence relation just defined. If α_1 and α_2 are homotopic, $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$ are homotopic, and if $\alpha_1(1) = \alpha_2(1) = \beta_1(0) = \beta_2(0)$, then $\alpha_1 \cdot \beta_1$ is homotopic to $\alpha_2 \cdot \beta_2$

The inverse of α is d efined $\alpha^{-1}(t) = \alpha(1-t)$

$$\text{i)} \quad \bar{A}(\gamma^{-1}) = (\bar{A}(\gamma))^{-1} \tag{L.-3}$$

$$\text{ii)} \quad \bar{A}(\gamma_2 \cdot \gamma_1) = \bar{A}(\gamma_2) \cdot \bar{A}(\gamma_1) \tag{L.-2}$$

$$\boxed{\text{i)} \quad \bar{A}(\gamma^{-1}) = (\bar{A}(\gamma))^{-1} \qquad \text{ii)} \quad \bar{A}(\gamma_2 \cdot \gamma_1) = \bar{A}(\gamma_2) \cdot \bar{A}(\gamma_1)} \tag{L.-2}$$

$$\boxed{\int d\mu_0[\overline{A}]\Psi_{\Gamma,f}(\overline{A}) := \int_{SU(2)^n} dg_1 \ldots dg_n f(g_1, \ldots, g_n)} \tag{L.-2}$$

connections that cannot be expressed as continuous fields on $\mathcal{M}$ but which all the same asign well defined holonomies on $\mathcal{M}$. It is called the quantum configuration space.

projective limit of the projective family of Hilbert space $\mathcal{H}_\Gamma$

Unfortunately the projective family itself does not have a largest element from which one can project to any other. However, such an element can in fact be obtained by a standard procedure called the "projective limit".

The situation is strikingly similar to ordinary quantum mechanics, where the Hilbert space of physical states is obtained by suitable completions of square integrable functions on the configuration space. In many ways, Cly is analogous to the space $C_0^\infty(\mathbb{R})$ of smooth fucntions of compact support on $\mathbb{R}^3$ which is dense in the Hilbert space of quantum mechanics. In field theory the situation is more involved. Not every physical state is a function on just the configuration space, but distributions on the time=constant hypersurface are also generically involved.

The quotient $\overline{\mathcal{A}/\mathcal{G}} := \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is the gauge invariant quantum configuration space.

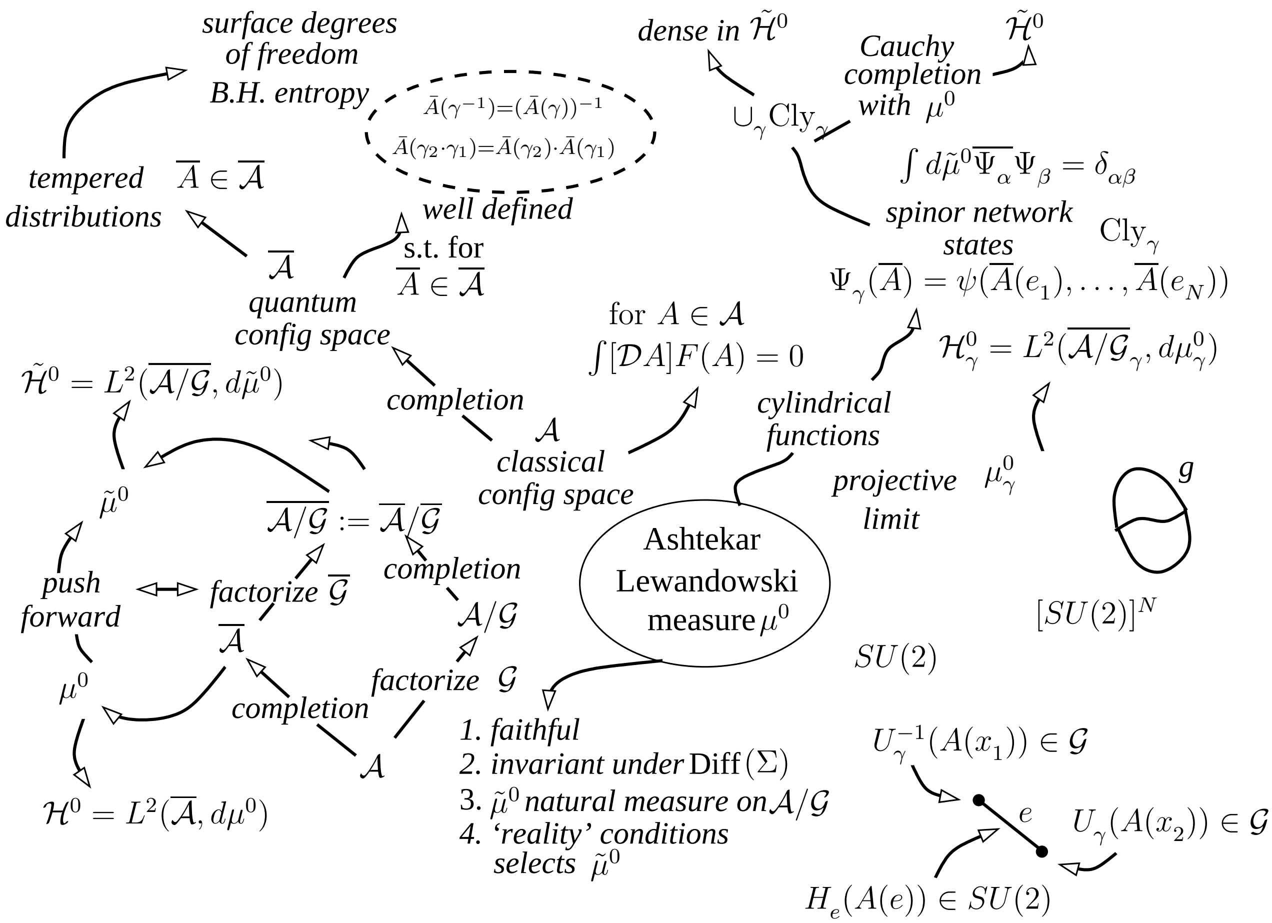


Figure L.5: Induce a measure on the projective limit. With the typical measures encountered in quantum field theory the set of smooth functions on the classical cofiguration space has measure zero. Tempered distributions not just needed but essential. Connections that cannot be expressed as continuous fields on $\mathcal{M}$ but which all the same asign well defined holonomies on $\mathcal{M}$. The quotient $\overline{\mathcal{A}/\mathcal{G}} := \overline{\mathcal{A}}/\overline{\mathcal{G}}$ is the gauge invariant quantum configuration space.

Classical configuration space $\mathcal{A}$ class of smooth functions of connections $A(x)$, such that they separate points in configuration space. When we say they separte points in configuration space, what we mean is that given the values of enough functions $F_1[A(x)], \ldots F_N[A(x)]$

we can determine uniquely $A(x)$ (??up to gauge transformations??).

Since $\overline{\mathcal{A}/\mathcal{G}}$ is compact, it admits regular (Borel, normalized) measures and can construct Hilbert space of L^2-functions.

It turns out that $\overline{\mathcal{A}}$ admits a measure μ^0 that is preferred both mathematically and physically

as $\mathcal{A}_\gamma$ is isomorphic to $[SU(2)]^n$, the Haar measure on $SU(2)$ induces a measure μ^0_γ on it

As we vary γ, we obtain a family of measures which turn out to be compatible and therefore induce a measure on the projective limit $\mathcal{A}$.

This measure has the nice properties (mathematically)

1. it is faithful; i.e., for any continuous, non-negative function f on $\overline{\mathcal{A}}$, $\int d\mu^0 f \geq 0$, equality holding if and only if f is identically zero.

2. it is invariant under the (induced) action of Diff[Σ], the diffeomorphism group of Σ

3. μ^0 induces a natural measure $\tilde{\mu}^0$ on $\mathcal{A}/\mathcal{G}$: $\tilde{\mu}^0$ simply the push-forward of μ^0 under the projection map that sends $\mathcal{A}$ to $\mathcal{A}/\mathcal{G}$.

(physically)

4. the classical phase space admits an (over)complete et of naturally defined configuration and momentum variables which are real, and the requirement that that the corresponding operators on the quantum Hilbert space be self-adjoint selects for us the measure $\tilde{\mu}^0$ [].

??the inner product was obtained on this set of states by requiring that the classical reality conditions be implemented as adjointness conditions on the corresponding quantum operators.??

L.5 The Space of Distributional Connections for Diffeomorphism Invariant Qunatum Gauge Theories

L.5.1 Introduction

This technical section involves category theory, covered in appendix X, and much of the maths on toplogy and measure theory needed is covered in appendix O.

Projective Limit

Ω_j is a topological Hausdorff space for every $j \in J$;

a **directed** set

J is a directed set of indexes, i.e. it is endowed with a partial order relationship $\preceq$ such that

if $i \preceq k$ $j \preceq k$ then $\pi_{ij} \circ \pi_{jk}$

if $i \preceq j$ then the maps for all $ij \in J$ there are are continuous surjective projections such that:

1. $\pi_{jj} = \mathrm{id}_{\Omega_j}$ for all $j \in J$

2. if $i \preceq j \preceq k$ then $\pi_{ij} \circ \pi_{jk} = \pi_{ik}$ (consistency relation).

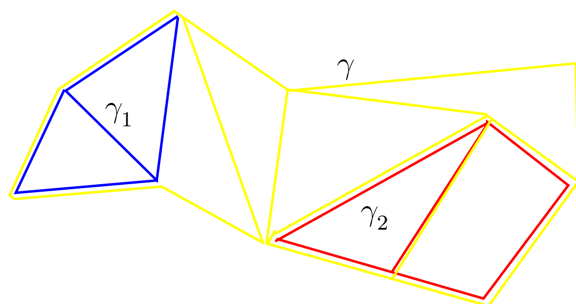


Figure L.6: direct set. The graphs $\gamma_1 \preceq \gamma_2$ if $\gamma_1 \subset \gamma_2$. Any two graphs γ_1 and γ_2 and there exists a graph γ such that $\gamma_1, \gamma_2 \subset \gamma$.

In the projective family there is, in general, no set $\overline{\chi}$ which can be regarded as the largest, from which we can project to any of the χ_S. Such a set emerges in an appropriate limit: The projective limit of Ω_j, π_{ij}, J is the subset of the cartesian product

$$\prod_{j \in J} \Omega_j \qquad \text{(L.-2)}$$

given by all its wires, this space is indicated by

$$\Omega \overset{\leftarrow}{\lim} \Omega_j. \qquad \text{(L.-2)}$$

The maps

$$\begin{aligned} \pi_j : \Omega &\rightarrow \Omega_j \\ \{\omega_i\}_{j\in J} &\mapsto \pi_j(\{\omega_i\}_{j\in J}) := \omega_j \end{aligned} \tag{L.-2}$$

are called the projections of Ω_j.

The projective limit Ω carries a natural topology,called initial topology , which is the smallest topology w.r.t.the projections j of are continuous.

A base of this topology is given by the sets $\prod_{j\in J} U_j$,where $U_j \in \Omega_j$ is an open set such that U j =j

L.5.2 The Label Set: Piecewise Analytic Paths

A groupoid is closed under binar operation, however, associativity, existence of identity, and inverse of each element is not required.

A groupoid is a special case of what is know as a **category** which is a general concept designed to encompass structures common in mathematics. The formal definintion of a category is the following:

Definition A category consists of a collection of **objects** $A, B, \ldots$ and **maps** between these objects. No restriction is placed on the objects, but the maps are required to satisfy the following conditions:

i) For any object A there is a map $\mathbf{1}_A : A \rightarrow A$, so that if $B \xrightarrow{f} A$ and $A \xrightarrow{g} C$ are maps in the category, the composite maps satisfy

$$g \cdot \mathbf{1}_A = g \quad \text{and} \quad \mathbf{1}_A \cdot f = f.$$

ii) If $A \xrightarrow{f} B$, $B \xrightarrow{g} C$ and $C \xrightarrow{h} D$ are maps in the category, we have associativity

$$(h \cdot g) \cdot f = h \cdot (g \cdot f).$$

One can define a map from one category to another as a pair of functions which takes objects to objects and maps to maps. Such a map $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ from category $\mathcal{C}_1$ to category $\mathcal{C}_2$ should satisfy

$$\begin{aligned} F(\mathbf{1}_A) &= \mathbf{1}_{F(A)} \\ F(g \cdot f) &= F(g) \cdot F(f). \end{aligned} \tag{L.-2}$$

Such a map from one category to another is called a **functor**.

Definition A morphism $f \in hom(x, y)$ is called an **isomorphism** provided there exists $g \in hom(y, x)$ such that

$$f \circ g = id_y$$

and

$$g \circ f = id_x$$

where id_x means the identity map from x to itself and ditto for id_y.

In other words, the maps f and g are inverses of one another. This leads to the categorical definition

Definition A category in which every morphism is an isomorphism is a **groupoid**.

Definition A **subcategory** is a category which contains a subclass of the class of objects and for each pair of objects (x, y) of the subcategory we have for the set of morphisms $hom'(x, y) \subset hom(x, y)$.

The definition of a category obviously applies to our situation with the following identifications:

Category: σ
Objects: points $x \in \sigma$.
Morphisms: paths between points $hom(x, y) := \{p \in \mathcal{P}; b(p) = x,\ f(p) = y\}$. Obviuously, every morphism is an isomorphism.
Collection of sets of morphisms: all paths $M(\sigma) = \mathcal{P}$.
Composition: composition of paths $p_{c_1} \circ p_{c_2} = p_{c_1 \circ c_2}$
Identities: $id_x = p \circ p^{-1}$ for any $p \in \mathcal{P}$ with $b(p) = x$.

L.5.3 The Topology: Tychonov Topology

For an element of $A \in \mathcal{A}$ its holonomy dependes only on p_c. To express this we use the notation

$$A(p_c) := h_c(A) \tag{L.-2}$$

We know that

$$A(p \circ p') = A(p)A(p'), \quad A(p^{-1}) = A(p)^{-1}, \tag{L.-2}$$

that is, every $A \in \mathcal{A}_p$ defines a **groupoid morphism**

(Or each A defines a functor between categories!)

Definition $Hom(\mathcal{P}, G)$ is the set of all groupoid morphisms from the set of paths in σ into the gauge group.

The set $Hom(\mathcal{P}, G)$ is larger than the classical space $\mathcal{A}$ as there are elements of $Hom(\mathcal{P}, G)$ that do not correspond to any smooth connection. We wish to equip $Hom(\mathcal{P}, G)$ with a topology, as measure theory becomes most powerful in the context of topology.

Definition The projective limit $\overline{X}$ of a projective family $(X_l, p_{ll'})_{l,l'\mathcal{L}}$ is the subset of the Cartesian product $\times_{l \in \mathcal{L}} X_l$ that satisfies certain consistency conditions:

$$\overline{X} := \{(x_l)_{l \in \mathcal{L}} \in \times_{l \in \mathcal{L}} X_l : \ l' \succeq l \Rightarrow p_{ll'} x_{l'} = x_l\}. \tag{L.-2}$$

The point of this definition is that in our application to gauge theory, this is the limit that gives us the continuum theory.

Definition Given a graph γ we denote by $l(\gamma)$ the subgroupoid generated by γ with $V(\gamma)$ as the set of objects and with the $e \in E(\gamma)$ together with their inverses and finite compositions as the set of homomorphisms.

The labels $\omega, 0$ in Γ_0^ω stand for "analytic" and "of compact support" respectively.

Definition The **tame subgroupoids** $l(\gamma)$ of $\mathcal{P}$ are those determined by graphs $\gamma \in \Gamma_0^\omega$.

Theorem L.5.1 *Let $\mathcal{L}$ be the set of all tame subgroupoids $l(\gamma)$ of $\mathcal{P}$. Then the relation $l \prec l'$ if and only if l is a subgroupoid of l' equips $\mathcal{L}$ with the structure of a partially ordered and directed set.*

Proof: Since l is a subgroupoid of l' if and only if all objects of l are objects of l' and all morphisms of l are morphisms of l' it is clear that $\prec$ defines a partial order. To see that $\mathcal{L}$ is a directed consider any two graphs $\gamma, \gamma' \in \Gamma_0^\omega$ and consider $\gamma'' := \gamma \cup \gamma'$. We must show that γ'' itself is an element of Γ_0^ω, that is, it has a finite number of edges.

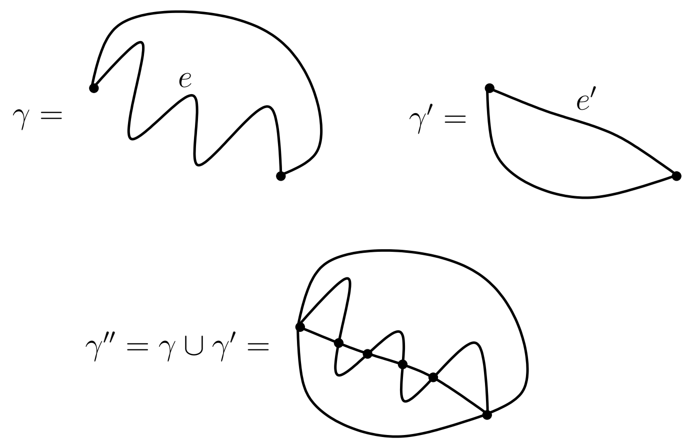


Figure L.7: The union of the two graphs γ and γ' has a finite number of edges. If the edge e were allowed to "oscillate" arbitarily rapidly the union of the two graphs would have an inifinte number of edges.

