Appendix P

The Semi-Classical Limit

P.1 Introduction

For graph changing operators such as the Hamiltonian constraints it turns out to be extremely diffcult to define coherent (or semiclassical) states. That is, states labelled by points in the classical phase space with respect to which the operator assumes an expecion value which reproduces the value of the corresponding classical function at that point in phase space and with respect to which the (relative) fluctuations are small. The reason for why this happens is that the existing coherent states for LQG [] are defined over a finite collection of finite graphs and these suppress very e ectively the fluctuations of those degrees of freedom that are labelled by the given graph. However, the Hamiltonian constraints add degrees of freedom to the state on which they act and the fluctuations of those are therefore no longer suppressed. Indeed, the semiclassical behaviour of the Hamiltonian constraints with respect to these coherent states is rather bad.

P.2 Quantum Mechanics: Schrödinger vs. Polymer particle

P.2.1 Weyl representation as Apposed to the Schrödinger representation

The first cast the standard fock description into a different form that will more convenient for comparision with the background independent theory. We will demostrate this alternative representation with the simple example of the one-particle Schrödinger system.

Schrödinger

- Hilbert space: $\mathcal{H} = L_2(R, dx)$
-
- operators:

$$\hat{U}(\lambda)\psi(x) = e^{i\lambda x}\psi(x) \tag{P.1}$$
$$\hat{V}(\mu)\psi(x) = \psi(x+\mu) \tag{P.2}$$

Both are unitary, and as such are well defined operators as unitary transformations on wavefunctions preserve the norm.

$$W(\xi) = e^{\frac{i}{2}\lambda\mu}U(\lambda)V(\mu) \qquad \text{(P.2)}$$

$$U(\lambda_1)U(\lambda_2) = U(\lambda_1+\lambda_2), \quad V(\mu_1)V(\mu_2) = V(\mu_1+\mu_2),$$
$$U(\lambda)V(\mu) = e^{\frac{-i}{2}\lambda\mu}V(\mu)U(\lambda) \qquad \text{(P.2)}$$

- Continuity in λ and μ: This means that have self-adjoint generators on H

$$\hat{U}(\lambda) = e^{i\lambda\hat{x}}, \quad \hat{V}(\mu) = e^{i\lambda\hat{k}} \qquad \text{(P.2)}$$

where

$$\hat{x}\psi(x) = x\psi(x), \quad \hat{k}\psi(x) = \frac{1}{i}\frac{d}{dx}\psi(x) \qquad \text{(P.2)}$$

$$[\hat{x}, \hat{k}] = i \qquad \text{(P.2)}$$

Polymer Particle quantum mechanics

We still want a representation of the Weyl-Heisenberg algebra. We want to mimic LQG

- Graphs γ are countable sets of points on R satisfying certain restrictions.
- For a given graph , define cyl

$$f(k) = \sum_{x_j \in \gamma} f_j e^{-ix_j k} \qquad \text{(P.2)}$$

• Put $CYL = \cup_\gamma CYL_\gamma$

• Introduce an inner product:

$$< e^{-ix_ik} | e^{-ix_jk} >= \delta_{x_i,x_j} \tag{P.2}$$

• Then $\mathcal{H}_{Poly} = L_2(R_d, d\mu_d)$ $f(k) \in \mathcal{H}_{Poly} \iff \sum_j |f_j|^2 < \infty$

Comparisons

Polymer particle		**Quantum Geometry**
• $CYL \subset \mathcal{H}_{Poly} \subset CYL^*$	$\iff$ CYL is based on "graphs" in each case	$CYL \subset \mathcal{H}_{Poly} \subset CYL^*$
• spin networks	$\iff$ Reflect the discrete nature	almost periodic functions.
• $\hat{x}$	$\iff$ Both are self-adjoint	$\hat{E}_S$
• $\hat{V}(\mu)$	$\iff$ Both are unitary	$\hat{h}_e$
• $\hat{k}$	$\iff$ Neither exists as an operator	$\hat{A}$

almost

Semi Classical States

A semiclassical state should have its expectation valuees for $\hat{x}$ and $\hat{p}$ peaked about $(0,0)$ with minimal uncertainty $(\Delta\hat{x})(\Delta\hat{p}) = \hbar/2$.

$$\hat{a}|\Psi_0 >= \frac{1}{\sqrt{2}}(\hat{x} + id^2\hbar\hat{p})|\Psi_0 >= 0 \tag{P.2}$$

$$(\Phi_0|e^{\sqrt{2}\alpha\hat{a}^\dagger}\hat{V}(-\alpha d)e^{\alpha^2/2} = (\Phi_0| \tag{P.2}$$

This has a unique solution in The dual space CYL^*

$$(\Phi_0| = \sum_{x \in R} e^{x^2/2d^2} (x| \tag{P.2}$$

Shadow States - Candidates for Semiclassical States?

