Appendix M

Hamiltonian Constraint

M.1 Solving Quantum Constraint Equations

Rovelli

" The space of physical states must have the structure of a Hilbert space, namely a scalar product, in order to be able to compute expectation values. This Hilbert structure is determined by the requirement that real physical observables correspond to self-adjoint operators. In order to define a Hilbert space of physical states, it is convenient to define first a Hilbert space of unconstrained states. This is because we have a much better knowledge of the unconstrained observables than the physical ones. If we choose a scalar product on the unconstrained state space which is gauge invaiant then there exist standard techniques to "bring it down" to the space of the physical states. Thus, we need a gauge and diffeomorphism invariant scalar product, with respect to which real observable are self-adjoint operators. "

M.2 Projection Operator

We firstly quantized the theory ingnoring the constraints (here the spatial diff and Hamiltonian constraints) to get a kinematical Hilbert space H_{Kin}. The constraints are imposed as self-adjoint operators on H_{Kin} and the physical states are those that are annihilated by the by them (mathematicions call the set of such states that are annihilated by an operator the *kernel* of the operator).

The symmetry group is compact, (finite group volume), so the solutions to this constraint are honest members of the Hilbert space.

This follows Rovelli's

The space of square integrable functions i.e. $\int_\Lambda d^2x|\Phi(x)|^2$ $(H = L^2(M))$

Before going any further we introduce a simple toy model of a constrained quantum mechnical system.

$$\hat{J} := \hat{J}_z = i(x\partial_y - y\partial_x) = J_\varphi = i\partial_\varphi \tag{M.0}$$

Consider a simple

$$\hat{\mathcal{J}}\Psi = 0 \tag{M.0}$$

$$\hat{\mathcal{J}}\Psi = 0 \tag{M.0}$$

The constraint equation defines a subspace $\mathcal{H}_{Phys}$ of $\mathcal{H}_{aux}$ - the space of physical states.(the set of such solutions are known as the *kernel* of the operator.

$$\Pi : \quad H \to H_{Phys}. \tag{M.0}$$

Note that the (finite, see appendix d.2) action of the constraint on a general state functional $\Psi(r,\varphi)$ is given by

$$e^{i\alpha\hat{\mathcal{J}}}\Psi(r,\varphi) = \Psi(r,\varphi+\alpha) \tag{M.0}$$

Hence, the progection operator Π on the physical states is defined as

$$\begin{aligned} \Pi &= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\ \Psi(r,\varphi) \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\alpha\ \Psi(r,\varphi+\alpha) = \tilde{\Psi(r)} \end{aligned} \tag{M.0}$$

resulting in a new function $\tilde{\Psi(r)}$ which is independent of φ, just as one might have expected for physical states, $\hat{\mathcal{J}}\Psi_{Phys}(r,\varphi) = 0$.

Using a scalar product

$$\langle\Psi|\Phi\rangle = \int_0^{2\pi}\int_0^{\infty} d\varphi\overline{\Psi(r,\varphi+\alpha)}\Phi(r,\varphi+\alpha) \tag{M.0}$$

in H, one arrives at the important result

$$\langle\Psi|\Phi\rangle_{Phys} \equiv < \Psi|\Pi|\Phi > . \tag{M.0}$$

The quadratic form $\langle\rangle_{Phys}$ in H_{Phys} is indeed expressed as a scalar product over states which lie in H. Thus, knowing the matrix elements (P.12) of the progection operator in the unconstraind Hilbert space is equivalent to having solved the constraint!

A similar scheme can be applied to operators (we talk more about this when we come to the Master constraint). Suppose there exists a non-gauge invariant operator $O = O(r, \varphi)$ on H. Then a fully gaue invariant operator $R = R(r)$ in H_{Phys} can be constructed by defining

$$R := \Pi O \Pi. \tag{M.0}$$

The calculation of the physical operator R is then reduced to a calculation in the unconstrained Hilbert space, which gives

$$\langle\Psi|O|\Phi\rangle_{Phys} \equiv < \Psi|\Pi O \Pi|\Psi > . \tag{M.0}$$

Some of the Jargon

the physical state functionsals are only those solutions that have the zero eigenstate of the constraint operator $\mathcal{J}$, (the set of such solutions are known as the *kernel* of the operator).

Let us summarize the previous section in more techniqual language.

$$\Phi \in \mathcal{H} \in \Phi^* \tag{M.0}$$

where $\mathcal{H}$ is a Hilbert space, Φ is a dense subset of the Hilbert space, and Φ^* is the dual space of Φ, i.e Φ^* is the space of antilinear functionals over the space Φ.

M.3 Rigged Hilbert Space

M.3.1 Inadequancies of Hilbert Space.

We firstly quantized the theory ingnoring the constraints (here the spatial diff and Hamiltonian constraints) to get a kinematical Hilbert space H_{Kin}. The constraints are imposed as self-adjoint operators on H_{Kin} and the physical states are those that are annihilated by the by them (mathematicions call the set of such states that are annihilated by an

operator the *kernel* of the operator). There is a problem here; such states are often not normalisible by the inner product of the unconstrained system, and hence do not belong to the kinematic Hilbert space! However, one should not be unduely alarmed as such "improper" solutions to quantum mechanical opretors are already evident in ordinary quantum mechanics. Infact, such issues come up whenever an operator has a continuous spectrum. For example,

Consider the space $\mathcal{K} = L^2[R^2, dxdy]$ and the self-adjoint operator

$H = -id/dx$. The solutions of $H\psi = 0$ or

$$-i\frac{d}{dx}\psi(x,y) = 0 \tag{M.0}$$

are functions $\psi(x,y)$ constants in the y and are non-normalizable in $\mathcal{K}$.