Although this seems intuiatively obvious, see fig. (Q.7), this is not so for paths of arbitrary differentiabilty; smooth curves can intersect in Cantor sets and thus define graphs which have an infinte number of edges. We prove, however, that it is not the case for peicewise analytic paths.

To prove this it is sufficient to show that any two edges $e, e' \in \mathcal{P}$ can only have a finite number of isolated intersections or $e \cap e'$ has a common finite segment. To prove this suppose then that $e \cap e'$ is an infinite discrete set of points. We may choose parameterizations of their representatives c, c' such that each of its component functions

$$f(t)^a := e'(t)^a - e(t)^a$$

vanishes in at least a countably infinite number of points t_m, $m = 1, 2, \ldots$. We show that for any function $f(t)$ which is real analytic in $[0, 1]$ implies $f = 0$. Since $[0, 1]$ is compact there is an accumulation point $t_0 \in [0, 1]$ of the t_m (here the compact support of the $c \in \mathcal{C}$ comes into play) and we assume that t_m converges to t_0. Since f is analytic we can write the absolutely convergent Taylor series

$$f(t) = \sum_{n=0}^{\infty} f_n (t - t_0)^n$$

(here the analyticity comes into play). We show that $f_n = 0$ by induction over $n = 0, 1, \ldots$. First we establish that $f_0 = 0$,

$$f_0 = f(t_0) = \lim_{m \to \infty} f(t_m) = \lim_{m \to \infty} 0 = 0.$$

Now suppose $f_1 = \cdots = f_n = 0$, we show that $f_{n+1} = 0$. Under this assumption we have

$$f(t) = f_{n+1}(t - t_0)^{n+1} + r_{n+1}(t)(t - t_0)^{n+2}$$

where $r_{n+1}(t)$ is uniformally bounded in $[0, 1]$, that is, there exists a number K such that $|r_{n+1}(t)| \le K$ for all $x \in [0, 1]$. Thus

$$0 = f(t_m)/(t_m - t_0)^{n+1} = f_{n+1} + r_{n+1}(t_m)(t_m - t_0)$$

for all m, hence

$$f_{n+1} = \lim_{m\to\infty} [f_{n+1} + r_{n+1}(t_m)(t_m - t_0)] = 0.$$

$\square$

Now that we have a partially ordered and directed index set $\mathcal{L}$ we must specify a projective family.

Definition For any $l \in \mathcal{L}$ define $X_l := Hom(l, G)$ the set of all homomorphisms from the subgroupoid l to G.

Definition For $l \prec l'$ define a projection by

$$p_{l'l} : X_{l'} \to X_l; \quad x_{l'} \mapsto (x_{l'})_l \tag{L.-2}$$

restriction of the homomorphism $x_{l'}$ defined on the groupoid l' to its subgroupid $l \prec l'$.

Lemma L.5.2 *The projections $p_{l'l}$, $l \prec l'$ are surjective, moreover, they are continuous.*

Proof:

$\square$

The direct product X_∞ is compact by Tychonov's theorem. From section J.10.6 we have that the projective limit $\overline{X}$ is also a compact Hausdorff space in the subspace topology.

Let us collect these results in the following theorem.

Theorem L.5.3 *The projective limit $\overline{X}$ of the spaces $X_l = Hom(l, G)$, $l \in \mathcal{L}$ where $\mathcal{L}$ denotes the set of all tame subgroupoids of $\mathcal{P}$ is a compact Hausdorff space in the induced Tychonov topology whenever G is a compact Hausdorff space.*

Theorem L.5.4 *The map*

$$\Phi : Hom(\mathcal{P}, G) \to \overline{X}; \quad H \mapsto (H_l)_{l \in \mathcal{L}}$$

is a bijection.

Proof:

$\square$

L.6 The C^* Algebraic Viewpoint

A basic result in the Gel'fand-Naimark representation theory assures us that every Abelian C^*-algebra $\overline{\mathcal{C}}$ with identity is realized as the C^*-algebra of continuous functions on a compact Hausdorff space, called the spectrum of $\overline{\mathcal{C}}$.

the spaces can be seen as the Gel'fand spectra of certain C^*-algebras, as such, we make contact with so-called cylindrical functions on these spaces explicit which helps to construct measures on them.

Suppose that we are given a partially ordered and directed index set $\mathcal{L}$ which label compact Hausdorff spaces X_l and that we have surjective and continuous projections $p_{l'l} : X_{l'} \to X_l$ for $l \prec l'$ satisfying the consistency condition $p_{l'l} \circ p_{l''l'} = p_{l''l}$ for $l \prec l' \prec l''$. Let $X_\infty, \overline{X}$ be the corresponding direct product and projective limit respectively with Tychonov topology with respect to which we know that they are Hausdorff and compact from the previous sections.

Definition Let $C(X_l)$ be the continuous, complex valued functions on X_l and consider their union

$$Cyl'(\overline{X}) := \cup_{l \in \mathcal{L}} C(X_l). \tag{L.-2}$$

Let us define the following equivalence relation. Given $f_{l_1} \in C(X_{l_1})$ and $f_{l_2} \in C(X_{l_2})$ we will say f_{l_1} and f_{l_2} are equivalent, denoted $f_{l_1} \sim f_{l_2}$ if

$$p^*_{l_3 l_1} f_1 = p^*_{l_3 l_2} f_2 \tag{L.-2}$$

for all $l_1,\ l_2 \prec l_3$, where $p^*_{l_3 l_1}$ denotes the pull-back map from the space of functions on X_{l_1} to the space of functions on X_{l_3}.

The space of cylindrical functions on the projective limit $\overline{X}$ is defined to be the space of equivalence classes

$$Cyl(\overline{X}) := Cyl'(\overline{X})/\sim \tag{L.-2}$$

We will denote the equivalence class of $f \in Cyl'(\overline{X})$ by $[f]_\sim$. The quotient just gets rid of a redundancy: pull-backs of functions from a smaller set to a larger set are now identified with the functions on the smaller set.

Note to check condition (L.6) it is sufficient to find just one single l_3. For suppose that $f_{l_1} \in C(X_{l_1}), f_{l_2} \in C(X_{l_2})$ are given and that we find some $l_1, l_2 \prec l_3$ such that $p^*_{l_3 l_1} f_1 = p^*_{l_3 l_2} f_2$. Now let any $l_1, l_2 \prec l_4$ be given. Since $\mathcal{L}$ is a directed set we find l_5 such that $l_1, l_2, l_3, l_4 \prec l_5$ and due to the consistency condition among projections we have

$$\begin{array}{rl} i) & p_{l_4 l_1} \circ p_{l_5 l_4} = p_{l_5 l_1} = p_{l_3 l_1} \circ p_{l_5 l_3} \\ ii) & p_{l_4 l_2} \circ p_{l_5 l_4} = p_{l_5 l_2} = p_{l_3 l_2} \circ p_{l_5 l_3} \end{array} \tag{L.-2}$$

from which follows

$$\begin{array}{rl} i) & p^*_{l_5 l_4} p^*_{l_4 l_1} f_{l_1} = p^*_{l_5 l_3} p^*_{l_3 l_1} f_{l_1} \\ ii) & p^*_{l_5 l_3} p^*_{l_3 l_2} f_{l_2} = p^*_{l_5 l_4} p^*_{l_4 l_2} f_{l_2}. \end{array} \tag{L.-2}$$

Equality of i) and ii) in (L.-2) follows from using (L.6), we conclude

$$p^*_{l_5 l_4}[p^*_{l_4 l_1} f_{l_1} - p^*_{l_4 l_2} f_{l_2}] = p^*_{l_5 l_4} g_{l_4} = 0. \tag{L.-2}$$

where $g_{l_4} := p^*_{l_4 l_1} f_{l_1} - p^*_{l_4 l_2} f_{l_2}$. Now for any $f_{l_4} \in C(X_{l_4})$ the condition $f_{l_4}(p_{l_5 l_4}(x_{l_5})) = 0$ for all $x_{l_5} \in X_{l_5}$ means that $f_{l_4} = 0$ because $p_{l_5 l_4} : X_{l_5} \to X_{l_4}$ is surjective (onto), therfore

$$p^*_{l_4 l_1} f_{l_1} = p^*_{l_4 l_2} f_{l_2}.$$

Lemma L.6.1

Lemma L.6.2 *Let $f, f' \in Cly(\overline{X})$ then the following operations are well defined (independent of the representatives)*

i)

$$
\begin{aligned}
f+f' &:= [f_l+f'_l]_\sim && \text{(L.-1)}\\
ff' &:= [f_lf'_l]_\sim && \text{(L.0)}\\
zf &:= [zf_l]_\sim && \text{(L.1)}\\
\overline{f} &:= [\overline{f}_l]_\sim && \text{(L.2)}
\end{aligned}
$$

where l, f_l, f'_l are as in the previous lemma, $z \in \mathbb{C}$ and $\overline{f}_l$ denotes complex conjugation.

ii)

$Cyl(\overline{X})$ contains the constant functions.

iii)

The sup-norm for $f = [f_l]_\sim$

$$\|f\| := \sup_{x_1 \in X_1} |f_l(x_l)| \tag{L.2}$$

is well defined.

Proof:

i)

We condiser only pointwise multiplication, the other cases are similar. Let l, f_{l_1}, f'_{l_1} and l', f_{l_2}, f'_{l_2} as in lemma L.6.1. We find $l_1, l_2 \prec l_3$ and have $p^*_{l_3l_1}f_{l_1} = p^*_{l_3l_2}f_{l_2}$ and $p^*_{l_3l_1}f'_{l_1} = p^*_{l_3l_2}f'_{l_2}$. Thus

$$p^*_{l_3l_1}(f_{l_1}f'_{l_1}) = p^*_{l_3l_1}(f_{l_1})p^*_{l_3l_1}(f'_{l_1}) = p^*_{l_3l_2}(f_{l_2})p^*_{l_3l_2}(f'_{l_2}) = p^*_{l_3l_2}(f_{l_2}f'_{l_2}) \tag{L.2}$$

thus $f_{l_1}f_{l_1} \sim f_{l_2}f'_{l_2}$.

ii)

iii)

If $[f_{l_1}]_\sim = [f_{l_2}]_\sim$ is given, choose any $l_1, l_2 \prec l_3$ so that we know that $p^*_{l_3l_1}f_{l_1} = p^*_{l_3l_2}f_{l_2}$. Then from the surjectivity of $p^*_{l_3l_1}, p^*_{l_3l_2}$ we have

$$\sup_{x_{l_1}\in X_{l_1}} |f_{l_1}(x_{l_1})| = \sup_{x_{l_3}\in X_{l_3}} |(p^*_{l_3l_1}f_{l_1})(x_{l_3})| = \sup_{x_{l_3}\in X_{l_3}} |(p^*_{l_3l_2}f_{l_2})(x_{l_3})| = \sup_{x_{l'_1}\in X_{l'_1}} |f_{l'_1}(x_{l_1})| \tag{L.2}$$

$\square$

Recall that a norm induces a metric on a liner space via $d(f, f') := \|f - f'\|$ and that metric space is complete if all its Cauchy sequences converge to an element of the space. Any metric space can be uniquely (up to an distance preserving mapping carrying every point of the original metric space into itself, theorem J.4.4) embedded into a complete metric space extending it by its non-converging Cauchy sequences. The original metric space is dense in the extended one. We can complete $Cyl(\overline{X})$ in the norm $\|.\|$ and obtain an Abelean, unital Banach $^*-$algebra

$$\overline{Cyl(\overline{X})}$$

But notice that not only the submulitiplicativity of the norm ($\|ff'\| \leq \|f\|\|f'\|$) holds but in fact the C^* property $\|f\overline{f}\| = \|f\|^2$. Thus $\overline{Cyl(\overline{X})}$ **is in fact a unital, Abelean $\mathbf{C^*}$−algebra**.

Denote by $\Delta(Cyl(\overline{X}))$ the **spectrum** of $Cyl(\overline{X})$, that is, the set of *all* (algebraic, i.e. not necessariliy continuous) homomorphisms from $Cyl(\overline{X})$ to the complex numbers and denote the Gel'fand isometric isomorphism by

$$\bigvee : \overline{Cyl(\overline{X})} \to C(\Delta(\overline{Cyl(\overline{X})})); \qquad f \mapsto \overset{\vee}{f} \quad \text{where} \quad \overset{\vee}{f}(\chi) := \chi(f) \tag{L.2}$$

where the space of continuous functions on the spectrum is equipped with the sup-norm.

....

It follows that $\chi(x)$ is a continuous linear (and therefore bounded) map from the normed linear space $Cyl(\overline{X})$ to the complete, normeed linear space $\mathbb{C}$. Hence, by the bounded linear transformation theorem (theorem J.4.6) each $\chi(x)$ can be uniquely extended to a bounded linear transformation (with the same bound) from the completetion $\overline{Cyl(\overline{X})}$ of $Cyl(\overline{X})$ to $\mathbb{C}$

....

Theorem L.6.3 *The map χ in ... is a homeomorphism.*

Proof:

Injectivity (one-to-oneness):

For χ to be one-to-one we must have $\chi(x) \neq \chi(x')$ whenever $x \neq x'$. Suppose then that $\chi(x) = \chi(x')$. In particular $[\chi(x)](f) = [\chi(x')](f)$ for any $f \in Cyl(\overline{X})$. Hence $f_l(x_l) = f_l(x'_l)$ for any $f_l \in C(X_l)$, $l \in \mathcal{L}$. Since X_l is a compact Hausdorff space, $C(X_l)$

separates points of X_l by the Stone-Weierstrass theorem (theorem J.11.4), hence $x_l = x'_l$ for all $l \in \mathcal{L}$. It follows that $x = x'$.

Surjectivity (onto):

Let $\chi \in Hom(\overline{Cyl(\overline{X})}, \mathbb{C})$ be given.

Continuity:

We have established that χ is a bijection. We must show that both, χ, χ^{-1} are continuous.

The topology on $\Delta(\overline{Cyl(\overline{X})})$ is the weakest topology such that the Gel'fand transforms $\overset{\vee}{f}, f \in \overline{Cyl(\overline{X})}$ are continuous while the toplogy on $\overline{X}$ is the weakest topology such that all projections p_l are continuous,

Recall a mapping f from one topological space X to another Y is continuous if and only if whenever $(x_\alpha)_I$ is a net convergent to x then the net $(f(x_\alpha))_I$ converges to $f(x)$.

Continuity of χ:

Let (x_α) be a net in $\overline{X}$ converging to x, that is, every net (x_l^α) converges to x_l.

hence $\chi(x^\alpha) \to \chi(x)$ in the Gel'fand topology.

Continuity of χ^{-1}:

Let (χ^α) be a net in $\Delta(\overline{Cyl(\overline{X})})$ converging to χ,

Hence $\chi^{-1}(\chi_\alpha) \to \chi^{-1}(\chi)$ in the Tychonov topology.

□

....

Corollary L.6.4 *The closure of the space of cylindrical functions* $\overline{Cyl(\overline{X})}$ *may be identified with the space of continuous functions* $C(\overline{X})$ *on the projective limit* $\overline{X}$.

Proof:

□

L.6.1 The spaces $\overline{X}/G$ and $\overline{X/G}$ are Homeomorphic

L.7 Regular Borel Measures on the Projective Limit: The Uniform Measure

Our spaces X_l are compact Hausdorff spaces and in particular topological spaces and are therefore naturally equipped with $\sigma-$algebra $\mathcal{B}_l$ of Borel sets (the smallest $\sigma-$algebra containing all open (equivalently closed) subsets of X_l).

regularity means that the measure of every measurable set can be approximately well by open and compact sets (hence closed since X_l is compact Hausdorff by lemma J.9.6)

Definition A family of measures $(\mu_l)_{l\in\mathcal{L}}$ on the projections X_l of a projection family $(X_l, p_{ll'})_{l\prec l'}$ where the $p_{l'l} : X_{l'} \to X_l$ are continuous and surjective projections is said to be consitent provided that

$$(p_{l'l})_* \mu_{l'} := \mu_{l'} \circ p_{l'l}^{-1} = \mu_l \tag{L.2}$$

for any $l \prec l'$. The measure $(p_{l'l})_* \mu_{l'}$ is called the push-forward of the measure $\mu_{l'}$.

....

Definition The Hilbert space $\mathcal{H}^0$ is defined as the space of square integrable functions over $\overline{\mathcal{A}}$ with respect to the uniform measure μ_0, that is

$$\mathcal{H}^0 := L_2(\overline{\mathcal{A}}, d\mu_0). \tag{L.2}$$

L.8 Operators

The specification of the topology in which the limit is taken is an integral part of the definition of the operator.

