Appendix O

The Master Constraint

O.1 Weak Dirac Observables

A function O is a weak Dirac observable if and only if

$$\{O,\{O,\mathbf{M}\}\}_{\mathbf{M=0}} \quad \text{Master Equation} \tag{O.0}$$

Proof:

It is a straight forward but tedious calculation, the end result is,

$$\begin{aligned} \{O,\{O,\mathbf{M}\}\} &= \int_X d\mu(x) \Big[q^{jk}(x)\{O,C_j(x)\}\{O,C_k(x)\} + q^{jk}(x)\{O,\{O,C_j(x)\}\}C_k(x) \\ &+ \{O,q^{jk}(x)\}\{O,C_j(x)\}C_k(x) + \frac{1}{2}\{O,\{O,q^{jk}(x)\}\}C_j(x)C_k(x) \Big] \end{aligned} \tag{O.0}$$

Restricting this expression to the constraint surface $\mathcal{C}$ is equivalent to setting $\mathbf{M} = 0$ hence

$$\{O,\{O,\mathbf{M}\}\}_{\mathbf{M=0}} = \int_X d\mu(x) q^{jk}(x)\{O,C_j(x)\}|_\rfloor \{O,C_k(x)\}|_\rfloor \tag{O.0}$$

Obviously, if $\{O,C_j(x)\}|_{\mathcal{C}=0} = 0$ then $\{O,\{O,\mathbf{M}\}\}_{\mathbf{M=0}} = 0$.

Now, since q is positive definite the condition $\{O,\{O,\mathbf{M}\}\}_{\mathbf{M=0}} = 0$ implies that

$$\{O,C_k(x)\}|_{\mathcal{C}=0} = 0 \quad \text{for } \forall\ x \in X. \tag{O.0}$$

Hence the conditions are equivalent. Eq.(O.1) can be reexpressed as

$$\{O, C_k(x)\}|_{\mathcal{C}=0} = 0 \tag{O.0}$$

for all smooth test functions of compact support.

Details: Use equation involving Poison brackets.

O.2 Regularization of the Master Constraint

rigging map

$$< \eta(T_s), \eta(T_{s'}) >= \eta(\eta(T_{s'}))[T_s]$$

$$\begin{aligned} < T_{[s]}, [\hat{\hat{C}}'(\Delta)]T_{[s_2]} >_{Diff} &= < \hat{\hat{C}}^\dagger(\Delta)T_{[s]}, T_{[s_2]} >_{Diff} \\ &= \eta(T_{s_2})[\hat{\hat{C}}^\dagger(\Delta)T_{s0(s)}] \\ &= T_{[s_2]}(\hat{\hat{C}}^\dagger(\Delta)T_{s0(s)}) \end{aligned} \tag{O.-1}$$

$$Q_M(T_{[s_1]}, T_{[s_2]}) = \lim_{\tau\Sigma} \sum_{\Delta\in\tau} \sum_{[s]} \overline{T_{[s_1]}(\hat{\hat{C}}^\dagger(\Delta)T_{s0([s])})}(T_{[s_2]}\hat{\hat{C}}^\dagger(\Delta)T_{s0([s])}), \tag{O.-1}$$

The part

$$\hat{\hat{C}}^\dagger(\Delta)T_{s0([s])}$$

has a finite number of terms so that

$$T_{[s_1]}\hat{\hat{C}}^\dagger(\Delta)T_{s0([s])}$$

obviously the number of $[s]$ contributing to (N.-19) is finite

O.3 Quantizing the Master Constraint

$$Q_{\mathbf{M}}(\Psi_{Diff}, \Psi'_{Diff}) := \lim_{\epsilon \to 0} \sum_{[s]} \frac{1}{2} \sum_{v \in V(\gamma)} \sum_{v' \in V(\gamma)} \frac{1}{C^3_{n(v)} C^3_{n(v')}} \sum_{v(\Delta)=v} \sum_{v(\Delta')=v'} \chi_\epsilon(v - v') \times \Psi_{Diff} \hat{h}^\epsilon_v \, \Psi'_{Diff} \hat{h}^\epsilon_{v'} \qquad \text{(O.-1)}$$

O.4 Testing the Master Constraint on Toy Models

$$p_1 = 0, \quad p_2 = 0$$
$$M := c_1 p_1^2 + c_2 p_2^2, \quad \text{where } c_1, c_2 \text{ are positive numbers.} \qquad \text{(O.-1)}$$

Quantization: Schrödinger representation on $L_2(R^2)$

$$\hat{M} := c_1 \hat{p}_1^2 + c_2 \hat{p}_2^2 \qquad \text{(O.-1)}$$

$$\int dp_1 dp_2 \overline{\psi(p_1, p_2)} \phi(p_1, p_2) = \int d\lambda \left[dp_1 dp_2 \delta(\lambda - c_1 \hat{p}_1^2 - c_2 \hat{p}_2^2) \overline{\psi(p_1, p_2)} \phi(p_1, p_2) \right] \qquad \text{(O.-1)}$$

$$\int d\lambda \int dp_1 dp_2 \delta(\lambda - c_1 p_1^2 - c_2 p_2^2) \psi(p_1, p_2) \qquad \text{(O.-1)}$$

$\lambda \int dx \delta(\lambda x) = 1 = \int_{-\infty}^{\infty} dx \delta(x)$ implies $\delta(\lambda x) \to \delta(x)/\lambda$

$$\int d\lambda \frac{\lambda^{1-1}}{\sqrt{c_1 c_2}} \int d\tilde{p}_1 d\tilde{p}_2 \delta(1 - \tilde{p}_1^2 - \tilde{p}_2^2) \psi \left(\frac{\sqrt{\lambda} \tilde{p}_1}{\sqrt{c_1}}, \frac{\sqrt{\lambda} \tilde{p}_2}{\sqrt{c_2}} \right) \qquad \text{(O.-1)}$$

$$d\Omega = \sin\theta d\theta \qquad \text{(O.-1)}$$

$$\int d\lambda \frac{\lambda^{1-1}}{\sqrt{c_1 c_2}} \int_{S^1} d\Omega \mathcal{H}^{\oplus} \left(\frac{\sqrt{\lambda} \cos\theta}{\sqrt{c_1}} \frac{\sqrt{\lambda} \sin\theta}{\sqrt{c_2}} \right) \qquad \text{(O.-1)}$$

$\lambda = \mathbf{0}$

$$\int dp_1 dp_2 \delta(c_1 p_1^2 + c_2 p_2^2)\psi(p_1, p_2) \tag{O.-1}$$

$$u = c_2 p_2^2 / c_1, \quad \text{so that} \quad p_2 = \sqrt{\frac{uc_1}{c_2}} \quad dp_2 = du\sqrt{\frac{uc_1}{uc_2}} \tag{O.-1}$$

$$\begin{aligned} &= \int dp_1 \frac{1}{c_2} \int \frac{du}{\sqrt{c_2}} \delta(1+u)\psi(p_1, \sqrt{\frac{c_1 u}{c_2}}) \\ &= \int dp_1 \frac{1}{c_2} \left[\int \frac{du}{\sqrt{c_2}} \psi(p_1, i\sqrt{\frac{c_1}{c_2}}) \right] \end{aligned} \tag{O.-1}$$

$$\mathcal{H}_{phys} = L_2(R). \tag{O.-1}$$

More general result:

$$\begin{aligned} &= \int_{R_+}^{\oplus} d\lambda \int_{R}^{\oplus} dp^{(n-k)} \delta(\lambda - \sum_{j>k} c_j p_j^2) \mathcal{H}^{\oplus}(k) \\ &= \frac{1}{\prod_{j>k} \sqrt{c_j}} \int_{R_+}^{\oplus} \lambda^{n-k-1} d\lambda \int_{S^{n-k-1}}^{\oplus} d\Omega \mathcal{H}^{\oplus} \left(\{ \frac{\sqrt{\lambda} n_j(\omega)}{\sqrt{c_j}} \} \right) \end{aligned} \tag{O.-1}$$

$$\mathcal{H}_{phys} = L_2(R^k) \tag{O.-1}$$

$$\{ \tilde{\Omega}_{sl} := \exp(-\frac{1}{2} \sum_{j=1}^{m} p_j^2) \psi_{s,l} | s \in \mathbf{N}, l \in l_s \} \tag{O.-1}$$

A more direct approach:

$$\frac{1}{4}\nabla^2 = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{O.-1}$$

General solution is

$$g(z, \overline{z}) = f(z) + \tilde{f}(\overline{z}) \tag{O.-1}$$

Maxwell's theory in Minkowskian spacetime

canonical pair of fields (A_a, E^a)
sympletic structure $\{E^a(x), A_b(y)\} = e^2 \delta^a_b \delta(x,y)$
fall off conditions $A \sim r^{-1}$, $E \sim r^{-2}$

$$G(\Lambda) = \int_{R^3} d^3x \Lambda(x) \partial_\mu E^\mu(x) \tag{O.-1}$$

The master constraint:

$$\mathbf{M} := \frac{1}{2} < \partial \cdot E, C \cdot \partial \cdot E >_{\mathcal{H}} \tag{O.-1}$$

where C is a positive definite operator on $\mathcal{H}$. $\mathcal{H} = L_2(R^3, d^3x)$

introduce

$$z^a = \frac{1}{\sqrt{2\hbar e^2}} [\sqrt{-\Delta} A_a - i \sqrt{\Delta}^{-1} E^a] \text{ and } \bar{z}^a \tag{O.-1}$$

$$\mathbf{M} = \frac{\hbar e^2}{4} \sum_{J,K} Q_{JK} (\bar{z}^{(3)}_J - z^{(3)}_J)(\bar{z}^{(3)}_K - z^{(3)}_K) \tag{O.-1}$$

$$Q_{JK} := < b_J, \sqrt{-\Delta}^{3/2} C \sqrt{-\Delta}^{3/2} b_K >_{\mathcal{H}} \tag{O.-1}$$

-choose Q such that $\mathbf{M}$ convergent.

$$\hat{\mathbf{M}} = \sum_{J,K} Q_{JK} [(\hat{z}^{(3)}_J)^\dagger - \hat{z}^{(3)}_J][\hat{z}^{(3)}_K - (\hat{z}^{(3)}_K)^\dagger] \tag{O.-1}$$

Master Constraint

$$\mathbf{M} := \int_\sigma d^3x \frac{G_J G_K \delta^{JK}}{\sqrt{\det(q)}} \tag{O.-1}$$

O.5 Functional Analytic Issues of the Master Constraint

semibounded form

the domain of the form

it needs to be densely defined

Removal of regulator we end up with a quadratic for $Q_{\mathbf{M}}$ on H_{Diff} which is by inspection is *positive*, hence semibounded

If we can prove that the form is closable, then there is a unique positive, self-adjoint operator $\hat{\mathbf{M}}$ such that

$$Q_{\mathbf{M}}(\Psi_{Diff}, \Psi'_{Diff}) =< \Psi_{Diff}|\hat{\mathbf{M}}|\Psi'_{Diff} >_{Diff} \quad \text{(O.-1)}$$

O.5.1 Closure

Definition. An operator is *closed* whenever a sequence of vectors φ_n in the domain of A converges to a limit vector ψ and a sequence of vectors $A\varphi_n$ converges to a limit vector ϕ, then φ is in the domain of A and $A\varphi = \phi$.