$$\tilde{f}(q_n) = \sum_{i=0}^{N} e^{ia_i q} f(a_i) \tag{M.1}$$

$$f_i = \sum_{qn2\pi/N} e^{-ia_i q_n} \tilde{f}(q_n) \tag{M.2}$$

finite intervals, say $-L/2$ to $L/2$

$$\tilde{f}(q_n) = \frac{1}{2\pi}\int_{-L/2}^{L/2} dx e^{ixq} f(x) \tag{M.3}$$

$$f(x) = \sum_{-\infty}^{\infty} e^{ixq_n} \tilde{f}(q_n) \tag{M.4}$$

$$\tilde{f}(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx e^{ixq} f(x) \tag{M.5}$$

$$f(x) = \int_{-\infty}^{\infty} dq e^{-ixq} \tilde{f}(q) \tag{M.6}$$

The "functions" $\psi(x) = e^{ixp}$ can only be normalized up to the delta function:

$$\int_{-\infty}^{\infty} dx \overline{\psi(p)}\psi(q) = \int_{-\infty}^{\infty} dx e^{ixp} e^{ixq} = 2\pi\delta(p-q) \tag{M.6}$$

We handle them according to the rules for using delta functions.

$$\hat{\mathcal{Q}}\psi(x) = x\psi(x). \tag{M.6}$$

The formal solution is the Dirac delta function

$$\phi(x) = \delta(x - \lambda). \tag{M.6}$$

$$\hat{\mathcal{Q}}|\lambda >= \lambda|\lambda >, \tag{M.6}$$

from which follows that

$$< \lambda'|\hat{\mathcal{Q}}|\lambda >= \lambda < \lambda'|\lambda > \tag{M.6}$$

On the other hand

$$< \lambda'|\hat{\mathcal{Q}}|\lambda >= \int_{-\infty}^{\infty} dx\delta(x - \lambda')\lambda\delta(x - \lambda) \tag{M.6}$$

Comparing we have

$$< \lambda'|\lambda >= \delta(\lambda' - \lambda) \tag{M.7}$$

Making use of the delta function property (), we can calculate the inner product of the position eigenstate $|\lambda >$ with arbitrary $|\psi >$.

$$< \lambda|\psi >= \int_{-\infty}^{\infty} dx\delta(x - \lambda)\psi(x) = \psi(\lambda). \tag{M.7}$$

$$\psi(x) =< x|\lambda >, \tag{M.8}$$

"projection operators" $|x >< x|$

$$\int_{-\infty}^{\infty} dx|x >< x|. \tag{M.8}$$

For any $|\phi >, |\psi > \in H$, we have

$$< \phi| \left(\int_{-\infty}^{\infty} dx|x >< x| \right) |\psi >= \int_{-\infty}^{\infty} dx < \phi|x >< x||\psi >= \int_{-\infty}^{\infty} dx\overline{\phi(x)}\psi(x) =< \phi|\psi > . \tag{M.8}$$

M.3.2 Introduction to Generalized Eigenfunctons and the RHS.

In undergaduate quantum mechanic courses we are told that wavefunctions are required to be square integrible in order to have a probabilistic interpretation. The Hilbert space is introduced as the space of squar integrable functions, $||\psi(x)||$, with respect to the usual inner product $\int dx\overline{\phi(x)}\psi(x)$. We are also told that plane waves are solutions to the Hamiltonian for free electrons and have energy $E = \hbar^2 k^2/2m$. However, as mentioned above, they cannot be normalized and so lie outside the Hilbert space.

In our undergraduate courses we circumvate this problem by considering electrons in a box of finite volume, V, with infinite barries. The wavefunctions that are eigenstates are obviously square integrable and so lie wihtin the Hilbert space. At the end we take the limit that the volume goes to infinity, $V \to \infty$. This is one way to avoid wavefunctions that are not normalizible. The justification given for taking this limit is that experiments are not going to be able to tell you the difference between a very large box of electrons and free electrons.

$$\begin{aligned} \mathcal{H}|E_n) &= E_n|E_n) \\ \mathcal{H}|E >&= E|E > \end{aligned} \tag{M.8}$$

These eigenstates are normalized according to:

$$(E_n|E_m) = \delta_{nm}, \quad < E|E' >= \delta(E - E'), \quad < E|E_n) = 0 \tag{M.8}$$

$$\varphi = \sum_n |E_n)(E_n|\varphi) + \int dE|E >< E||\varphi > . \tag{M.8}$$

kets $|\phi >$ are an astraction and must be understood to always be accompanated by a bra $< \psi|$.

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \tag{M.8}$$

$$< \phi(x)|\mathcal{H}\psi(x) >= \int_{-\infty}^{\infty} \phi^*(x)\mathcal{H}\psi(x) \tag{M.8}$$

In the case where $\psi(x) = e^{ikx}$ the vector $\mathcal{H}e^{ikx}$ is not defined. But we can define another Hamiltonian $\mathcal{H}^\times$ (that is a natural extension of $\mathcal{H}$) for which

$$< \phi(x)|\mathcal{H}^\times e^{ikx} >=< \mathcal{H}^\dagger\phi(x)|e^{ikx} > \tag{M.8}$$

where we integrated by parts in the third step

$$< \phi|\mathcal{H}\psi >=< \mathcal{H}^\times\phi|\psi > \tag{M.8}$$

where $\mathcal{H}^\times$ defines the dual action of $\mathcal{H}$.

$\mathcal{H}^\times$ is called the dual extension of $\mathcal{H}$.

experiments can never be complete

The inceptive motivation for introducing RHS in quantum mechanics was to provide Dirac's bra and ket formulism, already established calculational tool, with a proper mathematical content. Indicate how to make the above ideas more rigorous

$$(\varphi, H\varphi) \tag{M.8}$$

space of unconstrained observables

$$\Delta_\varphi H = \sqrt{(\varphi, H^2\varphi) - (\varphi, H\varphi)^2} \tag{M.8}$$

The expectation values cannot be computed for every element of the unconstrained Hilbert space, but only for those for which those $\varphi \in H$ that aslo belong to $\mathcal{D}(H)$

The solutions to the constraint are not square normalizable and so lie outside the Hilbert space. Therefore the unconstrained Hilbert space is not large enough to contain the non-normalizable physical states - a larger space is needed

$$\langle\varphi|\, H^\times\, |\rangle = \tag{M.8}$$

The dual space $\mathbf{\Phi}^\times$ contains the eigenkets associated with the solutions to the constraint operator.

