Appendix N

Spin Foams and Physical Observables.

N.1 Spin Foam

N.1.1 Spin Foam From Projector Technique Applied To the Hamiltonian Constraint

$$P\int[\mathcal{D}N]e^{i\hat{\mathcal{H}}[N]}=\int[\mathcal{D}N]e^{iN\hat{\mathcal{H}}}. \qquad \text{(N.0)}$$

In the spin network basis, the matrix elements of P are

$$<s|P|s'>=<s|\int[\mathcal{D}N]e^{iN\hat{\mathcal{H}}}|s'> \qquad \text{(N.0)}$$

It can be shown that a diffeomorphism invariant notion of integration exists for this functional integral.

$$<s|P|s'>\sim<s|s'>+\int[\mathcal{D}N]\left(N<s|\hat{\mathcal{H}}|s'>+NN<s|\hat{\mathcal{H}}\hat{\mathcal{H}}|s'>+\ldots\right) \qquad \text{(N.0)}$$

$$<s|P|s'>=\int_{Diff}\mathcal{D}[\phi]\int\mathcal{D}[N]<\mathcal{U}S|\left(\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}(H[N])^n\right)|s'> \qquad \text{(N.0)}$$

N.1.2 State Sum

We change the signature of the metric by changing the gauge group.

Putting it all together we see it has the form

$$\mathcal{Z} = \sum_J \mathcal{N}(J) \sum_N \prod_{f \in J} \Delta_{N_f} \prod_{v \in J} A_v(N), \tag{N.0}$$

The first sum is over all spin foams Γ interpolating between a given initial spin network s_i and a final spin network s_f. Δ_{j_f} is the dimension of the G representation labelling the face f of Γ. A_v is the amplitude on the vertex v of Γ, a given function of the labels on the faces and edges adjacent to v.

N.2 Barrett-Crane Model

$$S(\omega, B, \phi) = \int_{\mathcal{M}} B^{IJ} \wedge F_{IJ}(\omega) \tag{N.0}$$

$$S(\omega, B, \phi) = \int_{\mathcal{M}} d^4x \epsilon_{abcd} B^{IJab} F^{cd}_{IJ}(\omega) \tag{N.0}$$

$$S(\omega, B, \phi) = \int_{\mathcal{M}} \left[B^{IJ} \wedge F_{IJ}(\omega) - \frac{1}{2}\phi_{IJKL} B^{IJ} \wedge B^{KL} - \frac{1}{2}\mu \epsilon^{IJKL}\phi_{IJKL} \right] \tag{N.0}$$

which is a BF theory with variables a 2-form B^{IJ}_{ab} and a 1-form connection (with curvature F^{IJ}_{ab}), all with values in $so(3,1)$, but with a constraint on the B field enforced by the Lagrangian multiplier ϕ_{IJKL}.

$$\begin{aligned}
DB &= dB + [\omega, B] = 0, \quad \text{(var. of } \omega) & \text{(N.1)}\\
F(\omega) &= \phi^{IJKL} B_{KL}, \quad \text{(var. of } B) & \text{(N.2)}\\
B^{IJ} \wedge B^{KL} &= \mu \epsilon^{IJKL}, \quad \text{(var. of } \phi) & \text{(N.3)}\\
\epsilon^{IJKL}\phi_{IJKL} &= 0, \quad \text{(var. of } \mu) & \text{(N.4)}
\end{aligned}$$

contracting (N.3) with ϵ_{IJKL} we solve for μ:

$$\epsilon_{IJKL} B^{IJ} \wedge B^{KL} = \mu \epsilon_{IJKL} \epsilon^{IJKL}$$

implies

$$\mu = e := \frac{1}{4!}\epsilon_{IJKL}B^{IJ} \wedge B^{KL}. \quad \text{(N.4)}$$

For $e \neq 0$, equation (N.3) is equivalent to

$$\epsilon_{IJKL}B^{IJ}_{ab}B^{KL}_{cd} = e\epsilon_{abcd} \quad \text{(N.4)}$$

One can show that there are two types of solution to this equation and which are formed from co-triad fields e^I.

$$B^{IJ}_{ab} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL}e^K_a e^L_b \quad \text{or} \quad B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL}e^K \wedge e^L \quad \text{(N.5)}$$

$$B^{IJ}_{ab} = \pm e^I_{[a}e^J_{b]} \quad \text{or} \quad B^{IJ} = \pm e^I \wedge e^J \quad \text{(N.6)}$$

substituting any of these solutions into one obtains the Palanti action,

$$S \rightarrow S_{EH} = \frac{1}{2}\int_{\mathcal{M}} d^4x\epsilon^{abcd}\epsilon_{IJKL}e^K_a e^L_b F^{IJ}_{cd} \quad \text{(N.6)}$$

$$S \rightarrow S_{EH} = \int_{\mathcal{M}} \epsilon_{IJKL}e^K \wedge e^L \wedge F^{IJ} \quad \text{(N.6)}$$

i.e. it reduces to pure 1st order Einstein gravity. Which as $e \neq 0$, is equivalent to the Einstein-Hilbert action.

choice (N.5) corresponds to the graviational sector and (N.6) to the topological sector - [286].

The geometric information is resides in the labels. This is an important difference from lattice gauge theories with a background metric, where the discretization itself determines the edge lengths and hence how refined the lattice is.

N.2.1 Latice BF-Theory

$\sigma_0, \ldots, \sigma_n$ triangulation of $\mathcal{M}$
C_0, C_1, C_2

dual 2-complex

$$c_0 \leftrightarrow \sigma_n \qquad c_1 \leftrightarrow \sigma_{n-1} \qquad c_0 \leftrightarrow \sigma_{n-2}$$

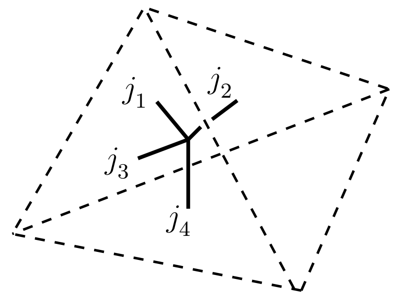


Figure N.1: The spins j_1, j_2, j_3, j_4 describe the areas of the triangles.

curvature associated with dual 2-cells

$$F(c_2)\int_{c_2} F \in g \tag{N.6}$$

B-field associated with $(n-2)-$simplicies

$$B(\sigma_{n-2})\int_{c_2} B \in g \tag{N.6}$$

The discretized action is:

$$S = \sum_{\sigma_{n-2}} Tr\big(B(\sigma_{n-2})F(\sigma^*_{n-2})\big) \tag{N.6}$$

Lattice path integral quantization:

$$\begin{aligned} \mathcal{Z} &= \text{``}\int \mathcal{DA}\int \mathcal{DB}e^{iS}\text{''} \\ &= \text{``}\int \mathcal{DA}\prod_{\sigma_{n-2}} \underbrace{\delta(F(\sigma_{n-2})}_{\text{enforces flatness}} \quad)\text{''} \end{aligned} \tag{N.6}$$

where we are integrating over all connections $\mathcal{A}$.

$$\mathcal{Z} = \left(\prod_{c_1}\int_G dg(c_1)\right)\ \delta_G\big(g\partial_1(c_1)(g\partial_2)(c_2)\cdots\big) \tag{N.6}$$

Latice gauge theory of flat connections on the dual 2-complex of a generic triangulation.

N.2.2 Spin Foam BF Theory

N.3 State Sums in 3-d Gravity

N.3.1 Regge Calculus

denote the lengths of its edges by l_i $(I = 1, 2, \ldots, 6)$ (Fig N.-19). The Regge action for the tetrahedron is

$$S_{Regge} = \sum_{I=1} l_i \theta_i, \tag{N.6}$$

where θ_i is the angle between the outward normals of two faces sharing the I-th edge.

$$W(j) = \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} \tag{N.6}$$

Witten's $ISO(3)$ Chern-Simons theory [E. Witten, Nucl. Phys. **B311**, 46 (1988).]

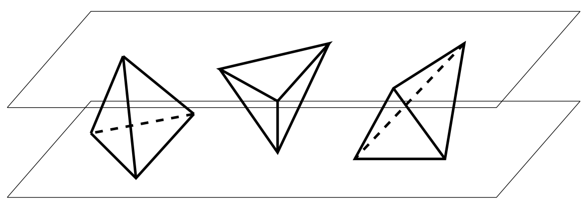


Figure N.2: The three types of tetrahedral building blocks used in 3d quantum gravity.

N.3.2 Ponzano-Regge-Turaev-Viro (PRTV)

Ponzano-Regge

$$\text{History amplitude} = \sum_{\{j\}} \prod_e (dim j_e) \prod_t (6j) \tag{N.6}$$

"triangulation independent"

It diverges. Turaev-Viromodel with regularization using q-deformed gauge group $SU(2)_q$, see ().

$$0 \leq j \leq \frac{k-1}{2} \tag{N.6}$$

k-level of $SU(2)_q$ is root of unitary i.e. $q = e^{2\pi/\kappa}$.

cellular decomposition $\triangle$ and associated dual 2-complex $\mathcal{F}_\triangle$.

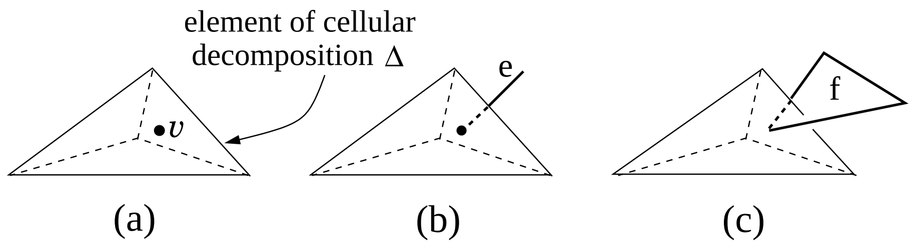


Figure N.3: (a) $v \in \mathcal{F}_\triangle$ - dual to 3-cells in $\triangle$; (b) $e \in \mathcal{F}_\triangle$ - dual to 2-cells in $\triangle$; (c) faces $f \in \mathcal{F}_\triangle$ - dual to 1-cells in $\triangle$

$su(2)$-valued 1-form field B represented by the assignment of a $B \in su(2)$ to each 1-cell. The connection field A is represented by the assignment of a group elements $g_e \in SU(2)$ to each edge in $\mathcal{F}_\triangle$.

$$B_a = B_a^i \tau_i, \quad [B_a, B_b] = f_{abc} B_c \tag{N.6}$$

$$S = \prod_{f \in \mathcal{F}_\triangle} B_{l_f} U_f \tag{N.6}$$

where $U_f = g_e^1 g_e^2 \ldots g_e^N$

vertices $v \in \mathcal{F}_\triangle$ (dual to 3-cells in $\triangle$)
edges $e \in \mathcal{F}_\triangle$ (dual to 2-cells in $\triangle$)
and faces $f \in \mathcal{F}_\triangle$ (dual to 1-cells in $\triangle$)

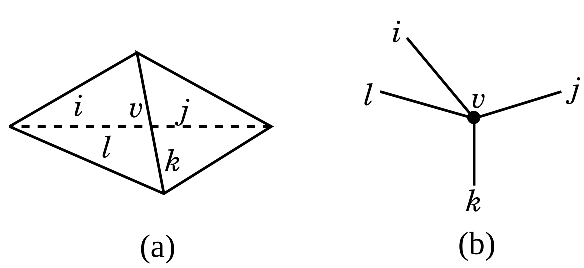


Figure N.4: $\triangle$dual3

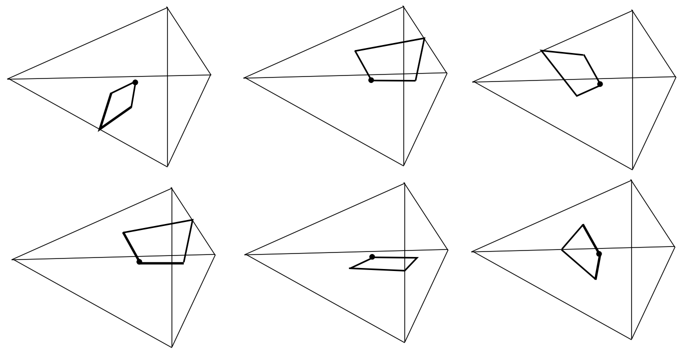


Figure N.5: atom2.

$$Z(\triangle) = \int \prod_{f\in\mathcal{F}} dB_f \prod_{e\in\mathcal{F}} dg_e e^{i\mathrm{tr}[B_f U_f]} \tag{N.6}$$

performing the integral over B_f

$$\int dB_f e^{i\mathrm{tr}[B_f U_f]} = \delta(U_f) = \delta(g_e^1 \dots g_e^N) \tag{N.6}$$

$$Z = \int \mathcal{D}A\delta[F], \tag{N.6}$$

namely an integral over flat $SU(2)$ connections.

$$\delta(g) = \sum_{j\in irrep(SU(2))} \Delta_j \mathrm{Tr}[\rho_j(g)] \tag{N.6}$$

$$Z(\triangle) = \sum_{C_f:\{f\}\to\{j\}} \int \prod_{e\in\mathcal{F}} dg_e \prod_{f\in\mathcal{F}} \Delta_{j_f} \mathrm{Tr}[\rho_{j_f}(g_e^1 \dots e_e^N)] \tag{N.6}$$

$$Z(\triangle) = \sum_{C_f:\{f\}\to\{j\}} \int \prod_{e\in\mathcal{F}} dg_e \prod_{f\in\mathcal{F}} \Delta_{j_f} \rho_{j_f}(g_e^1 \dots e_e^N)^\alpha{}_\alpha \tag{N.6}$$

$$Z(\triangle) = \sum_{C_f:\{f\}\to\{j\}} \sum_{C:\{e\}\to\{i\}} \prod_{f\in\mathcal{F}} \Delta_{j_f} \prod_{v\in\mathcal{F}} A_v(i_v, j_v) \tag{N.6}$$

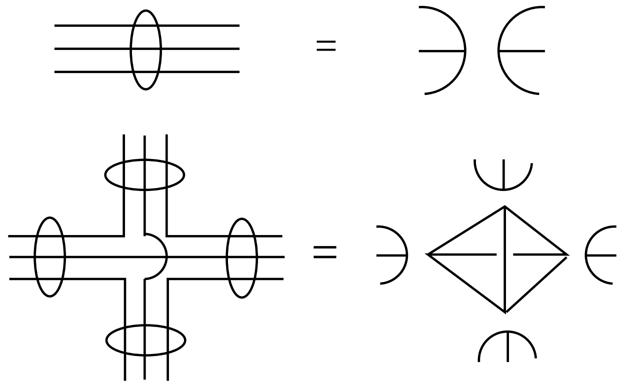


Figure N.6: How 6-j Symbols Appears.