For limits in the involved in the regularization of quantum field theoretical operators, the limit cannot be taken in the Hilbert space topology where, in general, it does not exist. The limit must be taken in the topology that "remembers" the topology in which the corresponding limit is taken.

We say a sequence of qunatum states Ψ_n converges to a state Ψ if $\Psi_n[A]$ converges to $\Psi[A]$ for all smooth connections A. We define a domain M as the set for which $\{\Psi_n\} \subset M$, $\Psi_n \to \Psi$ implies that $\Psi \in M$. We use the corresponding operator topology: $O_n \to O$ if $O_n \Psi \to TO$ for all Ψ in the domain.

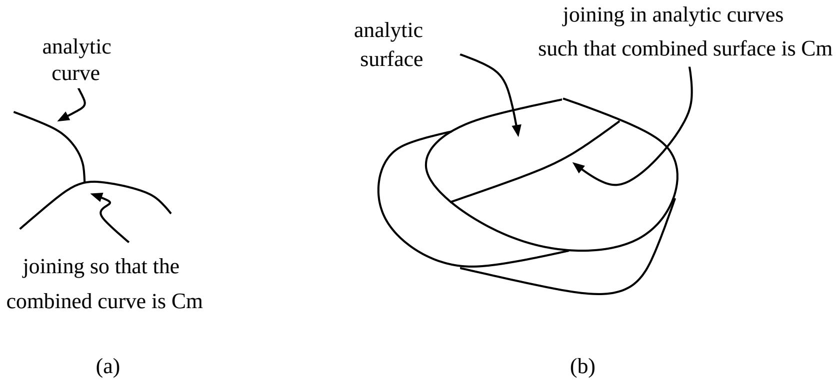


Figure L.8: (a) A semianalytic curve in $\mathbb{R}^3$. (b) A semianalytic surface in $\mathbb{R}^3$.

L.9 Functional Calculus on a Projective Limit

Functions

Differential Forms

Volume Forms

Vector Fields

Lie Brackets

Vector Field Divergences

L.10 Density and Support Properties of $\mathcal{A}, \mathcal{A}/\mathcal{G}$ with respect to $\overline{\mathcal{A}}, \overline{\mathcal{A}/\mathcal{G}}$

In this section we will see that $\mathcal{A}$ lies topologically dense, but measure theoretically thin in $\overline{\mathcal{A}}$ (similar results apply to $\mathcal{A}/\mathcal{G}$ with respest to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$) with respect to the uniform measure μ_0.

We have seen that every element of $A \in \mathcal{A}$ defines an element of $Hom(\mathcal{P}, G)$ and that this space can be identified with the projective limit $\overline{X} \equiv \overline{\mathcal{A}}$. Now via the C^*-algebraic framework we know that $\overline{Cyl(\overline{X})}$ can be identified with $C(\overline{X})$ and the latter space of functions separates the points of $\overline{X}$ by the **Stone-Weierstrass theorem** (thereom J.11.4) since it is Hausdorff and compact.

Now does the set of functions $Cyl(\overline{X})$ separates the set of points $\mathcal{A}$. Let $A \neq A'$ be given then there exists a point $x \in \sigma$ such that $A(x) \neq A'(x)$.

We therefore we have that a collection $\mathcal{C} = Cyl(\overline{X})$ of bounded complex valued functions on a set $X = \mathcal{A}$ including the constants which separates the points of X. The following result is an abstract property of Abelean unit C^*-algebras (in our case, $\overline{\mathcal{C}} = \overline{Cyl(\overline{\mathcal{A}})}$ and $\overline{X} = \overline{\mathcal{A}}$).

Theorem L.10.1 *Let $\mathcal{C}$ be a collection of real-valued, bounded functions on a set X which contains the constants and separates points of X. Let $\overline{\mathcal{C}}$ be the Abelean, unital C^*-algebra generated from $\mathcal{C}$ by pointwise addition, multiplication, scalar multiplication and complex conjugation, completed in the sup norm. Then the image of X under its natural embedding into teh Gel'fand spectrum $\overline{X}$ of $\overline{\mathcal{C}}$ is dense with respect to the Gel'fand topology on the spectrum.*

Proof:

.....

Let $\overline{J(X)}$ be the closure of $J(X)$ in the Gel'fand topology on $\overline{X}$ of pointwise convergence on $\overline{\mathcal{C}}$. Suppose that $\overline{X} - \overline{J(X)} \neq \emptyset$ and take $\chi \in \overline{X} - \overline{J(X)}$. Since $\overline{X}$ is a compact Hausdorff space we find $a \in C(\overline{X})$ such that $1 = a(\chi) \neq a(J_x) = 0$ for any $x \in X$ by Urysohn's lemma, lemma J.9.10. (Urysohn's lemma applies to normal spaces. Compact Hausdorff spaces are normal spaces (theorem J.9.8). In Hausdorff spaces one point sets are closed, hence $\{x\}$ and $\overline{J(X)}$ are disjoint closed sets).

Since the Gel'fand map $\bigvee : \overline{\mathcal{C}} \to C(\overline{X})$ is an isomorphic we find $f \in \overline{\mathcal{C}}$ such that $\overset{\vee}{f} = a$. Hence

$$0 = a(J_x) = \overset{\vee}{f}(J_x) = J_x(f) = f(x)$$

for all $x \in X$, hence $f = 0$, thus $a \equiv 0$ contracdicting $a(\chi) = 1$. Therefore χ infact does not exist whence $\overline{X} = \overline{J(X)}$.

□

That is, $\mathcal{A}$ is topologically dense in $\overline{\mathcal{A}}$.

L.11 Uniqueness Theorem for the Ashtekar-Lewandowski Representation

Fleischhack

Every physical theory requires fundamental mathematical assumptions at the very beginning. It is highly desirable to justify them by even more fundamental axioms that are both mathematically and physically as plausible as possible.

This measure is “natural”, since the Haar measure on a Lie group is “natural” as well. However, this is at most a mathematical statement or a statement of beauty. The deeper question behind is how one can justify this choice by mathematical physics arguments.

representation theory - diffeomorphism invariance.

physical selection, one case is, a unitary representation of the spacial diffeormorphism group (rather projective representation thereof as no representation of the infintesimal constraints cannot be well defined [96]). Remrkably it has been possible to show that such a representaion is unique ‘[6]’ . More precisely, in general ..... taken from [213]

Quantum Geometry: Representation on $L_2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, \mu_0)$ is unique if

1. diffeomorphism invariant;

2. semianalytic.;

A natural idea is to first look at irreducible or at least cyclic representations as the simple building blocks, out of which more complicated representations could eventually be built.

A simple formulation of these properties can be given by asking for a state (i.e. a positive, normalized, linear functional) on $\mathcal{U}$ that it is invariant under the classical symmetry automorphisms of $\mathcal{U}$. Given a state ω on $\mathcal{U}$ one can define a representation via the GNS construction. This representation will be cyclic by construction, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$,. If the state is invariant under some automorphism of $\mathcal{U}$, its action is automatically unitarily implemented in the representation.

Let G be a group of automorphisms of the $C*-$algebra $\mathcal{O}$ and ω a corresponding $G-$invariant state on $\mathcal{O}$. Then there is a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$

$$\pi_\omega(gA) = U_\omega(g)\pi_\omega(A)U_\omega(g)^{-1}, \quad U_\omega(g)\Omega_\omega = \Omega_\omega, \tag{L.2}$$

for all $g \in G$ and $A \in \mathcal{O}$.

Briefly, ‘semianalytic’ means ‘piecewise analytic’. For example, a semianalytic sub-manifold would be analytic except for on some lower dimensional sub-manifolds, which in turn have to be piecewise analytic. We have already met the idea of semianalycity, see fig (L.9) (a). To convey the general idea, fig (L.9) (b) depicts a semi-analytic surface in $\mathbb{R}^3$.

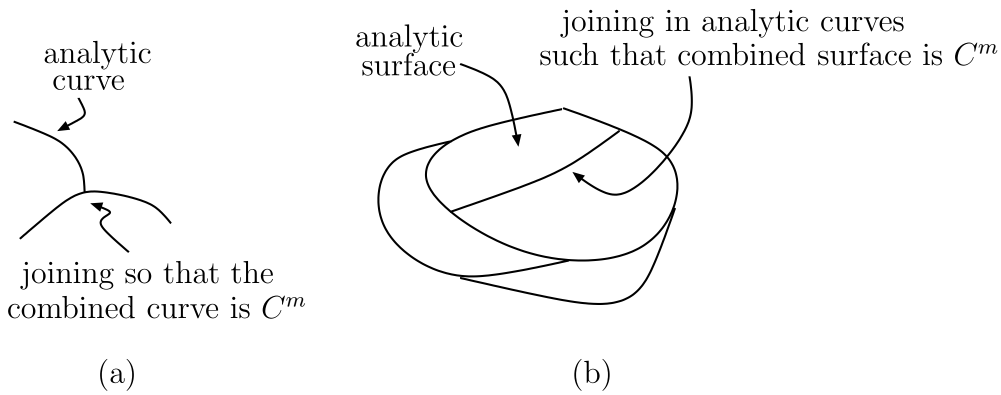


Figure L.9: (a) A semianalytic curve in $\mathbb{R}^3$. (b) A semianalytic surface in $\mathbb{R}^3$.

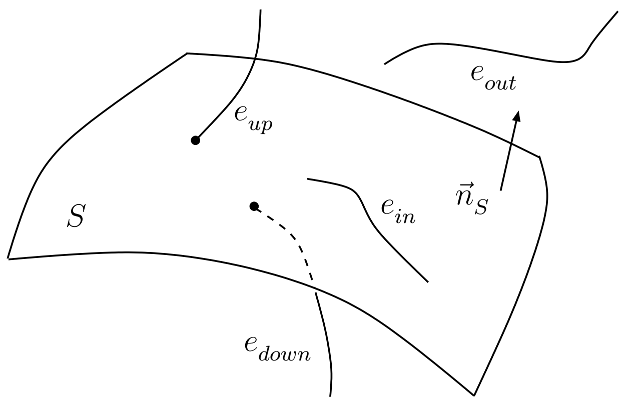


Figure L.10: Types of edges with respect to a face.

L.11.1 Flux Operators

The left invariant vector field in the $i-$th internal direction on the copy of G corresponds to the e-th edge

$$L_e^i \cdot \psi(h_{e_1}, \ldots, h_{e_N}) = (h_e \tau^i)^A_B \frac{\partial \psi}{\partial (h_e)^A_B} = \left(\frac{d}{dt}\right)_{t=0} \psi(h_{e_1}, \ldots, h_e e^{t\tau_i}, \ldots, h_{e_N})$$

$$R_e^i \cdot \psi(h_{e_1}, \ldots, h_{e_N}) = (\tau^i h_e)^A_B \frac{\partial \psi}{\partial (h_e)^A_B} = \left(\frac{d}{dt}\right)_{t=0} \psi(h_{e_1}, \ldots, e^{t\tau_i} h_e, \ldots, h_{e_N})$$

$$X_{S,n}[f] := \frac{1}{2} \sum_{p \in S \cap \gamma} \sum_{e_p} \sigma(e_p, S) n_i(p) X^i_{e_p}[f],$$

where the second sum is over the edges of γ adjacent to p,

$$\sigma(e_p, S) = \begin{cases} 1 & \text{if } e_p \text{ lies above } S \\ 0 & \text{if } e_p \cap S = \emptyset \text{ or } e_p \cap S = e_p \\ -1 & \text{if } e_p \text{ lies below } S \end{cases}$$

and $X^i_{e_p}$ is the ith left-invariant (right-invariant) vector field on $SU(2)$ acting on the argument of f corresponding to the holonomy h_{e_p} if e_p is pointing away from (towards) S.

Definition

$$W^n_t(S) := e^{t\beta\ell_p^2/2Y_n(S)} \tag{L.2}$$

$$e^L M e^{-L} = \sum_{n=0}^{\infty} \frac{1}{n!}[L, M]_n$$

$$\begin{aligned} W_\gamma(t_\gamma) f W_\gamma(t_\gamma)^{-1} &= e^{Y_\gamma(t_\gamma)} f e^{-Y_\gamma(t_\gamma)} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}[Y_\gamma(t_\gamma), f]_{(n)} \\ &= f + [Y_\gamma(t_\gamma), f] + \frac{1}{2!}[Y_\gamma(t_\gamma), [Y_\gamma(t_\gamma), f]] + \end{aligned} \tag{L.0}$$

$$\begin{aligned} W^n_t(S) W^{n'}_{t'}(S') (W^n_t(S))^{-1} &= e^{t\beta\ell_p^2/2Y_n(S)} \big(\sum_{m'}^{\infty} \frac{(t'\beta\ell_p^2/2)^{m'}}{m'!} (Y^{n'}_{t'}(S'))^{m'} \big) e^{-t\beta\ell_p^2/2Y_n(S)} \\ &= \sum_{m'}^{\infty} \frac{(t'\beta\ell_p^2/2)^{m'}}{m'!} \sum_{m=0}^{\infty} \frac{(t\beta\ell_p^2/2)^m}{m!} [Y_n(S), (Y^{n'}_{t'}(S'))^{m'}]_{(m)} \\ &= \end{aligned} \tag{L.-2}$$

L.11.2 Algebra of Cylindrical Functions and Space of Generalised Connections

As we have seen there are several complementary characterisations of the Kinematic Hilbert space.

C^*-algebraic characterisation.

Cyl^∞ is an algebra: We know that if Ψ is compatible with γ and if $\gamma' \geq \gamma$, then Ψ is compatible with γ' as well. First note that every element Ψ of the space of finite linear coombination of smooth cylindrical functions $\{\Psi_i\}$ $(i = 1, \dots, k)$ compatible, respectively, with graphs $\{\gamma_i\}$. There exists a graph γ such that $\gamma \geq \gamma_i$ for each i and so every function Ψ_i is compatible with γ. Hence Ψ is a smooth cylindrical function compatible with γ also. Now, given elements Ψ and Ψ' of Cyl^∞ we can find a graph γ' such that the two functions are compatible with it. Then $\Psi\Psi'$ is a smooth cylindrical function compatible with γ', thus an element of Cyl^∞.

The completion $\overline{\mathrm{Cyl}}$ of Cyl with respect to the sup norm $\|f\| := \sup_{A \in \mathcal{A}} |f(A)|$ defines an Abelean C^*-algebra. Define the space of generalsed connections $\overline{\mathcal{A}}$ as its Gel'fand spectrum $\Delta(\overline{\mathrm{Cyl}})$. By the Gel'fand isomorphism we can think of Cyl^∞ as the space $C(\mathcal{A})$ of continuous functions on the spectrum. The spectrum of an Abelean C^*-algebra is a compact Hausdorff space if equipped with the Gel'fand topology of pointwise convergence of nets. Hence, by the Riesz-Markov theorem the postive linear functional ω is in one to one correspondence with a regular Borel measure μ on $\overline{\mathcal{A}}$. The Hilbert space $\mathcal{H} := L_2(\overline{\mathcal{A}}, d\mu)$ is the space of square integrable functions on $\overline{\mathcal{A}}$ with respect to that measure.

L.11.3 Generalized Vector Fields Tangent to $\overline{\mathcal{A}}$

Definition The momentum variable space defined by a given space of smearing functions $\mathcal{F}$ is the real vector space spanned by the linear maps $\pi(f)$ such that $f \in \mathcal{F}$.

□

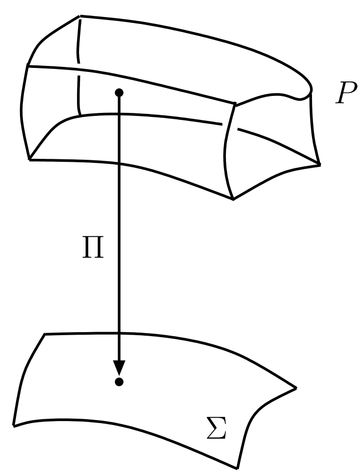


Figure L.11: Automorphisms of the bundle P implies action of the bundle automorphism group in the qunatum $*-$algebra .

Let S be a face. Consider the bundle

$$P_S := \Pi^{-1}(S) \subset P \tag{L.-2}$$

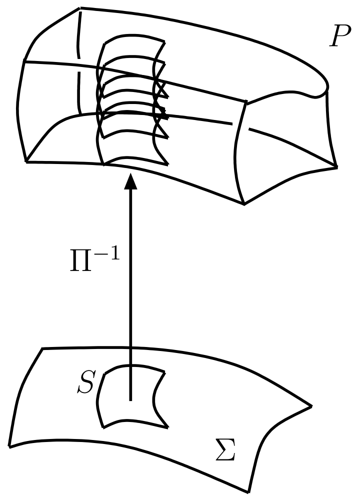


Figure L.12: .