Unitary representations of G on the rigged Hilbert space $H_{_R}$ instead on H. $U_{_R} : G \to \mathcal{U}(H)$ for normalizable states $|\psi >$,

$$U_{_R}(g) \equiv U(g)|\psi >, \tag{M.8}$$

and for non-normalizable states $|x>$

$$U_{_R}(g) := |gx> \tag{M.8}$$

$$U(g)|\psi>=|\psi'> . \tag{M.8}$$

Then,

$$\begin{aligned} U(g)\psi(x) &= \psi'(x) =< x|\psi'>=< U(g)|\psi> \\ &= < x|U_{_R}(g)|\psi>=< U_R^{-1}x|\psi>=< g^{-1}x|\psi>=\psi(g^{-1}x) \end{aligned} \tag{M.8}$$

satisfies the group homomorphism requirement $(g_1g_2)^{-1} = g_2^{-1}g_1^{-1}$

$$\begin{aligned} U(g_1)U(g_2)\psi(x) &= U(g_1)\psi(g_2^{-1}x) = \psi(g_1^{-1}g_2^{-1}x) \\ &= \psi((g_2g_1)^{-1}x) = U(g_1g_2)\psi(x) \end{aligned} \tag{M.8}$$

$$U(g_1)U(g_2)\psi(x) = U(g_1g_2)\psi(x) \tag{M.8}$$

M.3.3 Projector Technique with a Rigged Hilbert Space

We need to extend the progector (or group-averaging) technique to the caase where the symetry group is non-compact (has divergent group volume) where the solutions lie in the dual space.

An antilinear form $\eta : \Omega \to \Omega^*$ is provided by group averaging

$$(\psi| = \frac{1}{V}\int d\lambda \int dk \int d_{\mu_k} < \tag{M.8}$$

The physical inner product, denoted as $<,>$, is defined as

$$< \eta\psi', \eta\psi >_{phys}= (\psi|\psi'> . \tag{M.8}$$

M.4 Group-averaging

Projecting it onto the physical Hilbert space by gauge-orbit smearing.

Let us first suppose that the constraint operator $\hat{C}$ on the kinematical Hilbert space H_{kin} is self-adjoint, and that zero is a discrete point in its spectrum. Then, the kernel of $\hat{C}$ is a subspace of H_{kin}. Therefore, to extract physical states, one just has to project kinematical states to this subspace. In the case when the 1-parameter group $U(\lambda) = e^{-i\hat{C}}$ generated by $\hat{C}$ on H_{kin} provides a representation of $U(1)$, the projection procedure can be explicitly carried out through an integration: Given any $\Psi \in H_{kin}$, set

$$\Psi_{Phys} = \frac{1}{\Lambda}\int^{\Lambda} d\lambda e^{-i\hat{C}}\Psi \tag{M.8}$$

Since these physical states belong to H_{kin}, the scalar product between them is well-defined.

you have to always first of all you should define a subspace in the Hilbert space which should have stronger topology that the topology of the Hilbert space.

by Riesz' theorem, there is a unique vector $\eta'(\phi) \in \mathcal{H}_{Phys}$ which satisfies

$$\phi(f) := f[\phi] =< \eta'(\phi)|f >_{Phys} \tag{M.8}$$

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[268]:

The group averaging procedure [126], [127] need not result in a state which has finite and positive norm.

$$(\Psi_{Phy}| := \int d\lambda < e^{-i\lambda\hat{C}}\Psi|. \tag{M.8}$$

$$(\Psi_{Phy}|\Phi >= \int d\lambda < e^{-i\lambda\hat{C}}\Psi|\Phi > . \tag{M.8}$$

This procedure can be heuristically understood as follows. (M.4) extracts from Ψ a physical state $\Psi_{phy} \in \mathcal{S}^*$. This extractor $\hat{E}$ can be formally thought of as $\hat{E} = \delta(\hat{C})\Psi$. Therefore, the naive definition $< \Psi_{phy}|\Psi_{phy} >= (\delta(\hat{C})\Psi)|(\delta(\hat{C})\Psi)$ of the norm that one may first think of is divergent. In the correct definition, (M.4), one of the two delta-distributions is simply dropped, thereby removing the obvious infinity.

The norm of physical states is

$$\|\Psi_{Phy}\|^2 = \int d\lambda < e^{-i\lambda\hat{C}}\Psi|\Psi > . \tag{M.8}$$

For a general choice of the initial subspace $\mathcal{S}$, there is no guarantee that the norm would be finite and positive. The art in the group averaging procedure lies in selecting a dense subspace $\mathcal{S}$ of H_{kin} such that: i) the right side of (M.4) is well-defined for all $\Psi, \Phi \in \mathcal{S}$; i.e., Ψ_{Phy} is a well-defined distribution over S; and, ii) the norm (2.16) of each Ψ_{phy} is non-negative, vanishing if and only if Ψ_{phy} vanishes. The procedure succeeds in its goal of constructing the physical Hilbert space only if such a $\mathcal{S}$ can be located.