Consider an infinite linear combination $\varphi = \sum_{k=1}^{\infty} x_k \phi_k$ where all the vectors are in the domain of A. Then each partial sum $\varphi_n = \sum_{k=1}^{n} x_k \phi_k$ is in the domain of A and

$$A\varphi_n = \sum_{k=1}^{n} x_k A\phi_k \quad \text{(O.-1)}$$

The sequence of vectors φ_n converges to a limit vector φ. Suppose the sequence of vectors $A\varphi_n$ converges to a limit vector

$$\phi = \sum_{k=1}^{n} x_k A\phi_k. \quad \text{(O.-1)}$$

If A is closed, then $\sum_{k=1}^{n} x_k \phi_k$ is in the domain of A and

$$A\sum_{k=1}^{\infty} x_k \phi_k = \sum_{k=1}^{\infty} x_k A\phi_k \quad \text{(O.-1)}$$

Quadratic form $Q_M\{\varphi, \phi\}$ is closed whenever a sequence of vectors ψ_n in the domain of $Q_M\{,\}$ converges to a limit vector ψ and a sequence of numbers $Q_M\{\varphi, \psi_n\}$ converges to a limit $Q_M\{\varphi, \phi\}$, then ϕ is in the domain of $Q_M\{,\}$ and $\phi = \psi$

$Q_M\{\psi, \varphi_n\}$

$$Q_M\{\psi, \varphi_n\} \tag{O.-1}$$

$$0 = (\varphi, \hat{M}\psi_n) - (\varphi, \hat{M}'\psi) = (\varphi, (\hat{M} - \hat{M}')\psi_n) \tag{O.-1}$$

O.6 The Associated Master Constraint Operator

is manifestly positive and sesquilinear. It remains to show that it is closable.

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partly taken from Ouhabaz: Analysis of heat equations on domains

$u \in D(a)$ is in the domain $D(M)$ of M, if and only if there exists $v \in \mathcal{H}$ such that $a(u, \phi) = (v, \phi)$ for all $\phi \in D(a)$. Then we define the operator by

$$Mu := v. \tag{O.-1}$$

$D(M)$ is the set of vectors $u \in D(a)$ for which the mapping $\phi \mapsto a(u, v)$ is continuous on $D(a)$ with respect to the norm of $\mathcal{H}$.

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O.6.1 Master Constraint from Quadratic form

$Q_M\{\psi, \varphi_n\}$

$$Q_M\{\psi, \varphi_n\} \tag{O.-1}$$

$$0 = (\varphi, \hat{M}\psi_n) - (\varphi, \hat{M}'\psi) = (\varphi, (\hat{M} - \hat{M}')\psi_n) \tag{O.-1}$$

as the quadratic form is real the operator $\hat{M}$ in $(\psi, \hat{M}\psi = Q_M\{\psi, \varphi\})$ will be self-adjoint (see section N.4.3). As it is self-adjoint it is automatically dense and hence is uniquely defined $\hat{\mathbf{M}}$. Suppose $\hat{\mathbf{M}}$ and $\hat{\mathbf{M}}'$ are operators such that

$$(\phi, \hat{\mathbf{M}}\chi) = (\phi, \hat{\mathbf{M}}\chi) \tag{O.-1}$$

for ever vector ϕ in the domain of $\hat{\mathbf{M}}$ (or $\hat{\mathbf{M}}'$). Then, because the domain of $\hat{\mathbf{M}}$ is dense, there is a sequence of vectors ϕ_n such that $\phi_n \to \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi$ and

$$(\phi_n, \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi) = 0 \tag{O.-1}$$

Therefore $\hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi = 0$, because

$$(\phi_n, \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi) \to (\hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi, \hat{\mathbf{M}}\chi - \hat{\mathbf{M}}'\chi). \tag{O.-1}$$

Moreover $\hat{\mathbf{M}}$ is closed since $\hat{\mathbf{M}}^\dagger$ must be (see section).

In summary the operator would be densely defined and closed on H_{Diff}, so we really have pushed the constraint analysis one level up from H_{Kin}.

O.6.2 General Considerations

We assume that a judicious choice of ν, K has resulted in a positive, self-adjoint operaotr $\hat{\mathbf{M}}$ on some kinematic Hilbert space $\mathcal{H}_{Kin}$ which is assumed to be separable.

If zerois not in the spectrum of $\hat{\mathbf{M}}$ then compute the finite, positive number

$$\lambda_0 := \inf(\sigma(\hat{\mathbf{M}}))$$

and redifine $\hat{\mathbf{M}}$ by

$$\hat{\mathbf{M}} \to \hat{\mathbf{M}} - \lambda_0 \mathbf{1}_{\mathcal{H}_{Kin}}.$$

Here we assume that λ_0 vanishes in the limit $\hbar \to 0$ limit so that the modified operator still qualifies as a quantization of $\hat{\mathbf{M}}$. As we will see, this is justified in examples considered so far where λ_0 is usually related to some reordering of the operator. We will be assuming without loss of generality that $0 \in \sigma(\hat{\mathbf{M}})$.

Under these circumstances we can completely solve the Qunatum Master constraint equation

$$\hat{\mathbf{M}} = 0$$

and explicitly provide the phsical Hilbert space and its physical inner product.

O.6.3 Semi-Groups

Equations of the form

$$\frac{d}{dt}u(t) = Au(t), \quad t \geq 0, \qquad \text{with} \;\; u(0) = f, \tag{O.-1}$$

where u is a function of a (time) variable $t \geq 0$, with values in a "state space" X, and A an operator on X, are a mathematical modeling of time dependent dynamical systems.

given a Banach space X

is strongly continuous if

$$\lim_{t \to 0^+} T(t)u = u. \tag{O.-1}$$

theory of strongly continuous semigroups is used in the study of existence and uniqueness of solutions to the evolution equations:

where

if B is the generator of the strongly continuoious semigroup $(T(t))_{t \geq 0}$, then for every $f \in D(B)$, the Cauchy problem has a unique solution, given by $u(t) = T(t)f$.

Definitions Let X be a Banach space.

Definition A one-parameter family $(S(t))_{t \geq 0}$ of bounded linear operators from X into X is called a semigroup of bounded linear operators on X if

$S(0) = I$,

$S(t + s) = S(t)S(s)$, for all $t, s \geq 0$.

The linear operator $A : D(A) \to X$, defined on the domain

$$D(A) = \left\{ y \in X : \lim_{t \to 0_+} \frac{S(t)y - y}{t} \;\; \text{exists} \right\}$$

by

$$Ay = \lim_{t \to 0_+} \frac{S(t)y - y}{t}$$

for $y \in D(A)$, is called the infinitesimal generator of the semigroup $S(t)$.

Definition A semigroup S(t) of bounded linear operators is said

i) uniformly continuous if

$$\lim_{t \to 0_+} \|S(t) - I\| = 0;$$

ii) strongly continuous (or C_0 semigroup) if

$$\lim_{t \to 0_+} S(t)y = y,$$

for every $y \in X$.

Theorem O.6.1 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is bounded.*

For $\lambda \in \rho(A)$, let

$$R(\lambda, A) = (\lambda - IA)^{-1}$$

denote the resolvent of A....

Ergodic Semigroups

Let T be a bounded C_0−semigroup on a Banach space X. Denote by A the generator of T and by A^* the adjoint of A. We say that T is ergodic if

$$Px = \lim_{t \to \infty} \frac{1}{t} \int_0^t T(s)xds$$

exists for all $x \in X$.

O.7 Integral Decomposition and the Master Constraint.

$$(\phi, \hat{\mathbf{M}}\psi) = \int_{-\infty}^{\infty} dx\, d(\phi, E_x\psi) \qquad \text{(O.-1)}$$

$$\hat{\mathbf{M}}\psi = \int_0^\infty d\lambda\lambda\, d(E_\lambda\psi) \tag{O.-1}$$

$$\mathcal{H}_{\mathbf{Diff}} = \int_0^\infty d\lambda\lambda\, d(E_\lambda\mathcal{H}_{Diff}) \tag{O.-1}$$

$$\mathcal{H}_{\mathbf{Diff}} = \int_0^\infty d\lambda\mathcal{H}_{Diff}(\lambda) \tag{O.-1}$$

$$\mathcal{H}_{\mathbf{Diff}} = \int_R^\oplus d\mu(\lambda)\mathcal{H}_{Diff}(\lambda) \tag{O.-1}$$

$$\begin{aligned}
< \eta(f), \eta(f') >_{Phys} &= \int_R \frac{dt}{2\pi} < \hat{U}(t)f', f > \\
&= \int_R \frac{dt}{2\pi} \int_R d\mu(\lambda) < e^{i\lambda t} f'(\lambda), f(\lambda) >_{\mathcal{H}^\oplus_{Kin(\lambda)}} \\
&= \int_R \frac{dt}{2\pi} \int_R d\mu(\lambda) < \delta_R(\lambda) < f'(\lambda), f(\lambda) >_{\mathcal{H}^\oplus_{Kin(\lambda)}} \\
&= \mu(\delta) < f'(0), f(0) >_{\mathcal{H}^\oplus_{Kin(\lambda)}}
\end{aligned} \tag{O.-3}$$

O.7.1 Dirac observables

Let $\hat{O}_{Diff}$ be a bounded self-adjoint operator on $\mathcal{H}_{Diff}$. Since the Master constraint operator is self-adjoint, we may construct the strongly continuous oneparameter family of unitarities $\hat{U}(t) := \exp(it\hat{\mathbf{M}})$. Then, if the uniform limit exists, the operator

$$[\hat{O}_{Diff}] := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\hat{U}(t)\hat{O}_{Diff}\hat{U}(t)^{-1} \tag{O.-3}$$

Find out for which diffeomorphism invariant, bounded self-adjoint operators $\hat{O}_{Diff}$ the corresponding ergodic mean $[\hat{O}_{Diff}]$ converges (in the topology induced by the topology of $\mathcal{H}_{Diff}$).