Given a face S, a smearing

L.11.4 The Quantum $*-$algebra

$$\begin{aligned}(a_1, \ldots, a_n) \cdot (b_1, \ldots, b_m) &= (a_1, \ldots, a_n, b_1, \ldots, b_m) &\text{(L.-1)}\\ (a_1, \ldots, a_n)^* &= (\overline{a_n}, \ldots, \overline{a_1}). &\text{(L.0)}\end{aligned}$$

L.11.5 Symmetries of $\mathfrak{A}$

The group of semianalytic automorphisms of the principal fiber bundle P act naturally in the space $\mathcal{A}$ of connections.

L.11.6 Implementation of Piecewise Analytic Diffeomorphisms on $\mathcal{H}_{kin}$

It is straightforward to implement the action of piecewise analytic diffeomorphisms on $\mathcal{H}_{kin}$: This Hilbert space consists of functions $f : \mathcal{A} \to \mathbb{C}$, which are cylindrical over some graph γ. The space of quantum configurations $\mathcal{A}$, i.e. the space of (distributional) connections on Σ carries a natural action of the diffeomorphism group Diff Σ. An element

$\phi \in \mathrm{Diff}\,\Sigma$ simply acts by $A \to \phi^* A$ on a (distributional) connection A. With this, one can simplt define the action of Diff Σ on $\mathcal{H}_{kin}$ by

$$\alpha_\phi f(A) := f(\phi^* A)$$

where $\phi^* A$ is the pullback of the connection A under the diffeomorphism ϕ. Note that this defintion maps

$$\alpha_\phi \mathcal{H}_\gamma \to \mathcal{H}_{\phi(\gamma)}. \tag{L.0}$$

Let us discuss the action of the automorphisms/diffeomorphisms. One can build out of a map $\varphi : \Sigma \to \Sigma$ an induced map $B\varphi : P \to P$ of bundles by combining into one object all the maps of fibres induced by φ: that is, for each $v \in P$

$$B\varphi(v) = \varphi_{*\Pi(v)} v.$$

It cam be shown that

$$\Pi \circ B\varphi = \varphi \circ \Pi$$

For every bundle automorphism

$$\tilde{\varphi} : P \to P \tag{L.0}$$

there is a unique diffeomorphism

$$\varphi : \Sigma \to \Sigma \tag{L.0}$$

such that

$$\Pi \circ \tilde{\varphi} = \varphi \circ \Pi. \tag{L.0}$$

In our case both of them are semianalytic.

Definition Preservation of a fibre. The map $\tilde{\varphi}$ respects the bundle structure of P in the sense that if v and v' belong to the same fibre of P then their images $\tilde{\varphi}(v)$ and $\tilde{\varphi}(v')$ belong to the same fibre of P: this is the content of the property =.

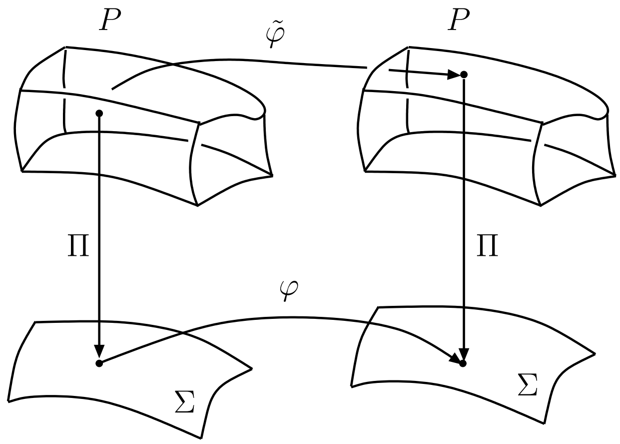


Figure L.13: A bundle map.

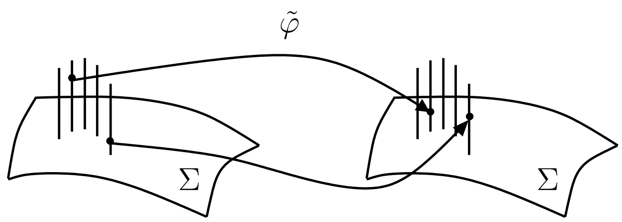


Figure L.14: Pictural illustraiton of a bundle map which preserves every fibre. The corresponding map on the base space is the identity - to be called a Yang-Mills gauge transformations.

A map $\tilde{\varphi} : P \to P$ which maps each fibre to itself is a particular case of a bundle map in which the corresponding map of the base is just the identity map of Σ.

In a sense the bundle automorphisms represent also the diffeomorphisms of Σ.

It is easy to check, that the new flow is the flow of the vector field $X_{\varphi(S),\tilde{\varphi}_* f}$.

L.11.7 Proof

In the Ashtekar-Lewandowski representation, ω_0, the quantum flux operator $X_{s,f}$ vanishes: $\pi_{\omega_0}[X_{s,f}] = 0$. The main part of the proof of the uniqueness theorem is to show that a consequence of diffeomorphism invaraince of any representation is the vanishing of the quantum flux operator, as it is fairly straightforward to show that the Ashtekar-Lewandowski representation is the only diffeomorphism invariant representation with this property.

Theorem L.11.1

$$[X_{S,f}] = 0 \tag{L.0}$$

Of course $[a] := \{a + b : b \in \mathfrak{A}$ *such that* $\omega(b^*b) = 0\}$ *is the equivalence class of* $a \in \mathfrak{A}$ *with respect to the Gel'fand ideal of null vectors.*

Proof:

For each point $p \in \mathrm{supp}(n)$, by the defnition of a face S we find a neighbourhood U_p of p and a chart x_p whose domain contains U_p such that

$$\chi_p(S \cap \mathcal{U}_p) = \{x \in \mathbb{R}^D : x^D = 0, 0 < x^1, \ldots, x^{D-1} < 1\}$$

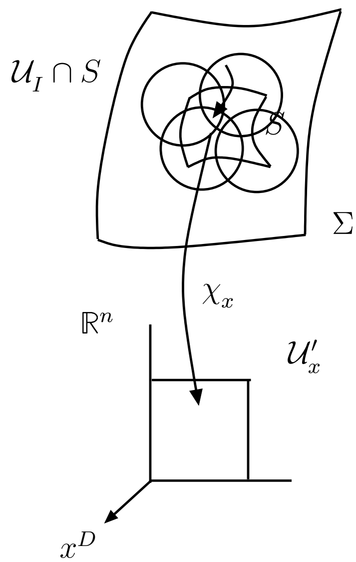


Figure L.15: x_I defined such tha the above is so.

Since the support of n is compact in P, choose a finite subcovering $\{\mathcal{U}_I\}_{I=1}^N$ of $\Pi(\mathrm{supp} f)$ with associated charts x_I. By the local character of semianalytic structures, there is a partition of unity subordinate to the covering $\{\mathcal{U}_I\}$, i.e., there exists a family of differentiable functions $\chi_I(x)$ such that

(i) $0 \leq \chi_I(x) \leq 1$

(ii) $\chi_I(x) = 0$ if $x \neq U_I$

(iii) $\sum_I^N \chi_1(x) = 1$ for any point $x \in \mathcal{M}$.

From (iii) it follows that

$$n(x) = \sum_I n(x)\chi_I(x) = \sum_I n_I(x)$$

where $n_I(x) \equiv n(x)\chi_I(x)$ vanishes outside of U_I by (ii).

Hence

$$n = \sum_{I-1}^{n} n_I$$

everwhere on Σ where

$$n_I = n \cdot \chi_I.$$

There doesn't exist global coordinates which ensure a basis for the collection of vector fields. The most that can be said is that for any point has a neighbourhood on which local vector fields are defined which form a basis of the tangent space at each point in the neighbourhood.

Furthermore, we may decompose

$$n_I = \sum_j n_I^j \tau_j \tag{L.0}$$

where τ_j is a basis in the Lie algebra of G and set $n_{Ij} = n_I^j \tau_j$ (no summation). It follows that

$$[X_n(S)] = \sum_{I=1}^{N} \sum_{j=1}^{\dim(G)} [X_{Ij}(S)]$$

and the result will follow from proving that $[X_{Ij}(S)] = 0$.

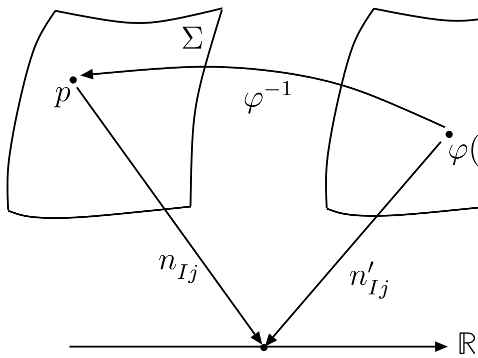
Consider for fixed I, j the following functional which assigns a number to any given pair of compactly supported functions $n_{Ij}, n'_{Ij} : S \cap U_I \to \mathbb{R}$,

$$(n_{Ij}, n'_{Ij})_S := < \left[X_{n_{Ij}}(S)^*\right], \left[X_{n'_{Ij}}(S)\right] > := \omega(X_{n_{Ij}}(S)^* X_{n'_{Ij}}(S)) \tag{L.0}$$

The product $(\cdot,\cdot)_S$ has the following properties:

(i) It is obviously bilinear and, due to the reality of the n, n', also symmetric.

(ii) It is invariant under semianalytic diffeomorphisms φ which preserve S and have support in U_I,



$$n'_{Ij}(\varphi(p)) := n_{Ij}(p)$$

Figure L.16: The pull-back $(\varphi)^* n_{Ij}$ of the scalar function $n_{Ij}(x)$ is given by $n'_{Ij} = n_{Ij} \circ \varphi^{-1}$.

The point of the decomposition (L.11.7) is that, if additionally φ preserves S, then the action of $\tilde{\varphi}$ on $X_{n_{Ij}}(S)$ amounts to

$$\alpha_{\tilde{\varphi}} X_{n_{Ij}}(S) = X_{(\varphi)^* n_{Ij}}(S) = X_{n_{Ij} \circ \varphi^{-1}}(S).$$

It follows that the product $(\cdot,\cdot)_S$ is invariant under the specified $\tilde{\varphi}$

$$\begin{aligned} (n_{Ij}, n'_{Ij})_S &= \omega\big(\alpha_\varphi\big[X_{n_{Ij}}(S)^* X_{n'_{Ij}}(S)\big]\big) \\ &:= \omega\big(X_{n_{Ij}\circ\varphi}(\varphi(S))^* X_{n'_{Ij}\circ\varphi}(\varphi(S))\big) \\ &= (n_{Ij} \circ \varphi^{-1}, n'_{Ij} \circ \varphi^{-1})_S. \end{aligned} \tag{L.-1}$$

For $n = n'$

$$\|[X_{n_{Ij}}(S)]\|^2 = \big\langle \big[X_{n_{Ij}}(S)^* X_{n'_{Ij}}(S)\big] \big\rangle = \omega(X_{n_{Ij}}(S)^* X_{n_{Ij}}(S)). \tag{L.-1}$$

The trick to proving $\omega([X_{n_{Ij}}]^*[X_{n_{Ij}}]) = 0$ is to construct (which will be done in the next two lemmas) a semianalytic diffeomorphism φ_t which reduces to identity outside U_I, a semianalytic function N_{Ij}, and another semianalytic function f with $f|_S = 1$ such that

$$(\varphi_t)^* N_{Ij} = N_{Ij} + t f n_{Ij} \tag{L.-1}$$

for all $0 < t < t_0$, to which we can then apply (L.-1). This results in

$$
\begin{aligned}
(N_{Ij}, N_{Ij})_S &= ((\varphi_t)^* N_{Ij}, (\varphi_t)^* N_{Ij})_S \\
&= (N_{Ij} + t f n_{Ij}, N_{Ij} + t f n_{Ij})_S \\
&= (N_{Ij}, N_{Ij})_S + 2t(N_{Ij}, n_{Ij})_S + t^2(n_{Ij}, n_{Ij})_S \qquad \text{(L.-2)}
\end{aligned}
$$

Since this holds for all $0 < t < t_0$ we may divide by $t > 0$ and find

$$2(N_{Ij}, n_{Ij})_S + t(n_{Ij}, n_{Ij})_S = 0 \qquad \text{(L.-2)}$$

for all $0 < t < t_0$. Subtracting this equation evaluated at $0 < t_1 < t_2 < t_0$ one easily sees that $(n_{Ij}, n_{Ij})_S = 0$.

□

Lemma L.11.2 *There is $t_0 > 0$ such that for every $0 < t < t_0$, φ'_t is a semianalytic diffeomorphism of $\mathbb{R}^D$ equal to the identity outside of $\mathcal{U}'_I$ and preserving $\mathcal{U}'_I$.*

Proof: Using the coordinate system x_I associated with U_I we set $U'_I = x_I(U_I)$,

$$S'_I = x_I(S \cap U_I) = \{x \in \mathbb{R}^D : x^D = 0, 0 < x^1, \ldots, x^{D-1} < 1\}$$

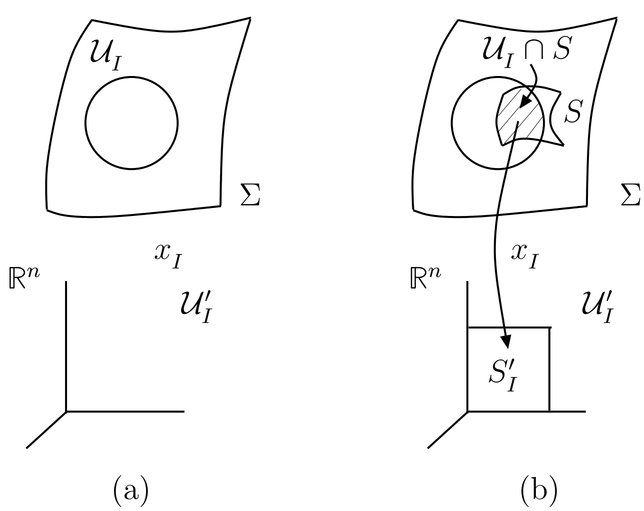


Figure L.17: x_I defined such tha the above is so.

and construct

$$n_{Ij} \circ x_I^{-1} : S'_I \to \mathbb{R}.$$

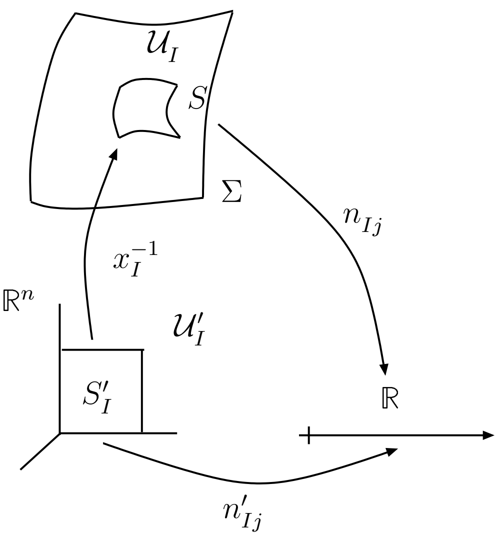


Figure L.18: .

To extend n'_{Ij} to U'_I, let $f' : \mathbb{R} \to \mathbb{R}$ be an arbitrary semianalytic function subject to $f'(0) := 1$ and such that

$$\tilde{n}'_{Ij}(x^1, \ldots, x^D) := n'_{Ij}(x^1, \ldots, x^{D-1}) f'(x^D)$$

From all this we can now define a map $\varphi'_t : \mathbb{R}^D \to \mathbb{R}^D$, where t is a real parameter by

$$\varphi'_t(x) = (x^1 + t\tilde{n}'_{Ij}(x^1, \ldots, x^D), x^2, \ldots, x^D). \tag{L.-2}$$

$$\det\left(\frac{\partial\varphi'_t(x)}{\partial x}\right) = 1 + t\frac{\partial\tilde{n}'_{Ij}(x)}{\partial x^1} = 1 + tf'(x^D)\frac{\partial n'_{Ij}(x^1, \ldots, x^{D-1})}{\partial x^1} \tag{L.-2}$$

The function $f'\partial n'_{Ij}/\partial x^1$ has compact support in U'_I and is at least continuous there. Thus, it is uniformally bounded whence there exists $t_0 > 0$ such that $1 + tf'\partial n'_{Ij}/\partial x^1 > 0$ for all $0 < t < t_0$.

Hence φ'_t is locally a semianalytic (since f', n'_{Ij}, x^k_I are semianalytic) diffeomorphism, provided $0 < t < t_0$. It is also a global diffeomorphism because outside of U'_I it acts as the identity.

A map which is identity in a subset will fail to preserve the complement only if it is not surjective and injective. That φ'_t preserves U'_I then follows from the fact that diffeomorphims are always bijective.

□

Lemma L.11.3

$$N_{Ij} \circ \varphi_t^{-1} = N_{Ij} + tfn_{Ij}.$$

Proof: We construct a semianalytic function N'_{Ij} with support in U'_I such that

$$N'_{Ij}(x^1,\ldots,x^D) = x^1 \qquad \text{whenever} \qquad (x^1,\ldots,x^D) \in \text{supp } n'_{Ij}f'. \tag{L.-2}$$

This is easily done by using an appropriate partition of unity.