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M.4.1 The Rigging Map

Assume the constraints are Abelian

$$\{F_\mu, F_\nu\} = 0 \tag{M.8}$$

$$\eta : \mathcal{H}_{kin} \to \mathcal{H}_{phys}$$

$$\psi \mapsto [\eta(\psi)](\phi, Q) = \int [d\beta/(2\pi)]\ [e^{i\beta^\mu F_\mu}](\phi, Q) \tag{M.8}$$

We can write this formally as

$$\eta(\psi) = \prod_\mu \delta(F_\mu)\psi \tag{M.8}$$

To prove this we use

$$e^{\beta^\mu \pi_\mu}\psi(\phi) = \psi(\phi - \beta)$$

Denote

$$\beta(t) := \beta_1 + t(\beta_2 - \beta_1)$$

It follows that

$$\begin{aligned} V(\beta_2) - V(\beta_1) &= \int_0^1 dt_1 \frac{d}{dt_1} V(\beta(t_1)) \\ &= i \int_0^1 dt_1 V(\beta(t_1))\ \dot{\beta}(0) h'_\mu(\phi - \beta(t_1)) \end{aligned} \tag{M.8}$$

We can find an iterative formula for $V(\beta(t))$. Note

$$\beta_{t_1}(t_2) = \beta_1 + t_2(\beta(t_1) - \beta_1) = \beta_1 + t_1 t_2(\beta_2 - \beta_1) = \beta(t_1 t_2) \tag{M.8}$$

so that

$$\begin{aligned} V(\beta_2(t_1)) - V(\beta_1) &= \int_0^1 dt_2 \frac{d}{dt_2} V(\beta_{t_1}(t_2)) = i \int_0^1 dt_2 \frac{d}{dt_2} V(\beta(t_1 t_2)) t_1 \dot{\beta}(0) h'_\mu(\phi - \beta(t_1 t_2)) \\ &= i \int_0^{t_1} dt_2 V(\beta(t_2))\ \dot{\beta}(0) h'_\mu(\phi - \beta(t_2)) \end{aligned} \tag{M.8}$$

$$V(\beta(t_1)) = V(\beta_1) + i \int_0^{t_1} dt_2 V(\beta(t_2)) \dot{\beta}^\mu(0) h'_\mu(\phi - \beta(t_2))$$

or

$$V(\beta_1)^{-1} V(\beta(t_1)) = 1 + i \int_0^{t_1} dt_2 V(\beta_1)^{-1} V(\beta(t_2))\ \dot{\beta}^\mu(0) h'_\mu(\phi - \beta(t_2)) \tag{M.8}$$

Now set

$$\begin{aligned} U(t,0) &:= V(\beta_1)^{-1} V(\beta(t)) \\ \Phi(t) &:= \dot{\beta}^\mu(0) h'_\mu(\phi - \beta(t)) \\ &= [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta(t)) \end{aligned} \tag{M.7}$$

Then () becomes

$$U(t,0) = 1 + i \int_0^t dt_1 U(t_1, 0) \Phi(t_1) \tag{M.7}$$

From this equation we obtain successive approximations:

$$
\begin{aligned}
U_0(t,0) &= 1 \\
U_1(t,0) &= 1 + i\int_0^{t_1} dt_1 \Phi(t_1) \\
U_2(t,0) &= 1 + i\int_0^{t_1} dt_1 U_1(t_1,0)\Phi(t_1) \\
&= 1 + i\int_0^{t} dt_1 \Phi(t_1) + (i)^2 \int_0^{t} dt_1 \int_0^{t_1} dt_2 \Phi(t_2)\Phi(t_1). \qquad \text{(M.5)}
\end{aligned}
$$

The $N-th$ order approximation being

$$
U_N(t,0) = 1 + \sum_{n=1}^{N} U^{(n)}(t,0)
$$

where

$$
U^{(n)}(t,0) = (i)^n \int_0^{t} dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \Phi(t_n)\ldots\Phi(t_1)
$$

We define the left-time-ordered product as

$$
T_l[\Phi(t_2)\Phi(t_1)] \equiv \left\{ \begin{array}{ll} \Phi(t_2)\Phi(t_1) & t_1 > t_2 \\ \Phi(t_1)\Phi(t_2) & t_2 > t_1 \end{array} \right.
$$

Thus

$$
U^{(2)}(t,0) = \frac{i^2}{2} \int_0^{t} dt_1 \int_0^{t} dt_2 T_l[\Phi(t_2)\Phi(t_1)]
$$

for n operators we obtain

$$
U^{(n)}(t,0) = \frac{i^n}{n!} \int_0^{t} dt_1 \int_0^{t} dt_2 \ldots \int_0^{t} dt_n T_l[\Phi(t_n)\ldots\Phi(t_1)] \qquad \text{(M.5)}
$$

$$
\begin{aligned}
U(t,0) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \dots \int_0^t dt_n T_l[\Phi(t_n) \dots \Phi(t_1)] \\
&= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \dots \int_0^t dt_n \Phi(t_n) \dots \Phi(t_1) \\
&= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int_0^t d\tau \Phi(\tau) \right)^n \qquad \text{(M.4)}
\end{aligned}
$$

We can formally write the expression for $U(t,0)$ as

$$
V(\beta_1)^{-1} V(\beta(t_1)) = T_l \exp(i \int_0^{t_1} dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))
$$

Setting $t_1 = 1$ we get

$$
V(\beta_1)^{-1} V(\beta_2) = T_l \exp(i \int_0^{1} dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))
$$

Theorem M.4.1 *Equation () holds pointwise in ϕ spaceon a dense set of analytic vectors for the operator $h'(\phi) = \phi^\mu h'_\mu(\phi)$.*

Proof: Let $V_0(\phi) := 1$ for $N > 0$

$$
V_N(\phi) = 1 + \sum_{n=1}^{N} i^n \int_0^1 dt_1 \ \phi^{\nu_1} h'_{\nu_1}(t_1 \phi) \dots \int_0^{t_{n-1}} dt_n \ \phi^{\nu_n} h'_{\nu_n}(t_n \phi) \qquad \text{(M.4)}
$$

We define for $N > 0$

$$
\begin{aligned}
R_N(\phi) &= -i^{N-1} \sum_{n=1}^{N} \int_0^1 dt_1 \ \phi^{\nu_1} h'_{\nu_1}(t_1 \phi) \dots \int_0^{t_{n-1}} dt_n \ t_n \phi^{\nu_n} \{[h'_\mu, h'_{\nu_n}]\}(t_n \phi) \cdots \times \\
&\quad \times \dots \int_0^{t_{N-1}} dt_N \ \phi^{\nu_N} h'_{\nu_N}(t_N \phi) \qquad \text{(M.4)}
\end{aligned}
$$

and prove by induction for $N > 0$ that

$$
\partial_\mu V_N(\phi) = i h'_\mu(\phi) V_{N-1}(\phi) + R_N(\phi) \qquad \text{(M.4)}
$$