How 6-j Symbols Appears:

N.4 10j symbols

$$\int_{(S^3)^5} \prod_{k<l} K^R_{2j_{kl}+1}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \cdots \frac{dh_5}{2\pi^2} \tag{N.6}$$

$$K^R_a(\phi) = \frac{\sinh a\phi}{\sin \phi} \tag{N.6}$$

Lorentzian $10j$ symbols - same sort of integral

$$\int_{(H^3)^5} \prod_{k<l} K^L_{a_{kl}}(\phi_{kl}) \, \frac{dh_1}{2\pi^2} \frac{\ldots dh_5}{2\pi^2} \tag{N.6}$$

where

$$H^3 = \{t^2 - x^2 - y^2 - z^2 = 1, \; t > 0\} \tag{N.6}$$

$$K^L_a(\phi) = \frac{\sin a\phi}{\sinh \phi}. \tag{N.6}$$

N.5 4D State Sum

spin foams live on the 2-skeleton

each dual 0-cell has incident on it five endpoints of 1-cells and ten corners of 2-cells with each incident 2-cell corner bounded by two of the incident 1-cells. This corresponds to the fact that a 4-simplex (dual to a 0-cell) has in its boundary five 3-simplicies (dual to 1-cells) and ten 2-simplices (dual to 2-cells).

N.6 Group Field Theory for 2D Gravity

a model in two dimensions in which we perform a path integral quantization of a (interacting) group field theory defined on $SU(2) \times SU(2)$, and expand perturbatively the transition amplitudes defined in the theory in power of the coupling constant λ. We explicitly perform the main calculation to present in concrete terms the relationship between group field theory and spin foam models. The partition function obtained has the same form of the generic one that defines a spin foam model. Such identity makes clear the relation between field theory over a group and the spin foam formalism.

Consider the field theory defined by the following action

$$\begin{aligned} S[\Phi] \;=\; & \int_{SU(2)\times SU(2)} dg_1 dg_2 \Phi^2(g_1, g_2) + \\ & \frac{\lambda}{3!}\int_{SU(2)\times SU(2)\times SU(2)} dg_1 dg_2 dg_3 \Phi(g_1, g_2)\Phi(g_2, g_3)\Phi(g_3, g_1) \end{aligned} \tag{N.6}$$

for the real scalar field $\Phi(g_1, g_2)$ defined on the group manifold $SU(2) \times SU(2)$ and having the following properties

$$\Phi(g_1, g_2) = \Phi(g_1 g, g_2 g), \qquad \text{for all } g \in SU(2) \tag{N.6}$$

and

$$\Phi(g_1, g_2) = \Phi(g_2, g_1). \tag{N.6}$$

Using the Peter-Weyl theorem we expand the field in terms of irreducible representations of the group (repeated indices are summed over),

$$\Phi(g_1, g_2) = \phi^{a_1 b_1 a_2 b_2}_{j_1 j_2} \overline{R^{j_1}_{a_1 b_1}(g_1)} R^{j_2}_{a_2 b_2}(g_2) \tag{N.6}$$

where $R^{j_1}_{a_1 b_1}(g_1)$ are the matrix elements of the group element g_1 in the spin j representation of $SU(2)$. The symmetry property (N.-19) of the field can be used to simplify the expansion of $\Phi(g_1, g_2)$ in terms of R^j_{ab}. The symmetry property implies

$$\begin{aligned}\Phi(g_1,g_2) &= \int dg\ \Phi(g_1 g, g_2 g)\\ &= \int dg\ \phi^{a_1 b_1 a_2 b_2}_{j_1 j_2}\overline{R^{j_1}_{a_1 b_1}(g_1 g)}R^{j_2}_{a_2 b_2}(g_2 g)\\ &= \phi^{a_1 b_1 a_2 b_2}_{j_1 j_2}\overline{R^{j_1}_{a_1 c_1}(g_1)}R^{j_2}_{a_2 c_2}(g_2)\int dg\overline{R^{j_1}_{c_1 b_1}(g)}R^{j_2}_{c_2 b_2}(g).\end{aligned} \tag{N.5}$$

Using the orthogonality relation

$$\int dU\overline{R^{(j)}_{ab}(U)}R^{(k)}_{cd}(U) = \frac{1}{2j+1}\delta^{jk}\delta_{ac}\delta_{bd}, \tag{N.5}$$

we can write

$$\Phi(g_1,g_2) = \sqrt{2j+1}\ \phi^{a_1 a_2}_j\ \overline{R^j_{a_1 c}(g_1)}R^j_{a_2 c}(g_2) \tag{N.5}$$

where we have defined

$$\phi^{a_1 a_2}_j := \frac{1}{\sqrt{2j+1}}\ \phi^{a_1 b_1 a_2 b_2}_{j j_2}\ \delta^{j j_2}\delta_{b_1 b_2}. \tag{N.5}$$

The reality and the symmetry properties of the field Φ imply $\phi^{a_1 a_2}_j = \overline{\phi^{a_2 a_1}_j}$. This hermitian matrix has dimension $(2j+1)\times(2j+1)$ and represents the Fourier transform of the field Φ. Writing the action (N.6) in terms of these modes, we obtain for the kinetic term

$$\int \Phi^2(g_1,g_2)dg_1 dg_2 = \phi^{a_1 a_2}_j \phi^{a_2 a_1}_j \tag{N.5}$$

and for the potential term

$$\frac{\lambda}{3!}\int dg_1 dg_2 dg_3 \Phi(g_1,g_2)\Phi(g_2,g_3)\Phi(g_3,g_1) = \frac{\lambda}{3!}\frac{1}{\sqrt{2j+1}}\phi^{ab}_j\phi^{bc}_j\phi^{ca}_j \tag{N.5}$$

We can express the action in terms of the matrices ϕ_j of the Peter-Weyl expansion of the field

$$S[\Phi] = \sum_j\left(\frac{1}{2}Tr(\Phi^2_j) + \frac{\lambda}{3!}\frac{1}{\sqrt{2j+1}}Tr(\Phi^3_j)\right), \tag{N.5}$$

where Φ_j is an hermitian matrix of dimension $N_j = 2j+1$, defined by $(\Phi_j)_{ab} = \phi_j^{ab}$. The action is a sum over j of terms which have the standard form for what are known as the matrix models action [195]. As such, the field theory on the groups leads to a generalization of the matrix models: the action (N.6) is the sum over all the values of the dimension N_j of the matrix Φ_j.

Counterparts with the $U(1)$ Case

for the real scalar field $\Phi(g_1, g_2)$ defined on the group manifold $U(1) \times U(1)$ and having the following properties

where $R^{j_1}(g_1)$ are the elements of the group element g_1 in the j representation of $U(1)$. Say $g_1 = e^{i\alpha_1}$

$$\Phi(g_1, g_2) = \Phi(g_1 g, g_2 g), \qquad \text{for all } g \in U(1) \tag{N.5}$$

and

$$\Phi(g_1, g_2) = \Phi(g_2, g_1). \tag{N.5}$$

Using the Peter-Weyl theorem we expand the field in terms of irreducible representations of the group (repeated indices are summed over),

$$\Phi(g_1, g_2) = \phi_{j_1 j_2} \overline{R^{j_1}(g_1)} R^{j_2}(g_2) \tag{N.5}$$

The Peter-Weyl theorem applied to $U(1)$ gives the Fourier series $f(\theta) = \sum_{n=-\infty}^{\infty} f_n e^{in\theta}/\sqrt{2\pi}$

$$\Phi(g_1, g_2) = \sum_{j_1, j_2 = 0}^{\infty} \phi_{j_1 j_2} e^{-ij_1\alpha_1} e^{ij_2\alpha_2} \tag{N.5}$$

The symmetry property implies explicitly

$$\begin{aligned}
\Phi(g_1, g_2) &= \int_0^{2\pi} d\alpha \; \Phi(e^{i\alpha_1 + i\alpha}, e^{i\alpha_2 + i\alpha}) \\
&= \int_0^{2\pi} d\alpha \sum_{j_1, j_2 = 0}^{\infty} \phi_{j_1 j_2} e^{-ij_1(\alpha_1 + \alpha)} e^{ij_2(\alpha_2 + \alpha)} \\
&= \sum_{j_1, j_2 = 0}^{\infty} \phi_{j_1 j_2} e^{-ij_1\alpha_1} e^{ij_2\alpha_2} \int_0^{2\pi} d\alpha e^{-i(j_1 - j_2)\alpha}
\end{aligned} \tag{N.4}$$

Using the orthogonality relation

$$\int dU \overline{R^{(j)}(U)} R^{(k)}(U) = \delta_{jk}, \tag{N.4}$$

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta e^{-in\theta} e^{im\theta} = \delta_{nm}$$

$$\begin{aligned}
\Phi(g_1, g_2) &= \sum_{j_1, j_2=0}^{\infty} \phi_{j_1 j_2} e^{-ij_1\alpha_1} e^{ij_2\alpha_2}\ \delta_{j_1 - j_2} \\
&= \sum_{j_1=0}^{\infty} \phi_{j_1, -j_1} e^{-ij_1(\alpha_1 - \alpha_2)} \\
&= \sum_{j=0}^{\infty} \phi_j\ e^{ij}\ \overline{R^j(g_1)} R^j(g_2) \qquad \text{(N.3)}
\end{aligned}$$

$$\phi_j\ \overline{R^j(g_1)} R^j(g_2) = \int dk e^{-ik} \Phi(g_1, g_2)$$

$$\phi_j = e^i \int dk e^{-ik} \Phi(g_1, g_2)$$

$$\begin{aligned}
\int dk e^{-ik} \Phi(g_1, g_2) &= \sum_k e^{-ik} \sum_{j=0}^{\infty} \phi_j\ e^{ij}\ \overline{R^j(g_1)} R^j(g_2) \\
&= \sum_{j=0}^{\infty} \phi_j\ \int dk e^{i(j-k)}\ \overline{R^j(g_1)} R^j(g_2) \\
&= \sum_{j=0}^{\infty} \phi_j\ \int dk e^{i(j-k)}\ \overline{R^j(g_1)} R^j(g_2) \qquad \text{(N.2)}
\end{aligned}$$

so that ϕ_j represents the Fourier transform of Φ.

N.7 GFT Dual to State Sums

N.7.1 TOCY model as a QFT over a group manifold

$$\begin{aligned}
S[\phi] &= \int dg_1 \ldots dg_4 \phi^2(g_1, g_2, g_3, g_4) + \frac{\lambda}{5!} \int dg_1 \ldots dg_4 \phi(g_1, g_2, g_3, g_4) \\
&\quad \phi(g_4, g_5, g_6, g_7) \phi(g_7, g_3, g_8, g_9) \phi(g_9, g_6, g_2, g_{10}) \phi(g_{10}, g_8, g_5, g_1). \qquad \text{(N.2)}
\end{aligned}$$

In fig(N.7) we give an mnemonic to remember the second term of (N.2), which also serves in perturbation theory we will develop later on. Here $g_i \in SO(4)$ and the field ϕ is a function of $SO(4)$. All the integrals are in the normalized Haar measure. The field ϕ is required to be invariant under any permutation of its arguments; that is, $\phi(g_1, g_2, g_3, g_4) = \phi(g_{\sigma(1)}, g_{\sigma(2)}, g_{\sigma(3)}, g_{\sigma(4)})$, where; and under simultaneous right multiplication by any element g of $SO(4)$:

$$\phi(g_1, g_2, g_3, g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g) \tag{N.2}$$

if given any function $\tilde{\phi}$

$$\phi(g_1, g_2, g_3, g_4) = \int dg \tilde{\phi}(g_1\gamma, g_2\gamma, g_3\gamma, g_4\gamma) \tag{N.2}$$

(verify exercise) This condition is analogous to the "translational invariance" and leads to compatibility conditions on the representations, the analogy of "momentum conservation". Since vertices are 5-valent in our discretization the interaction term should contain the product of five field operators.

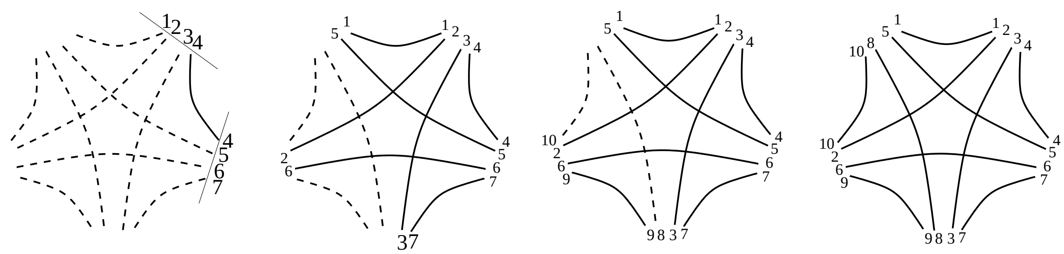


Figure N.7: Structure of interaction vertex. (a) $\phi(g_1, g_2, g_3, g_4)\phi(g_4, g_5, g_6, g_7)$. (d) Reading off the numbers clockwise gives $\phi(g_7, g_3, g_8, g_9)\phi(g_9, g_6, g_2, g_{10})\phi(g_{10}, g_8, g_5, g_1)$.

$$\phi(g) = \sum_{\Lambda} \Phi^{\Lambda}_{\alpha\beta} D^{\Lambda}_{\alpha\beta}(g) \tag{N.2}$$

$$\phi(g_1, g_2, g_3, g_4) = \sum_{(N_1 \ldots N_4)} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} D^{(N_1)\beta_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\beta_4}_{\alpha_4}(g_4) \tag{N.2}$$

$$\int_G dg D^a_{\alpha\beta}(g) D^{a'}_{\alpha'\beta'}(g) = \frac{1}{\dim a} \delta_{\alpha\alpha'} \delta_{\beta\beta'} \delta^{aa'} \tag{N.2}$$

$$\int_G dg D^{(N_1)}_{\alpha_1\beta_1}(g) \ldots D^{(N_4)}_{\alpha_4\beta_4}(g) = \sum_{\Lambda} C^{N_1 \ldots N_4\ \Lambda}_{\alpha_1 \ldots \alpha_4} C^{N_1 \ldots N_4\ \Lambda}_{\beta_1 \ldots \beta_4} \tag{N.2}$$

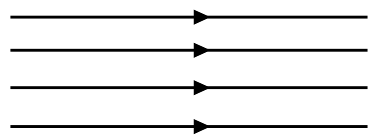


Figure N.8: Structure of interaction vertex.