Then compute the induced operator $\hat{O}_{Phys}$ on $\mathcal{H}_{Phy}$ which is automatically self-adjoint.

answer if the integral diverges. use L'hopital's rule

$$[O] := \lim_{T\to\infty} [\alpha_T^{\mathbf{M}}(O) + \alpha_{-T}^{\mathbf{M}}(O)] \tag{O.-3}$$

Details:

uniform operator topology is:

Consider linear bounded operators from a Hilbert space $\mathcal{H}$ to itself, denoted as $\mathcal{L}(\mathcal{H},\mathcal{H})$. The norm of an operator is defined as

$$\|T\| := \sup_{\|x\|=1} \|Tx\| \tag{O.-3}$$

where x is an ellement of the Hilbert space, i.e. $x \in \mathcal{H}$ (the domain of T $\hat{T}$ which is the set of elements for which Tx exists, $\mathcal{D}(T)$). The induced topology on $\mathcal{H}$ is called the **uniform operator topology**.

Uniform convergence defined as

$$\|T_n - T\| \to 0, \text{ as } n \to 0. \tag{O.-3}$$

the norm $\|.\|_{Diff}$ provides a topology on the Hilbert space $\mathcal{H}_{Diff}$ by open sets

$$U_\epsilon(0) = \{\hat{O} \mid \|\hat{O}\|_{Diff} < \epsilon\} \tag{O.-3}$$

where $\|.\|_{Diff}$ is the norm induced by the scalar product on $< .,. >_{Diff}$.

This topology induces a topology on

uniform operator convergence is when in Banach spaces

$$\|\hat{O}_n - \hat{O}\| \to 0 \tag{O.-3}$$

(ii) It is automatically selfadjoint??

Proof:

O.7.2 Direct Integral Decomposition and Rigging Maps

Let Φ^*_{Kin} be the algebraic dual equipped with the topology of poinwise convergence (this is the space of all functionals on Φ_{kin}, not just the bounded functionals, and so no continuity assumptions are made and the definition does not involve a norm).

generalized eigenvector $l \in \Phi^*_{Kin}$ with eigenvalue λ with respect to the closable operator $\hat{\mathbf{M}}$ which together with its adjoin is is densely defined on the (invariant) domain Φ_{Kin} provided that

$$\hat{\mathbf{M}}' l = \lambda l \Longleftrightarrow l(\hat{\mathbf{M}} f) = \lambda l(f) \text{ for all} f \in \Phi_{kin} \tag{O.-3}$$

Here $\hat{\mathbf{M}}'$ is called the dual representation on Φ^*_{Kin}. The subspace of generalized eigenvectors with eigenvalue λ is denoted by $\Phi^*_{Kin}(\lambda) \subset \Phi^*_{Kin}$ and $(\Phi^*_{Kin})_{Phys} := \Phi^*_{Kin}(0)$ is the physical subspace.

A countably Hilbert space is always metrizable, i.e., we can always define a metric on it that yields the original topology. In terms of the norms, the meric is given by

$$d(\phi, \phi') := \sum_{n=1}^{\infty} 2^{-n} \frac{\|\phi - \phi'\|_n}{1 + \|\phi - \phi'\|_n} \tag{O.-3}$$

(Exercise show that (O.7.2) satisfies the conditions for a metric). Thus one can apply all the results for the well studied meric spaces to the countably Hilbert spaces.

It is easy to verify that $\Phi = \cap_{n=1}^{\infty} \Phi_n$ and the inclusion $\Phi_{n+1} \subset \Phi_n$ holds, (Exercise??).

Let Φ' be the topological dual of Φ (continuousl linear functionals) and Φ'_n the topological dual of Φ_n.

By Riez lemma Φ'_n is isometric isomorphic with Φ_n

and

$$\|F\|_{-n} := \sup_{0 \neq \phi \in \Phi_n} \frac{|F(\phi)|}{\|\phi\|_n} = \|\phi_F^{(n)}\|_n \tag{O.-3}$$

A rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi'$ is given by a Nuclear space Φ and a Hilbert space $\mathcal{H}$ which is the cauchy completion of Φ in yet another scalar product $< .,. >:=< .,. >_0$

Given a *positive* self-adjoint operator $\hat{\mathbf{M}}$ on a Hilbert space $\mathcal{H}_{Kin}$, a corresponding Rigged Hilbert space as follows:

Let $\mathcal{D}$ be a dense, invariant domain for $\hat{\mathbf{M}}$, generically some space of smooth functions of compact support. Define positive sesquilinear forms $< ;\dot{>}_n$ on $\mathcal{D}$ defined by

$$< \phi, \phi' >_n := \sum_{k=0}^{n} < \phi, (\hat{\mathbf{M}})^k \phi' > \tag{O.-3}$$

Theorem

A self-sdjoint operator $\hat{\mathbf{M}}$ on a separable Rigged Hilbert space $\Phi_{Kin} \subset \mathcal{H}_{Kin} \subset \Phi'_{Kin}$ has a complete set of generalized eigenvectors corresponding to real eigenvalues. More precisely:

Let

$$\mathcal{H}_{Kin} = \int_R^\oplus d\mu(\lambda)\mathcal{H}^\oplus_{Kin}(\lambda) \tag{O.-3}$$

be the direct integral representation of $\mathcal{H}_{Kin}$. There is an integer n such that for $\mu -$ $a.a.$ $\lambda \in R$ there is a trace operator $T_\lambda : \Phi_n \to \mathcal{H}^\oplus_{Kin}(\lambda)$ which restricts to Φ_{Kin}

O.8 Toy Model III Free Field Theory: Maxwell Theory

O.8.1 Free Field Theories

O.8.2 Recap of Standard Canonical Quantization of Maxwell Theory

The canonical formulation of Maxwell theory on $\mathbb{R}^4$ consists of an infinite-dimensional phase space $\mathcal{M}$ with canonically conjugate coordinates (A_a, E^a) and symplectic structure

$$\{A_a(x), A_b(y)\} = \{E_a(x), E_b(y)\} = 0, \quad \{E_a(x), A_b(y)\} = e^2\delta^a_b\delta(x,y) \tag{O.-3}$$

where e is the electric charge.

Maxwell's theory in Minkowskian spacetime

canonical pair of fields (A_a, E^a)
sympletic structure $\{E^a(x), A_b(y)\} = e^2\delta^a_b\delta(x,y)$
fall off conditions $A \sim r^{-1}$, $E \sim r^{-2}$

$$G(\Lambda) = \int_{R^3} d^3x\Lambda(x)\partial_\mu E^\mu(x) \tag{O.-3}$$

In order to quantize the theory we want to introduce canonically conjugate coordinates (like x and p_x in non-relativistic quantum mechanics) for each degree of freedom and subject to these commutation relations.

We simplify the problem by considering radiation inside a large cubic box, of side L and volume $V = L^3$, and imposing periodic boundary conditions on the vector potential $\mathbf{A}$. The vector potential can then be represented as a Fourier series. The Fourier analysis corresponds to finding the normal modes of the radiation field.

One quantizes the radiation field by taking over the quantization of the harmonic oscillator from non-relativistic quantum mechanics.

With the boundary conditions

$$\mathbf{A}(0, y, z, t) = \mathbf{A}(L, y, z, t)$$

etc, the functions

$$\frac{1}{\sqrt{V}}\epsilon_r(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$$

form a complete set of transverse orthonormal vector fields. Here the wave vectors $\mathbf{k}$ must be of the form

$$\mathbf{k} = \frac{2\pi}{L}(n_1, n_2, n_3), \qquad n_1, n_2, n_3 = 0, \pm 1, \ldots,$$

so the fields satisfy the periodicity conditions. $\epsilon_1(\mathbf{k})$ and $\epsilon_2(\mathbf{k})$ are two mutually perpendicular real unit vectors which are also orthogonal to $\mathbf{k}$:

$$\epsilon_r(\mathbf{k}) \cdot \epsilon_s(\mathbf{k}) = \delta_{rs}, \qquad \epsilon_r(\mathbf{k}) \cdot \mathbf{k} = 0, \quad r, s = 1, 2.$$

The last of these conditions ensures that the fields are transverse, satisfying the Coulomb gauge condition.

O.8.3 Master constraint

The master constraint:

$$\mathbf{M} := \frac{1}{2}(\partial \cdot E, K \cdot \partial \cdot E)_{\mathcal{H}} \qquad \text{(O.-3)}$$

where K is a positive definite operator on $\mathcal{H}$. $\mathcal{H} = L_2(R^3, d^3x)$

Obviously $\mathbf{M} = 0$ if and only if $\partial \cdot E = 0$ a.e., that is, if and only if $G(\Lambda) = 0$ for all test functions of rapid decrease.