□

M.4.2 More on RAQ: Open Algebras

Say in the classical theory the constraints C_I are real and the structure functions ${f_{IJ}}^K$ are imaginary then

$$\hat{C}_I^\dagger = \hat{C}_I, \qquad (\hat{{f_{IJ}}}^K)^\dagger = -\hat{{f_{IJ}}}^K$$

Take the classical brackets

$$\{C_I, C_J\} = {f_{IJ}}^K C_K \tag{M.4}$$

and replace them by the arbitrary ordering (that depending on the constants λ_{IJ}),

$$[\hat{C}_I, \hat{C}_J] = \lambda_{IJ} \hat{{f_{IJ}}}^K \hat{C}_K + (1-\lambda_{IJ}) \hat{C}_K \hat{{f_{IJ}}}^K$$

as can easily be seen by taking the congujate of both sides

$$[\hat{C}_I, \hat{C}_J]^\dagger = \lambda_{IJ} \hat{C}_K^\dagger (\hat{{f_{IJ}}}^K)^\dagger + (1-\lambda_{IJ}) (\hat{{f_{IJ}}}^K)^\dagger \hat{C}_K^\dagger$$

which gives

$$[\hat{C}_I, \hat{C}_J] \;\; = \;\; (1-\lambda_{IJ}) \hat{{f_{IJ}}}^K \hat{C}_K + \lambda_{IJ} \hat{C}_K \hat{{f_{IJ}}}^K$$

therefore we must have $\lambda_{IJ} = 1/2$ and we would have to have

$$[\hat{C}_I, \hat{C}_J] = \frac{1}{2} (\hat{{f_{IJ}}}^K \hat{C}_K + \hat{C}_K \hat{{f_{IJ}}}^K) \tag{M.3}$$

It turns out that this is disasterous for solving the constraints. to see this impose

$$[(\hat{C}_I)' l](f) := l(\hat{C}_I^\dagger) = 0 \quad \text{for all } I \in \mathcal{I},\ f \in \mathcal{D}_{kin}$$

on (M.4.2) we obtain

$$((\hat{{f_{IJ}}}^K)' \hat{C}_K)' + (\hat{C}_K)' (\hat{{f_{IJ}}}^K)') l = ([(\hat{C}_I K)', (\hat{{f_{IJ}}}^K)']) l = 0 \tag{M.3}$$

implying that l is not only annihilated by the dual constraint operators but also by $[(\hat{C}_I K)', (\hat{f}_{IJ}{}^K)']$, which is not necessarily proportional to a dual constraint, implying that the physical Hilbert space will be too small.

Therefore, in the case of an open constraint algebra the constraints should not be chosen to be self-adjoint operators. Notice that this is no contraction because self-adjointness is usually required to ensure that the spectrum (measurement values) of the operator lies in the real line, however, for constraint operators this is not a requirement because we are only intereseted in the kernel and the only requirement is that the point zero belongs to the spectrum at all.

In order to allow for non-self-adjoint constraints, in what follows we will assume that the set

$$\mathcal{C} := \{\hat{C}_I : I \in \mathcal{I}\}$$

is self-adjoint (i.e., contains with $\hat{C}_I$ also $\hat{C}_I^\dagger = \hat{C}_J$ for some J)

Consider the commutant of $\mathcal{C}$ within $\mathcal{O}_{kin}$, that is,

$$\mathcal{C}' := \{O \in \mathcal{O}_{kin} : [C, O] = \text{ for all } C \in \mathcal{C}\} \tag{M.3}$$

$I_{\mathcal{Z}}$ is a two-sided ideal in $\mathcal{C}'$ if

$$[I_{\mathcal{Z}}, \mathcal{C}'\,] \in I_{\mathcal{Z}} \quad \text{and} \quad [\mathcal{C}'\,, I_{\mathcal{Z}}] \in I_{\mathcal{Z}}$$

The set of commutators $[\mathcal{C}', \mathcal{C}']$ is a subalgebra of $\mathcal{C}'$.

M.5 The Direct Integral Decomposition of the Hilbert Space

Consider the space $\mathcal{K} = L^2[\mathbb{R}^2, dxdy]$ and the self-adjoint operator $H = -id/dx$. The solutions of $H\psi = 0$ or

$$-i\frac{d}{dx}\psi(x, y) = 0 \tag{M.3}$$

are functions $\psi(x, y)$ constants in the y and are non-normalizable in $\mathcal{K}$. However, the decomposition

$$\mathcal{K} = \int_{\mathbb{R}} dy H_y. \tag{M.3}$$

where $H(y) = L^2[\mathbb{R}, dx]$.

$$(\psi, \phi)_{\mathcal{K}} = \int_{\mathbb{R}^2} dx \overline{\psi(x,y)} \phi(x,y) = \int_{\mathbb{R}} (\psi_y, \phi_y)_{H_y}, \tag{M.3}$$

where $\psi_y(x) = \psi(x,y)$ and

$$(\psi_y, \phi_y)_{H_y} = \int_{\mathbb{R}} dx \overline{\psi_y(x)} \phi_y(x). \tag{M.3}$$

The space of solutions of (M.5) is $\mathcal{H}(0)$ and has the natural Hilbert structure $\mathcal{H}(0) = L^2[\mathbb{R}, dx]$.

M.5.1 The Direct Integral Decomposition Theorem

Theorem M.5.1 *(Direct Integral Decomposition (DID)). Let a be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there is a unitary operator U such that*

$$U\mathcal{H} = \mathcal{H}^{\oplus} = \int_{\mathbb{R}}^{\oplus} d\mu(\lambda) \mathcal{H}^{\oplus}(\lambda)$$

where μ is a probability measure and UaU^{-1} is represented on $\mathcal{H}^{\oplus}(\lambda)$ by multiplication by λ. Moreover, the measure class $< \mu >$ and the Hilbert spaces $\mathcal{H}^{\oplus}(\lambda)$ areuniquely determined.