We compute

$$\begin{aligned}
[(\varphi_t'^*)N'_{Ij}](x^1,\ldots,x^D) &= N'_{Ij}(x^1 + t\tilde{n}'_{Ij}(x^1,\ldots,x^D), x^2,\ldots,x^D) \\
&= \begin{cases} N'_{Ij}(x^1 + t\tilde{n}'_{Ij}, x^2,\ldots,x^D) & x \in \text{supp } (\tilde{n}'_{Ij}) \\ N'_{Ij}(x^1, x^2,\ldots,x^D) & x \notin \text{supp } (\tilde{n}'_{Ij}) \end{cases} \\
&= \begin{cases} x^1 + t\tilde{n}'_{Ij}(x) & x \in \text{supp } (\tilde{n}'_{Ij}) \\ N'_{Ij}(x^1, x^2,\ldots,x^D) & x \notin \text{supp } (\tilde{n}'_{Ij}) \end{cases} \\
&= \begin{cases} N'_{Ij}(x) + t\tilde{n}'_{Ij}(x) & x \in \text{supp } (\tilde{n}'_{Ij}) \\ N'_{Ij}(x^1, x^2,\ldots,x^D) & x \notin \text{supp } (\tilde{n}'_{Ij}) \end{cases} \\
&= N'_{Ij}(x) + t\tilde{n}'_{Ij}(x)
\end{aligned} \tag{L.-5}$$

Let us deonte by $N_{Ij}, n_{Ij}, f, \varphi_t$ the pull-back by x_I of $N'_{Ij}, n'_{Ij}, f', \varphi'_t$. Since x_I is a bijection and N'_{Ij}, n'_{Ij} have compact support in U'_I, it follows that $N_{Ij}, \tilde{n}_{Ij} = fn_{Ij}$ have compact support $U_I = x^{-1}(U'_I)$. We may thus extend them to all of Σ by setting them equal to zero outside of U_I. Likewise, φ_t equals the identity outside of U_I and preserves U_I for $0 < t < t_0$. Furthermore (L.-5) translates into

$$(\varphi_t)^* N_{Ij} = N_{Ij} + tfn_{Ij}.$$

Notice that

$$[\varphi'_t(x)]^D = x^D$$

preserves $x^D = 0$, hence it preserves S'_I and therefore φ_t preserves $S_I = U_I \cap S$. Since it is the identity outside of U_I, φ_t and its inverse are diffeomorphisms which preserve S. Also we see that $f = 1$ on S_I since $f' = 1$ when $x^D = 0$.

□

$$\int_{\overline{\mathcal{A}}} X_{S,f}(\Psi) d\mu = 0 \tag{L.-5}$$

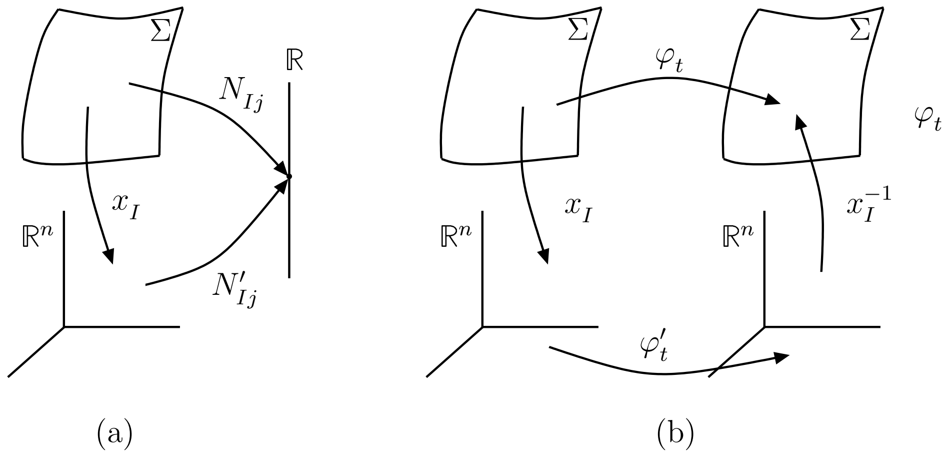


Figure L.19: (a) Pull-back N_{Ij} of N'_{Ij} under x_I is given by $N'_{Ij} \circ x_I$. (b) Pull-back φ_t of φ'_t under x_I is given by $x_I^{-1} \circ \varphi'_t \circ x_I$

Let us recall, that any regular, Borel, probability measure μ on the space $\overline{\mathcal{A}}$ is uniquely determined by its projections μ_γ on the spaces $\overline{\mathcal{A}}_\gamma$. Therefore, to prove the lemma it is enough to find out what restrictions are imposed by ()

Lemma L.11.4 *Every compact connected Lie group, G is isomorphic to a quotient $\tilde{G}/M$, where M is a central discrete subgroup of $\tilde{G}$, and $\tilde{G}$ is a simple product*

$$\tilde{G} = T \times P,$$

(that is, any $h \in \tilde{G}$ can be written as tp where $t \in T$ and $p \in P$), of an abelian group T and a semisimple group P.

Proof:

Recall that each Lie group possesses a Lie algebra g isomorphic to the tangent vector space at the identity element of the Lie group. An ideal in a Lie algebra is a Lie subalgebra $h \subset g$ such that $[X, Y] \in h$ for all $X \in h$, $Y \in g$. An ideal is said to be an invariant subalgebra.

An ideal is the Lie algebra equivalent of a closed, normal subgroup of a connected Lie group.

A connected Lie group can be defined to be simple if its Lie algebra is simple, or equivalently, if it contains no non-trivial, closed, connected normal subgroups. Under this definition, a simple connected Lie group can possess non-trivial, closed, normal subgroups, but if they exist they must be discrete.

A semisimple Lie algebra can be defined as a Lie algebra which has no non-trivial abelian ideals, but here we wish to characterise it as a Lie algebra which is the direct sum of simple Lie algebras. Semisimple Lie groups arae the direct products of simple Lie groups.

Clearly, a simple Lie algebra is semisimple.

Lemma L.11.5

Proof:

Let us consider an arbitrary graph γ consisting of edges $\{e_1, \dots, e_N\}$. Divide each edge e_I

$$e_I = e_{I,1} \circ e_{I,2}$$

see fig (L.20)

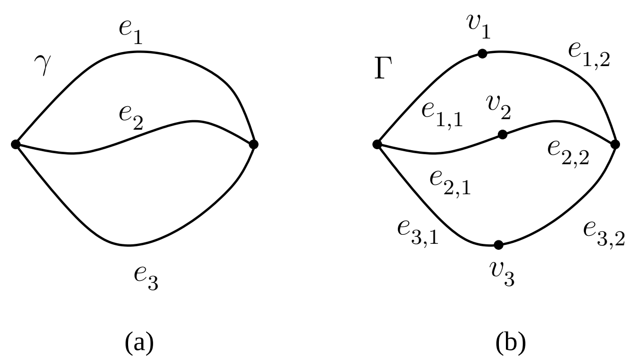


Figure L.20: $e_1 = e_{1,1} \circ e_{1,2}$

$$e_1 = e_{1,1} \circ e_{1,2}$$

where:

$$g_i(\overline{A}) := (\overline{A}(e_{1,i}), \dots, \overline{A}(e_{1,N})) \in G^N$$

$$\Psi_\Gamma = \psi(g_1(), g_2)$$

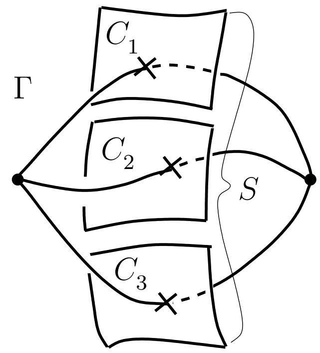


Figure L.21: We let S consist of 3 disjoint cubes C_1, C_2, C_3.

Semisimple case

Suppose the function f is defined on S in the following way

$$f\Big|_{C_I} := const_I f_I \in P'$$

Consider the operator

$$\hat{X}_{S,f} = \sum_{I=1}^{N} \hat{X}_{C_I,f_I}.$$

Assuming that Ψ_Γ is a smooth cylindrical function,

$$\begin{aligned}
0 &= \int_{\overline{A}} \hat{X}_{S,f}\Psi_\Gamma \, d\mu \\
&= \int_{\overline{A}_\Gamma} \hat{X}_{S,f}\psi \, d\mu_\Gamma \\
&= \sum_{I=1}^{N} \int_{\overline{A}_\Gamma} \hat{X}_{C_I,f_I}\psi \, d\mu_\Gamma \\
&= -\frac{i}{2}\int_{G^{2N}} \frac{d}{ds}\Big|_{s'=0} \psi(g_1 \exp(\vec{f}s'), \exp(\vec{f}s')g_2) \, d\mu_\Gamma \\
&= -\frac{i}{2}\frac{d}{ds'}\Big|_{s'=0} \int_{G^{2N}} \psi(g_1 \exp(\vec{f}s'), \exp(\vec{f}s')g_2) \, d\mu_\Gamma
\end{aligned} \quad \text{(L.-8)}$$

where

$$\vec{f} := (f_1, \ldots, f_N)$$

and

$$\exp(\vec{f}s) = (\exp(f_1 s), \ldots, \exp(f_N s)) \in P^N.$$

using this equality on the function

$$\tilde{\psi}(g_1, g_2) := \psi(g_1 \exp(\vec{f}s), \exp(\vec{f}s) g_2)$$

and that

$$\frac{d}{ds'}\Big|_{s'=0} \tilde{\psi}(s' + s) = \frac{d}{ds}\tilde{\psi}(s)$$

results in

$$\begin{aligned} 0 &= \frac{d}{ds'}\Big|_{s'=0} \int_{G^{2N}} \tilde{\psi}(g_1 \exp(\vec{f}s'), \exp(\vec{f}s') g_2)\; d\mu_\Gamma \\ &= \frac{d}{ds'}\Big|_{s'=0} \int_{G^{2N}} \psi(g_1 \exp(\vec{f}(s+s')), \exp(\vec{f}(s'+s)) g_2)\; d\mu_\Gamma \\ &= \frac{d}{ds} \int_{G^{2N}} \psi(g_1 \exp(\vec{f}s), \exp(\vec{f}s) g_2)\; d\mu_\Gamma \end{aligned} \quad \text{(L.-9)}$$

As the group is connected

$$\int_{G^{2N}} \psi(g_1 b, b g_2)\; d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\; d\mu_\Gamma \quad \text{(L.-9)}$$

for every $b \in P^N$.

$$\zeta(a, a') := \int_{G^{2N}} \psi(g_1 a, a' g_2)\; d\mu_\Gamma$$

Equation (L.11.7) implies

$$\zeta(ba, a'b) = \zeta(a, a') \quad \text{(L.-9)}$$

for every $b \in P^N$. Let $b = a^{-1}$. Then

$$\zeta(a, a') = \zeta(\mathbb{I}_P, a'a^{-1}) =: \xi(a'a^{-1}),$$

where $\mathbb{I}_P$ is the identity element of P^N. Now, () reads

$$\xi(a'ba^{-1}b^{-1}) = \xi(a'a^{-1}).$$

The substitution $b_0 = a'a^{-1}$ gives an identity

$$\xi(b_0 aba^{-1}b^{-1}) = \xi(b_0), \tag{L.-9}$$

which holds for every $a, b, b_0 \in P^N$. We now use this to prove that the function ξ, and consequently the function ζ, is constant. Let L_1, L_2 be arbitrary left invariant vector fields on P^N, then the vector $[L_1, L_2]_{b_0}$ tangent to P^N at the point b_0 can be generated by a curve of the form

$$b_0 a(t) b(t) a^{-1}(t) b^{-1}(t)$$

for t small has the geometric interpretation of fig (L.22). Thus (L.11.7)

$$[L_1, L_2]_{b_0} \xi = 0$$

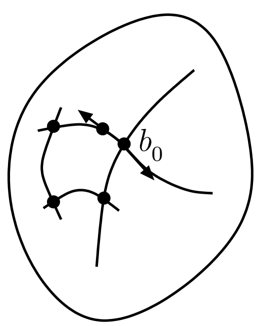


Figure L.22:

The commutator of two left invariant vector fields is also a left invariant vector field. This can easily be seen by fig (). The importance of this is that left invariant vector fields form a Lie algebra are isomorphic to the Lie algebra P'^N as the Lie algebra g is isomorphic to the tangent vector space at the identity element of the Lie group..

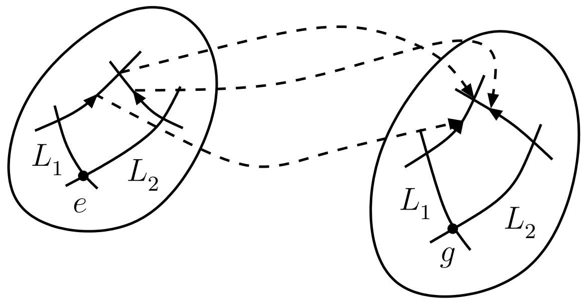


Figure L.23:

As the group P^N is semisimple, the algebra satisfies $[P'^N, P'^N] = P'^N$. So for arbitrary left invariant vector field L

$$L_{b_0}\xi = 0$$

Hence the function ξ and consequently the function ζ are both constant. Thus

$$\int_{G^{2N}} \psi(g_1 b_1, b_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every smooth function ψ on G^{2N} and for every $b_1, b_2 \in P^N$.

Abelian case

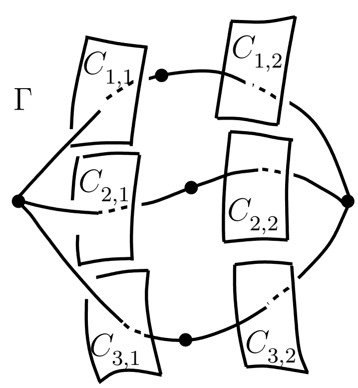


Figure L.24: We let S consist of 6 disjoint cubes $C_{1,1}, C_{2,1}, C_{3,1}, C_{1,2}, C_{2,2}, C_{3,2}$.

the operator

$$\hat{X}_{S,f} = \sum_{I=1}^{N} \hat{X}_{C_{I,i},f_{I,i}}.$$

$$\int_{G^{2N}} \psi(g_1 t_1, t_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every smooth function ψ on G^{2N} and for every $t_1, t_2 \in T^N$.

Combined

$$\int_{G^{2N}} \psi(g_1 t_1 b_1, t_2 b_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma$$

for every $t_i b_i \in T^N \times P^N$, that is

$$\int_{G^{2N}} \psi(g_1 h_1, h_2 g_2)\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_\Gamma \tag{L.-9}$$

for every $h_i \in \tilde{G}^N$

Obviously the space $C^\infty(G^{2N}, \mathbb{C})$ separates the points of G^{2N} and includes the constant functions, so the Stone-Weiestrass theorem applies, showing that the closure $C^\infty(G^{2N}, \mathbb{C})$ with respect to the sup-norm is $C^0(G^{2N}, \mathbb{C})$ and therefore equation (L.11.7) holds for every $\psi \in C^0(G^{2N}, \mathbb{C})$.

If we swap the function $\psi(g_1, g_2)$ by the function $\psi(g_1, g_2^{-1})$ then

$$\int_{G^{2N}} \psi(g_1 h_1, (h_2 g_2)^{-1})\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1 h_1, g_2^{-1} h_2^{-1})\, d\mu_\Gamma = \int_{G^{2N}} \psi(g_1, g_2^{-1})\, d\mu_\Gamma$$

consider the map

$$\omega(g_1, g_2) \mapsto \omega(g_1, g_2) := (g_1, g_2^{-1})$$

with the push forward measure

$$\mu_\Gamma^* := \omega^* \mu_\Gamma.$$

then

$$\int_{G^{2N}} \psi(g_1, g_2)\, d\mu^*_\Gamma = \int_{G^{2N}} \psi(g_1 h_1, g_2 h_2^{-1})\, d\mu^*_\Gamma \tag{L.-8}$$

for every $(h_1, h_2) \in G^N \times G^N$.