Recall that the Maxwell-Hamilonian is given by

$$H = \frac{1}{2e^2} \int d^3x \delta_{ab}(E^a E^b + B^a B^b) \simeq \hbar \int d^3x \delta_{ab} \overline{z^a} P^{\perp}_{ab} z^b \tag{O.-3}$$

where

$$z^a = \frac{1}{\sqrt{2\hbar e^2}}[\sqrt{-\Delta} A_a - i\sqrt{\Delta}^{-1} E^a] \text{ and } \bar{z}^a \tag{O.-3}$$

$$\mathbf{M} = \frac{\hbar e^2}{4} \sum_{J,K} Q_{JK} (\bar{z}^{(3)}_J - z^{(3)}_J)(\bar{z}^{(3)}_K - z^{(3)}_K) \tag{O.-3}$$

$$Q_{JK} := (b_J, \sqrt{-\Delta}^{3/2} K \sqrt{-\Delta}^{3/2} b_K)_{\mathcal{H}} \tag{O.-3}$$

-choose Q such that $\mathbf{M}$ convergent.

$$\hat{\mathbf{M}} = \sum_{J,K} Q_{JK} [(\hat{z}^{(3)}_J)^\dagger - \hat{z}^{(3)}_J][\hat{z}^{(3)}_K - (\hat{z}^{(3)}_K)^\dagger] \tag{O.-3}$$

Master Constraint Quantization

$$\mathbf{M} := \int_\sigma d^3x \frac{G_J G_K \delta^{JK}}{\sqrt{\det(q)}} \tag{O.-3}$$

$$\begin{aligned} [\hat{z}^j_J, \hat{z}^k_K] = [(\hat{z}^j_J)^\dagger, (\hat{z}^k_K)^\dagger] &= 0 \\ [\hat{z}^j_J, (\hat{z}^k_K)^\dagger] &= \alpha \delta^{jk} \delta_{JK} \end{aligned} \tag{O.-3}$$

$$\hat{z}^j_J \Omega = 0. \tag{O.-3}$$

In terms of creation and annihilation operators, the Master constraint operator becomes

$$\hat{\mathbf{M}} = \frac{\alpha}{4} \sum_{J,K} Q^{JK} \left[\hat{z}^3_J - (\hat{z}^3_J)^\dagger\right]^\dagger \left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right]. \tag{O.-3}$$

$$
\begin{aligned}
\hat{\mathbf{M}}(\hat{z}^3_J)^\dagger &= \frac{\alpha}{4}\sum_{J,K} Q^{JK}\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right](\hat{z}^3_L)^\dagger \\
&= \frac{\alpha}{4}\sum_{J,K} Q^{JK}(\alpha\delta_{KL} - \alpha\delta_{JL} + (\hat{z}^3_L)^\dagger)\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right] \\
&= \frac{\alpha^2}{4}\sum_{J,K} Q^{JK}(\delta_{KL} - \delta_{JL})\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[\hat{z}^3_K - (\hat{z}^3_K)^\dagger\right] + (\hat{z}^3_L)^\dagger\hat{\mathbf{M}}.
\end{aligned}
$$

As

$$
\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]\left[(\hat{z}^3_K)^\dagger - \hat{z}^3_K\right] = \left[(\hat{z}^3_K)^\dagger - \hat{z}^3_K\right]\left[(\hat{z}^3_J)^\dagger - \hat{z}^3_J\right]
$$

and Q_{IJ} is symmetric

$$
\hat{\mathbf{M}}(\hat{z}^3_J)^\dagger = (\hat{z}^3_J)^\dagger\hat{\mathbf{M}}.
$$

Therefore in general we have

$$
\hat{\mathbf{M}}(\hat{z}^j_J)^\dagger = (\hat{z}^j_J)^\dagger\hat{\mathbf{M}}. \tag{O.-6}
$$

The action of the Master constraint can be extended by linearity to the (dense) set of finite linear combinations of finite excitations of the vacuum Ω, thus it is densely defined operator on the Fock space, provided, that is, if $\hat{\mathbf{M}}\Omega$ has finite norm.

Theorem O.8.1 *The Master constraint operator is densely defined if and only if Q is a trace class operator.*

Proof: The Master constraint operator is densely defined if and only if $\hat{\mathbf{M}}\Omega$ has finite norm.

$$
\begin{aligned}
\|\hat{\mathbf{M}}\Omega\|^2 &= (\frac{\alpha}{2})^4 \sum_{J,K,M,N} Q_{JK}Q_{MN}\left(\left[\hat{z}^3_J - (\hat{z}^3_J)^\dagger\right](\hat{z}^3_K)^\dagger\Omega, \left[\hat{z}^3_M - (\hat{z}^3_M)^\dagger\right](\hat{z}^3_N)^\dagger\Omega\right) \\
&= (\frac{\alpha}{2})^4 \sum_{J,K,M,N} Q_{JK}Q_{MN}\left(\left[\alpha\delta_{JK} - (\hat{z}^3_J)^\dagger(\hat{z}^3_K)^\dagger\right]\Omega, \left[\alpha\delta_{MN} - (\hat{z}^3_M)^\dagger(\hat{z}^3_N)^\dagger\right]\Omega\right) \\
&= (\frac{\alpha}{2})^4 \sum_{J,K,M,N} Q_{JK}Q_{MN}\left[\alpha^2\delta_{JK}\delta_{MN} + (\Omega, \hat{z}^3_K\hat{z}^3_J(\hat{z}^3_M)^\dagger(\hat{z}^3_N)^\dagger\Omega)\right] \\
&= (\frac{\alpha}{2})^4\,[2\mathrm{Tr}(Q^2) + (\mathrm{Tr}(Q))^2].
\end{aligned} \tag{O.-8}
$$

The first term in the last line is finite if Q is a Hilbert-Schmidt operator, the second if Q is trace-class. Since every trace class operator is Hilbert-Schmidt, it is necessary and sufficient that Q be trace class.

□

O.8.4 Physical Hilbert Space

The Master Constraint operator acts as an identity on the Hilbert space of transversal modes so that we need only to consider the action of the Master Constraint operatoron the longitudinal Hilbert space.

O.9 Master Constraint for Gravity

The following theorem [214]

Theorem O.9.1 *The quadratic form $Q_{\mathbf{M}}(\ ,\ )$ is a closed quadratic form on $\mathcal{H}_{Diff}$. Hence there exists a unique densely deifned, positive self-adjoint operator $\hat{\mathbf{M}}$ on $\mathcal{H}_{Diff}$, leaving $\mathcal{H}_{Diff}$ invariant, such that:*

$$Q_{\mathbf{M}}(\Psi_{Diff}, \Psi'_{Diff}) = <\Psi_{Diff}|\hat{\mathbf{M}}|\Psi'_{Diff}>_{Diff} \,. \tag{O.-8}$$

Proof:

$$\hat{\mathcal{H}}^{\epsilon}_{C}\ f_{\gamma} = \sum_{v \in V(\gamma)} \frac{\chi_C(v)}{C^3_{n(v)}} \sum_{v(\Delta)=v} h^{\epsilon,\Delta}_{v}\ f_{\gamma} \tag{O.-8}$$

where $\chi_C(v)$ is the characteristic function of the cell C. The Master constraint operator, $\hat{\mathbf{M}}$

$$\boxed{\hat{\mathbf{M}} := \lim_{\mathcal{P}\to\Sigma} \sum_{C\in\mathcal{P}} \frac{1}{2} \hat{\mathcal{H}}'^{\dagger}_{C} \hat{\mathcal{H}}'_{C}} \tag{O.-8}$$

where

$$\begin{aligned}
(\hat{\mathcal{H}}'_{C}\Psi)[f_{\gamma}] &:= \lim_{\epsilon\to 0} \Psi[\hat{\mathcal{H}}^{\epsilon}_{C}\ f_{\gamma}] \equiv \lim_{\epsilon\to 0} (\Psi|\hat{\mathcal{H}}^{\epsilon}_{C}|f_{\gamma}> \\
(\hat{\mathcal{H}}'^{\dagger}_{C}\Psi)[f_{\gamma}] &:= \lim_{\epsilon\to 0} \Psi[\hat{\mathcal{H}}^{\epsilon\dagger}_{C}\ f_{\gamma}]
\end{aligned} \tag{O.-8}$$

$$(\hat{\mathbf{M}}\Psi_{Diff})[f_\gamma] := \lim_{\mathcal{P}\to\sigma,\epsilon,\epsilon'\to\sigma} \Psi_{Diff}[\sum \hat{H}^\epsilon_C \hat{H}^{\epsilon'\dagger}_C f_\gamma] \tag{O.-8}$$

Theorem O.9.2 (Closability of quadratic form $Q_{\mathbf{M}}(\ ,\ )$ - Thiemann's proof).
i. The positive quadratic form $Q_{\mathbf{M}}$ is closable and includes a unique, positive self-adjoint operator $\hat{\mathbf{M}}$ on $\mathcal{H}_{Diff}$.
ii. Moreover, the point zero is contained in the point spectrum of $\hat{\mathbf{M}}$.

Thiemann's proof

Thus, the heuristic idea is to define the quadratic form on $\mathcal{D}^*_{Diff}$ by

$$Q_M(l,l') := \lim_{\tau\to\Sigma} \sum_{\Delta\in\tau} < l, [\hat{\hat{C}}'(\Delta)]^* [\hat{\hat{C}}'(\Delta)] l' >_{Diff} \tag{O.-8}$$

where the prime denotes the operator dual as usual and * denotes the adjoint on $\mathcal{H}_{Diff}$. Unfortunately (N.-19) is ill defined as it stands because the operators $\hat{\hat{C}}'(\Delta)$ do not preserve $\mathcal{H}^*_{Diff}$. The cure is to extend $< .,. >_{Diff}$ to an inner product $< .,. >_*$ on all of $\mathcal{D}^*$. The final result turns out to be insensitive to the details of the extension because in the limit $\tau \to \Sigma$ the Riemann sum becomes well defined on $\mathcal{D}^*_{Diff}$.

$$\hat{\mathbf{M}}\, T_{[s_2]} := \sum_{T_{[s_1]}} Q_{\mathbf{M}}(T_{[s_1]}, T_{[s_2]})\, T_{[s_1]} \tag{O.-8}$$

$$\|\hat{\mathbf{M}}\, T_{[s_2]}\|^2 = \sum_{T_{[s_1]}} |Q_{\mathbf{M}}(T_{[s_1]}, T_{[s_2]})|^2 < \infty \tag{O.-8}$$

Let us fix $[s_1], [s_2]$ and consider theterm corresponding to $[s]$. In order that it does nto vanish

$$\sum_{v\in V(\gamma(s_0[s]))} \overline{T_{[s_1]}(\hat{C}^\dagger_v T_{s_0([s])})} T_{[s_2]}(\hat{C}^\dagger_v T_{s_0([s])})$$

must be non-zero. Hence the spin network decomposition of $\hat{C}^\dagger_v T_{s_0([s])}$ must contain a term diffeomorphic to T_{s_1} and a term diffeomorphic to T_{s_2} for at least one vertex $v \in V\big(\gamma(s_0[s])\big)$.