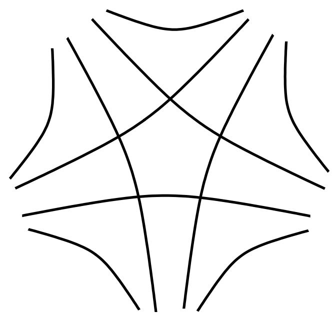


Figure N.9: Structure of interaction vertex.

defining

$$\phi^{\alpha_1 \dots \alpha_4}_{N_1 \dots N_4, \Lambda} := \frac{\Phi^{\alpha_1 \dots \alpha_4}_{(N_1 \dots N_4), \beta_1 \dots \beta_4} C^{N_1 \dots N_4, \Lambda}_{\beta_1 \dots \beta_4}}{\Delta_{N_1} \Delta_{N_2} \Delta_{N_3} \Delta_{N_4}} \tag{N.2}$$

$$\begin{aligned} &\phi(g_1, g_2, g_3, g_4) = \\ &\phi^{\alpha_1 \dots \alpha_4}_{N_1 \dots N_4, \Lambda} \left((\Delta_{N_1} \dots \Delta_{N_4})^2 D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \dots D^{(N_4)\gamma_4}_{\alpha_4}(g_4) C^{N_1 \dots N_4 \ \Lambda}_{\gamma_1 \dots \gamma_4} \right) \end{aligned} \tag{N.2}$$

A "Free" theory

Every n-point function of the field theory can be calculated as a functional derivatives of the generating function $\mathcal{W}(J)$

$$\mathcal{W}(J) = \int \mathcal{D}\phi \exp \left(iS[\phi] + J^{\alpha_1 \dots \alpha_4}_{N_1 \dots N_4, \Lambda} \phi^{N_1 \dots N_4, \Lambda}_{\alpha_1 \dots \alpha_4} \right). \tag{N.2}$$

The moments of the

$$< x^n >= \int_{-\infty}^{\infty} dx x^n \exp \left(i \alpha^2 x^2 \right) \tag{N.2}$$

by taking derivatives of

$$P(J) = \int_{-\infty}^{\infty} dx \exp\left(i\alpha^2 x^2 + Jx\right) = C \exp\left(\frac{1}{2}\frac{J^2}{\alpha}\right), \tag{N.2}$$

i.e.

$$< x^n >= \frac{d^n}{dJ^n} P(J)\bigg|_{J=0}. \tag{N.2}$$

$$\mathcal{W}(J) = \mathcal{C} \exp\left(\frac{1}{2}\frac{J^{\alpha_1\ldots\alpha_4}_{N_1\ldots N_4,\Lambda} J^{N_1\ldots N_4,\Lambda}_{\alpha_1\ldots\alpha_4}}{\Delta_{N_1}\ldots\Delta_{N_4}}\right) \tag{N.2}$$

$$W(s_1, s_2) = \left\{\frac{\delta}{\delta J^{\alpha_1\ldots\alpha_4}_{N_1\ldots N_4,\Lambda}} \frac{\delta}{\delta J^{\alpha_1\ldots\alpha_4}_{N_1\ldots N_4,\Lambda}} \frac{\delta}{\delta J^{\beta_1\ldots\beta_4}_{M_1\ldots M_4,\Lambda}} \frac{\delta}{\delta J^{\beta_1\ldots\beta_4}_{M_1\ldots M_4,\Lambda}} \mathcal{W}(J)\right\}_{J=0}, \tag{N.2}$$

N.7.2 Mode Expansion

The requirement of invariance under the right $SO(3)$ action can be written

$$\phi(g) = \int_{SO(3)} dh \phi(gh). \tag{N.2}$$

Expanding this into the modes, we have

$$\phi(g) = \sum_{\Lambda} \phi^{\Lambda}_{\alpha\beta} D^{(\Lambda)}_{\alpha\beta}(g) = \int_{SO(3)} dh\, \phi(gh) = \sum_{\Lambda} \int_{SO(3)} dh\, \phi^{\Lambda}_{\alpha\beta} D^{(\Lambda)}_{\alpha\gamma}(g) D^{(\Lambda)}_{\gamma\beta}(h). \tag{N.2}$$

$$\phi(g_1 \ldots g_4) = \sum_{N} \phi^{N}_{\alpha} D^{(\Lambda)}_{\alpha\beta}(g) w_{\beta}. \tag{N.2}$$

$SO(4)/SO(3)$

$$g'x = ghx = gx \tag{N.2}$$

The coset $[g]$ then consists of all the elements of in $SO(3)$ which take the point $(0, 0, 1)$ to the point $(0, 0, 1)$. This point is specified by the polar coordinates (θ, ϕ). As such, each point on the unit circle corresponds to a coset and we have that

$$SO(3)/SO(2) \sim S^2 \tag{N.2}$$

Since $SO(2)$ is not a normal subgroup of $SO(3)$, S^2 does not admit a groups structure.

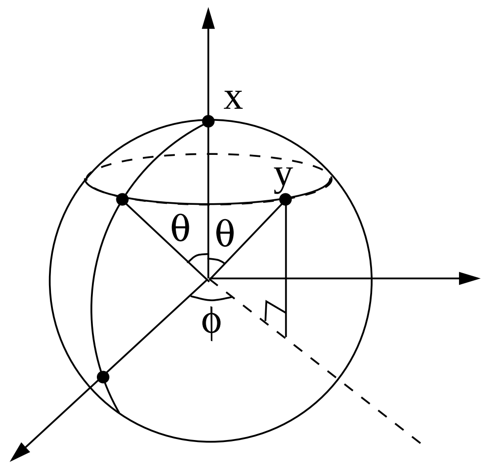


Figure N.10: $SO(3)/SO(2)$.

$SO(3)$ acts on S^2 transitively and we have $SO(3)/SO(2) \sim S^2$.

It is easy to generalise this result to higher dimensional rotation groups and we have the result

$$SO(n)/SO(n-1) \sim S^{n-1} \tag{N.2}$$

where S^{n-1} is the $(n-1)$-sphere.

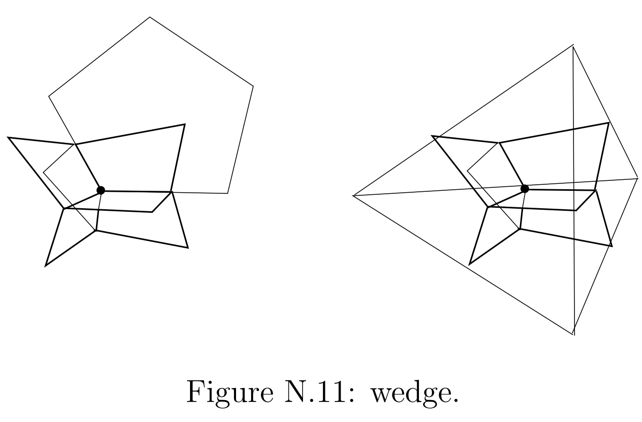


Figure N.11: wedge.

N.7.3 S^3 Spherical Functions

Often:

$$S^3 \cong SO(4)/SO(3) \cong Spin(4)/SU(2) \tag{N.2}$$

here:

$$S^3 \cong SU(2) \tag{N.2}$$

basis of the algebra $C_{alg}(S^3)$

$$?? \tag{N.2}$$

Transitive action of $SU(2) \times SU(2)$ on S^3:

$$(h_1, h_2) \cdot g := h_1 g h_2^{-1} \tag{N.2}$$

$$\text{stab}_{SU(2)\times SU(2)}(g) = \{(ghg^{-1}, h) : h \in SU(2)\} \tag{N.3}$$

Transitive action of $SO(4) \times SU(2)$ on S^3: $(SO(4) \cong SU(2) \times SU(2)/\mathbb{Z}_2)$

N.7.4 $SO(4)$ Barrett-Crane Model

$$\Sigma^{IJ} = \frac{1}{2}\Sigma^{IJ}_{ab}\; dx^a \wedge dx^b$$

defined

$$\Sigma^{IJ} = e^I \wedge e^J \tag{N.3}$$

$\Sigma^{IJ}_{ab} = e^I_a e^J_b - e^I_b e^J_a$

$$|\Sigma_{ab}|^2 = |B_{ab}|^2 \tag{N.3}$$

$$\begin{aligned} |B_{ab}|^2 &= \frac{1}{4}\epsilon^{IJ}{}_{KL}\Sigma^{KL}_{ab}\ \epsilon_{IJMN}\Sigma^{MN}_{ab} \\ &= \frac{1}{4}(\epsilon^{IJ}{}_{KL}\epsilon_{IJMN})\Sigma^{KL}_{ab}\Sigma^{MN}_{ab} \\ &= \frac{2}{4}(\delta_{KM}\delta_{LN}-\delta_{KN}\delta_{LM})\Sigma^{KL}_{ab}\Sigma^{MN}_{ab} \\ &= \frac{1}{2}(\Sigma^{KL}_{ab}\Sigma_{abKL}-\Sigma^{KL}_{ab}\Sigma_{abLK}) \\ &= \Sigma^{IJ}_{ab}\Sigma_{abIJ} \end{aligned} \tag{N.0}$$

$$|\Sigma_{ab}|^2 = g_{aa}g_{bb} - g_{ab}g_{ab} \equiv 2A^2_{ab} \tag{N.0}$$

$$\begin{aligned} |\Sigma_{ab}|^2 &= \frac{1}{2}(e^I_a e^J_b - e^J_a e^I_b)\frac{1}{2}(e_{aI}e_{bJ} - e_{aJ}e_{bI}) \\ &= \frac{1}{4}(e^I_a e_{aI}\ e^J_b e_{bJ} - e^I_a e_{bI}\ e^J_b e_{aJ} - e^J_a e_{bJ}\ e^I_b e_{aI} + e^J_a e_{aJ}\ e^I_b e_{bI}) \\ &= \frac{1}{2}(g_{aa}g_{bb} - g_{ab}g_{ab}) \end{aligned} \tag{N.-1}$$

and

$$\Sigma_{ab}\cdot\Sigma_{ac} = B_{ab}\cdot B_{ac} \tag{N.-1}$$

$$\begin{aligned} B_{ab}\cdot B_{ac} &= \frac{1}{4}\epsilon^{IJ}{}_{KL}\Sigma^{KL}_{ab}\ \epsilon_{IJMN}\Sigma^{MN}_{ac} \\ &= \frac{1}{4}(\epsilon^{IJ}{}_{KL}\epsilon_{IJMN})\Sigma^{KL}_{ab}\Sigma^{MN}_{ac} \\ &= \Sigma^{IJ}_{ab}\Sigma_{acIJ} \end{aligned} \tag{N.-2}$$

$$\Sigma_{ab}\cdot\Sigma_{ac} = g_{aa}g_{bc} - g_{ab}g_{ac} \equiv 2J_{aabc} \tag{N.-2}$$

$$\begin{aligned} \Sigma_{ab}\cdot\Sigma_{ac} &= \frac{1}{2}(e^I_a e^J_b - e^J_a e^I_b)\frac{1}{2}(e_{aI}e_{cJ} - e_{aJ}e_{cI}) \\ &= \frac{1}{2}(g_{aa}g_{bc} - g_{ab}g_{ac}) \end{aligned} \tag{N.-2}$$

$$\int_S |\Sigma| = \int_S |B| = \int_S |\Sigma_{ab}| dx^a dy^b = \sqrt{2} Area(S) \tag{N.-2}$$

the 4-form

$$V \equiv \frac{1}{4!}\epsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL} = \frac{1}{4!}\epsilon_{IJKL}B^{IJ} \wedge B^{KL} \tag{N.-2}$$

proof

$$\begin{aligned} \epsilon_{IJKL}B^{IJ} \wedge B^{KL} &= \frac{1}{4}\epsilon^{IJ}{}_{I'J'}\Sigma^{I'J'}(\epsilon^{KL}{}_{K'L'}\Sigma^{K'L'}) \\ &= \frac{1}{4}(\epsilon_{IJKL}\epsilon^{IJ}{}_{I'J'}\epsilon^{KL}{}_{K'L'})\Sigma^{I'J'}\Sigma^{K'L'} \\ &= \frac{1}{2}(\delta_{KI'}\delta_{LJ'} - \delta_{KJ'}\delta_{LI'})\epsilon^{KL}{}_{K'L'}\Sigma^{I'J'}\Sigma^{K'L'} \\ &= \epsilon_{IJKL}\Sigma^{IJ} \wedge \Sigma^{KL} \end{aligned} \tag{N.-4}$$

Using the Plebanski field, the action can be written in the BF-like form

$$\begin{aligned} S[e,\omega] &= \frac{1}{2}\int \epsilon_{IJKL}\, \Sigma^{IJ}[e] \wedge F^{KL}[\omega] \\ &= \int B_{IJ}[e] \wedge F^{KL}[\omega]. \end{aligned} \tag{N.-4}$$

This condition can be expressed as a constraint equation for Σ:

$$\Sigma^{IJ} \wedge \Sigma^{KL} = V\ \epsilon^{IJKL} \tag{N.-4}$$

Say Σ^{IJ} is of the form $e^I \wedge e^J$, then

$$\begin{aligned} \Sigma^{IJ} \wedge \Sigma^{KL} &= e^I \wedge e^J \wedge e^K \wedge e^L \\ &= e_a^{[I} e_b^J e_c^K e_d^{L]} dx^a \wedge \cdots \wedge dx^d \\ &= \alpha\epsilon^{IJKL} \\ &= \left(\frac{1}{4!}\epsilon_{I'J'K'L'}e^{I'} \wedge e^{J'} \wedge e^{K'} \wedge e^{L'}\right)\epsilon^{IJKL} \\ &= V\ \epsilon^{IJKL} \end{aligned} \tag{N.-7}$$

Conversly if Σ^{IJ} satifies

$$\Sigma^{IJ} \wedge \Sigma^{KL} = \frac{1}{4!}(\epsilon_{I'J'K'L'}\Sigma^{I'J'} \wedge \Sigma^{K'L'})\epsilon^{IJKL} \tag{N.-7}$$

it must be of the form $e^I \wedge e^J$:

Condition (N.7.4) is equivalent to

$$^*\Sigma_{ab} \cdot \Sigma_{cd} = \frac{1}{2}\tilde{V}\epsilon_{abcd} \tag{N.-7}$$

or

$$\frac{1}{2}\epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd} = \frac{1}{2}\tilde{V}\epsilon_{abcd} \tag{N.-7}$$

where $V = \frac{1}{4!}\tilde{V}\epsilon_{abcd}dx^a \wedge \cdots \wedge dx^d$. To prove this, take this and use $\epsilon_{IJKL}\epsilon^{IJKL} = 4!$

$$\epsilon_{IJKL}(4!\Sigma^{IJ}_{[ab}\Sigma^{KL}_{cd]}) = \tilde{V}\epsilon_{abcd}\epsilon_{IJKL}\epsilon^{IJKL}$$

The summation on the left hand side is

$$\begin{aligned}
&\epsilon_{1234}(4!\Sigma^{12}_{[ab}\Sigma^{34}_{cd]}) + \epsilon_{2134}(4!\Sigma^{21}_{[ab}\Sigma^{34}_{cd]}) + \epsilon_{1324}(4!\Sigma^{13}_{[ab}\Sigma^{24}_{cd]}) + \dots \\
&\quad = 4!(4!\Sigma^{12}_{[ab}\Sigma^{34}_{cd]})
\end{aligned}$$

so that

$$4!\Sigma^{12}_{[ab}\Sigma^{34}_{cd]} = \tilde{V}\epsilon_{abcd}$$

it is easy to see this implies

$$\Sigma^{IJ}_{[ab}\Sigma^{KL}_{cd]} = \frac{1}{4!}\tilde{V}\epsilon_{abcd}\epsilon^{IJKL}$$

which becomes upon applying $dx^a \wedge \cdots \wedge dx^d$

$$\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd}dx^a \wedge \cdots \wedge dx^d = \frac{1}{4!}\tilde{V}\epsilon_{abcd}dx^a \wedge \cdots \wedge dx^d\epsilon^{IJKL} \tag{N.-9}$$

giving (N.7.4)

Now contract (N.7.4) with ϵ_{IJKL} we find

$$\epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd}(dx^a \wedge \cdots \wedge dx^d) = \frac{1}{4!}\tilde{V}\epsilon_{abcd}\ \epsilon_{IJKL}\epsilon^{IJKL}(dx^a \wedge \cdots \wedge dx^d)$$

which implies

$$\frac{1}{2}\epsilon_{IJKL}\Sigma^{IJ}_{ab}\Sigma^{KL}_{cd} = \frac{1}{2}\tilde{V}\epsilon_{abcd} \tag{N.-9}$$

giving (N.7.4).