The Haar measure μ^*_Γ on G^{2N} is a product of two copies of the Haar measure μ_H on G^N. Since G is compact, left and right invaraint measures on the group coincide. Therefore we can write

$$\begin{aligned} \int_{G^{2N}} \psi(g_1, g_2)\, d\mu_H^{(1)} d\mu_H^{(2)} &= \\ &= \int_{G^{2N}} \psi(g_1, g_2^{-1})\, d\mu_H^{(1)} d\mu_H^{(2)} \end{aligned} \tag{L.-8}$$

and so

$$\mu^*_\Gamma = \mu_\Gamma.$$

Then gives for every $(h_1, h_2) \in G^N \times G^N$

$$\int_{G^{2N}} \psi(g_1, g_2) d\mu_\Gamma = \int_{G^{2N}} \psi(g_1 h_1, g_2 h_2) d\mu_\Gamma.$$

Every graph Γ is obtained by a subdivision of some graph γ, hence every cylindrical function Ψ compatible with γ is also compatible with Γ. Recall that if Ψ is compatible with two grpahs γ, γ', then

$$\int_{\overline{A}_\gamma} \psi d\mu_\gamma = \int_{\overline{A}_{\gamma'}} \psi' d\mu_{\gamma'},$$

and we conclude the push forward measure is the Haar measure on $G^{|E(\gamma)|}$. As the graph γ is arbitrary,

$$\mu = \mu_{AL}.$$

$\square$

L.12 Irreduciblity of the Ashtekar-Lewandowski Representation

First let γ be a graph and split each edge $e \in E(\gamma)$ into two halves $e = e'_1 \circ (e'_2)^{-1}$ and replace the e's the e'_1, e'_2. We obtain a graph γ' which occupies the same points in Σ as γ but changes the set of edges of γ in such a way that each edge is outgoing from the vertex $b(e') = v \in V(\gamma)$. We call a graph refined in this way a standard graph. The reason for using this freedom is to simplify the following discussion. For notational simplicity we denote standard graphs as γ from now on.

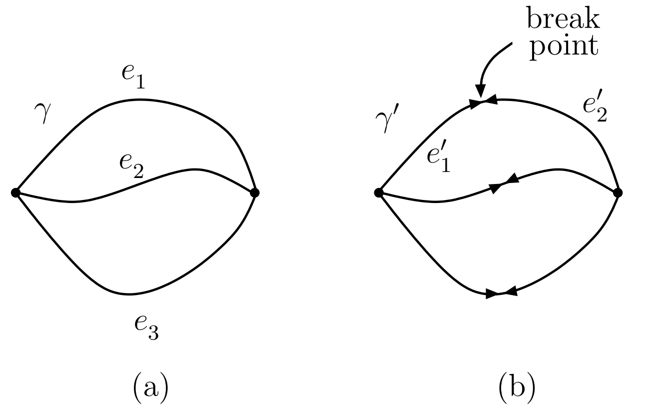


Figure L.25: The graph γ' is the standard graph associated with the original graph γ.

Lemma L.12.1 *Let γ be a standard graph. Assign to each $e \in E(\gamma)$ a vector $t_e = (t_e^j)_{j=1}^{dim(G)}$ and collect them into a label $t_\gamma = (t_e)_{e\in E(\gamma)}$.*

Then there exists a vector field $Y(t_\gamma, \gamma)$ in the Lie algebra of the flux fields $Y_{S,f}$ such that for any cylindrical function $f = p^*_\gamma f_\gamma$ over γ we have

$$Y_\gamma(t_\gamma) p^*_\gamma f_\gamma = p^*_\gamma \sum_{e\in E(\gamma)} t_e^j R_j^e f_\gamma. \tag{L.-8}$$

Any compact connected Lie group G has the structure $G/Z = A \times S$ where Z is a discrete subgroup, A is an abelean Lie group and S is a semisimple Lie group.

Abelean case:

Consider any $e \in E(\gamma)$ and take any surface S_e which intersects γ only

$$Y_j(S_e) p^*_\gamma f_\gamma = p^*_\gamma [R^j_{e_2} - R^j_{e_1}] f_\gamma \tag{L.-8}$$

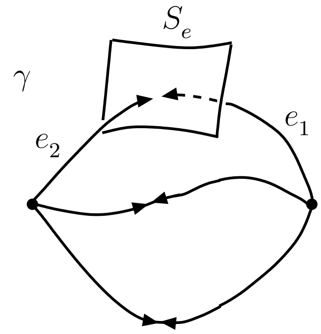


Figure L.26: Abelean case

Due to gauge invariance

$$(\tau^i h_{e_1})^A_B \frac{\partial f}{\partial (h_{e_1})^A_B} + (\tau^i h_{e_2})^A_B \frac{\partial f}{\partial (h_{e_2})^A_B} = 0$$

$$[R^j_{e_1} + R^j_{e_2}] f_\gamma = 0,$$

thus

$$Y^j_e p^*_\gamma f_\gamma = \frac{1}{2} Y_j(S_e) p^*_\gamma f_\gamma \tag{L.-8}$$

is an appropriate choice.

Non-Abelean case:

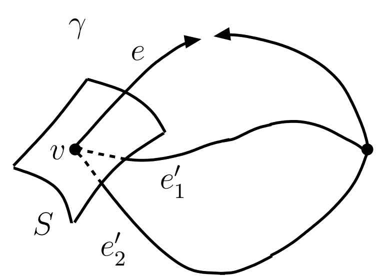


Figure L.27: Non-Abelean case

An analytic surface S is completely determined by its Taylor coefficients in the expansion of its parameterisation

$$S(u,v) = \sum_{m,n=0}^{\infty} \frac{u^m v^n}{m!n!} S^{(m,n)}(0,0) \tag{L.-8}$$

e is determined

$$e(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{(n)}(0) \tag{L.-8}$$

In order that $s \subset S$ we just need to choose a parameterisation of S such that $S(t,0) = e(t)$ which fixes the Taylor coefficients

$$S^{(m,0)}(0,0) = e^{(m)}(0) \tag{L.-8}$$

for all m. Say that the other edges $e'_1, \ldots e'_n$ were to have a beginning segment s_k of e'_k in S then there would be an analytic function $v_k(t)$, such that

$$s_k(t) = S(t, v_k(t)).$$

Obviously, we can not have $v_k(t) = 0$ in an arbitrary small neighbourhood of $t = 0$ otherwise $s_k = s_e$. For each k let n_k be the first derivative such that $v_k^{(n_k)}(0) \neq 0$. By relabeling the edges we may arrange that $n_1 \leq n_2 \leq \cdots \leq n_N$. Consider $k = 1$ and take the n_1-th derivative at $t = 0$. We find

$$\frac{d^{n_1}}{dt^{n_1}} s_1(0) = S^{(n_1,0)}(0,0) + S^{(0,1)}(0,0) \frac{d^{n_1}}{dt^{n_1}} v_1(0)$$

Since $v_1^{(n_1)} \neq 0$ we can arrange the surface S, by using the freedom in $S^{(n_1,0)}(0,0)$, so that this equation does not hold and hence s_k is not in S.

$$\frac{d^{n_2+1}}{dt^{n_2+1}} s_1(0) = S^{(n_1,0)}(0,0) + 2S^{(1,1)}(0,0) \frac{d^{n_1}}{dt^{n_1}} v_2(0) + S^{(0,1)}(0,0) \frac{d^{n_2+1}}{dt^{n_2+1}} v_2(0)$$

Since $v_2^{(n_2)} \neq 0$ we can use the freedom in $S^{(1,1)}$ in order to violate this equation. Proceeding in this way we can use the coefficients $S^{(k-1,1)}$ in order for the edges to be transversal to S.

Having constructed the surfaces $S_{v,e}$ we can compute the associated vector field applied to a cylindrical function over γ

$$Y_j(s_{v,e})p_\gamma^* f_\gamma = p_\gamma^* \sum_{e'\in E(\gamma)-\{e\},b(e')=v} \sigma(s_{v,e},e')R^j_{e'} f_\gamma \tag{L.-8}$$

Taking the commutator

$$\begin{aligned}
[Y_j(s_{v,e}),Y_k(s_{v,e})]p_\gamma^* f_\gamma &= Y_j(s_{v,e})p_\gamma^* \sum_{e'} \sigma(s_{v,e},e')R^k_{e'} f_\gamma - (j\leftrightarrow k) \\
&= p_\gamma^* \sum_{e'} \sigma(s_{v,e},e') \sum_{e''} \sigma(s_{v,e},e'')R^l_{e''}R^k_{e'} f_\gamma - (j\leftrightarrow k) \\
&= f_{jkl}\, p_\gamma^* \sum_{e'} R^l_{e'} f_\gamma
\end{aligned} \tag{L.-9}$$

where we used

$$[R^j_{e''},R^k_{e'}] = \delta_{e'',e'} f_{jkl} R^l_{e'}.$$

$$R^j_v := \sum_{e'\in E(\gamma),b(e')=v} R^j_{e'}$$

we get

$$f_{jkl}[Y_k(s_{v,e}),Y_l(s_{v,e})]p_\gamma^* f_\gamma = p_\gamma^*[R^j_v - R^j_e] f_\gamma$$

This, if n_v is the valence of v

$$\begin{aligned}
Y^j_e p_\gamma^* f_\gamma &:= \Big(-f_{jkl}[Y_k(s_{v,e}),Y_l(s_{v,e})] + \frac{1}{n_v-1}\sum_{e'\in E(\gamma)} f_{jkl}[Y_k(s_{v,e'}),Y_l(s_{v,e'})]\Big)p_\gamma^* f_\gamma \\
&= -p_\gamma^*[R^j_v - R^j_e]f_\gamma + \frac{1}{n_v-1}\sum_{e'\in E(\gamma),b(e')=v} p_\gamma^*[R^j_v - R^j_{e'}]f_\gamma \\
&= -p_\gamma^*[R^j_v - R^j_e]f_\gamma + (\frac{n_v}{n_v-1} - 1)R^j_v p_\gamma^* f_\gamma \\
&= p_\gamma^* R^j_e f_\gamma
\end{aligned} \tag{L.-11}$$

Collecting the vector fields Y^j_e for the Abelean and non-Abelean labels j respectively and contracting them with t^j_e and summing over $e \in E(\gamma)$ yields an appropriate vector field

$$Y_\gamma(t_\gamma) = \sum_{e\in E(\gamma)} t_e^j Y_e^j. \qquad \text{(L.-11)}$$

Actually, here we have impiciyly assumed that where no $e \in E(v)$ is (a segment of) the analytic extension through v of another edge $e' \in E(v)$. We also need to consider the case where there is at least one pair of edges $e, \tilde{e} \in E(v)$ that are (segments of) analytic continuations of each other through v. See [31] for details.

$\square$

Recall that the Hilbert space $\mathcal{H}_0$ has an orthonormal basis of particular cylindrical functions - the spin network functions - labeled by a spin network $s = (\gamma, \{\pi_e\}, \{m_e\}, \{n_e\})_{e\in E(\gamma)}$ defined by

$$T_s(A) = \prod_{e\in E(\gamma)} \{\sqrt{d_{\pi_e}}[\pi_e(h_e)]_{m_e n_e}\}$$

where π denotes an irreducible representation of G. Later we will need the right action R_e^j on T_s which is easily computed

$$\begin{aligned} R_e^j T_s &= [\tau^j \pi(h_e)]_{mn} \frac{\partial T_s}{\partial[\pi(h_e)]_{mn}} \\ &= \sqrt{d_{\pi_{e_1}}}[\pi_{e_1}(h_{e_1})]_{m_{e_1} n_{e_1}} \ldots\ [\tau_j]_{m_e l_e} \sqrt{d_{\pi_e}}[\pi_e(h_e)]_{l_e n_e} \ldots \sqrt{d_{\pi_{e_N}}}[\pi_{e_N}(h_{e_N})]_{m_{e_N} n_{e_N}} \end{aligned}$$

We now define for any two $\psi, \psi' \in \mathcal{H}_0$ the function

$$M_{\psi,\psi'}(t_\gamma, I_\gamma) := <\psi, T_{\gamma,I_\gamma}\ W_\gamma(t_\gamma)\psi'>_{\mathcal{H}_0} \qquad \text{(L.-13)}$$

We exploit that for a compact connected Lie group the exponential map is onto.

Thus, there exists a region $D_G \subset \mathbb{R}^{\dim(G)}$ such that $\exp : D_G \to G;\ t \mapsto \exp(t^j \tau_j)$ is a bijection. Consider the measure μ on D_G defined by $d\mu(t) = d\mu_H(\exp(t^j \tau_j))$ where μ_H is the Haar measure on G. Finally, let $D_\gamma = \prod_{e\in E(\gamma)} D_G$ and let L_γ be the space of the I_γ.

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma := \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \overline{M_{\psi_1,\psi_1'}(t_\gamma, I_\gamma)}\ M_{\psi_2,\psi_2'}(t_\gamma, I_\gamma) \qquad \text{(L.-13)}$$

where $d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e)$.

Lemma L.12.2 *i) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ we have*

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \le \|\psi_1\| \, \|\psi_1'\| \, \|\psi_2\| \, \|\psi_2'\| \tag{L.-13}$$

ii) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_{0,\gamma}$ we have

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma = <\psi_2, \psi_1>_{\mathcal{H}_0} <\psi_1', \psi_2'>_{\mathcal{H}_0} \tag{L.-13}$$

where $\mathcal{H}_{0,\gamma}$ denotes the closure of the cylindrical functions over γ.

Proof:

$$\begin{aligned}
(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma &= \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') \overline{T_{\gamma,I_\gamma}(A)} T_{\gamma,I_\gamma}(A') \\
&\quad \psi_1(A) \overline{[W_\gamma(t_\gamma)\psi_1'](A)\psi_2(A')} [W_\gamma(t_\gamma)\psi_2'](A') \\
&= \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') [\sum_{I_\gamma} \overline{T_{\gamma,I_\gamma}(A)} T_{\gamma,I_\gamma}(A')] \ldots \\
&= \int_{\overline{\mathcal{A}}} d\mu_0(A) \int_{\overline{\mathcal{A}}} d\mu_0(A') \int_{D_\gamma} d\mu(t_\gamma) \delta_\gamma(A, A') \ldots
\end{aligned} \tag{L.-15}$$

where we have defined the cylidrical $\delta-$distribution

$$\delta_\gamma(A, A') = \prod_{e \in E(\gamma)} \delta_{\mu_H}(h_e[A], h_e[A'])$$

which comes from the Plancherel formula

$$\delta_{\mu_H}(g, g') = \sum_{\pi,m,n} \overline{T_{\pi,m,n}(g)} T_{\pi,m,n}(g').$$

$$f(A) = F(A_{|\overline{\gamma}}, A_{|\gamma}) \tag{L.-15}$$

the (effective) measure on $\overline{A}_{\overline{\gamma}}$ by

$$\int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \left[\int_{\overline{A}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) F(A_{|\overline{\gamma}}, A_{|\gamma}) \right] = \int_{\overline{A}} d\mu_0(A) f(A). \tag{L.-15}$$

all occuring f are countable linear combinations of spin network functions

$$T_s = \prod_{e' \in E(\gamma\cup\gamma(s))-E(\gamma)} \{\sqrt{d_{\pi_{e'}}}[\pi_{e'}(h_{e'})]_{m_{e'}n_{e'}}\} \prod_{e\in E(\gamma)} \{\sqrt{d_{\pi_e}}[\pi_e(h_e)]_{m_e n_e}\}.$$

Thus either integral can be written as a countable linear combination of integrals over spin-network functions T_s and then the prescription is to integrate only either over the degrees of freedom $A(e), e \in E(\gamma)$ or $A(e'), e' \in E(\gamma(s)\cup\gamma) - E(\gamma)$ for each individual integral with the corresponding product Haar measure.

$$\begin{aligned}(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma &= \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}}) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\ &\times \Psi_1(A_{|\overline{\gamma}}, A_\gamma)\overline{[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}}, A_\gamma)\Psi_2(A'_{|\overline{\gamma}}, A_\gamma)} \\ &\times [W_\gamma(t_\gamma)\Psi_2'](A'_{|\overline{\gamma}}, A_\gamma)\end{aligned} \tag{L.-17}$$

In order to evaluate the Weyl operators, consider a spin network function T_s cylindrical over $\gamma(s)$ which we write in the form

$$T_s(A) = F(\{h_{e'}\}_{E(\gamma\cup\gamma(s))-E(\gamma)}, \{h_e\}_{e\in E(\gamma)}) \tag{L.-17}$$

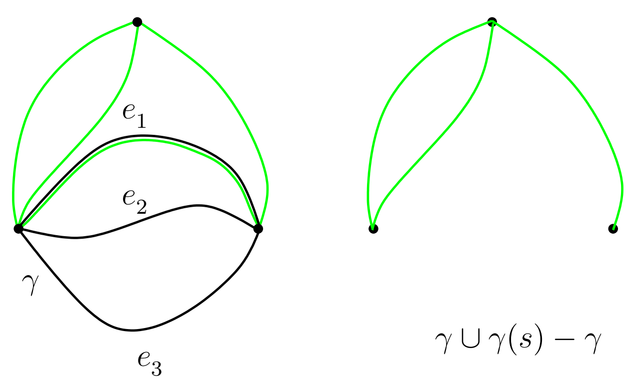


Figure L.28:

We know how the vector field $Y_\gamma(t_\gamma)$ acts on functions cylindrical over γ, (L.12.1), but how does it act on $\gamma(s)\cup\gamma-\gamma$?

it is easy to see that the action of $Y_\gamma(t_\gamma)$ on T_s is given by

$$Y_\gamma(t_\gamma)T_s = p^*_{\gamma(s)\cup\gamma}[\sum_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)} t_j^{e'}(t_\gamma)R^j_{e'} + \sum_{e\in E(\gamma)} t^e_j R^j_e]F \tag{L.-17}$$