The first term adds an arc in between any possible pair of edges with two possible orientations and changes the spin of the two corresponding adjacent segments by $\pm 1/2$. Therefore it adds two more vertices.

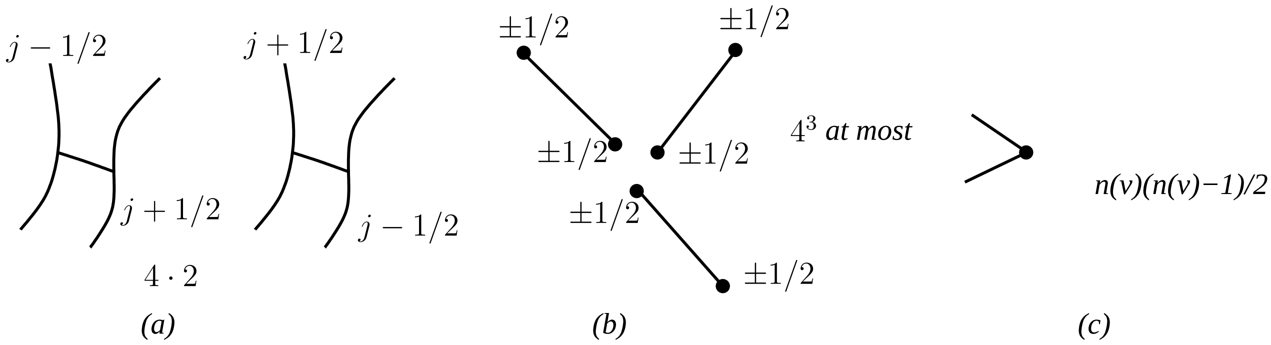

Figure O.1: MastatmostF. .

$$4 \cdot 2 \cdot 4^3 n(v)(n(v) - 1)/2 = 4^4 n(v)(n(v) - 1) \quad \text{(O.-8)}$$

The second term

$$4^8 n(v)^2 (n(v) - 1)^2 \quad \text{(O.-8)}$$

O.10 Worked Exercises

O.10.1 Toy Models

Worked example:

(c) prove the spaces generated from the different $\tilde{\Omega}_{s,l}$ are mutually orthogonal.

Worked example: Differentiating series.

O.10.2 Functional Analysis Issues

Worked example: Absolute converge and interchanging series summation and differentiation

$$|s_i| \leq M. \quad \text{(O.-8)}$$

Worked example: Verify $\mathbf{\Phi} = \cap_{n=1}^{\infty} \mathbf{\Phi_n}$ and $\mathbf{\Phi_{n+1}} \subset \mathbf{\Phi_n}$.

A sequence (φ_k) is a Cauchy sequence in a norm $\|.\|_n$ if given $\epsilon > 0$ there exists N such that $j, k \geq N$ imply $\|\varphi_j - \varphi_k\|_n < \epsilon$. $\mathbf{\Phi}_n$ is the Cauchy completetion of Φ with respect to the norm $\|.\|_n$.

$\Phi = \cap_{n=1}^{\infty} \Phi_n$ and the inclusion $\Phi_{n+1} \subset \Phi_n$

there exists some $C > 0$ such that

$$\|\varphi\|_1 \leq C\|\varphi\|_2, \quad \text{for all } \varphi \in \mathbf{\Phi}. \tag{O.-8}$$

Every sequence (φ_j) that is Cauchy with respect to $\|\varphi\|_2$ is a Cauchy sequence with respect to $\|\varphi\|_1$. For $\epsilon > 0$ there exists N such that

$$\|\varphi_j - \varphi_k\|_2 < \epsilon/C \tag{O.-8}$$

$$\|\varphi_j - \varphi_k\|_1 < \epsilon \tag{O.-8}$$

The converse is not in general true: there may be sequencies Cauchy with respect to $\mathbf{\Phi_1}$ but which are not Cauchy with respect to $\mathbf{\Phi_2}$. The completetion with respect to $\|.\|_1$ so that $\mathbf{\Phi_2} \subset \mathbf{\Phi_1}$.

$$\mathbf{\Phi_1} \supset \mathbf{\Phi_2} \supset \mathbf{\Phi}. \tag{O.-8}$$

Worked example:

Given that the norms $\|\varphi\|_n, \ n = 1, 2, \ldots$ have the following proerties

$$\|\varphi\|_n \geq 0, \quad \text{for all } \varphi \in \Phi \tag{O.-8}$$

verify that $d(.,.)$ has satisfies the conditions for being a metric, i.e., show that

(i) $d(\varphi, \phi) \leq d(\varphi, \phi) + d(\phi, \psi)$,

(ii) $d(\varphi, \psi) = d(\psi, \varphi)$,

(iii) $d(\varphi,\psi) \geq 0$, $d(\varphi,\psi) = 0$ implies $\varphi = \psi$.

Solution:

(i) Triangle inequality

$$f(x) = \frac{x}{1+x} \tag{O.-8}$$

Differentiation gives

$$f'(x) = \frac{1}{(1+x)^2} \geq 0. \tag{O.-8}$$

showing that it is monotonically increasing. Given that

$$\|a+b\|_n \leq \|a\|_n + \|b\|_n \tag{O.-8}$$

it means that

$$f(\|a+b\|_n) \leq f(\|a\|_n + \|b\|_n). \tag{O.-8}$$

$$\begin{aligned} \frac{\|a+b\|_n}{1+\|a+b\|_n} &\leq \frac{\|a\|_n + \|b\|_n}{1+\|a\|_n + \|b\|_n} \\ &= \frac{\|a\|_n}{1+\|a\|_n + \|b\|_n} + \frac{\|b\|_n}{1+\|a\|_n + \|b\|_n} \\ &\leq \frac{\|a\|_n}{1+\|a\|_n} + \frac{\|b\|_n}{1+\|b\|_n} \end{aligned} \tag{O.-9}$$

Making the substitution $a = \varphi - \phi$, $b = \phi - \psi$

$$\frac{\|\varphi-\psi\|_n}{1+\|\varphi-\psi\|_n} \leq \frac{\|\varphi-\phi\|_n}{1+\|\varphi-\phi\|_n} + \frac{\|\phi-\psi\|_n}{1+\|\phi-\psi\|_n} \tag{O.-9}$$

Multiplying both sides of this inequality by 2^{-n} and summing over n from zero to infinity, we have the inequality

$$d(\varphi,\psi) \leq d(\varphi,\phi) + d(\phi,\psi). \tag{O.-9}$$

(ii) $d(\varphi,\psi) = d(\psi,\varphi)$ easily follows from

$$\|\varphi-\psi\|_n = \|\psi-\varphi\|_n. \tag{O.-9}$$

(iii) $d(\varphi, \psi) \geq 0$: Since $\|\varphi - \psi\|_n \geq 0$ each term in the summation is ≥ 0.

$d(\varphi, \psi) = 0$

$$\sum_{n=1}^{\infty} 2^{-n} \frac{\|\varphi - \psi\|_n}{1 + \|\varphi - \psi\|_n} = 0, \tag{O.-9}$$

since $\|\varphi - \psi\|_n \geq 0$ for all $\varphi, \psi \in \Phi_n$????, this implies

$$\frac{\|\varphi - \psi\|_n}{1 + \|\varphi - \psi\|_n} = 0 \quad \Rightarrow \quad \|\varphi - \psi\|_n = 0. \tag{O.-9}$$

$$\|\varphi - \psi\|_n = 0, \quad \text{for all } n = 1, 2, \dots \tag{O.-9}$$

This leads us to conclude $\varphi = \psi$.????

Worked example: Verify tha the collection of norms defined by the positive operator $\hat{\mathbf{M}}$ satisfy $\|.\|_n \leq \|.\|_{n+1}$.

positive sesquilinear forms $< .,. >_n$

or iteratively

$$< \phi, \phi' >_{n+1} := < \phi, \hat{\mathbf{M}} \phi' > + < \phi, \phi' >_n \tag{O.-9}$$

$$\|\phi\|_{n+1}^2 := < \phi, \hat{\mathbf{M}} \phi > + \|\phi\|_n^2 \tag{O.-9}$$

Since $\hat{\mathbf{M}}$ is positive $< \phi, \hat{\mathbf{M}} \phi > \geq 0$, $\|\phi\|_{n+1}^2 \geq \|\phi\|_n^2$ and so

$$\|\phi\|_{n+1} \geq \|\phi\|_n. \tag{O.-9}$$

Worked example:

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, C_k\} = f_{Jk}{}^l C_l, \quad \{C_j, C_k\} = f_{jk}{}^L C_L \tag{O.-9}$$

Master constraint

$$M := \frac{1}{2}\sum_{j,k} Q_{jk} C_j C_k \tag{O.-9}$$

We define new constraints

$$\tilde{C}_k := \{M, T_k\} \approx \sum_{k,l} Q_{kl} C_k A_{lj} \tag{O.-9}$$

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l \tag{O.-9}$$

Establish

(a) $\{C_J, \tilde{C}_k\} = 0$
(b) $\{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l$

Proof:

(a)

$$\begin{aligned} \{C_J, M\} &= \{C_J, Q_{jk} C_j C_k\} \\ &= \{C_J, Q_{jk}\} C_j C_k + \{C_J, C_j\} Q_{jk} C_k + \{C_J, C_k\} Q_{jk} C_l \\ &= \{C_J, Q_{jk}\} C_j C_k + 2\{C_J, C_k\} Q_{jk} C_j \qquad \text{as } Q_{jk} = Q_{kj}. \end{aligned} \tag{O.-10}$$

The condition that the Master constraint M be diffeomorphism invariant, i.e. $\{C_L, M\} = 0$, requires

$$\begin{aligned} \{C_J, Q_{jk}\} C_k &= -2\{C_J, C_j\} Q_{jk} \\ &= -2 f_{Jj}{}^l C_l Q_{jk} \end{aligned} \tag{O.-10}$$