The system of constraints (N.7.4) can be decomposed into three parts:

$$\begin{aligned}
(a) \quad {}^*\Sigma_{ab} \cdot \Sigma_{ab} &= 0, && \text{(N.-8)}\\
(b) \quad {}^*\Sigma_{ab} \cdot \Sigma_{ac} &= 0, && \text{(N.-7)}\\
(c) \quad {}^*\Sigma_{ab} \cdot \Sigma_{cd} &= \pm 2\tilde{V}. && \text{(N.-6)}
\end{aligned}$$

where the indices $abcd$ are all different, and the sign in the last equation is determined by the sign of their permutation. These are the simplicity constraints.

GR can be written as an $SO(4)$ BF theory whose B field satisfies the simplicity constraints (N.-8-N.-7-N.-6).

Proof:

$$\begin{aligned}
B^{IJ} \wedge B^{KL} &= \frac{1}{4}(\epsilon^{IJ}{}_{I'J'}\Sigma^{I'J'})(\epsilon^{KL}{}_{K'L'}\Sigma^{K'L'})\\
&= \frac{1}{4}(\epsilon^{IJ}{}_{I'J'}\epsilon^{KL}{}_{K'L'})\Sigma^{I'J'}\Sigma^{K'L'} && \text{(N.-6)}
\end{aligned}$$

N.7.5 Simplicity Constraints for 4-Simplicies

N.7.6 Self-dual structure of $SO(4)$

$$\hat{X} = X^{IJ}J_{IJ}, \tag{N.-6}$$

$$[\hat{J}_{IJ}, \hat{J}_{KL}] = i\delta_{IK}\hat{J}_{JL} - i\delta_{JK}\hat{J}_{IL} - i\delta_{IL}\hat{J}_{JK} - i\delta_{JL}\hat{J}_{IK}. \tag{N.-6}$$

where $\hat{J}_{IJ}$ are generators of $SO(4)$

$X^{\pm i} = 1/2(X^i \pm X^{0i})$, where $X^i \equiv 1/2\epsilon^i{}_{jk}X^{jk}$ and

$$\hat{\mathbf{X}} = \hat{X}_+ + \hat{X}_-, \quad \hat{X}^\pm = X^{\pm i}\hat{J}_i^\pm. \tag{N.-6}$$

for Lie algebra

$$[\hat{J}_i^\pm, \hat{J}_j^\pm] = i\epsilon_{ij}{}^k\hat{J}_k^\pm, \quad [\hat{J}_i^+, \hat{J}_j^-] = 0. \tag{N.-6}$$

We can write each $U \in SO(4)$ can be written in the form

$$U = (g_+, g_-) \text{ where } g_+ \in SU(2) \quad \text{and} \quad g_- \in SU(2) \quad \text{and} \quad UU' = (g_+g'_+, g_-g'_-). \tag{N.-6}$$

Define selfdual and anti-selfdual generators

$$J_\pm := *J \pm J,$$

that satisfy

$$J_\pm = \pm * J_\pm.$$

then

$$[J_+, J_-] = 0.$$

J_+ span a three dimensional subalgebra $su(2)_+$ of $so(4)$, and the J_- span a three dimensional subalgebra $su(2)_-$ of $so(4)$, both isomorphic to $su(2)$.

N.8 Coherent State Formulation of New BC Models

N.8.1 New Geometric Criterion

N.8.2 Coherent States

$$1_j = \sum_m |j, m >< j, m|, \tag{N.-6}$$

$$\delta_{mm'} = d_j \int_{SU(2)} dg\, t^j_{mj}(g)\overline{t^j_{m'j}(g)} \qquad \text{(N.-6)}$$

$t^j_{mj}(g)$ and $t^j_{mj}(gh)$ differ only by a phase for any group element h from the $U(1)$ subgroup of $SU(2)$. The $U(1)$ subgroup being of the form

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \qquad \text{(N.-6)}$$

$$1_j = d_j \int_{G/H} dn|j, n >< j, n|. \qquad \text{(N.-6)}$$

$$< j, n|\hat{J}^i|j, n > \sigma_i = jn\sigma_3 n^{-1} \equiv jn^i\sigma_i \qquad \text{(N.-6)}$$

or

$$< j, \hat{n}|\hat{J}^i|j, \hat{n} > = < j, j|\hat{J}^{i\prime}|j, j > \qquad \text{(N.-6)}$$

where $J^{i\prime} = g(\hat{n})^{-1}J^i g(\hat{n})$ is the rotated generator.

Thus the state $|j, n >$ describes a vector in $\mathbb{R}^3$ of length j and of direction...

$$\Delta J^2 = j + j^2 - m^2 \qquad \text{(N.-6)}$$

$|j, \hat{n} > = g(\hat{n})|j, j >$, where $\hat{n}$ is a unit vector defining a direction on the sphere S^2 and $g(\hat{n})$ an $SU(2)$ group element rotating the direction $\hat{z} \equiv (0, 0, 1)$ into the direction $\hat{n}$.

Just as $|j, j >$ has direction z with minimal uncertainty, $|j, \hat{n} >$ has direction $\hat{n}$ with minimal uncertainty.

Thus, the highest and lowest states $m = \pm j$ minimize the uncertainty relation correspond to cohernt states.

$$|j, j > \quad \text{and} \quad |j, -j >$$

N.8.3 Partition Function

N.9 Perturbation Theory

N.9.1 Diagrammatic Perturbation Theory

In this section we investigate general rules for the perturbative calculation of correlation functions, rules designed to yield the result in the form of an expansion in powers of g,

$$G = G_0 + gG_1 + g^2 G_2 + g^3 G_3 + ... + g^n G_n + ... \tag{N.-6}$$

Where G_0 is the correlation function of the Gaussian model, (non-interacting model). These rules are easily represented in diagrammatic form. These diagrams are the so-called *Feynman Diagrams*. As a simple example we examine the Ginzburg-Landau theory (see eq.(**??**)). It is impossible to find an exact closed formula for Z(0), but if g is small on can expand $\exp(-g \int d^d x \phi^4(x)/4!)$.

The calculation of $\mathbf{G^{(2)}}$ to order g

First we calculate the 2-point greens function to order g. One must evaluate the integral

$$I(x, y) = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H} = \int \mathcal{D}\phi \phi(x)\phi(y) e^{-H_0} \left[1 - \frac{g}{4!} \int d^d z \phi^4(z) + \cdots \right]. \tag{N.-6}$$

The first term in the square brackets merely yields

$$\mathcal{N} \langle \phi(x)\phi(y) \rangle_0 = \mathcal{N} G_0(x - y) \quad \text{where } \mathcal{N} = Z_0(j = 0). \tag{N.-6}$$

To evaluate the integral in the second term,

$$\int \mathcal{D}\phi \phi(x)\phi(y) e^{-H_0} \int d^d z \phi^4(z), \tag{N.-6}$$

we use Wick's theorem (**??**). There are two types of result from the contractions:

$$(a) \quad \langle \phi(x)\phi(y) \rangle_0 \left\langle \phi^4(z) \right\rangle \quad \text{and} \quad (b) \quad \langle \phi(x)\phi(z) \rangle_0 \left\langle \phi^2(z) \right\rangle_0 \langle \phi(y)\phi(z) \rangle_0 \tag{N.-6}$$

in the wick expansion there were $4\times = 12$ terms of type (a) and 3 terms of type (b). It is convenient to represent these contractions as diagrams, by drawing two "external" points x and y (”external” means that they refer to the arguments of the correlation function), and ”internal” point or ”vertex” z, which stems from the expansion of exp(-V), and over which we integrate. Every contraction is represented by a line joining arguments of ϕ. The two types of terms possible in (N.9.1) are drawn

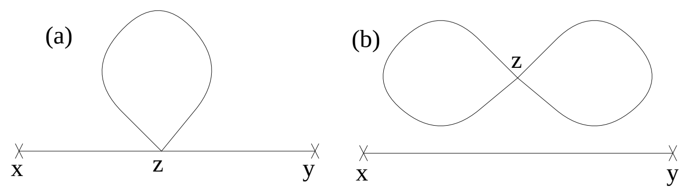


Figure N.12: The two diagrams of order g

These diagrams are called *Feynman diagrams (or graphs)*; one such diagram corresponds to every distinct group of terms of the perturbation expansion. The integral I reads

$$I(x,y) = \mathcal{N}\left[G_0(x-y) - \frac{1}{2}g\int d^dz G_0(x-z)G_0(0)G_0(z-y) - \frac{1}{8}gG_0(x-y)(G_0(0))^2\int d^dz\right] \tag{N.-6}$$

In order to obtain the correlation function, we must divide by Z(0):

$$Z(0) = \int \mathcal{D}\phi e^{-H_0}\left(1 - \frac{g}{4!}\int d^dz\phi^4(z) + \cdots\right) = \mathcal{N}\left[1 - \frac{g}{8}(G_0(0))^2\int d^dz + \cdots\right]. \tag{N.-6}$$

The second term in the square brackets is represented by the diagram.

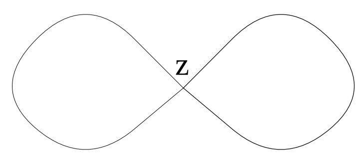


Figure N.13: The vacuum-fluctuation diagram

Dividing (N.9.1) by (N.9.1) we obtain the correlation function to order g

$$G^{(2)}(x-y) = \frac{I(x,y)}{Z(0)} = G_0(x-y) - \frac{1}{2}g\int d^dz G_0(x-z)G_0(0)G_0(z-y) + \mathcal{O}(g^2). \tag{N.-6}$$

The graph(b) from fig.(N.12) does not feature in the perturbation expansion of G. Diagrams of this type are called ”vacuum-fluctuation” (sub)diagrams, meaning a subgraph that is completely disconnected from the "external" points x and y. The sum of all vacuum-fluctuation diagrams is equal to $Z(0) = \mathcal{D}\phi e^{-H}$. **Division by $Z(0)$ cancels all graphs containing "vacuum-fluctuations" parts disconnected from the rest of the diagram**. A proof is given in citeBellac (p 160).

On taking the Fourier transform, eq.(N.9.1) becomes

$$G^{(2)}(k) = G_0(k) - \frac{1}{2} g G_0(k) \left[\int \frac{d^d q}{(2\pi)^d} G_0(q) \right] G_0(k). \tag{N.-6}$$

The factor in front of the second term on the r.h.s. is called the *symmetry factor* of the diagram. To become familiar with the "Feynman rules", i.e. the rules for associating diagrams with the perturbation expansion, we move to the calculation of $G^{(2)}$ to order g^2.

The calculation of $\mathbf{G^{(2)}}$ to order $\mathbf{g^2}$

We use Wick's theorem to compute the expression

$$\left\langle \phi(x)\phi(y) \int d^d z d^d u \phi^4(z)\phi^4(u) \right\rangle_0 . \tag{N.-6}$$

Eliminating the terms that contain vacuum-fluctuation parts, one finds three types of graphs shown in fig.(N.14), with their symmetry factors given in brackets:

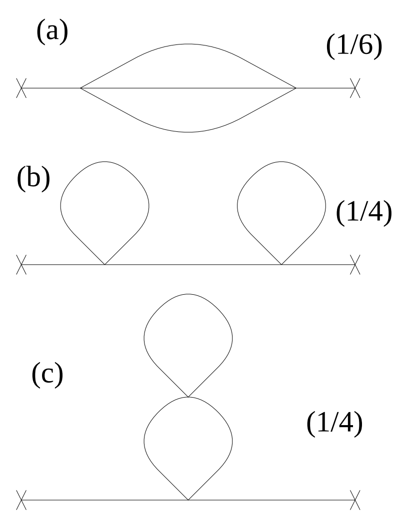


Figure N.14: The vacuum-fluctuation diagram

The vertices z and u may be permuted, which yields a multiplicative factor 2!; however this is exactly cancelled by the factor 1/2! from the expansion of the exponential. This is the same kind of cancellation happens in the nth order.

We shall settle for examining the contribution $\bar{G}(x-y)$ to the correlation function from graph (a) in fig.(N.14). Thus

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int d^dzd^duG_0(x-z)[G_0(z-u)]^3G_0(u-y). \tag{N.-6}$$

Let us write $\bar{G}(x-y)$ as a Fourier transform, by replacing every factor G_0 by its Fourier representation

$$\begin{aligned}\bar{G}(x-y) = \frac{1}{6}g^2 \int d^dzd^du\frac{d^dk}{(2\pi)^d}\frac{d^dk'}{(2\pi)^d}\prod_{l=1}^{3}\left\{\frac{d^dq_l}{(2\pi)^d}e^{i\sum_{l=1}^{3}q_l\cdot(z-u)}\right\} \\ \times e^{ik\cdot(x-z)}e^{ik'\cdot(u-y)}G_0(k)G_0(k')\prod_{l=1}^{3}G_0(q_l).\end{aligned} \tag{N.-6}$$

The integration over z and u yield a product of two delta functions

$$2\pi)^d\delta^d(k-q_1-q_2-q_3)\times(2\pi)^d\delta^d(k'-q_1-q_2-q_3) \tag{N.-6}$$

which represent "momentum conservation" at the two vertices. Hence

$$\bar{G}(x-y) = \frac{1}{6}g^2 \int \frac{d^dk}{(2\pi)^d}e^{ik\cdot(x-y)}[G_0(k)]^2 \times \int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}G_0(q_1)G_0(q_2)G_0(k-q_1-q_2). \tag{N.-6}$$

The last expression shows that $\bar{G}(x-y)$ is the Fourier transform of the function $\bar{G}(k)$,

$$\bar{G}(k) = \frac{1}{6}g^2G_0(k)\left[\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}G_0(q_1)G_0(q_2)G_0(k-q_1-q_2)\right]G_0(-k), \tag{N.-6}$$

(where we have used $G_0(k) = G_0(-k)$). (N.9.1) can be represented diagrammatically in fig.(N.15). The graph shown there has two external propagators $G_0(k)$ and $G_0(-k)$, and three internal propagators; because of the delta-functions $\delta^d(\ldots)$ ("momentum conservation"), only two of the three internal lines are independent. By following the internal propagators one can describe three different closed loops, but because of "momentum conservation" only two of these are independent; i.e. there are only two integration variables in (N.9.1).