Proof:

Let the projection-valued measure of a be denoted by $E(\lambda)$. Consider

$$W_t = \exp(ita) = \int_{\mathbb{R}} dt e^{it\lambda} E(\lambda)$$

this is bounded and weakly continuous

$$\psi_f := \int_{\mathbb{R}} dt \; f(t) W_t \psi \tag{M.3}$$

Stone's theorem regards one-parameter unitary groups which establishes a one-to-one correspondence between self-adjoint operators on a Hilbert space $\mathcal{H}$ and one parameter families of unitary operators

$$\{U_t\}_{t\in\mathbb{R}}$$

which are strongly continuous, that is

$$\lim_{t\to t_0} U_t\psi = U_{t_0}\psi \quad \text{for all} \quad t_0 \in \mathbb{R}, \psi \in \mathcal{H}$$

and are homomorphisms

$$U_{t+s} = U_t U_s.$$

Choose any vector ψ_1 and function $f_1 \in C_0^\infty(\mathbb{R})$ and set

$$\Omega_1 = \int_{\mathbb{R}} dt\ f_1(t) W_t \psi_1$$

Set

$$p(W) = \sum_{k=1}^{N} z_k W_{t_k}$$

Denote by $\mathcal{H}_1$ the closure of the finite linear span of the W_t, that is

$$\mathcal{H}_1 := \overline{\{p(W)(\int_{\mathbb{R}} dt\ f_1(t) W_t\psi_1) :\ \text{for } N<\infty, z_k \in \mathbb{C}\}}$$

If $\mathcal{H}_1 \neq \mathcal{H}$ choose $\psi_2 \in \mathcal{H}_1^\perp$ and $f_2 \in C_0^\infty(\mathbb{R})$. Then

$$W_{-t}\sum_{k=1}^{N} z_k W_{t_k}\Omega_1 = \sum_{k=1}^{N} z_k W_{t_k - t}\Omega_1 \in \mathcal{H}_1.$$

Hence

$$\begin{aligned} < \Omega_2, \Omega_1 > &= \int_{\mathbb{R}} dt\ \overline{f_2(t)} < W_t \psi_2, \Omega_1 > \\ &= \int_{\mathbb{R}} dt\ \overline{f_2(t)} < \psi_2, W_{-t} \Omega_1 > \\ &= 0 \end{aligned}$$

so also $\Omega_2 \in \mathcal{H}_1^{\perp}$. Iterating, since $\mathcal{H}$ is separable we arrive at the direct sum

$$\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n \tag{M.0}$$

A dense set of vectors can be presented in the form

$$(p_n(W)\Omega_n)_{n=1}^{\infty}.$$

How $a^m \Omega_n$ are elements of $\mathcal{H}_n$

Thus, a dense set of vectors can be given in form

$$(p_n(a)\Omega_n)_{n \in \mathbb{N}}$$

where p_n is a polynomial in a and the Ω_n are C^{∞}-vectors for a.

Borel measures

Consider the probability Borel measure (all the μ_n are Borel measures)

$$\mu(\lambda) := \sum_{n=1}^{\infty} c_n\ \mu_n(\lambda) \tag{M.0}$$

Radon-Nikodym Derivative

$$\rho_n = \frac{d\mu_n}{d\mu}$$

□

Example:

Two commuting constraints

$$C_1 = p_1, \quad C_2 = p_2$$

for a particle moving in the plane.

□

M.5.2 Comparing the RAQ and DID Programmes

M.6 Anomaly-freeness

Anomaly-freeness WAS HERE

M.7 Quantum Dirac Algebra

We may now compute the commutator $[\hat{C}(N), \hat{C}(N')]$ on Φ_{Kin} corresponding to the Poisson bracket $\{C(N), C(N')\}$ which is proportional to the spatial diffeomorphism constraint C_a. This commutator turns out to be non-vanishing on Φ_{Kin} as it should be, however,

$$\Psi([\hat{C}(N), \hat{C}(N')]f) = 0$$

for all $f \in \Phi_{Kin}, \Psi \in (\Phi^*_{Kin})_{Diff}$. This is precisely how we would expect it in the absence of an anomaly. Note that this is sometimes paraphrased..... in the strict sense, the commutator is defined on Φ_{Kin}, where it would not vanish, and not on $(\Phi^*_{Kin})_{Diff}$. On the other hand, the right hand side of the commutator on Φ_{Kin} does not obviously resemble the quantization of the classical expression $\int d^3x(NN'_{,a} - N_{,a}N')q^{ab}C_b$ so there are doubts, expressed in [], [] whether the quantization of $C(N)$ produces the correct quantum dynamics.

M.8 Quantization Ambiguity

source of ambiguity is associated with the choice of representation j of the new edge added by the action of the Hamiltonian constraint.

using again the expansion of the holonomy,

$$h^{(m)}_{s_k} \approx I^{(m)} + A^j_a \tau^{(m)}_j s^a_k(\Delta) \quad \text{(M.1)}$$
$$h^{(m)}_{\alpha_{ij}} \approx I^{(m)} + F^j_{ab} \tau^{(m)}_k s^a_i(\Delta) s^b_j(\Delta) \quad \text{(M.2)}$$

$$\mathrm{tr}\left(\tau^{(m)}_i \tau^{(m)}_j\right) = -\frac{1}{12} m(m+1)(m+2)\delta_{ij} \quad \text{(M.2)}$$

$$C(m) = \frac{1}{12} m(m+1)(m+2) \quad \text{(M.2)}$$

then

$$\mathcal{H}^m_T[N] := \sum_{\Delta \in T} \mathcal{H}^m_\Delta[N] \quad \text{(M.2)}$$

Are the results on the spectrum of area and volume operators. Loop quantum cosmology.

However, starting with LQG in 2+1 dimensions, Perez [120] argues that any choice other than the fundamental representation leads to unphysical local degres of freedom also in 3+1 dimensions.