As T_i is a strong Dirac observable with respect to the constraints C_J, i.e., $\{C_J, T_i\} = 0$. Using this and the Jacobi identity we find

$$\begin{aligned} \{C_J, A_{lj}\} \equiv \{C_J, \{C_l, T_j\}\} &= -\{C_l, \{T_j, C_J\}\} - \{T_j, \{C_J, C_l\}\} \\ &= \{\{C_J, C_l\}, T_j\} \\ &= f_{Jl}{}^k \{C_k, T_j\} \qquad \text{remember } f_{Jl}{}^k \text{ are structure } \textbf{constants}, \\ &= f_{Jl}{}^k A_{kl} \end{aligned} \tag{O.-12}$$

$$
\begin{aligned}
\{C_J, \tilde{C}_k\} &= \sum_{k,l} [\{C_J, Q_{jk}\} C_k A_{lj} + \{C_J, C_k\} Q_{kl} A_{lj} + \{C_J, A_{lj}\} Q_{jk} C_k] \\
&= \sum_{k,l} \left[-2 f_{Jj}{}^l C_l Q_{jk} A_{lj} + f_{Jk}{}^l C_l Q_{kl} A_{lj} + f_{Jl}{}^k A_{kl} Q_{jk} C_k \right] \\
&= 0. \qquad \text{(O.-13)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
\{\tilde{C}_j, \tilde{C}_k\} &= \\
&= \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l \qquad \text{(O.-13)}
\end{aligned}
$$

Proof of the Mellin-Barnes integral.

$$
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n \qquad \text{(O.-13)}
$$

Proof:

Definitions of the Gamma function $\Gamma(z)$

We first define the Gamma function $\Gamma(z)$ as the function of a complex variable by the integral

$$
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt
$$

with Re$z > 0$. Using

$$
e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n
$$

we get

$$
\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt \\
&= \lim_{n\to\infty} n^z \int_0^1 (1-\tau)^n \tau^{z-1} d\tau \qquad \text{(O.-13)}
\end{aligned}
$$

Integration by parts now gives

$$\begin{aligned}
\int_0^1 (1-\tau)^n \tau^{z-1} d\tau &= \left[\frac{1}{z}\tau^z(1-\tau)^n\right]_0^1 + \frac{n}{z}\int_0^1 (1-\tau)^{n-1}\tau^z d\tau \\
&= \frac{n(n-1)\dots 2}{z(z+1)\dots(z+n-1)}\int_0^1 \tau^{z+n} d\tau \\
&= \frac{n(n-1)\dots 1}{z(z+1)\dots(z+n)} \qquad \text{(O.-14)}
\end{aligned}$$

so that

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)} \\
&= \frac{1}{z}\prod_{n=1}^{\infty}\left[\left(1+\frac{1}{n}\right)^z \left(1+\frac{z}{n}\right)^{-1}\right] \qquad \text{(O.-14)}
\end{aligned}$$

We see that the poles of the Gamma function are at

We take this as our definition of $\Gamma(z)$.

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\dots\left(1+\frac{z}{n}\right)\left(\frac{2}{1}\right)^{-z}\left(\frac{3}{2}\right)^{-z}\dots\left(\frac{n}{n-1}\right)^{-z}\left(\frac{n+1}{n}\right)^{-z}\right] \\
&= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\dots\left(1+\frac{z}{n}\right)e^{-z\ln n}\right] \\
&= \lim_{n\to\infty}\left[z\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-1/2z}\dots\left(1+\frac{z}{n}\right)e^{-(1/n)z}\right. \\
&\qquad\qquad \left. e^{-(1/n)z} e^{[1+1/2+\dots+(1/n)-\ln n]z}\right] \\
&= z e^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-(1/n)z}\right] \qquad \text{(O.-17)}
\end{aligned}$$

Using this we form the product

$$\frac{1}{\Gamma(z)}\frac{1}{\Gamma(-z)} = -z^2\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = -\frac{z}{\pi}\sin(\pi z)$$

Since by $\Gamma(z) = -(1/z)\Gamma(1+z)$ we have

$$\Gamma(-s)\Gamma(1+s) = -\pi \text{cosec}(\pi s)$$

Take the logarithm of ()

$$\begin{aligned}
\frac{\Gamma'(a)}{\Gamma(a)} &= \frac{d}{da}[\ln\Gamma(a)] \\
&= \frac{d}{da}[-\ln a - \gamma a - \sum_{n=1}^{\infty}\ln\left(1+\frac{a}{n}\right)+\sum_{n=1}^{\infty}\frac{a}{n}] \\
&= -\gamma - \frac{1}{a} + \sum_{n=1}^{\infty}\frac{a}{n(a+n)}
\end{aligned}$$

The generalsed zeta function $\zeta(s,a)$

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(a+n)^s}$$

Since

$$(a+n)^{-s}\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-(n+a)x}dx$$

we have

$$\begin{aligned}
\Gamma(s)\zeta(s,a) &= \lim_{N\to\infty}\sum_{n=0}^{N}\int_0^{\infty} x^{s-1}e^{-(n+a)x}dx \\
&= \left\{\int_0^{\infty}\frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx - \int_0^{\infty}\frac{x^{s-1}}{1-e^{-x}}e^{-(N+1+a)x}dx\right\}
\end{aligned}$$

Now, when $x \geq 0$, $e^x \geq 1+x$ and then

$$\begin{aligned}
\int_0^{\infty}\frac{x^{\sigma-1}e^{-(N+1+a)x}}{1-e^{-x}}dx &= \int_0^{\infty}\frac{x^{\sigma-1}e^{-(N+a)x}}{e^x-1}dx \\
&\leq \int_0^{\infty} x^{\sigma-2}e^{-(N+1+a)x}dx \\
&= (N+a)^{1-\sigma}\Gamma(\sigma-1)
\end{aligned}$$

which (when $\sigma \geq 1+\delta$) tends to 0 as $N \to \infty$.

$$\zeta(s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty}\frac{x^{\sigma-1}e^{-(N+1+a)x}}{1-e^{-x}}dx$$

Consider the contour integral

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz$$

where the contour $\mathcal{C}$ is as in fig (). On the real axis in the first part of this path we have $\arg(-z) = -\pi$, so that $(-z)^{s-1} = e^{-i\pi(s-1)}z^{s-1}$; and on the last part of the path $(-z)^{s-1} = e^{i\pi(s-1)}z^{s-1}$. On the circle write $-z = \delta e^{i\theta}$. We get

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz = \lim_{\rho\to\infty}\int_{\rho}^{\delta} \frac{e^{-i\pi(s-1)}z^{s-1}e^{-az}}{1-e^{-z}}dz + \lim_{\rho\to\infty}\int_{\delta}^{\rho} \frac{e^{i\pi(s-1)}z^{s-1}e^{-az}}{1-e^{-z}}dz + + \int_{-\pi}^{\pi} \frac{(\delta e^{i\theta})^{s-1}e^{a\delta(\cos\theta+i\sin\theta)}}{1-e^{\delta(\cos\theta+i\sin\theta)}}\delta e^{i\theta}id\theta$$

$$\int_{-\pi}^{\pi} \frac{(\delta e^{i\theta})^{s-1}e^{a\delta(\cos\theta+i\sin\theta)}}{1-e^{\delta(\cos\theta+i\sin\theta)}}\delta e^{i\theta}id\theta$$

$$\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz = 2i\sin\pi(s-1)\int_0^{\infty} \frac{x^{s-1}e^{-ax}}{1-e^{-x}}dx$$

Therefore

$$\zeta(s,a) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\mathcal{C}} \frac{(-z)^{s-1}e^{-az}}{1-e^{-z}}dz$$

Take s to be zero or a negative integer $(= -m)$

$$\zeta(-m,a) = -\frac{\Gamma(1+m)}{2\pi i}\int_{\mathcal{C}} \frac{(-z)^{-m-1}e^{-az}}{1-e^{-z}}dz$$

By Cauchy's theorem, this is equal to the residue of the integrand at $z = 0$.

differentiate

$$-z\frac{e^{-az}-1}{e^{-z}-1} = \sum_{n=1}^{\infty} \frac{(-1)^n\phi_n(a)z^n}{n!}$$

a power series

$$\frac{z^2e^{-az}}{e^{-z}-1} = \sum_{n=1}^{\infty} \frac{(-1)^n\phi_n'(a)z^n}{n!}$$

so that

$$\frac{e^{-az}}{z^{m+1}(e^{-z}-1)} = \sum_{n=1}^{\infty} \frac{(-1)^n \phi_n'(a) z^{n-m-3}}{n!}$$

we want the coefficient of z^{-1}, that is we want the coefficient corresponding to the term $n = m+2$:

$$\frac{(-1)^{m+2}\phi_{m+2}'(a)}{(m+2)!}$$

Therefore

$$\zeta(-m,a) = \frac{-\phi_{m+2}'(a)}{(m+1)(m+2)} \tag{O.-28}$$

Writing $s = 0$ in , we see that

Hermite's formula for $\zeta(s,a)$

We need the formula by Plana: If x is an integer, and $\phi(z)$ is a function which is analytic and bounded for all values of z such that $x_1 \leq R(z) \leq x_2$, then

$$\sum_{n=x_1}^{x_2} \phi(n) = \frac{1}{2}\phi(x_1)+\frac{1}{2}\phi(N)+\int_{x_1}^{x_2} \phi(\xi)d\xi+\frac{1}{i}\int_0^{\infty} \frac{\phi(x_2+iy)-\phi(x_1+iy)-\phi(x_2-iy)+\phi(x_1-iy)}{e^{2\pi y}-1}dy \tag{O.-28}$$

integrate

$$\int \frac{\phi(z)dz}{e^{\pm 2\pi i z}-1}$$

round rectangles whose corners are $x_1, x_2, x_2 \pm i\infty, x_1 \pm i\infty$

Put

$$\phi(\zeta) = \frac{1}{(a+\zeta)^s}$$

$$
\begin{aligned}
\sum_{n=0}^{N} \frac{1}{(a+n)^s} &= \frac{1}{2}a^{-s} + \frac{1}{2}\frac{1}{(a+N)^s} + \int_0^N (a+y)^{-s} dy \\
&+ \frac{1}{i}\int_0^\infty \left[(a+N+iy)^{-s} - (a+iy)^{-s} - (a+N-iy)^{-s} + (a-iy)^{-s}\right] \frac{dy}{e^{2\pi y}-1} \\
&= \frac{1}{2}a^{-s} + \frac{1}{2}\frac{1}{(a+N)^s} + \int_0^N (a+y)^{-s} dy + 2\int_0^\infty (q(N,y)-q(0,y))\frac{dy}{e^{2\pi y}-1}
\end{aligned}
\tag{O.-30}
$$

where the function $q(x,y)$ is defined by

$$
q(x,y) = \frac{1}{2i}[(a+x+iy)^{-s} - (a+x-iy)^{-s}]
$$

Is it legitimate to let $N \to \infty$ on the RHS? We will need the formula

$$
\sin\left\{s\ \tan^{-1} x\right\} = \frac{1}{2i}\frac{(1-ix)^s - (1+ix)^s}{(x^2+1)^{s/2}}.
$$