Our experience with the previous examples suggest the following "Feynman rules" in k-space ("momentum space"):

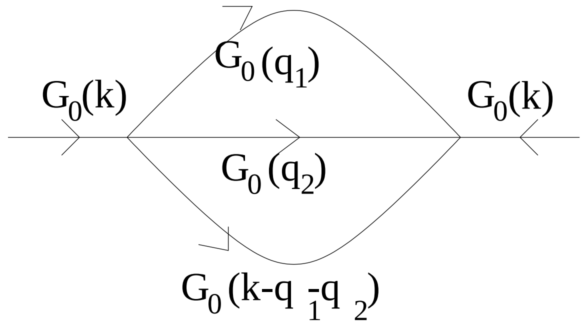


Figure N.15: Diagrammatic representation of (N.9.1)

1. We draw the Feynman diagram with a momentum assigned to each line. We must have overall momentum conservation and conservation at each vertex.

2. To every vertex we assign a factor -g

3. To every line we assign a factor $G_0(k)$

4. To every independent loop there corresponds an integration $\int d^dq/(2\pi)^d$.

5. Finally, every graph is multiplied by a symmetry factor.

N.9.2 The Generating Functional of Connected Diagrams

We start with an example, by investigating the correlation function $G^{(4)}$. It subdivides into one connected and three disconnected diagrams,

$$G^{(4)}(1,2,3,4) = G_c^{(4)}(1,2,3,4) + \{G_c^{(2)}(1,2)G_c^{(2)}(3,4) + permutations\}, \qquad \text{(N.-6)}$$

where G_c denotes a connected correlation function. (note $G_c^{(2)} = G^{(2)}$. In terms of graphs this is represented as in fig.(refbubble0)

The number of disconnected terms is $3 = 4!/[(2!)^2 \times (2!)]$. 4! is the number of permutations of the external points (1,2,3,4); but the result is unaffected by permuting (1,2), or (3,4), or the two bubbles (A) and (B), hence a factor $(2!)^2 \times 2!$.

We have been only considering theories where the n-point correlation functions with n odd vanish: $G^{(2k+1)} = 0$. For more generality, we shall assume that the interaction contains terms inφ^{2p+1}. Consider a disconnected diagram of $G^{(N)}$ corresponding to the subdivision into connected diagrams (fig N.17):

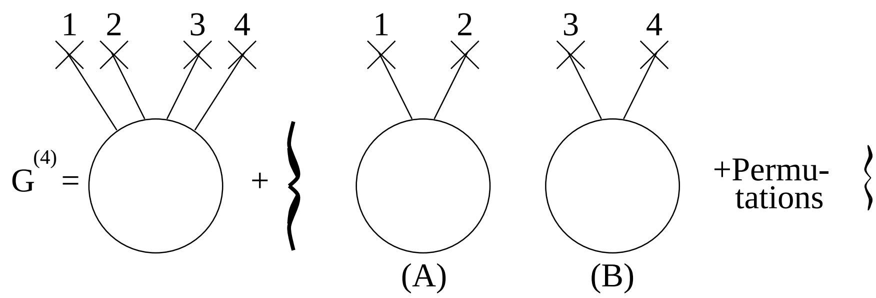


Figure N.16:

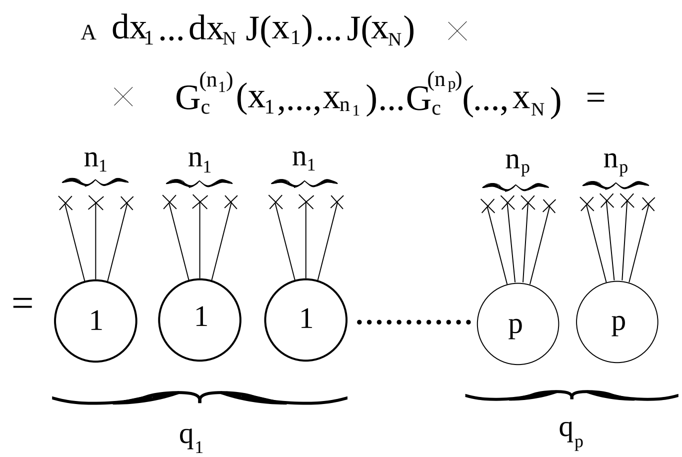


Figure N.17:

There are q_l bubbles connected to n_l external points,...., q_p bubbles connected to n_p external points, with

$$q_1 n_1 + \cdots + q_p n_p = N. \tag{N.-6}$$

The number of independent terms is

$$\frac{N!}{[(n_1!)^{q_1} q_1!] \ldots [(n_p!)^{q_p} q_p!]} \tag{N.-6}$$

It is found that the Functional that generates just connected diagrams is the logarithm of the normalised Generating functional. Hence, consider the exponential of the generating functional of connected diagrams:

$$\exp \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1 \ldots dx_N j(x_1) \ldots j(x_N) G_c^N(x_1 \ldots x_N) \tag{N.-6}$$

This should give the expansion for the generating function of all possible diagrams. When the exponential is expanded it is obvious that the amplitude for every possible disconnected diagram will be produced. To complete the proof that this is the correct Generating Functional we need to check each diagram comes with the correct prefactor.(i.e. equation (N.9.2). So expanding equation (N.9.2)

$$\sum_{q=0}^{\infty} \frac{1}{q!} \left(\sum_{n=1}^{\infty} \int dx_1....dx_n j(x_1)....j(x_n) G_c^{(n)}(x_1....x_n) \right)^q \tag{N.-6}$$

We convert this sum into a summation over N, the number of legs of the disconnected diagrams (figure).

$$\sum_{N=0}^{\infty} \sum_{q_1 n_1 + ... + q_p n_p = N} \prod_{i=1}^{p} \frac{1}{q_i!} \left[\frac{\int dx_1....dx_{n_i} j(x_1)....j(x_{n_i}) G_c^{n_i}(x_1....x_{n_i})}{n_i!} \right]^{q_i} \tag{N.-6}$$

Now we use (N.9.2) and the symmetry of G_c with respect to its arguments to rewrite the above equation as

$$\sum_{N=0}^{\infty} \frac{1}{N!} \int dx_1....dx_N j(x_1)....j(x_N) \sum_{q_1 n_1 + ... + q_p n_p = N} G_c^{n_i}(x_1....x_{n_i})....G_c^{n_p}(....x_N) \tag{N.-6}$$

Which is the correct form for the generating functional. Thus we have found that the generating functional of connected diagrams W(j) is indeed ln[Z(j)/Z(0)],

$$W(j) = ln \frac{Z(j)}{Z(0)} = \sum_{N=1}^{\infty} \frac{1}{N!} \int dx_1... dx_N j(x_1)...j(x_N) G_c^N(x_1...x_N) \tag{N.-6}$$

N.9.3 1PI n-point functions

1PI $n-$point functions play a prominent role in the process of renormalization, as it is enough to renormalize 1PI $n-$point functions. This can be seen from their intmate relation to the effective action.

N.9.4 Rernormalization

since the observed magnitude of physical quantities (such as the charge of the electron) is finite, this number should arise from the addition of a “bare” (unobservable) value and the quantum corrections. Even though both of these quantities were divergent as only its (finite) sum that can be observed.

N.10 Coupling Matter to Spin Foams

group field theory

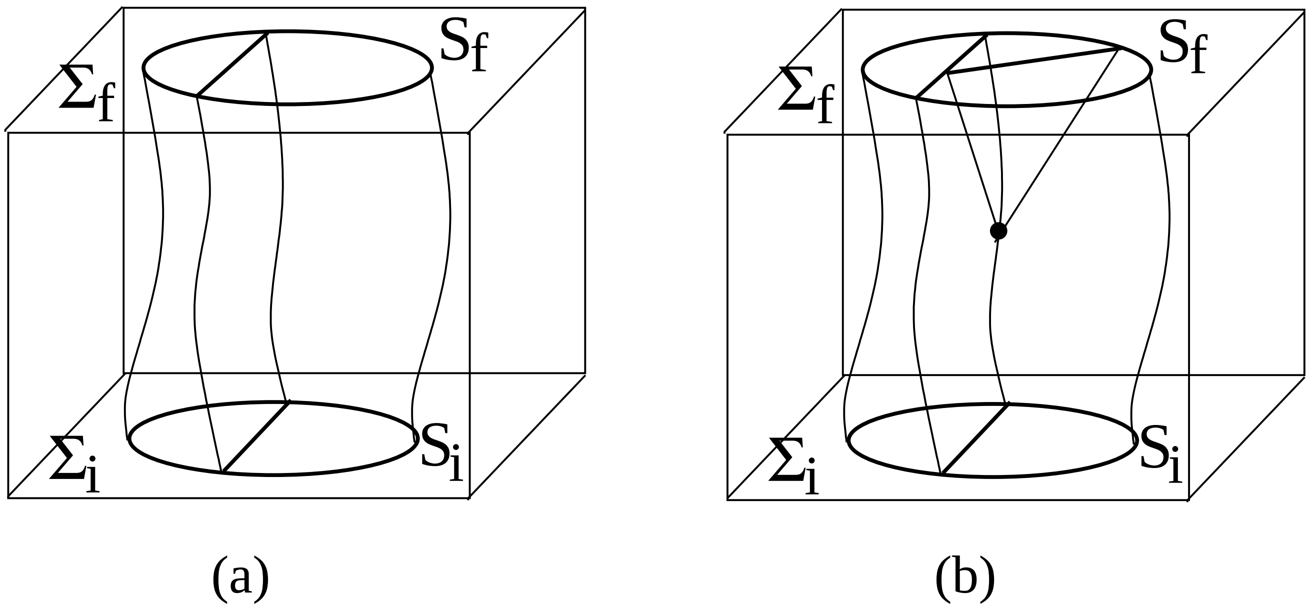


Figure N.18: Spin foam map.

N.11 Background-Independent Renormalization

Tentative ideas have been formulated by Markopoulou [122], [123] and Oeckl [124].

N.11.1 Background-Independent Renormalization a la Markopoulou

The partition function

$$Z_N[K] = Tre^{\mathcal{H}} \tag{N.-6}$$

An RG transformation

$$\exp(\mathcal{H}_N[K'], S'_I) = Tr' \exp(\mathcal{H}_N[K], S_i) \tag{N.-6}$$

This is accomplished by making a partial trace over the degrees of freedom $\{S_i\}$.

The lattice doesn't sit in a preexisting background geometry, the lattice itself represents the (quantized) geometry. The manifold the spinfoam is sitting in isn't equipped with a metric with respect to which scales can be defined; no lattice spacing is associated with the edges of the spin foam.

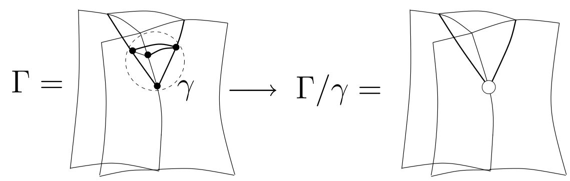


Figure N.19:

A nice example of a Hopf algebra.

background-independent course graining *a la* Fontini-Markopoulou [122], [**?**]

$$Z(s_i, s_f) = \sum_{\Gamma} N(\Gamma) \sum_{\text{labels on }\Gamma} \prod_{f \in \Gamma} \dim_{j_f} \prod_{v \in \Gamma} A_v(j) \tag{N.-6}$$

Two steps:

1. The calculation of a typical block transformation,

2. repeatedly apply it on the entire spin foam Γ to obtain a course grained one, Γ'.

If spin foams are highly irregular this makes the second step a non-trivial combinatorial problem.

Hopf Algebras

algebra: An algebra is simply a vector space over $\mathbb{C}$ (or over $\mathbb{R}$) in which there is defined a distributive and associative multiplication:

(i) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$;

(ii) $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$ for every scalar α, a complex (or real) number.

(iii) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ associativity.

A multiplication associates to each pair of elements of A to an element from A. Mathematicians give this a formal description:

$$m \circ (a \otimes b) = a \cdot b \tag{N.-6}$$

Let's illustrate the notion

$$m \circ (m \otimes id_A)(a \otimes b \otimes c) = m \circ ((ab) \otimes c) = (ab)c \tag{N.-6}$$

(iii) a condition on the multiplication operation: $m \circ (m \otimes \mathrm{id}_A)(a \otimes b \otimes c) = m \circ (id_A \otimes m) \circ (a \otimes b \otimes c)$

$$m \circ (m \otimes \mathrm{id}_A) = m \circ (id_A \otimes m) \tag{N.-6}$$

A unit operation ϵ

The comultiplication does the opposite: it associates a pair of elements from the set C with a single element from C - its coproduct, $\Delta(a) = b \otimes c, \quad b, c \in C$. a compatibility with m,

$$\Delta(a \cdot b) = \Delta(a)\Delta(a) \tag{N.-6}$$

There is a counit $\bar{\epsilon}$. This maps an element of C to a scalar $k : \bar{\epsilon}(a) = k$

A bialgebra is formed by combining an algebra and coalgebra for which $A = C$. There are conditions required of the multiplication and comultiplication so that they are compatible

$\gamma_1 \subset \gamma_2$ $\gamma_2 \subset \gamma_1$ $\gamma_1 \cap \gamma_2 = \emptyset$

let γ denote a proper sublattice of Γ, namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that is left after we "cut out" γ the **remainder** and denote it Γ/γ.

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p \tag{N.-6}$$

These are the **primitive elements** of the Hofp algebra.

The **counit** is an operation which annihilates every lattice except e.

$\gamma_1 \subset \gamma_2$ $\gamma_2 \subset \gamma_1$ $\gamma_1 \cap \gamma_2 = \emptyset$

let γ denote a proper sublattice of Γ, namely $\gamma \neq e$ and $\gamma \neq \Gamma$. We call the lattice that is left after we "cut out" γ the **remainder** and denote it Γ/γ.