M.9 Constructing the Solution's

More precisely, the equation

$$(\Psi|\hat{\mathcal{C}}(N) = 0, \quad \text{(M.2)}$$

is equivalent to the following hierarchy of equations

$$\begin{aligned}
(\Psi_{(1)[\alpha,j]}|\hat{\mathcal{T}}(N) &= 0 \\
(\Psi_{(2)[\alpha,j]}|\hat{\mathcal{T}}(N) &= (\Psi_{(1)[\alpha,j]}|\hat{\mathcal{C}}^{Eucl}(N) \\
&\cdots \\
(\Psi_{(n+1)[\alpha,j]}|\hat{\mathcal{T}}(N) &= (\Psi_{(n)[\alpha,j]}|\hat{\mathcal{C}}^{Eucl}(N) \\
&\cdots
\end{aligned} \quad \text{(M.-1)}$$

M.10 The Generalized Wick Transformation

$$\begin{aligned} f &\rightarrow W(1)\cdot f \\ W(t)\cdot f &\equiv \sum_{n\geq 0}\frac{t^n}{n!}\{f,T\}_n \\ T &\equiv i\frac{\pi}{2}\int_\Sigma A_i^a E_a^i \end{aligned} \tag{M.-2}$$

where $\{f,T\}_2=\{f,\{f,T\}\}$ and $\{f,T\}_3=\{f,\{f,T\}_2\},\ldots$ etc.

$$\{E_a^i,T\}=\{E_a^i,\int_\Sigma dx^3 A_j^b E_b^j\}=iE_a^i \tag{M.-2}$$

$$W(1)\circ E_a^i=\sum_{n\geq 0}\frac{t^n}{n!}\{E_a^i,T\}_n \tag{M.-2}$$

$$W(1)\circ A_i^a=\sum_{n\geq 0}\frac{t^n}{n!}\{A_i^a,T\}_n \tag{M.-2}$$

$$W(1)\circ S_{_E}=\sum_{n\geq 0}\frac{t^n}{n!}\{S_{_E},T\}_n \tag{M.-2}$$

M.11 Inclusion of Matter

$$H_{tot}=\Lambda^i G_i+N^a C_a+NC, \tag{M.-2}$$

where Λ^i, N^a and N are Lagrange multiplyers, and the three constraints in the Hamiltonian are

$$\begin{aligned} G_i &= \mathcal{D}_a\tilde{E}_i^a:=\partial_a\tilde{E}_i^a+\epsilon_{ij}{}^k A_a^i\tilde{E}_i^a, && \text{(M.-1)}\\ C_a &= \tilde{E}_i^b F_{ab}^i-A_a^i G_i+\tilde{\pi}\partial_a\phi, && \text{(M.0)}\\ C &= \frac{\kappa\gamma^2}{2\sqrt{|\det q|}}\tilde{E}_i^a\tilde{E}_j^b[\epsilon^{ij}{}_k F_{ab}^k-2(1+\gamma^2)K_{[a}^i K_{b]}^j] \\ &+ \frac{1}{\sqrt{|\det q|}}[\frac{\kappa^2\gamma^2\alpha_{_M}}{2}\delta^{ij}\tilde{E}_i^a\tilde{E}_j^b(\partial_a\phi)\partial_b\phi+\frac{1}{2\alpha_{_M}}\tilde{\pi}^2], && \text{(M.0)} \end{aligned}$$

$\tilde{\pi}$ is conjugate to ϕ

$$\tilde{\pi} := \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\alpha_{_M}}{N}\sqrt{|\det q|}(\dot{\phi} - N^a \partial_{_a}\phi). \tag{M.0}$$

M.11.1 The "habitat" of functions

Another objection that was raised by Lewandowski and Marolf has to do with the fact that it is easy to construct a more general "habitat" of functions where Thiemann's Hamiltonian is well defined. Consider any function of a spin net with n vertices. Multiply i times a scalar function with n entries, evaluated at each vertex,

$$|s, f >= \int d^3x_{_1} \ldots d^3x_{_n} f(x_{_1}, \ldots, x_{_n}) \tag{M.0}$$

These functions are invariant under diffeomorphisms that leave the vertices of the spin network fixed. Otherwise diffeomorphisms are correctly implemented geometrically: $C(N)|s, f >= |s, L_{_N} f >$

It is obvious that on these states one can implement Thiemann's Hamiltonian. It is also obvious that on these states the Hamiltonian will have an abelian algebra too.

Is the theory inconsistent? For that we should evaluate the right hand side of the commutator, on these states.

$$\{H(N), H(M)\} = \int d^3x \omega_{_a}(x) q^{ab}(x) \tilde{C}_{_b}(x) \tag{M.0}$$

where $\omega_{_a}(x) = (N\partial_{_a}M - M\partial_{_a}N)$.

Which in turn requires computing the doubly-contravariant metric. Remeber that we need to write it in terms of the fundamental variables. Using Thiemann's construction:

$$q^{ab}(x) = \frac{1}{4}\epsilon^{abc}\epsilon_{ijk}\epsilon^{bef}\epsilon_{ilm}\frac{e^j_c e^k_c}{\sqrt{det(q)}}\frac{e^l_e e^m_f}{\sqrt{det(q)}} \tag{M.0}$$

And it is not hard so see that this operator vanishes identically.

The theory is therefore consistent. But it apears (to some) that one is paying too high a price: the contravariant metric is a highly non-linear combinatin of the fundamental variables and its relation to operators that are well defined and non-vanishing like the area and volume.

So is it a problem or not? At the moment the issue is debated. It is interesting that the same problem arises in 2+1 dimensions and one nevertheless recovers Witten's correct quantization.

R. Gambini, J. Griego, C, Di Bartolo, JP (2000)

Habitats are unphysical and completely irrelevant in LQG

Habitats were introduced in [??, ??]. The idea was to take the limit $\epsilon \to 0$ for the duals of the Hamiltonian constraints on such a habitat in the sense of pointwise convergence. The habitat ambiguity is that there maybe zillions of habitats on which a limit of this kind can be performed. As was shown in those papers, there exists at least one such habitat and it has the property that the limit dual operators are Abelian.