The proof is straightforward. Set $y = s\tan^{-1} x$ ($x = \tan(y/s)$), then

$$
\begin{aligned}
\sin\left\{s\ \tan^{-1} x\right\} &= \sin y \\
&= \frac{1}{2i}(e^{iy} - e^{-iy}) \\
&= \frac{1}{2i}\left[\left(\frac{e^{iy/s}}{e^{-iy/s}}\right)^{s/2} - \left(\frac{e^{-iy/s}}{e^{iy/s}}\right)^{s/2}\right] \\
&= \frac{1}{2i}\left[\left(\frac{\cos(y/s)+i\sin(y/s)}{\cos(y/s)-i\sin(y/s)}\right)^{s/2} - \left(\frac{\cos(y/s)-i\sin(y/s)}{\cos(y/s)+i\sin(y/s)}\right)^{s/2}\right] \\
&= \frac{1}{2i}\left[\left(\frac{1+ix}{1-ix}\right)^{s/2} - \left(\frac{1-ix}{1+ix}\right)^{s/2}\right] \\
&= \frac{1}{2i}\frac{(1+ix)^s - (1-ix)^s}{(x^2+1)^{s/2}}
\end{aligned}
$$

Using this we can rewrite $q(x,y)$ as

$$
\begin{aligned}
q(x,y) &= \frac{1}{2i}[(a+x+iy)^{-s}-(a+x-iy)^{-s}] \\
&= \frac{1}{2i}\frac{(a+x-iy)^{-s}-(a+x+iy)^{-s}}{[(a+x)^2+y^2]^s} \\
&= -[(a+x)^2+y^2]^{-s/2}\ \frac{1}{2i}\frac{(a+x+iy)^{-s}-(a+x-iy)^{-s}}{[(a+x)^2+y^2]^{s/2}} \\
&= -[(a+x)^2+y^2]^{-s/2}\ \sin\left\{s\ \tan^{-1}\frac{y}{x+a}\right\}
\end{aligned}
$$

$$
\int_0^\infty q(x,y)(e^{2\pi y}-1)^{-1}dy
$$

is convergent when $x \geq 0$ and tends to zero as $x \to \infty$

It is legitimate to let $N \to \infty$ and we have

$$
\zeta(s,a) = \frac{1}{2}a^{-s} + \int_0^\infty (a+x)^{-s}dx + 2\int_0^\infty (a^2+y^2)^{-\frac{1}{2}s}\left\{\sin\left(s\ \tan^{-1}\frac{y}{a}\right)\right\}\frac{dy}{e^{2\pi y}-1}
$$

We arrive at Hermite's formula

$$
\zeta(s,a) = \frac{1}{2}a^{-s} + \frac{a^{1-s}}{s-1} + 2\int_0^\infty (a^2+y^2)^{-\frac{1}{2}s}\left\{\sin\left(s\ \tan^{-1}\frac{y}{a}\right)\right\}\frac{dy}{e^{2\pi y}-1} \tag{O.-40}
$$

Special values of $\zeta(s,a)$ and its derivative from Hermite's formula

Writing $s=0$ in Hermite's formula (O.10.2), we have

$$
\zeta(0,a) = \frac{1}{2} - a. \tag{O.-40}
$$

We will need the value of $\zeta'(0,a)$.

$$
\begin{aligned}
\left.\frac{d}{ds}\zeta(s,a)\right|_{s=0} &= \lim_{s\to 0}\left[-\frac{1}{2}a^{-s}\ln a - \frac{a^{-s}\ln a}{s-1} - \frac{a^{1-s}}{(s-1)^2}\right. \\
&\quad +2\int_0^\infty -\frac{1}{2}\ln(a^2+y^2)\ (a^2+y^2)^{-\frac{1}{2}s}\sin\left(s\ \tan^{-1}\frac{y}{a}\right) \\
&\quad \left.+(a^2+y^2)^{-\frac{1}{2}s}\tan^{-1}\frac{y}{a}\cos\left(s\ \tan^{-1}\frac{y}{a}\right)\frac{dy}{e^{2\pi y}-1}\right] \\
&= \left(a-\frac{1}{2}\right)\ln a - a + 2\int_0^\infty \frac{\tan^{-1}(y/a)}{e^{2\pi y}-1}dy
\end{aligned} \tag{O.-42}
$$

How to evaluate this integral?

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}$$

By ()

$$\sum_{n=0}^{N} = \frac{1}{2z^2} + \frac{1}{2}\frac{1}{(z+N)^2} + \int_0^N \frac{d\xi}{(z+\xi)^2} + 2\int_0^\infty \frac{q(t,z+N)-q(t,z)}{e^{2\pi y}-1}$$

where

$$q(t,z) = \frac{1}{2i}\left[\frac{1}{(z-it)^2} - \frac{1}{(z+it)^2}\right] = \frac{2tz}{(z^2+t^2)^2}$$

Hence

$$\frac{d^2}{dz^2}\ln\Gamma(z) = \frac{1}{z^2} + \frac{1}{z} + \int_0^\infty \frac{4yz}{(z^2+y^2)^2}\frac{dy}{e^{2\pi y}-1}$$

Integrate from 1 to z

$$\frac{d}{dz}\ln\Gamma(z) = -\frac{1}{2z} + \ln z + C - \int_0^\infty \frac{ydy}{(z^2+y^2)(e^{2\pi y}-1)}$$

where C is a constant. Integrating again,

$$\ln\Gamma(z) = \left(z-\frac{1}{2}\right)\ln z + (C-1)z + C' + 2\int_0^\infty \frac{\tan^{-1}(y/z)dt}{(e^{2\pi y}-1)}$$

where C' is a constant.

We see that $C=0$ and $C' = \frac{1}{2}\ln(2\pi)$. Therefore

$$2\int_0^\infty \frac{\tan^{-1}(y/z)dt}{(e^{2\pi y}-1)} = \ln\Gamma(z) - \left(z-\frac{1}{2}\right)\ln z + z - \frac{1}{2}\ln(2\pi)$$

Substituting this into (O.-42) we finally obtain

$$\left.\frac{d}{ds}\zeta(s,a)\right|_{s=0} = \ln\Gamma(a) - \frac{1}{2}\ln(2\pi). \qquad \text{(O.-42)}$$

Now

$$\lim_{s\to 1}\left\{\zeta(s,a)-\frac{1}{s-1}\right\}=-\frac{\Gamma'(a)}{\Gamma(a)} \tag{O.-42}$$

Asymptotic expansion

$$\begin{aligned}
\frac{e^{-\gamma z}\Gamma(a)}{\Gamma(z+a)} &= e^{-\gamma z}\lim_{n\to\infty}\frac{n!n^a}{n!n^{z+a}}\frac{(z+a)(z+a+1)\dots(z+a+n)}{a(a+1)\dots(a+n)} \\
&= e^{-\gamma z}\lim_{n\to\infty} n^{-z}\left(1+\frac{z}{a}\right)\left(1+\frac{z}{a+1}\right)\left(1+\frac{z}{a+2}\right)\dots\left(1+\frac{z}{a+n}\right) \\
&= e^{-\gamma z}\left(1+\frac{z}{a}\right)\lim_{n\to\infty} e^{z(1+(1/2)+\cdots+(1/n)-\ln n)}\left(1+\frac{z}{a+1}\right)e^{-z}\times \\
&\qquad \times\left(1+\frac{z}{a+2}\right)e^{-z/2}\dots\left(1+\frac{z}{a+n}\right)e^{-z/n} \\
&= \left(1+\frac{z}{a}\right)\prod_{n=1}^{\infty}\left(1+\frac{z}{a+n}\right)e^{-z/n}
\end{aligned}$$

the principal values of the logarithms

$$\begin{aligned}
& \ln\left(1+\frac{z}{a}\right)+\ln\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{a+n}\right)e^{-z/n}\right\} \\
=& \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{a^m}+\sum_{n=1}^{\infty}\left\{\ln\left(1+\frac{z}{a+n}\right)-\frac{z}{n}\right\} \\
=& \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{a^m}+\sum_{n=1}^{\infty}\left\{\left(\frac{-az}{n(a+n)}\right)+\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{(a+n)^m}\right\}
\end{aligned}$$

Take the absolute value of the double sum

$$\begin{aligned}
\left|\sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{(-1)^{m-1}}{m}\frac{z^m}{(a+n)^m}\right| &= \sum_{n=1}^{\infty}\sum_{m=2}^{\infty}\frac{1}{m}\frac{|z|^m}{(a+n)^m} \\
&= \sum_{n=1}^{\infty}\ln\left(1-\frac{|z|}{a+n}\right)-\frac{|z|}{a+n}
\end{aligned}$$

Consequently

$$\begin{aligned}
\ln \frac{e^{-\gamma z}\Gamma(a)}{\Gamma(z+a)} &= \frac{z}{a} - \sum_{n=1}^{\infty} \frac{az}{n(a+n)} + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} z^m \left\{ \sum_{n=1}^{\infty} \frac{1}{(a+n)^m} + \frac{1}{a^m} \right\} \\
&= \frac{z}{a} - \sum_{n=1}^{\infty} \frac{az}{n(a+n)} + \sum_{m=2}^{\infty} \frac{(-1)^{m-1}}{m} z^m \zeta(m,a)
\end{aligned}$$

Now consider

$$-\frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s,a) ds$$

the contour of integration being that in fig (O.2) enclosing the points $s = 2, 3, 4, \ldots$ but not the points $1, 0, -1, -2, \ldots$. To find the residue we will need by L'hopital's rule

$$\lim_{s \to m} \frac{s-m}{\sin(\pi s)} = \lim_{s \to m} \frac{\frac{d}{ds}(s-m)}{\frac{d}{ds}\sin(\pi s)} = \lim_{s \to m} \frac{1}{\pi \cos(\pi s)} = \frac{(-1)^m}{\pi}$$

The residue of the integrand at $s = m (m \geq 2)$ is

$$\frac{1}{2\pi i} \frac{(-1)^m}{m} z^m \zeta(m,a).$$

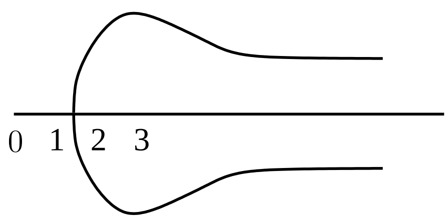

Figure O.2: .