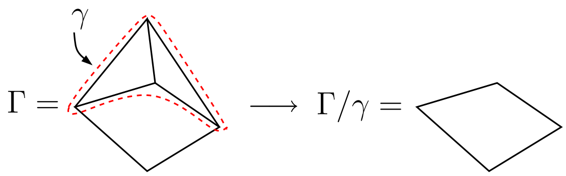


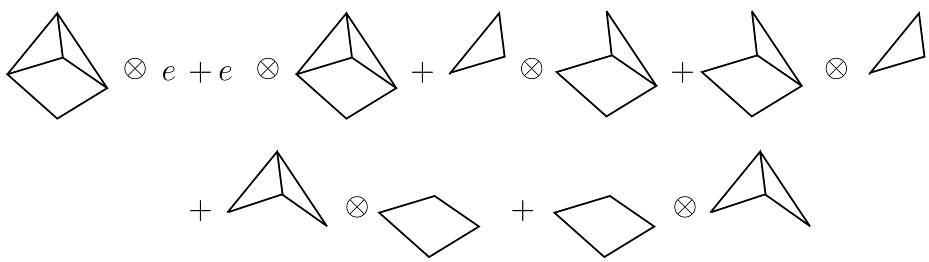
Figure N.20:

$$\begin{aligned} \Delta(\Gamma) &= \Gamma \otimes e + e \otimes \Gamma + \sum_\gamma \gamma \otimes \Gamma/\gamma & \text{(N.-5)} \\ \Delta(e) &= e \otimes e & \text{(N.-4)} \\ \Delta(\Gamma_1 \cdot \Gamma_2) &= \Delta(\Gamma_1)\Delta(\Gamma_2) & \text{(N.-3)} \end{aligned}$$

$$\Delta(\gamma_p) = \gamma_p \otimes e + e \otimes \gamma_p \qquad \text{(N.-3)}$$

These are the **primitive elements** of the Hofp algebra.

$$\Delta(\Gamma) = \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \Gamma/\gamma_1 + \gamma_2 \Gamma/\gamma_2 + \gamma_3 \Gamma/\gamma_3 + \gamma_4 \Gamma/\gamma_4$$



$$= \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \Gamma/\gamma_1 + \gamma_2 \Gamma/\gamma_2 + \gamma_3 \otimes \gamma_4 + \gamma_4 \otimes \gamma_3.$$

Figure N.21: Markoprenorm2

The **counit** is an operation which annihilates every lattice except e.

$$\bar{\epsilon}(\Gamma) = \begin{cases} 0 & \text{for } \Gamma \neq e, \\ 1 & \text{for } \Gamma = e. \end{cases} \qquad \text{(N.-3)}$$

$$\begin{aligned} S(\Gamma) &= -\Gamma - \sum_\gamma S(\gamma)\Gamma/\gamma & \text{(N.-2)} \\ S(\gamma_p) &= -\gamma_p & \text{(N.-1)} \\ S(e) &= e & \text{(N.0)} \\ S(\Gamma_1 \cdot \Gamma_2) &= -S(\Gamma_2) \cdot -S(\Gamma_1) & \\ &= S(\Gamma_1)S(\Gamma_2) \qquad \text{as } \Gamma_1\Gamma_2 = \Gamma_2\Gamma_1. & \text{(N.0)} \end{aligned}$$

$S(\Gamma)$ is an iterative equation that stops when a primative lattice is reached.

(taken from hep-th/9805098) The full set of properties of a Hopf algebra canonly be guaranteed if equivalence works for products, in a certain sence

$$R\left(\prod_i R(\Gamma_1^i)\prod_j \Gamma_2^j\right) = \prod_i R(\Gamma_1^i)\prod_j R(\Gamma_2^j) \tag{N.0}$$

N.11.2 Background-Independent Renormalization a la Oeckl

renormalization *a la* R. Oeckl [124]

N.12 Reduced Phase Space Path-Integral

N.13 Operator Constraint Quantisation Path-Integral

Denote

$$\beta(t) := \beta_1 + t(\beta_2 - \beta_1)$$

It follows that

$$\begin{aligned} V(\beta_2) - V(\beta_1) &= \int_0^1 dt_1 \frac{d}{dt_1} V(\beta(t_1)) \\ &= i\int_0^1 dt_1 V(\beta(t_1))\ \dot{\beta}(0) h'_\mu(\phi - \beta(t_1)) \end{aligned} \tag{N.0}$$

We can find an iterative formula for $V(\beta(t))$. Note

$$\beta_{t_1}(t_2) = \beta_1 + t_2(\beta(t_1) - \beta_1) = \beta_1 + t_1 t_2(\beta_2 - \beta_1) = \beta(t_1 t_2) \tag{N.0}$$

so that

$$\begin{aligned} V(\beta_2(t_1)) - V(\beta_1) &= \int_0^1 dt_2 \frac{d}{dt_2} V(\beta_{t_1}(t_2)) = i\int_0^1 dt_2 \frac{d}{dt_2} V(\beta(t_1 t_2)) t_1 \dot{\beta}(0) h'_\mu(\phi - \beta(t_1 t_2)) \\ &= i\int_0^{t_1} dt_2 V(\beta(t_2))\ \dot{\beta}(0) h'_\mu(\phi - \beta(t_2)) \end{aligned} \tag{N.0}$$

$$V(\beta(t_1)) = V(\beta_1) + i\int_0^{t_1} dt_2 V(\beta(t_2))\dot{\beta}^\mu(0)h'_\mu(\phi - \beta(t_2))$$

or

$$V(\beta_1)^{-1}V(\beta(t_1)) = 1 + i\int_0^{t_1} dt_2 V(\beta_1)^{-1}V(\beta(t_2))\ \dot{\beta}^\mu(0)h'_\mu(\phi - \beta(t_2)) \qquad \text{(N.0)}$$

Now set

$$\begin{aligned} U(t,0) &:= V(\beta_1)^{-1}V(\beta(t)) \\ \Phi(t) &:= \dot{\beta}^\mu(0)h'_\mu(\phi - \beta(t)) \\ &= [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta(t)) \qquad \text{(N.-1)} \end{aligned}$$

Then () becomes

$$U(t,0) = 1 + i\int_0^t dt_1 U(t_1,0)\Phi(t_1) \qquad \text{(N.-1)}$$

From this equation we obtain successive approximations:

$$\begin{aligned} U_0(t,0) &= 1 \\ U_1(t,0) &= 1 + i\int_0^{t_1} dt_1 \Phi(t_1) \\ U_2(t,0) &= 1 + i\int_0^{t_1} dt_1 U_1(t_1,0)\Phi(t_1) \\ &= 1 + i\int_0^t dt_1\Phi(t_1) + (i)^2\int_0^t dt_1\int_0^{t_1} dt_2\Phi(t_2)\Phi(t_1). \qquad \text{(N.-3)} \end{aligned}$$

The $N-th$ order approximation being

$$U_N(t,0) = 1 + \sum_{n=1}^N U^{(n)}(t,0)$$

where

$$U^{(n)}(t,0) = (i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} dt_n \Phi(t_n) \ldots \Phi(t_1)$$

We define the left-time-ordered product as

$$T_l[\Phi(t_2)\Phi(t_1)] \equiv \begin{cases} \Phi(t_2)\Phi(t_1) & t_1 > t_2 \\ \Phi(t_1)\Phi(t_2) & t_2 > t_1 \end{cases}$$

Thus

$$U^{(2)}(t,0) = \frac{i^2}{2} \int_0^t dt_1 \int_0^t dt_2 T_l[\Phi(t_2)\Phi(t_1)]$$

for n operators we obtain

$$U^{(n)}(t,0) = \frac{i^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \ldots \int_0^t dt_n T_l[\Phi(t_n) \ldots \Phi(t_1)] \tag{N.-3}$$

$$\begin{aligned} U(t,0) &= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n T_l[\Phi(t_n) \ldots \Phi(t_1)] \\ &= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \int_0^t dt_1 \ldots \int_0^t dt_n \Phi(t_n) \ldots \Phi(t_1) \\ &= T_l \sum_{n=0}^{\infty} \frac{i^n}{n!} \left(\int_0^t d\tau \Phi(\tau) \right)^n \end{aligned} \tag{N.-4}$$

We can formally write the expression for $U(t,0)$ as

$$V(\beta_1)^{-1} V(\beta(t_1)) = T_l \exp(i \int_0^{t_1} dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))$$

Setting $t_1 = 1$ we get

$$V(\beta_1)^{-1} V(\beta_2) = T_l \exp(i \int_0^1 dt [\beta_2 - \beta_1]^\mu h'_\mu(\phi - \beta_1 - t(\beta_2 - \beta_1)))$$

N.14 Biblioliographical notes

In this chapter I have relied on the following refferences:

"On some aspects of canonical and covariant approaches to quantum gravity", D. Colosi.

N.15 Worked Exercies and Details

Details: Usefull formula.

Abelian subgroup - Cartan subgroup

$$h_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{N.-4}$$

Weyl integration formula

$$\int_G dg f(g) = \int_H \frac{\Delta(\theta)^2}{|W|} \left(\int_{G/H} f(x h_\theta x^{-1}) dx \right) d\theta \tag{N.-4}$$

where $\Delta(\theta) = \sin\theta$ and $|W|$ is the order of the Weyl group.

$$\int_G \delta_\theta(g) f(g) dg = \int_{G/H} \delta_\theta(g) f(x h_\theta x^{-1}) dx \tag{N.-4}$$

$$\delta_\theta(g) = \sum_j a_j \chi_j(g) = \sum_j \left(\delta_\theta \cdot \chi_j\right) \chi_j(g) \tag{N.-4}$$

where

$$\delta_\phi \cdot \chi_j = \int_G \delta_\phi(g) \chi_j(g) dg = \int_{G/H} \delta_\phi(g) \chi_j(x h_\phi x^{-1}) dx = \chi_j(h_\phi) \int_{G/H} dx = \chi_j(h_\phi) V_H \tag{N.-4}$$

$$\delta_\theta(g) = \sum_j \chi_j(h_\theta) \chi_j(g) = \sum_j \frac{\sin d_j \theta}{\sin\theta}(h_\theta) \chi_j(g) \tag{N.-4}$$

Details: 2D GFT.

(i)

$$
\begin{aligned}
\int \Phi^2(g_1,g_2)dg_1dg_2 &= \int \Phi(g_1,g_2)\Phi(g_2,g_1)dg_1dg_2 \\
& \int \left[\sqrt{2j+1}\phi_j^{a_1a_2}\overline{R^j_{a_1c}(g_1)}R^j_{a_2c}(g_2)\right]\times \\
& \qquad \left[\sqrt{2j'+1}\phi_{j'}^{b_1b_2}\overline{R^{j'}_{b_1d}(g_2)}R^{j'}_{b_2d}(g_1)\right]dg_1dg_2 \\
&= \sqrt{2j+1}\sqrt{2j'+1}\;\phi_j^{a_1a_2}\phi_{j'}^{b_1b_2}\left[\int \overline{R^j_{a_1c}(g_1)}R^{j'}_{b_2d}(g_1)dg_1\right]\times \\
& \qquad \left[\int \overline{R^{j'}_{b_1d}(g_2)}R^j_{a_2c}(g_2)dg_2\right] \\
&= \sqrt{2j+1}\sqrt{2j'+1}\phi_j^{a_1a_2}\phi_j^{b_1b_2}\delta_{jj'}\delta_{j'j}\left[\frac{\delta_{a_1b_2}\delta_{cd}}{2j+1}\right]\left[\frac{\delta_{b_1a_2}\delta_{dc}}{2j+1}\right] \\
&= \phi_j^{a_1a_2}\phi_j^{a_2a_1} \qquad \text{(N.-9)}
\end{aligned}
$$

where we used $\delta_{cd}\delta_{dc} = 2j+1$.

(ii)

$$
\begin{aligned}
& \int \Phi(g_1,g_2)\Phi(g_2,g_3)\Phi(g_3,g_1)dg_1dg_2dg_3 \\
= & \int \left[\sqrt{2j_1+1}\phi_{j_1}^{a_1a_2}\overline{R^{j_1}_{a_1d}(g_1)}R^{j_1}_{da_2}(g_2)\right]\times\left[\sqrt{2j_2+1}\phi_{j_2}^{b_1b_2}\overline{R^{j_2}_{b_1e}(g_2)}R^{j_2}_{eb_2}(g_3)\right]\times \\
& \left[\sqrt{2j_3+1}\phi_{j_3}^{c_1c_2}\overline{R^{j_3}_{c_1f}(g_3)}R^{j_3}_{fc_2}(g_1)\right]dg_1dg_2dg_3 \\
= & \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}\;\phi_{j_1}^{a_1a_2}\phi_{j_2}^{b_1b_2}\phi_{j_3}^{c_1c_2}\left[\int \overline{R^{j_1}_{a_1d}(g_1)}R^{j_3}_{fc_2}(g_1)dg_1\right]\times \\
& \left[\int \overline{R^{j_2}_{b_1e}(g_2)}R^{j_1}_{da_2}(g_2)dg_2\right]\times\left[\int \overline{R^{j_3}_{c_1f}(g_3)}R^{j_2}_{eb_2}(g_3)dg_3\right] \\
= & \sqrt{(2j_1+1)(2j_2+1)(2j_3+1)}\;\phi_{j_1}^{a_1a_2}\phi_{j_2}^{b_1b_2}\phi_{j_3}^{c_1c_2}\;\delta_{j_1j_3}\delta_{j_2j_1}\delta_{j_3j_2}\left[\frac{\delta_{a_1c_2}\delta_{df}}{2j_1+1}\right]\left[\frac{\delta_{b_1a_2}\delta_{ed}}{2j_2+1}\right]\left[\frac{\delta_{c_1b_2}\delta_{fe}}{2j_3+1}\right] \\
= & \frac{1}{\sqrt{2j+1}}\;\phi_j^{ab}\phi_j^{bc}\phi_j^{ca} \qquad \text{(N.-14)}
\end{aligned}
$$

Details:

(a):

$$
\begin{aligned}
\phi(g_1g,g_2g,g_3g,g_4g) &= \int dg\tilde{\phi}(g_1g\gamma,g_2g\gamma,g_3g\gamma,g_4g\gamma) \\
&= \qquad \text{(N.-14)}
\end{aligned}
$$