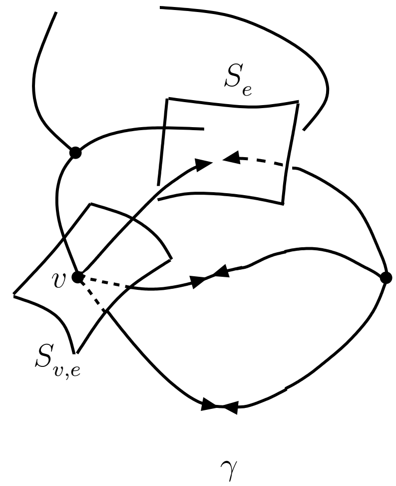


Figure L.29: Action of $Y_\gamma(t_\gamma)$ on T_s

where $t_j^{e'}(t_\gamma)$ is a certain linear combination of the t_j^e depnding on e' and the concrete surfaces S_e, $S_{v,e}$ used in the construction of $Y_\gamma(t_\gamma)$.

$$Y_\gamma(t_\gamma)\, T_s \;=\; F[\{t_j^{e'}(t_\gamma)\tau_j h_{e'}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)} + \{t_j^e\}_{e\in E(\gamma)}\tau_j h_e] \qquad \text{(L.-16)}$$

so that

$$\begin{aligned} W_\gamma(t_\gamma)T_s &= \sum_{m=0}^\infty \frac{1}{m!} Y_\gamma(t_\gamma)^m\, T_s \\ &= F(\{e^{t_j^{e'}(t_\gamma)\tau_j} h_{e'}\}_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)}, \{e^{t_j^e\tau_j} h_e\}_{e\in E(\gamma)}) \qquad \text{(L.-16)} \end{aligned}$$

the map $\alpha_{t_\gamma} : \overline{\mathcal{A}} \to \overline{\mathcal{A}};\, A \mapsto W_\gamma(t_\gamma) A W_\gamma(t_\gamma)$ is just some right or left translation.

$$\begin{aligned} |(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \;\le\; & \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{A}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\ & \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|\; |\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))| \\ & \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}}) |\Psi_2(A'_{|\overline{\gamma}}, A_\gamma)|\; |\Psi_2'(\alpha_{t_\gamma}(A'_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|. \end{aligned} \qquad \text{(L.-18)}$$

Consider the second line on the R.H.S., by the Cauchy-Schwarz inequality applied to functions $[\Psi_1(A_\gamma)](A_{|\overline{\gamma}}) = \Psi_1(A_{|\overline{\gamma}}, A_{|\gamma})$ in $L_2(\overline{\mathcal{A}}_{\overline{\gamma}}, d\mu_{0\overline{\gamma}})$ we can estimate

$$\begin{aligned}&\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|\,|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|\\ \leq\ & \Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|^2\Big)^{1/2}\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|^2\Big)^{1/2}.\end{aligned} \tag{L.-19}$$

A similar result holds for functions $[\Psi_2(A_\gamma)](A'_{|\overline{\gamma}}) = \Psi_2(A'_{|\overline{\gamma}}, A_{|\gamma})$ in $L_2(\overline{\mathcal{A}}'_{\overline{\gamma}}, d\mu_{0\overline{\gamma}})$. From now on let us use the notation

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1(A_{|\overline{\gamma}}, A_\gamma)|^2 = \|\Psi_1(A_\gamma)\|^2_{|\overline{\gamma}}.$$

We can simplify the second factor on the R.H.S. of (L.-19) from the fact that

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|^2 = \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_\gamma))|^2 = \|\Psi_1'(\alpha_{t_\gamma}(A_\gamma))\|^2_{|\overline{\gamma}}$$

To prove this expand ψ_1' into spin-network functions

$$\psi_1'(A) = \Psi_1'(A_{|\overline{\gamma}}, A_{|\gamma}) = \sum_{n=1}^\infty z_n T_{s_n}(A).$$

Then the integral becomes

$$\begin{aligned}&\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})|\Psi_1'(\alpha_{t_\gamma}(A_{|\overline{\gamma}}), \alpha_{t_\gamma}(A_\gamma))|^2\\ &= \sum_{m,n=1}^\infty \overline{z}_m z_n \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\overline{T_{s_m}(\alpha_{t_\gamma}(A))}T_{s_n}(\alpha_{t_\gamma}(A)).\end{aligned} \tag{L.-19}$$

The integration with measure $d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})$ over $\overline{\mathcal{A}}_{\overline{\gamma}}$ reduces to integration with the Haar measure over the space $G^{|E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma)-E(\gamma)|}$. Note first that by the bi-invariance of the Haar measure, for any $e' \in E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma) - E(\gamma)$ it follows then, writing $E_{mn}(\gamma) \equiv E(\gamma(s_m)\cup\gamma(s_n)\cup\gamma) - E(\gamma)$, that

$$
\begin{aligned}
& \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \overline{T_{s_m}(\alpha_{t_\gamma}(A))}\, T_{s_n}(\alpha_{t_\gamma}(A)) \\
= & \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{G^{|E_{mn}(\gamma)|}} \prod_{e'\in E_{mn}(\gamma)} d\mu_H(h_{e'}) \overline{T_{s_m}(\{e^{t_j^{e'}(t_\gamma)\tau_j} h_{e'}\}, \{e^{t_j^e \tau_j} h_e\})}\, T_{s_n}\{e^{t_j^{e'}(t_\gamma)\tau_j}(h_{e'}\}, \{e^{t_j^e \tau_j} h_e\}) \\
= & \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{G^{|E_{mn}(\gamma)|}} \prod_{e'\in E_{mn}(\gamma)} d\mu_H(h_{e'}) \overline{T_{s_m}(\{h_{e'}\}, \{e^{t_j^e \tau_j} h_e\})}\, T_{s_n}(h_{e'}\}, \{e^{t_j^e \tau_j} h_e\}) \\
= & \sum_{m,n=1}^{\infty} \overline{z}_m z_n \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \overline{T_{s_m}(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_{|\gamma}))}\, T_{s_n}(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_{|\gamma})) \\
= & \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi'_1(A_{|\overline{\gamma}}, \alpha_{t_\gamma}(A_{|\gamma}))|^2 \qquad \text{(L.-22)}
\end{aligned}
$$

We now exploit that

$$\alpha_{t_\gamma}(A_{|\gamma}) = \{e^{t_j^e \tau_j} A(e)\}_{e\in E(\gamma)}$$

and introduce new integration variables $A'(e) := g(t_e)A(e)$ where $g(t_e) = \exp(t_j^e \tau_j)$. Since by definition

$$d\mu(t_\gamma) = \prod_{e\in E(\gamma)} d\mu(t_e) = \prod_{e\in E(\gamma)} d\mu_H(g(t_e))$$

we can estimate further

$$
\begin{aligned}
|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \;\leq\; & \int_{G^{|E(\gamma)|}} \prod_{e\in E(\gamma)} d\mu_H(g_e) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\
\times\; & \|\Psi_1(A_{|\gamma})\|_{|\overline{\gamma}}\, \|\Psi'_1(\{g_e A(e)\}_{e\in E(\gamma)})\|_{|\overline{\gamma}} \\
\times\; & \|\Psi_2(A_{|\gamma})\|_{|\overline{\gamma}}\, \|\Psi'_2(\{g_e A(e)\}_{e\in E(\gamma)})\|_{|\overline{\gamma}} \\
=\; & \left[\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\; \|\Psi_1(A_{|\gamma})\|_{\overline{\gamma}}\;\; \|\Psi_2(A_{|\gamma})\|_{\overline{\gamma}}\right] \\
\times\; & \left[\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma})\; \|\Psi'_1(A'_{|\gamma})\|_{\overline{\gamma}}\;\; \|\Psi'_2(A'_{|\gamma})\|_{\overline{\gamma}}\right] \\
\leq\; & \|\, \|\Psi_1\|_{\overline{\gamma}}\,\|_\gamma\, \|\, \|\Psi'_1\|_{\overline{\gamma}}\,\|_\gamma \|\, \|\Psi_2\|_{\overline{\gamma}}\,\|_\gamma\, \|\, \|\Psi'_2\|_{\overline{\gamma}}\,\|_\gamma \qquad \text{(L.-26)}
\end{aligned}
$$

where we have used Fubini's theorem and have again applied the Cauchy-Schwarz inequality to functions in $L_2(\overline{\mathcal{A}}_\gamma, d\mu_{0\gamma})$. But

$$
\begin{aligned}
\| \, \|\Psi_1\|_{\overline{\gamma}} \, \|^2_{\gamma} &= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \mid \|\Psi_1(A_{|\gamma})\|_{\overline{\gamma}} \mid^2 \\
&= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi_1(A_{|\overline{\gamma}}, A_{|\gamma})|^2 \\
&= \int_{\overline{\mathcal{A}}} d\mu_0(A) |\psi_1(A)|^2 = \|\psi_1\|^2_{\mathcal{H}_0}
\end{aligned}
\tag{L.-28}
$$

ii)

If all functions in question are cylindrical L_2-functions over γ then the integrals over $\overline{\mathcal{A}}_{|\overline{\gamma}}$ are trivial and () simplifies to

$$
\begin{aligned}
(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma &= \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \Psi_1(A_\gamma) \overline{[W_\gamma(t_\gamma)\Psi_1'](A_\gamma)\Psi_2(A_\gamma)} [W_\gamma(t_\gamma)\Psi_2'](A_\gamma) \\
&= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma}) \Psi_1(A_\gamma) \overline{\Psi_1'(A'_\gamma)\Psi_2(A_\gamma)} \Psi_2'(A'_\gamma) \\
&= [\int_{\overline{\mathcal{A}}} d\mu_0(A) \overline{\psi_2(A)} \psi_1(A)] \, [\int_{\overline{\mathcal{A}}} d\mu_0(A') \overline{\psi_2'(A')} \psi_1'(A')] \\
&= < \psi_2, \psi_1 >_{\mathcal{H}_0} < \psi_1', \psi_2' >_{\mathcal{H}_0}
\end{aligned}
\tag{L.-30}
$$

□

Theorem L.12.3

Proof:

Proof was given in chapter 3.

□

L.12.1 Fleischhack

Regular: Weyl representation is weakly continuous - said to be regular.

Stone-von Neuman theorem says that if a representation is regular and irreducible then the representation is unique.

Quantum geometry:

1. diffeomorphism invariant;
2. regular;
3. irreducible;
4. semianalytic - stratified diffeomorphisms.

L.12.2 Properties of the Kinematic Hilbert Space

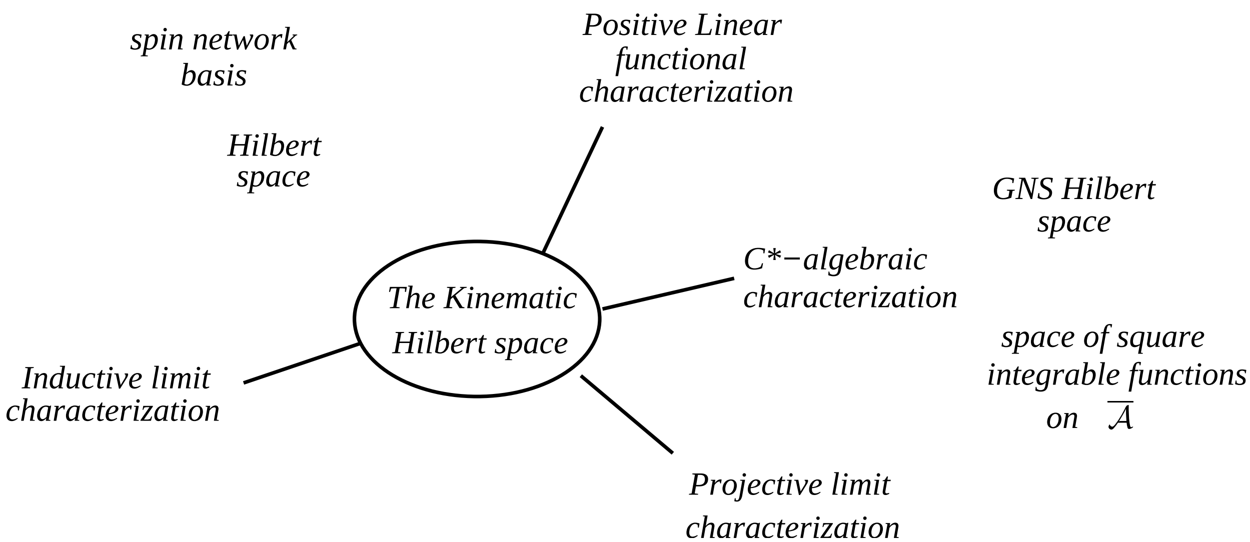


Figure L.30: KinHilbFig.

Spin networks states provide a natural decomposition of $\tilde{\mathcal{H}}^0$ into finite dimensional subspaces each of which can be identified with the space of states of a spin-system. This simplifies various constructions and calculations enermoursly.

L.13 Grassmann Integration

Grassmann Algebra

We consider a set of anticommutating Grassmann variables $\{\zeta_i\}_{i=1,\dots,n}$, with complex linear coefficients, where n is the dimension of the algebra. The decisive relation defining the structure of the algebra is the anticommutation relation

$$\zeta_i \zeta_j + \zeta_j \zeta_i = 0 \tag{L.-30}$$

for all i and j. As a particular consequence of this condition the square and all higher powers of a variable vanish,

$$\zeta_i^2 = 0 \tag{L.-30}$$

The Grassmann algebra generate a Grassmann algebra of functions which have the form

$$f(\zeta) = f^{(0)} + \sum_i f_i^{(1)} + \sum_{i_1<i_2} f_{i_1 i_2}^{(2)} \zeta_{i_1} \zeta_{i_2} + ... + f^{(n)} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_n} \tag{L.-30}$$

where the coefficients $f^{(k)}$ are ordinary complex numbers.

On this algebra we will need to define the derivative. We first consider a simple Grassmann algebra of order n=2 with the variables ζ_1 and ζ_2.

$$f(\zeta_1, \zeta_2) = f^{(0)} + f_1^{(1)} \zeta_1 + f_2^{(1)} \zeta_2 + f^{(2)} \zeta_1 \zeta_2$$

$$\frac{\partial f}{\partial \zeta_1} = f_1^{(1)} + f^{(2)} \zeta_2, \qquad \frac{\partial f}{\partial \zeta_1} = f_2^{(1)} - f^{(2)} \zeta_1. \tag{L.-30}$$

Note the minus sign in the last equation of (L.13). The general rule for differentiation of a product is given by

$$\frac{\partial}{\partial \zeta_j} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_m} = \delta_{j i_1} \zeta_{i_2} ... \zeta_{i_m} - \delta_{j i_2} \zeta_{i_1} \zeta_{i_3} ... \zeta_{i_m} + ... + (-1)^{m-1} \delta_{j i_2} \zeta_{i_1} \zeta_{i_2} ... \zeta_{i_{m-1}} \tag{L.-30}$$

The respective factor ζ_{i_k} is anticommuted to the left until the derivative operator can be directly applied. We may prove the following properties of the derivatives

$$\frac{\partial}{\partial \zeta_i} \frac{\partial}{\partial \zeta_j} + \frac{\partial}{\partial \zeta_j} \frac{\partial}{\partial \zeta_i} = 0 \tag{L.-30}$$

$$\frac{\partial}{\partial \zeta_i} \zeta_j + \zeta_j \frac{\partial}{\partial \zeta_i} = 0 \tag{L.-30}$$

Grassmann integration

An attempt to to introduce an indefinite integral as the inverse of differentiation is bound to fail. This illustrated by the fact that according to (L.13) the second derivative of any Grassmann function vanishes, so that the inverse operation does not exist.

We must be content with some formal definition. One way to arrive at it is to require that integration be translationally invariant. For an arbitrary function $g(\zeta) = g_1 + g_2\zeta$ we have

$$\begin{aligned}\int d\zeta g(\zeta+\eta) &= \int d\zeta\,[g_1 + g_2(\zeta+\eta)] = \int d\zeta\,[g_1 + g_2\zeta] + \int d\zeta g_2\eta \\ &= \int d\zeta g(\zeta) + \left[\int d\zeta 1\right] g_2\eta = \int d\zeta g(\zeta) \qquad \text{(L.-30)}\end{aligned}$$

The translational invariance requires the integral of 1 is zero. The following postulates uniquely fix the value of any integral.

$$\int d\zeta 1 = 0, \qquad \text{(L.-30)}$$

$$\int d\zeta\zeta = 1. \qquad \text{(L.-30)}$$

Eq. (L.13) comes from the condition of translational invariance. The sole non-vanishing integral $\int d\zeta\zeta$ arbitrarily is assigned the value 1. This is a convenient normalisation condition.