Since as the real part of s tends to infinity,

$$\begin{aligned}
\ln \frac{\Gamma(a)}{\Gamma(z+a)} &= -z[-\gamma - \frac{1}{a} + \sum_{n=1}^{\infty} \frac{a}{n(a+n)}] - \frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s,a) ds \\
&= -z \frac{\Gamma'(a)}{\Gamma(a)} - \frac{1}{2\pi i} \int_C \frac{\pi z^s}{s \sin(\pi s)} \zeta(s,a) ds
\end{aligned}$$

$$\begin{aligned}
|z^s| &= |z^{\sigma+it}| \\
&= |(|z| e^{iargz})^{\sigma+it}| \\
&= ||z|^{\sigma+it}| \, |e^{iargz(\sigma+it)}| \\
&= |z|^{\sigma} e^{-t \, argz}
\end{aligned}$$

$$
\begin{aligned}
\frac{\pi}{s\sin(\pi s)} &= \frac{\pi}{(\sigma+it)\sin(\pi(\sigma+it))} \\
&= \frac{2i\pi}{(\sigma+it)[e^{i(\pi(\sigma+it))}-e^{-i(\pi(\sigma+it))}]}
\end{aligned}
$$

$$
\int_{\mathcal{D}} \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds
$$

but see fig (O.3)

$$
\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} + \int_{\mathcal{D}} + \int_{\mathcal{C}} = 0
$$

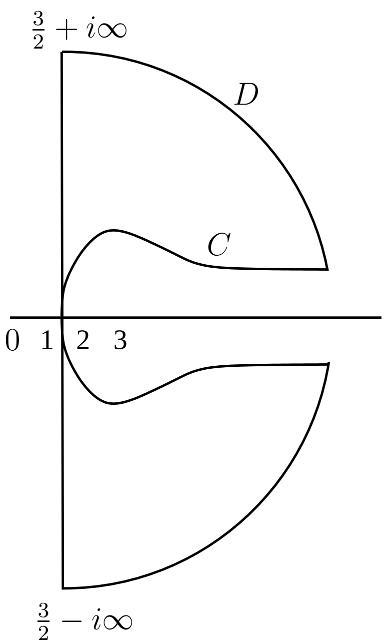

Figure O.3:

so

$$
\ln\frac{\Gamma(a)}{\Gamma(z+a)} = -z\frac{\Gamma'(a)}{\Gamma(a)} + \frac{1}{2\pi i}\int_{\frac{3}{2}-i\infty}^{\frac{3}{2}+i\infty} \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds
$$

We wish to write the RHS in terms of the integral

$$
\frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty} \frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds
$$

where n is a fixed integer. We have by Cauchy's theorem

$$\frac{1}{2\pi i}\left\{\int_{\frac{3}{2}-iR}^{\frac{3}{2}+iR}-\int_{-n-\frac{1}{2}+iR}^{\frac{3}{2}+iR}+\int_{-n-\frac{1}{2}-iR}^{\frac{3}{2}-iR}-\int_{-n-\frac{1}{2}-iR}^{-n-\frac{1}{2}+iR}\right\}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds=\sum_{m=-1}^{n}R_m$$

where R_m is the residue of the integrad at $s=-m$. So we must consider the integrals

$$\frac{1}{2\pi i}\int_{-n-\frac{1}{2}\pm iR}^{\frac{3}{2}\pm iR}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds.$$

Therefore

$$\ln\frac{\Gamma(a)}{\Gamma(z+a)}=-z\frac{\Gamma'(a)}{\Gamma(a)}+\frac{1}{2\pi i}\int_{-n-\frac{1}{2}-i\infty}^{-n-\frac{1}{2}+i\infty}\frac{\pi z^s}{s\sin(\pi s)}\zeta(s,a)ds+\sum_{m=-1}^{n}R_m$$

Consequently,

$$\ln\frac{\Gamma(a)}{\Gamma(z+a)}=-z\frac{\Gamma'(a)}{\Gamma(a)}+\sum_{m=-1}^{n}R_m+O(z^{-n-\frac{1}{2}})$$

when $|z|$ is large.

Now, when m is a positive integer,

$$R_m=\frac{(-1)^m z^{-m}\zeta(-m,a)}{-m}$$

and so by ()

$$R_m=\frac{(-1)^m z^{-m}\phi'_{m+2}(a)}{m(m+1)(m+2)}$$

R_0 is the residue at $s=0$

$$\begin{aligned}
\frac{\pi}{\sin(\pi s)} &= \frac{2i}{e^{i\pi s} - e^{-i\pi s}} \\
&= \frac{2i}{\left((i\pi s) + \frac{(i\pi s)^2}{2!} + \frac{(i\pi s)^3}{3!}\right) - \left((-i\pi s) + \frac{(-i\pi s)^2}{2!} + \frac{(-i\pi s)^3}{3!}\right)} \\
&= \frac{1}{\pi s}\frac{1}{1 - \frac{(\pi s)^2}{6} + \dots} \\
&= \frac{1}{\pi s}(1 + \frac{(\pi s)^2}{6} + \dots) \\
z^s &= e^{s\ln z} = 1 + s\ln z + \dots \\
\zeta(s,a) &= \zeta(0,a) + s\zeta'(0,a) + \dots \\
&= \left(\frac{1}{2} - a\right) + s\zeta'(0,a) + \dots
\end{aligned}$$

The residue is the s^{-1} coefficient of

$$\frac{1}{s} \times \frac{1}{s}\left(1 + \frac{(\pi s)^2}{6} + \dots\right)(1 + s\ln z + \dots)\left(\frac{1}{2} - a + s\zeta'(0,a)\right)$$

and so

$$\begin{aligned}
R_0 &= \left(\frac{1}{2} - a\right)\ln z + \zeta'(0,a) \\
&= \left(\frac{1}{2} - a\right)\ln z + \ln\Gamma(a) - \frac{1}{2}\ln(2\pi)
\end{aligned}$$

Consequently if $|argz| \leq \pi - \delta$ and $|z|$ large is large,

$$\ln\Gamma(z+a) = \left(z + a - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + \sum_{m=1}^{\infty}\frac{(-1)^m z^{-m}\phi'_{m+2}(a)}{m(m+1)(m+2)} + \mathcal{O}(z^{-n-\frac{1}{2}})$$

R_{-1} is the residue at $s = 1$. Set $s = S + 1$

$$
\begin{aligned}
\frac{1}{1+S} &= (1 - S + S^2 - \dots) \\
\frac{\pi}{\sin(\pi(1+S))} &= \frac{2i}{e^{i\pi(1+S)} - e^{-i\pi(1+S)}} \\
&= -\frac{2i}{e^{i\pi S} - e^{-i\pi S}} \\
&= -\frac{1}{\pi S}(1 + \frac{(\pi S)^2}{6} + \dots) \\
z^{1+S} &= e^{(S+1)\ln z} = z(1 + S\ln z + \dots) \\
\zeta(1+S,a) &= \zeta(1,a) + S\zeta'(1,a) + \dots \\
&= + \dots
\end{aligned}
$$

We finally have the asymptotic formula

$$
\ln\Gamma(z+a) = \left(z + a - \frac{1}{2}\right)\ln z - z + \frac{1}{2}\ln(2\pi) + o(1) \tag{O.-79}
$$

where the term $o(1)$ tends to zero as $|z| \to \infty$.

Proof of the formula for $|z| < 1$

With these preparations we can move onto the proof of the formula. Consider

$$
\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds
$$

The poles of $\Gamma(a+s)\Gamma(b+s)$ are $s = -a-n$, $-b-n$ $(n = 1, 2, 3\dots)$

Now noting the relation $\Gamma(-s)\Gamma(1+s) = -\pi\text{cosec}(s\pi)$, consider

$$
\frac{1}{2\pi i}\int_C \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)}\frac{\pi(-z)^s}{\sin(s\pi)}ds
$$

the integrand tends to zero sufficiently rapidly to ensure

$$
\int_{\mathcal{C}} \to 0 \quad \text{as} \quad N \to \infty.
$$

Now

$$
\int_{-i\infty}^{i\infty} - \left\{ \int_{-i\infty}^{-i(N+\frac{1}{2})} + \int_{\mathcal{C}} + \int_{i(N+\frac{1}{2})}^{i\infty} \right\}
$$

by Cauchy's theorem, is equal to minus $2\pi i$ times times the sum of the residues of the integrand at the points $s = 0, 1, 2, \ldots, N$.

We easily find the residue of the integrand of () for the pole at $s = n$

$$\begin{aligned} \lim_{s\to n} \quad & \frac{1}{2\pi i}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)} \cdot \frac{(s-n)}{\sin(s\pi)} \cdot \pi(-z)^s \\ = \quad & \frac{1}{2\pi i}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n \end{aligned} \tag{O.-79}$$

Let $N \to \infty$. In the case $|z| < 1$

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)n!} z^n$$

We prove the integrand is an analytic function

$$\begin{aligned} \ln \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)} &= \ln\Gamma(a+s)\ln\Gamma(b+s) - \ln\Gamma(c+s) \\ &= \left(s + (a+b-c) - \frac{1}{2}\right)\ln s - s + \frac{1}{2}\ln(2\pi) + o(1) \end{aligned} \tag{O.-79}$$

We prove the integrand tends to zero sufficiently rapidly

Proof of the formula for $|z| > 1$

□