(b):

$$D^{(N)}_{\alpha\beta}(g_1 g) = D^{(N)\gamma}_{\alpha}(g_1) D^{(N)}_{\gamma\beta}(g) \tag{N.-14}$$

$$\begin{aligned}
\int \quad dg \quad & \phi(g_1 g, g_2 g, g_3 g, g_4 g) = \\
= & \int dg \sum_{N_1 \ldots N_4} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} D^{(N_1)}_{\alpha_1\beta_1}(g_1 g) \ldots D^{(N_4)}_{\alpha_4\beta_4}(g_4 g) \\
= & \sum_{N_1 \ldots N_4} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\gamma_4}_{\alpha_4}(g_4) \underbrace{\int dg D^{(N_1)}_{\gamma_1\beta_1}(g) \ldots D^{(N_4)}_{\gamma_4\beta_4}(g)}_{Eq} \\
= & \left[\sum_{N_1 \ldots N_4, \Lambda} \Phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4)\beta_1 \ldots \beta_4} C^{N_1 \ldots N_4\ \Lambda}_{\beta_1 \ldots \beta_4} C^{N_1 \ldots N_4\ \Lambda}_{\gamma_1 \ldots \gamma_4} \right] D^{(N_1)\gamma_1}_{\alpha_1}(g_1) \ldots D^{(N_4)\gamma_4}_{\alpha_4}(g_4)
\end{aligned} \tag{N.-17}$$

$$\phi(x_1, x_2) = \phi(x_1 + y, x_2 + y) \quad \Longleftrightarrow \quad \Phi(g_1, g_2) = \Phi(g_1 g, g_2 g) \tag{N.-17}$$

and

$$\phi(x_1, x_2) = \phi(x_2, x_1) \tag{N.-17}$$

$$\begin{aligned}
\phi(x_1, x_2) &= \int dy \phi(x_1 + y, x_2 + y) \\
&= \int dy \sum_{k_1, k_2} \phi_{k_1, k_2} e^{(x_1 k_1 + x_2 k_2) + iy(k_1 + k_2)} \\
&= \sum_{k_1, k_2} \phi_{k_1, k_2} e^{(x_1 k_1 + x_2 k_2)} \int dy e^{iy(k_1 + k_2)} \\
&= \sum_{k_1, k_2} \phi_{k_1, k_2} e^{(x_1 k_1 + x_2 k_2)} \delta(k_1 + k_2)
\end{aligned} \tag{N.-19}$$

so

the orthogonality relations

$$\int dy e^{ikx} e^{iky} = \delta(k_1 + k_2) \tag{N.-19}$$

$$\phi(x) = \sum_{n_1, n_2} \phi_{k_1, k_2} e^{ik_1 x_1 + ik_1 x_2} \quad \Longleftrightarrow \quad \Phi(g_1, g_2) = \sum_{ab}^{j} \phi^{ac}_j \ R^j_{ab}(g) R^j_{cd}(g) \tag{N.-19}$$

$$\phi_{k_1,k_2} \quad \Longleftrightarrow \quad \phi_j^{a_1 a_2} \tag{N.-19}$$

$$\begin{aligned}
\phi(x_1,x_2,x_3,x_4) &= \int dy \phi(x_1+y, x_2+y, x_3+y, x_4+y) \\
&= \int dy \sum_{k_1,\ldots,k_4} \phi_{k_1,\ldots,k_4} e^{i(x_1k_1+x_2k_2+k_3x_3+x_4k_4)+iy(k_1+k_2+k_3+k_4)} \\
&= \sum_{k_1,\ldots,k_4} \phi_{k_1,\ldots,k_4} e^{i(x_1k_1+x_2k_2+k_3x_3+x_4k_4)} \underbrace{\int dy e^{iy(k_1+k_2+k_3+k_4)}} \\
&= \sum_{k_1,\ldots,k_4} \phi_{k_1,\ldots,k_4} e^{i(x\cdot k)} \delta(k_1+k_2+k_3+k_4)
\end{aligned} \tag{N.-22}$$

$$\begin{aligned}
\phi(x_1,x_2,x_3,x_4) = \phi(x_1+a, x_2+a, x_3+a, x_4+a) \rightarrow \\
\phi(g_1,g_2,g_3,g_4) = \phi(g_1 g, g_2 g, g_3 g, g_4 g)
\end{aligned} \tag{N.-22}$$

$$\begin{aligned}
\phi(x_1,x_2,x_3,x_4) = \phi(x_1-x_4, x_2-x_4, x_3-x_4, 0) \Rightarrow \\
\phi(g_1,g_2,g_3,g_4) = \phi(g_1 g_4^{-1}, g_2 g_4^{-1}, g_3 g_4^{-1}, I)
\end{aligned} \tag{N.-22}$$

$$\phi(x,y) = \phi(x-y) \Rightarrow \phi(g,\gamma) = \phi(g\gamma^{-1}) \tag{N.-22}$$

$$f(g) = \sum_{\Lambda}^{\alpha\beta} D_{\alpha\beta}^{(\Lambda)}(g) \Rightarrow f(x) = \sum_n f_n e^{inx} \tag{N.-22}$$

(c):

$$S[\phi] = \frac{i}{2} \int_G dg_1 \ldots dg_4 \phi^2(g_1,g_2,g_3,g_4) \tag{N.-22}$$

$$\begin{aligned}
S[\phi] &= \frac{i}{2} \sum_{N_1 \ldots N_4} \phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4),\Lambda} (\Delta_{N_1} \ldots \Delta_{N_4}) C^{N_1 \ldots N_4, \Lambda}_{\gamma_1 \ldots \gamma_4} \\
&\quad \sum_{M_1 \ldots M_4} \Phi^{\beta_1 \ldots \beta_4}_{(M_1 \ldots M_4), \delta_1 \ldots \delta_4} \int_G dg_1 D^{(N_1)\gamma_1}_{\alpha_1}(g_1) D^{(M_1)\delta_1}_{\beta_1}(g_1) \ldots \int_G dg_4 D^{(N_4)\gamma_4}_{\alpha_4}(g_4) D^{(M_4)\delta_4}_{\beta_4}(g_4) \\
&= \frac{i}{2} \sum_{N_1 \ldots N_4} \phi^{\alpha_1 \ldots \alpha_4}_{(N_1 \ldots N_4),\Lambda} (\Delta_{N_1} \ldots \Delta_{N_4}) \left[\sum_{M_1 \ldots M_4} \Phi^{\beta_1 \ldots \beta_4}_{(M_1 \ldots M_4), \delta_1 \ldots \delta_4} C^{M_1 \ldots M_4, \Lambda}_{\gamma_1 \ldots \gamma_4} \delta^{\gamma_1 \delta_1} \ldots \delta^{\gamma_4 \delta_4} \right] \\
&\quad \delta_{\alpha_1 \beta_1} \ldots \delta_{\alpha_4 \beta_4} \delta^{N_1 M_1} \ldots \delta^{N_4 M_4} \delta^{\Lambda \tilde{\Lambda}}
\end{aligned} \tag{N.-24}$$

where we $C^{N_1\dots N_4,\Lambda}_{\gamma_1\dots\gamma_4} \to C^{M_1\dots M_4,\Lambda}_{\gamma_1\dots\gamma_4}$ because of the term $\delta^{N_1 M_1}\dots\delta^{N_4 M_4}$.

$$S[\phi] = \frac{i}{2}\phi^{\alpha_1\dots\alpha_4}_{(N_1\dots N_4),\Lambda}\phi^{\beta_1\dots\beta_4}_{(M_1\dots M_4),\tilde{\Lambda}}\Big((\Delta_{N_1}\dots\Delta_{N_4})^2\delta_{\alpha_1\beta_1}\dots\delta_{\alpha_4\beta_4}\delta^{N_1 M_1}\dots\delta^{N_4 M_4}\delta^{\Lambda\tilde{\Lambda}}\Big) \qquad \text{(N.-24)}$$

SO(4)

$$U(g) = 1 - \frac{1}{2}J^j_i\epsilon^i_j + \mathcal{O}, \qquad \text{(N.-24)}$$

where the J^j_i are $N \times N$ matrices

$$[J^j_i, J^l_k] = \delta^j_k J^l_i - \delta^i_l J^j_k + \delta^i_k J^l_j - \delta^j_l J^k_i\ , \quad i,j,kl = 1,\dots,n. \qquad \text{(N.-24)}$$

Hopf algebra of rooted trees.

$$\Delta(t_n) \;=\; t_n \otimes e + e \otimes t_n + \sum_{i=1}^{n-1} t_i \otimes t_{n-i} \qquad \text{(N.-23)}$$

$$S(t_n) \;=\; -t_n - \sum_{i=1}^{n-1} S(t_i)t_{n-i} \qquad \text{(N.-22)}$$

Proofs

Verify the vector space of partitioned lattices form a Hopf algebra under the operations (N.-19), (N.-19).

(a) Verify:

$$\Delta(\Gamma_1 \cdot \Gamma_2) = \Delta(\Gamma_1)\cdot\Delta(\Gamma_2). \qquad \text{(N.-22)}$$

(b) Verify: Definition of the antipode

$$m(S \otimes \mathrm{id})\Delta(\Gamma) = \overline{\epsilon}\epsilon(\Gamma) = \begin{cases} e & \text{for } \Gamma = e \\ 0 & \text{for } \Gamma \neq e \end{cases} \qquad \text{(N.-22)}$$

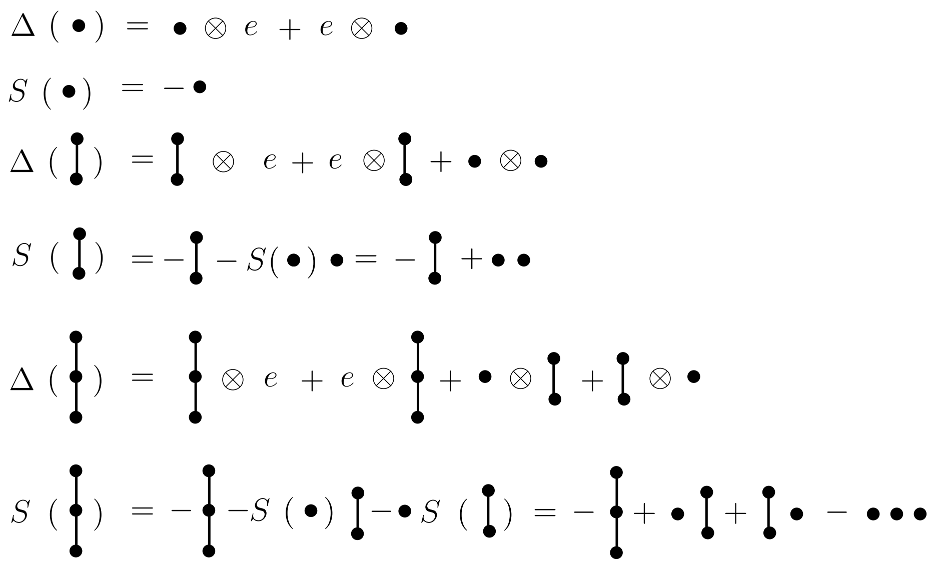
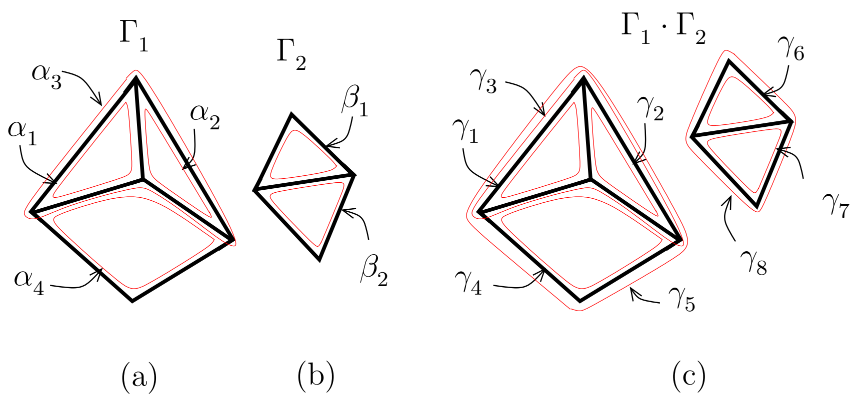


Figure N.22: Example of a Hopf algebra - rooted trees.

Verify: Antipode anti-homomorphism. Note that because the multiplication $\cdot$ is commutative ($\Gamma_1 \cdot \Gamma_2 = \Gamma_2 \cdot \Gamma_1$), the anti-homomorphism is equivalent to a homomorphism.

$$S(\Gamma_1 \cdot \Gamma_2) = S(\Gamma_1) \cdot S(\Gamma_2). \qquad \text{(N.-22)}$$



$\gamma_9 = \gamma_1 \cdot \gamma_6$
$\gamma_{10} = \gamma_2 \cdot \gamma_6$
$\gamma_{11} = \gamma_3 \cdot \gamma_6$
$\gamma_{12} = \gamma_4 \cdot \gamma_6$
$\gamma_{13} = \gamma_5 \cdot \gamma_6$
$\gamma_{14} = \gamma_1 \cdot \gamma_7$
$\gamma_{15} = \gamma_2 \cdot \gamma_7$
$\gamma_{16} = \gamma_3 \cdot \gamma_7$
$\gamma_{17} = \gamma_4 \cdot \gamma_7$
$\gamma_{18} = \gamma_5 \cdot \gamma_7$
$\gamma_{19} = \gamma_1 \cdot \gamma_8$
$\gamma_{20} = \gamma_2 \cdot \gamma_8$
$\gamma_{21} = \gamma_3 \cdot \gamma_8$
$\gamma_{22} = \gamma_4 \cdot \gamma_8$
g23

Figure N.23: Γ_1 and Γ_1 (c) all 22 of them.