We see that integration is equivalent to differentiation. Generalising integration rules to higher dimensions straightforward

$$\int d\zeta_i 1 = 0, \qquad \text{(L.-30)}$$

$$\int d\zeta_i\zeta_j = \delta_{ij}. \qquad \text{(L.-30)}$$

Note that the differentials $d\zeta_i$ must anticommute with all other Grassmann variables as integration is equivalent to differentiation. In order to obtain analog results of conventional integration we introduce complex Grassmann variables. Let us start with two disjoint sets of Grassmann variables $\zeta_1^*, ..., \zeta_n^*$ and $\zeta_1, ..., \zeta_n$, which are all mutually anticommutating

$$\{\zeta_i, \zeta_j\} = \{\zeta_i^*, \zeta_j^*\} = \{\zeta_i, \zeta_j^*\} = 0 \tag{L.-30}$$

The two sets are related, using complex cunjugation, according to

$$\begin{aligned}
(\zeta_i)^* &= \zeta_i^*, \\
(\zeta_i^*)^* &= -\zeta_i, \\
(\zeta_{i_1}\zeta_{i_2}...\zeta_{i_m})^* &= \zeta_{i_m}^*...\zeta_{i_2}^*\zeta_{i_1}^* \\
(\lambda\zeta_i)^* &= \lambda^*\zeta_i^*
\end{aligned} \tag{L.-32}$$

where λ is a complex number.

In order to develop functional integral formalism for Grassmann fields we need to solve *Gaussian integrals.*

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp\left\{-\sum_{k,l=1}^{N} \zeta_k^* M_{kl} \zeta_l\right\} \tag{L.-32}$$

To simplify the notation, let us write this as

$$I = \int [d\zeta^* d\zeta] e^{-\zeta^* M \zeta} \tag{L.-32}$$

The calculation in principle is very simply because grassmann functions can at worst be linear in each variable, causing the series expansion of the exponential function to terminate. On the other hand , according to the rules for Grassmann integration, the integrand must contain as many different Grassmann variables as there are integrals or else the overall integration vanishes. For the case of two pairs of variables one effectively has

$$\begin{aligned}
e^{\zeta^* M \zeta} &\rightarrow \frac{1}{2!}(\zeta^* M \zeta)^2 \\
&\rightarrow \frac{1}{2!}(\zeta_1^* M_{11}\zeta_1\zeta_1^* M_{12}\zeta_2\zeta_2^* M_{21}\zeta_1\zeta_2^* M_{22}\zeta_2) \\
&\rightarrow (M_{11}M_{22} - M_{12}M_{21})\zeta_1^*\zeta_1\zeta_2^*\zeta_2
\end{aligned} \tag{L.-33}$$

where the last line follows from the anticommutating character of the Grassmann numbers. The integration of $\zeta_1^*\zeta_1\zeta_2^*\zeta_2$, gives unity, and so for this case

$$\int [d\zeta^* d\zeta] e^{-\zeta^* M\zeta} = detM \tag{L.-33}$$

One should suspect that this result holds in general. For the case of N pairs of variables, only the term of order $(\zeta^* M\zeta)^N$ survives in the expansion of the exponential. Moreover, only the terms which are multilinear in all the ζ_k^*, ζ_k can contribute and, in view of the anticommutativity of the Grassmann variables, these terms contain the appropriately signed products of matrix elements which define the determinant. But rather than go through this combinatorial exercise we will follow the method given in (Brown QFT).(page83) which is presented in Appendix(B). We do obtain the expected result:

$$\boxed{I = \int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) e^{-\zeta^* M\zeta} = detM} \tag{L.-33}$$

This should be compared to the ordinary integration where the corresponding integral gives $\det M^{-1}$.

L.13.1 Grassmann generating Functional

It is not surprising that the Gaussian integral formula (L.13) can be generalised to the case of general bilinear forms in the exponent:

$$\int \prod_{k=1}^{N} (d\zeta_k^* d\zeta_k) \exp -\zeta^* M\zeta = detM e^{-\frac{1}{2}\rho^T A^{-1}\rho}. \tag{L.-33}$$

Here ρ is an n-component vector of Grassmann variables. Equation (L.13.1) is obtained by translating the integration variable, $\zeta' = \zeta + A^{-1}\rho$.

The construction of functional integration in section (4.1.2) did not make use of any special properties of the integration over field variables which might restrict the validity to ordinary c-numbers.

$$\boxed{\begin{gathered}\int \mathcal{D}\bar{\chi}\mathcal{D}\chi \exp\left[-\int\ d^d x'\, d^d x \bar{\chi}(x')A(x',x)\chi(x) + \int\ d^d x (\bar{\rho}(x)\chi(x) + \bar{\chi}(x)\rho(x))\right] \\ = detA \exp\left[\int\ d^d x'\, d^d x \bar{\rho}(x')A^{-1}(x',x)\rho(x)\right].\end{gathered}} \tag{L.-33}$$

in which the measure is $\propto \prod_{\mathbf{r}} d\bar{\varphi}(r) d\varphi(r)$ and $Z(\rho = 0) = detA$. Note that to normalise the functional we divide by $detA$ as apposed to $det(A^{-1})$ in the bosonic case (**??**).

It is rather straightforward to extend the results of esction 4.1 to the fermionic case: The Grassmann functional derivative is defined

$$\frac{\delta G[\chi(y)]}{\delta \chi(x)} = \lim_{\Delta V_i \to 0} \frac{\partial G}{\partial \chi_i} \quad \text{where } \mathbf{x} \text{ is located in cell } \Delta V_i \tag{L.-33}$$

The (2n)-point correlators

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \langle \chi(y_n), \ldots, \chi(y_1); \bar{\chi}(x_1), \ldots, \bar{\chi}(x_n) \rangle \tag{L.-33}$$

can now be obtained by forming derivatives of the generating functional [1]

$$G^{(2n)}(y_1, \ldots, y_n; x_1, \ldots, x_n) = \left. \frac{\delta^{2n} Z[\rho, \bar{\rho}]}{\delta\rho(x_n) \cdots \delta\rho(x_1) \delta\bar{\rho}(y_1) \cdots \delta\bar{\rho}(y_n)} \right|_{\rho=\bar{\rho}=0} . \tag{L.-33}$$

We could in fact use the Grassmann formalism instead of the Bosonic functional integral with the replica trick to do our calculations without too much adjustment. But we introduced the Grassmann function integral here to help form the supersymmetric formalism.

What about operators that act on this Hilbert space? All operators that are well defined on that Hilbert space arise from consistent family of operators. These operators on each of these individual finite Hilbert spaces fit together in a certain way. If it is well defined on here then it can be shown that that they come from something that fits together in this way.

L.14 Biblioliographical notes

In this chapter I have relied on the following refferences:

infinite product measures: Probability theory, S.R.S. Varadhan downloaded from www.math.nyu.edu/facult

L.15 Worked Exercises and Details

□

[1] The order of the derivatives was chosen in such that we get agreement with the bosionic case. This is not a trivial matter as the Grassmann derivatives $\delta/\delta\rho(x)$ and $\delta/\delta\rho(x)$ anticommute with the field variables $\chi(x)$ and $\bar{\chi}(x)$. One can show, however, that there is an even number of commutations when we carry out the differentiations of (L.13.1) and write the result in the form (L.13.1).

Curves (Thiemann)

a)

Despite the name, composition and inversion does not equip $\mathcal{C}$ with a group structure for many reasons.

i) Verify that composition is not associative and that the curve $c \circ c^{-1}$ is not simply $b(c)$ but rather a retracing.

ii) Moreover, contemplate that $\mathcal{C}$ does not have a unit and that not every two elements can be composed.

b)

Define composition and inversion of paths by taking the equivalence class of the compositions and inversions of any of their representatives and check that this definition is well defined.

Check that then composition of paths is associtive and that $p \circ p{-}1 = b(p)$. However, $\mathcal{P}$ still does not have a unit and still not every two elements can be composed.

c)

Let Obj:= σ and for each $x, y \in \Sigma$ let $\mathrm{Mor}(x, y) := \{p \in \mathcal{P} : b(p) = x, f(p) = y\}$. Recall the mathematical definition of a category (section **??**) and conclude that $\mathcal{P}$ is a category in which every morphism is invertible, that is, a groupoid.

d)

Define the relation $\prec$ on Γ by saying that $\gamma \prec \gamma'$ if and only if every $e \in E(\gamma)$ is a finite composition of the $e' \in E(\gamma')$ and their inverses.

Verify that $\prec$ equips Γ with the structure of a directed set, that is, for each $\gamma, \gamma' \in \Gamma$ we find $\gamma'' \in \Gamma$ such that $\gamma, \gamma' \prec \gamma''$.

For this to work, analyticity of the curve representatives is crucial. Smooth curves can intersect in Cantor sets and thus define graphs which are no longer finitely generated. Show first that this is not possible for analytic curves.

Answers:

a)

Bochner-Minlos

The characteristic function $\tilde{P}_m$ the Fourier transformation of the probability function:

$$\tilde{P}_m = \sum_l P_l e^{ilm} \quad \text{(L.-33)}$$

$$P_l = \sum_m \tilde{P}_m e^{-iml} \quad \text{(L.-33)}$$

say we also have $\tilde{P}'_m$

$$P_l = \sum_m \tilde{P}'_m e^{-iml} \quad \text{(L.-33)}$$

$$0 = P_l - P_l = \sum_m (\tilde{P}_m - \tilde{P}'_m) e^{-iml} = \sum_m a_m e^{-iml} \quad \text{(L.-33)}$$

We wish to show that the above condition can only hold if the coefficients $a_m = \tilde{P}_m - \tilde{P}'_m$ vanish. Multiply both sides by $e^{m'l}$ and sum over l,

$$0 = \sum_l \sum_m a_m e^{-i(m-m')l} = \sum_m a_m (\sum_{l'} e^{-i(m-m')} l) = \sum_m a_m \delta_{mn} = a_n \quad \text{(L.-33)}$$

Details: Operator identity

Prove the operator equation

$$e^{-\hat{B}} \hat{A} e^{\hat{B}} \quad \text{(L.-33)}$$

$$e^{-t\hat{B}} \hat{A} e^{t\hat{B}} = I + t\hat{C}_1 + \frac{t^2}{2!}\hat{C}_2 + \cdots \quad \text{(L.-33)}$$

$$e^{-(t+\delta t)\hat{B}} \hat{A} e^{(t+\delta t)\hat{B}} - e^{-t\hat{B}} \hat{A} e^{t\hat{B}} = \delta t (e^{-t\hat{B}} \hat{A}\hat{B} e^{t\hat{B}} - e^{-t\hat{B}} \hat{B}\hat{A} e^{t\hat{B}}) \quad \text{(L.-33)}$$

$$\hat{C}_1 = \frac{d}{dt}\left(e^{-t\hat{B}} \hat{A} e^{t\hat{B}}\right)|_{t=1} = \{\hat{A}, \hat{B}\} \quad \text{(L.-33)}$$

$$\frac{d^2}{dt^2}\left(e^{-t\hat{B}} \hat{A} e^{t\hat{B}}\right) = \frac{d}{dt}\left(e^{-t\hat{B}} \{\hat{A}, \hat{B}\} e^{t\hat{B}}\right) = \left(e^{-t\hat{B}} \{\{\hat{A}, \hat{B}\}, \hat{B}\} e^{t\hat{B}}\right) \quad \text{(L.-33)}$$

$$\frac{d^n}{dt^n}\left(e^{-t\hat{B}}\hat{A}e^{t\hat{B}}\right) = \left(e^{-t\hat{B}}\{\hat{A},\hat{B}\}_n e^{t\hat{B}}\right) \tag{L.-33}$$

$$\hat{C}_n = \{\hat{A},\hat{B}\}_n \tag{L.-33}$$

Details: The integral

$$I = \int_{-\infty}^{\infty} dx \left(\frac{\sin x}{x}\right)^2 \tag{L.-33}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} dx \left(\frac{\sin x}{x}\right)^2 &= \Big[-\sin^2 x\frac{1}{x}\Big]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} dx \frac{2\sin x\cos x}{x} \\
&= \int_{-\infty}^{\infty} dx \frac{\sin 2x}{x} \\
&= \int_{-\infty}^{\infty} dy \frac{\sin y}{y}
\end{aligned} \tag{L.-34}$$

Can be evaluated using complex...

$$\begin{aligned}
I &= \left(\int_C dz \frac{e^{iz}}{2iz} - \int_C dz \frac{e^{-iz}}{2iz}\right) \\
&= \left(\int_{C_1} dz \frac{e^{iz}}{2iz} - \int_{C_2} dz \frac{e^{-iz}}{2iz}\right)
\end{aligned} \tag{L.-34}$$

Poisson's formula.

Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta \tag{L.-34}$$

From Cauchy's integral formula, an explicit solution for the Dirichlet problem for a cirular region can be obtained. Without loss of generality, the circle can be taken to be of unit radius and centred at the origin. Let $z = e^{i\alpha}$, $\zeta = e^{it}$, $r < 1$.

$$f(z) = u(r,\alpha) + iv(r,\alpha)$$

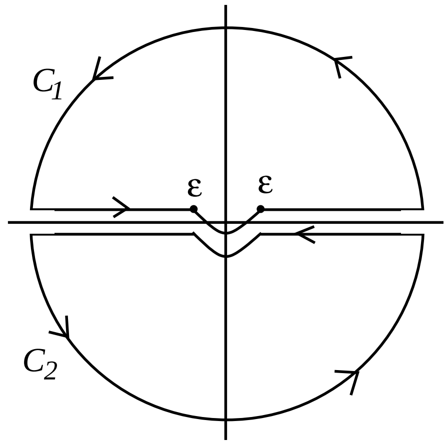


Figure L.31: $\sin x/x$.

Then since $d\zeta = i\zeta dt$, (L.15) gives

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - z}dt \tag{L.-34}$$

But since z is inside the unit circle, $1/z^*$ is outside it, so that

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/z^*}dt \tag{L.-34}$$

or

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\zeta)}{\zeta - 1/z^*}d\zeta \tag{L.-34}$$

Consequently using $\eta = 1/\zeta^*$, (note $\eta = e^{it} = 1/(e^{it})^* = 1/\eta^*$)

$$\begin{aligned}
\frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\zeta)\zeta}{\zeta - 1/z^*}dt &= \frac{1}{2\pi i}\int_{2\pi}^{0} \frac{f(1/\eta^*)(1/\eta^*)z^*}{z^*/\eta^* - 1}dt \\
&= -\frac{1}{2\pi i}\int_{2\pi}^{0} \frac{f(1/\eta^*)z^*}{\eta^* - z^*}dt \\
&= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\eta)z^*}{\eta^* - z^*}dt
\end{aligned} \tag{L.-35}$$

we arrive at

$$0 = \frac{1}{2\pi}\int_0^{2\pi} f(\zeta)\frac{z^*}{\zeta^* - z^*}dt \tag{L.-35}$$

adding or subtracting this to (L.15)

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \left(\frac{\zeta}{\zeta - z} \pm \frac{z^*}{\zeta^* - z^*} \right) dt \tag{L.-35}$$

Taking first the positive sign in (L.15),

$$\frac{\zeta}{\zeta - z} + \frac{z^*}{\zeta^* - z^*} = \frac{\zeta(\zeta^* - z^*) + z^*(\zeta - z)}{|\zeta - z|^2} = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2},$$

we obtain

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\zeta) \frac{1 - |\zeta|^2}{|\zeta - z|^2} dt \tag{L.-35}$$

But the factor multiplying $f(\zeta)$ is purely real, so that the process of taking the real part gives

$$\begin{aligned} u(r, \alpha) &= \frac{1}{2\pi} \int_0^{2\pi} u(t) \, \frac{1 - r^2}{|e^{ti} - re^{i\alpha}|^2} \, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(t) \, \frac{1 - r^2}{1 - 2r\cos(\alpha - t) + r^2} \end{aligned} \tag{L.-35}$$

where $u(t)$ is the value of the harmonic function u on the boundary. This result, which solves the Dirichlet problem for the circle, is known as Poison's formula.

If in (L.15) we take the negative sign,

$$\frac{\zeta}{\zeta - z} - \frac{z^*}{\zeta^* - z^*} = \frac{\zeta(\zeta^* - z^*) - z^*(\zeta - z)}{|\zeta - z|^2} = \frac{|\zeta|^2 + |z|^2 - 2\zeta z^*}{|\zeta - z|^2}$$

we shall obtain the conjugate function v in terms of $u(t)$,

$$\begin{aligned} v(r, \alpha) &= v(0) + \frac{1}{\pi} \int_0^{2\pi} f(\zeta) \frac{1 - 2\zeta z^* + |z|^2}{|\zeta - z|^2} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(1 + \frac{2iIm(z\zeta^*)}{|\zeta - z|^2} \right) dt \end{aligned} \tag{L.-35}$$

Consequently,

$$v(r, \alpha) = \frac{1}{2\pi} \int_0^{2\pi} u(t) \, \frac{r\sin(\alpha - t)}{1 - 2r\cos(\alpha - t) + r^2} dt \tag{L.-35}$$

Finally we obtain $f(z)$ in terms of $u(t)$, by combining the two results for u and v,

$$f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} u(t)\, \frac{\zeta + z}{\zeta - z}\, dt \tag{L.-35}$$