We now show that this habitat ambiguity is actually absent: Namely, the habitat spaces must be genuine extensions of $\mathcal{D}^*_{Diff}$. Hence these spaces are not in the kernel of the spatial di eomorphism constraint and are therefore unphysical. Hence the only domain where to define the Hamiltonian constraints (rather than their duals) is on $\mathcal{D}$ i.e. on a dense subspace of the kinematical Hilbert space $\mathcal{H}$. This is the same domain as for the spatial di eomorphism constraints which thus treats both types of constraints democratically. This fact is widely appreciated in the LQG community and not a matter of debate, the habitat construction presented in [??] is outdated. Habitats are unphysical and completely irrelevant in LQG.

M.12 Analysis of the Volume Operator

$$V(\Omega) = \int_\Omega \sqrt{h} d^3x = \int_\Omega \sqrt{E} d^3x \tag{M.0}$$

E being

$$E = \frac{1}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a E^j_b E^k_c.$$

Given a graph α and a function $h_\alpha(A)$ of parallel transports along its edges e, the operator E acts as follows

$$\begin{aligned} E(x)h_\alpha(A) &= \frac{1}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a(x)E^j_b(x)E^k_c(x)h_\alpha(A) \\ &= \frac{8i\pi\gamma l_P^2}{3!}E^i_a(x)E^j_b(x)\left[\int_0^1 dt \frac{d\alpha^k}{dt}(U_\alpha)^t_0 \tau_c (U_\alpha)^1_t \delta^3(x-\alpha(t))\right] \end{aligned}$$

And then by acting with E^j_b

$$
\begin{aligned}
&= \frac{(8i\pi\gamma l_P^2)^2}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a(x) \\
&= \frac{(8i\pi\gamma l_P^2)^2}{3!}\epsilon_{ijk}\epsilon^{abc}E^i_a(x)\left[\int_0^1 dt\int_0^t ds\frac{d\alpha^j}{ds}(U_\alpha)^s_0\tau_b(U_\alpha)^t_s\delta^3(x-\alpha(s))\ \frac{d\alpha^k}{dt}\ \times\right. \\
&\quad (U_\alpha)^t_0\tau_b(U_\alpha)^1_t\delta^3(x-\alpha(t))\ ] \\
&\quad \left.+\int_0^1 dt\frac{d\alpha^j}{dt}(U_\alpha)^t_0\tau_c(U_\alpha)^1_t\delta^3(x-\alpha(t))\quad\int_t^1 ds\frac{d\alpha^j}{dt}(U_\alpha)^s_t\tau_b(U_\alpha)^1_s\delta^3(x-\alpha(s))\right]
\end{aligned}
$$

The third E^i_a operator...

The last condition produces a term

$$\epsilon_{ijk}\frac{d\alpha^i}{dt}\frac{d\alpha^j}{dt}\frac{d\alpha^k}{dt},$$

which is not vanishing if and only if there exists a point in the graph α where tangent vectors form a set of three independent vectors.

Let us consider the case in which only one vertex v with n out-going edges e_i , $i = 1, \ldots, n$ is contained in the region Ω we obtain for the volume operator

A consistency check [100] on the equivalence between a quantization based on triads and one based on fluxes implies that the latter is the correct one.

We consider the case in which only one vertex v with n out-going edges e_i, $i = 1, \ldots, n$ is contained in Ω.

$$V(\Omega)h_\alpha = (8\pi\gamma)^{3/2}l_P^3\sqrt{|q|}h_\alpha \tag{M.-6}$$

$$h_\alpha = \epsilon^{abc}\sum_{e,e',e''}o(v,e,e',e'')\tau_a^{j_e}\tau_b^{j_{e'}}\tau_c^{j_{e''}}h_\alpha, \tag{M.-6}$$

M.13 Non-Hermitian Hamiltonian in the Rovelli-Reisenberg Spin Foam

Is there a difference in using $\frac{1}{2}(C(N) + C^\dagger(N))$ or using $C(N)$?

M.13.1 Spin Foam

M.14 Spacially Diffeomorphism Invariant Hamiltonian Constraints

An unfortunate feature of the Hamiltonian constraint was that it could not be implemented on the spatially diffeomorpism invariant Hilbert space because it would map spatially diffeomorpism invariant states onto non-diffeomorpism invariant states - i.e. it did not close on the space.

Thus the inner product structure of H_{Diff} cannot be employed, via the same powerful technies used to construct the inner product H_{Diff} from the kinematic inner product structure ... , in the construction of the physical inner product.

In [300] it was shown that if one is given a constraint algebra of the form

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, C_k\} = f_{Jk}{}^l C_l, \quad \{C_j, C_k\} = f_{jk}{}^L C_L \tag{M.-6}$$

$$A_{lj} := \{C_l, T_j\}. \tag{M.-6}$$

$$\tilde{C}_j := \{M, T_j\} \approx \sum_{k,l} Q_{kl} C_k A_{lj} \tag{M.-6}$$

the constraint algebra can be simplified to

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l. \tag{M.-6}$$

Diffeomorphism Invariant Hilbert Space

M.15 Biblioliographical notes

In this chapter I have relied on the following references: .

M.16 Worked Exercies and Details

Proofs

Questions

1. Check that there are 8 of them.

Answers

1. A tetrahedal cannot have e_i and e_i^r edges.

The list reads:

$$\begin{array}{lll} e_1, e_2, e_3 & e_1, e_2, e_3^r & e_1^r, e_2^r, e_3 \\ e_1^r, e_2^r, e_3^r & e_1, e_3^r, e_2 & e_1^r, e_3^r, e_2 \\ & e_2, e_3, e_1^r & e_2^r, e_3^r, e_1 \end{array} \tag{M.-7}$$

Proofs. 3+1 decomposition and Legendre transformation

Questions

1. (Eq. M.11)