$\Delta(\Gamma_1 \cdot \Gamma_2) = (\Gamma_1 \cdot \Gamma_2 \otimes e + e \otimes \Gamma_1 \cdot \Gamma_2 + \sum_\gamma \gamma \otimes \Gamma_1/\gamma)$

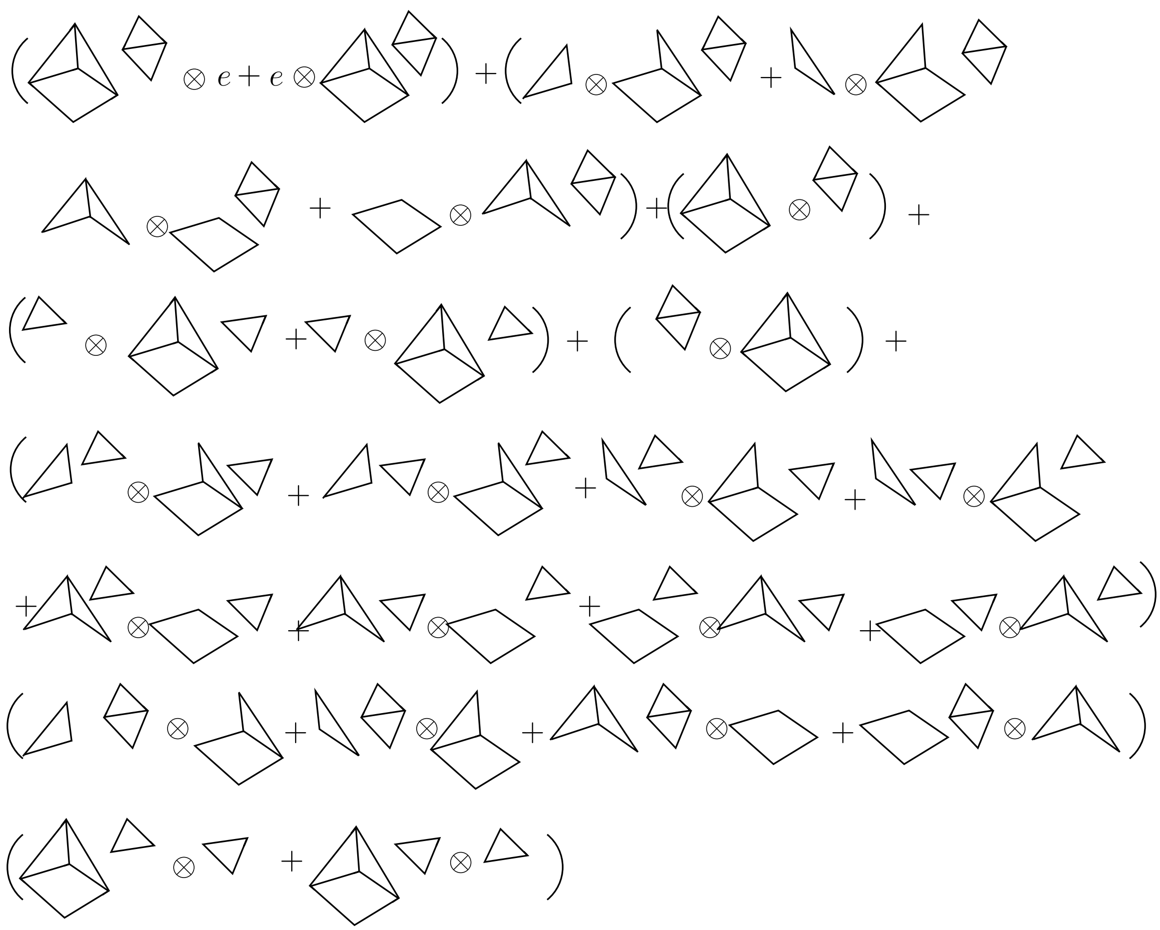


Figure N.24: Homomorphism of Δ.

$$
\begin{aligned}
\Delta(\Gamma_1 \cdot \Gamma_2) &= \Gamma_1 \cdot \Gamma_2 \otimes e + e \otimes \Gamma_1 \cdot \Gamma_2 + \sum_\gamma \gamma \otimes \Gamma_1 \cdot \Gamma_2/\gamma \\
&= \Gamma_1 \cdot \Gamma_2 \otimes e + e \otimes \Gamma_1 \cdot \Gamma_2 + \\
&+ \sum_\alpha \alpha \otimes (\Gamma_1/\alpha) \cdot \Gamma_2 + \Gamma_1 \otimes \Gamma_2 \\
&+ \sum_\beta \beta \otimes (\Gamma_2/\beta) \cdot \Gamma_1 + \Gamma_2 \otimes \Gamma_1 \\
&+ \sum_{\alpha,\beta} \alpha \cdot \beta \otimes (\Gamma_1/\alpha) \cdot (\Gamma_2/\beta) + \\
&+ \sum_\alpha \alpha \cdot \Gamma_2 \otimes \Gamma_1/\alpha + \sum_\beta \Gamma_1 \cdot \beta \otimes \Gamma_2/\beta \\
&= (\Gamma_1 \otimes e + e \otimes \Gamma_1 + \sum_\alpha \alpha \otimes \Gamma_1/\alpha)\ (\Gamma_2 \otimes e + e \otimes \Gamma_2 + \sum_\beta \beta \otimes \Gamma_2/\beta) \\
&= \Delta(\Gamma_1)\Delta(\Gamma_2) \qquad \text{(N.-28)}
\end{aligned}
$$

(b) Anitipode definition.

$$(S \otimes \mathrm{id})\Delta(\gamma_p) = m(S \otimes \mathrm{id})(\gamma_p \otimes e + e \otimes \gamma_p) = S(\gamma_p) + \gamma_p = 0 \tag{N.-28}$$

$$\begin{aligned}(S \otimes \mathrm{id})\Delta(\Gamma) &= m(S \otimes \mathrm{id})(\Gamma \otimes e + e \otimes \Gamma + \sum_\gamma \gamma \otimes \Gamma/\gamma) \\ &= S(\Gamma) + \Gamma + \sum_\gamma S(\gamma) \cdot \Gamma/\gamma \end{aligned} \tag{N.-28}$$

Anitipode anti-homomorphism.

$$\begin{aligned} S(\gamma_p \cdot \gamma_{p'}) &= -\gamma_p \cdot \gamma_{p'} - [S(\gamma_p)\gamma_{p'} + S(\gamma_{p'})\gamma_p] \\ &= \gamma_p \gamma_{p'} \\ &= S(\gamma_p)S(\gamma_{p'}). \end{aligned} \tag{N.-29}$$

$$\begin{aligned} S(\Gamma_1 \cdot \Gamma_2) &= -\Gamma_1 \cdot \Gamma_2 - \sum_\gamma S(\gamma) \cdot (\Gamma_1 \cdot \Gamma_2)/\gamma \\ &= (-\Gamma_1 - \sum_\alpha S(\alpha) \cdot \Gamma_1/\alpha) \cdot (-\Gamma_2 - \sum_\beta S(\beta) \cdot \Gamma_2/\beta) \end{aligned} \tag{N.-29}$$

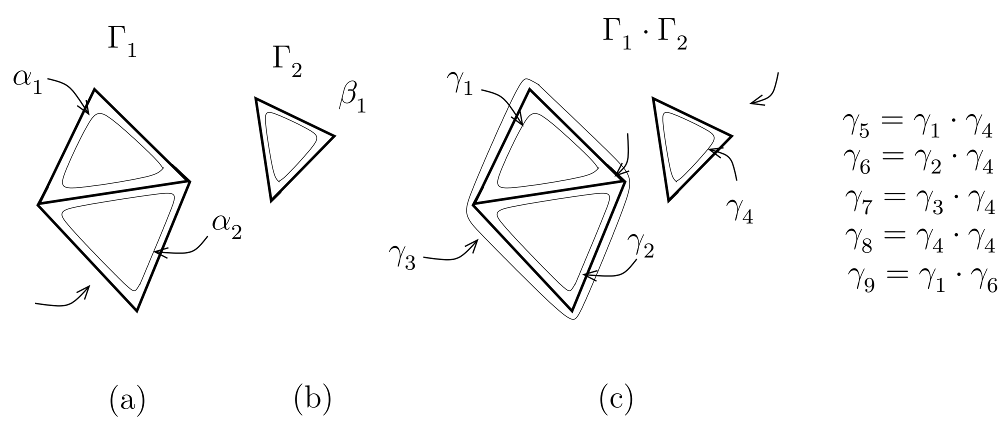


Figure N.25: Anti-Homomorphism of S.

$$\begin{aligned} S(\Gamma_1 \cdot \Gamma_2) &= -\Gamma_1 \cdot \Gamma_2 - \sum_\gamma S[\gamma](\Gamma_1 \cdot \Gamma_2)/\gamma \\ &= -\Gamma_1 \cdot \beta - S(\alpha_1) \cdot \alpha_2 \cdot \beta \\ &= [2\alpha_1 \cdot \alpha_2 \cdot -S(\Gamma_1)] \cdot \beta \\ &= S(\Gamma_1) \cdot \beta \end{aligned} \tag{N.-31}$$

where we have used

$$S(\Gamma_1) = -\Gamma_1 - S(\alpha_1)\cdot\alpha_2 - S(\alpha_2)\cdot\alpha_1 = -\Gamma_1 + 2\alpha_1\cdot\alpha_2. \qquad \text{(N.-31)}$$

Ising chain

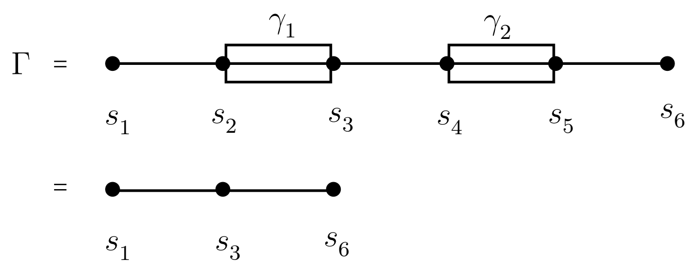


Figure N.26: Markoprenorm5.

$$\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i\sigma_{i+1} - h\sum_{i=1}^{N}\sigma_i \qquad \text{(N.-31)}$$

with $\sigma_i = \pm 1$ and $\sigma_{N+1} = \sigma_1$.

The partition function is given by

$$Z = Tr_{\sigma_i} e^{\mathcal{H}} = \sum_{\{\sigma_i\}=\pm 1}^{N} \exp\left\{\sum_{i=1}^{N}\left[K\sigma_i\sigma_{i+1} + \frac{1}{2}(\sigma_i+\sigma_{i+1})\right]\right\}. \qquad \text{(N.-31)}$$

Carry out sum over odd numbered degrees of freedom

$$\sum_{\sigma_1=\pm 1} e^{\mathcal{H}} = \sum_{\sigma_2=\pm 1}\sum_{\sigma_4=\pm 1}\cdots\sum_{\sigma_N=\pm 1}\left[\sum_{\sigma_1=\pm 1}\sum_{\sigma_3=\pm 1}\cdots\sum_{\sigma_{N-1}=\pm 1} e^{\mathcal{H}}\right] \qquad \text{(N.-31)}$$

$$\begin{aligned}
\sum_{\sigma_1=\pm 1} e^{K\sigma_1(\sigma_N+\sigma_2)+h\sigma_1} &= 2\cosh[K(\sigma_N+\sigma_2)+h] \\
\sum_{\sigma_3=\pm 1} e^{K\sigma_3(\sigma_N+\sigma_4)+h\sigma_3} &= 2\cosh[K(\sigma_N+\sigma_4)+h] \\
&\vdots
\end{aligned} \qquad \text{(N.-32)}$$

$$\sum_{\sigma_1=\pm1}\sum_{\sigma_3=\pm1}\cdots\sum_{\sigma_{N-1}=\pm1} e^{\mathcal{H}} = 2\cosh[K(\sigma_N+\sigma_2)+h]\times\cdots\times 2\cosh[K(\sigma_N+\sigma_2)+h] \times e^{K\sigma_2(\sigma_1+\sigma_3)+h\sigma_2}\times\cdots\times e^{K\sigma_{N-1}(\sigma_1+\sigma_3)+h\sigma_2} \quad \text{(N.-32)}$$

$$2e^{h(\sigma_N+\sigma_2)/2}\cosh[K(\sigma_N+\sigma_2)+h] \exp\{2g+K'\sigma_N\sigma_2+\frac{1}{2}h'(\sigma_N+\sigma_2)\} \quad \text{(N.-32)}$$

where

$$K' = \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h} \quad \text{(N.-32)}$$

$$h' = h + \frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)} \quad \text{(N.-32)}$$

and

$$g = \frac{1}{8}\ln[16\cosh(2K+h)\cosh(2K-h)\cosh^2 h]. \quad \text{(N.-32)}$$

$$\sum_{\sigma_1=\pm1}\sum_{\sigma_3=\pm1}\cdots\sum_{\sigma_{N-1}} e^{\mathcal{H}} = \exp\left\{Ng(K,h)+K'\sum_i\sigma_{2i}\sigma_{2i+2}+h'\sum_i\sigma_{2i}\right\} \quad \text{(N.-32)}$$

sum is over remaining even numbered sites.

Algebraic block transform

The coproduct on Γ is

$$\Delta[\Gamma] = \Gamma\otimes e + e\otimes\Gamma + \gamma_1\otimes\gamma_2 + \gamma_2\otimes\gamma_1 \quad \text{(N.-32)}$$

$$\Delta[\Gamma] = \Gamma + \gamma_1\gamma_2 \quad \text{(N.-32)}$$

The shrinking antipode S'_R on w_Γ, gives

$$S'_R(w_\Gamma) = -R(w_\Gamma) + R(w_{\gamma_1})w_{\gamma_2} + R(w_{\gamma_2})w_{\gamma_1}$$
$$= w_\Gamma + 2w_{\gamma_1}w_{\gamma_2}, \quad \text{(N.-32)}$$

where $w_{\Gamma'} = R(w_\Gamma)$

define the operation R

$$R(\gamma) = \partial\gamma, \quad \text{(N.-31)}$$
$$R(w_\gamma) = \sum_{\text{Internal spins } \gamma=\pm} w_\gamma \quad \text{(N.-30)}$$

Ising/Potts model in two dimensions.

$$w_\Gamma = \exp\left(\sum_{<i,j>} \kappa_{ij} s_i s_j\right) \quad \text{(N.-30)}$$

where means that i and j are adjacent sites in the lattice.

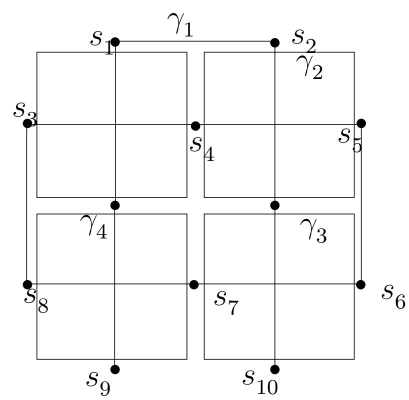


Figure N.27: Homomorphism of Δ.

The coproduct on Γ is

$$\Delta[\Gamma] = \Gamma \otimes e + e \otimes \Gamma + \gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1 \quad \text{(N.-30)}$$

$$\Delta[\Gamma] = \Gamma + \gamma_1\gamma_2 \quad \text{(N.-30)}$$

The shrinking antipode S'_R on w_Γ, gives

$$
\begin{aligned}
S'_R(w_\Gamma) &= -R(w_\Gamma) + R(w_{\gamma_1})w_{\gamma_2} + R(w_{\gamma_2})w_{\gamma_1} \\
&= w_\Gamma + 2w_{\gamma_1}w_{\gamma_2}, \qquad \text{(N.-30)}
\end{aligned}
$$

where $w_{\Gamma'} = R(w_\Gamma)$

define the operation R

$$
\begin{aligned}
R(\gamma) &= \partial\gamma, \qquad \text{(N.-29)} \\
R(w_\gamma) &= \sum_{\text{Internal spins } \gamma=\pm} w_\gamma \qquad \text{(N.-28)}
\end{aligned}
$$

Proofs