$$(x_i | x_j >:= \delta_{x_i, x_j} \tag{P.2}$$

any state $(\Psi| \in CYL^*$ can be written

$$(\Psi| = \sum_{x \in R} \Psi(x)(x| \tag{P.2}$$

Let $(\Phi|$ be any state in CLY^*, and let γ be a graph. The shadow state $|\Phi_\gamma^{Shad}$ is the unique state in CYL_γ such that

$$< \Psi_\gamma^{Shad} |`\Phi >= (\Psi|\Phi > \tag{P.2}$$

What will we use for a momentum operator? formally

$$e^{-ik\mu} = 1 - ik\mu - \frac{k^2\mu^2}{2} + \dots \tag{P.2}$$

then

$$\frac{e^{-ik\mu} - e^{ik\mu}}{-2i\mu} = k + \mathcal{O}(k^2\mu) \tag{P.2}$$

$$\hat{k}_{\mu_0} := \frac{i}{2\mu_0} (\hat{V}(\mu_0) - \hat{V}(-\mu_0)) \tag{P.2}$$

Relation between polymer and Fock excitations [gr-qc/0107043]

Note: This stage of quantum geometry - Cyl* does not have a natural inner product with respect to which both eigenstates $< \mathcal{N}_{\alpha,\vec{n}}|$ of the electric flux-operators and the (images of the) Fock states are normalizable.

P.3 Varadarajan's Polymer Version of Maxwell's Theory

http://golem.ph.utexas.edu/ distler/blog/archives/000855.html:

As "traditional canonical" LQG(or LQG type quantizations) essentially relies on choosing a particular Poisson subalgebra(the so called holonomy-flux algebra) and quantizing it using a peculiar GNS functional (one that is spatially diffeomorphism invariant) one can also apply this procedure to field theories on flat spacetime. So for example take $U(1)$ theory on flat spacetime and quantize it this way. One will get a "loopy" Hilbert space with a spatially diffeomorphism invariant measure on which holonomies and smeared electric fields are well defined operators. Now one can ask the question, how is this representation (with holonomies as well defined operators) related to the usual Fock representation on which holonomies(without smearing) are not even well defined. This question was answered by Varadarajan in his 2000-2001 papers. The upshot is that, there is a so called r-Fock representation which is very closely tied to the usual Fock representation and whose states are distributions over the loopy Hilbert space. One can play the same game for scalar fields (Ashtekar et al. 2001).

Varadarajan's polymer states:

Graph changing polymer states:

the GNS Hilbert space $\mathcal{H}$. $\mathcal{D}$ is the linear subspace of $\mathcal{H}$ spanned by elements of the form

$$\hat{A}_a(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{k}} (e^{i\vec{k}\cdot\vec{x}} \hat{a}_a(\vec{k}) + e^{-i\vec{k}\cdot\vec{x}} \hat{a}_a^\dagger(\vec{k})) \tag{P.2}$$

where

The image of the Poincare condition in the r-Fock representation is

$$(\Gamma_F([\alpha, \{q\}]))^2 e^{-\int \frac{d^3x}{4q_0^2} X^a_{\alpha,\{q\}(r)}(x))E_{ra}(x)} \tag{P.2}$$

More General Version

The basic operators can be taken to be

$$U(f) \exp i \int d^3x \hat{\phi} f(x) \tag{P.3}$$

$$\hat{\pi} = \int d^3x \hat{\pi} g(x), \tag{P.4}$$

where f and g are test functions

$$[U(f)\Psi](\tilde{\phi}). \tag{P.4}$$

P.4 Coherent States

Let $a(a^\dagger)$ be the annihilation (creation) operator of the harmonic oscillator. If we set $N := a^\dagger a$ (the number operator), then

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1. \tag{P.4}$$

Let H be a Fock space generated by a and $a^\dagger$. The actions of a and $a^\dagger$ on H are given by

$$\begin{aligned} a|n> &= \sqrt{n}|n-1> \\ a^\dagger|n> &= \sqrt{n+1}|n+1> \\ N|n> &= n|n> . \end{aligned} \tag{P.3}$$

$$|n>= \frac{(a^\dagger)^n}{\sqrt{n}}|0> . \tag{P.3}$$

These states satisfy the orthogonality and completeness conditions

$$< m|n >= \delta_{mn}, \quad \sum_{n=0}^{\infty} |n >< n| = I. \tag{P.3}$$

the following three conditions are equivalent

$$(i) \quad a|z >= z|z > \text{ and } < z|z >= 1 \tag{P.4}$$

$$(ii) \quad |z >= e^{-|z|^2/2} \sum_{-\infty}^{\infty} \frac{z^n}{\sqrt{n!}}|n >= e^{-|z|^2/2} e^{za^\dagger}|0 > \tag{P.5}$$

$$(iii) \quad |z >= e^{za^\dagger - \bar{z}a}|0 > . \tag{P.6}$$

In going from (P.6) to (P.6) we make use of the *Baker-Campbell-Hausdorff formula*

$$e^{A+B} = e^{\frac{1}{2}[A,B]}e^A e^B \tag{P.6}$$

which holds whenever $[A,[A,B]] = [B,[A,B]] = 0$.

$$|\Psi_\alpha >= e^{-|\alpha|/2} \sum_{n_1,\ldots,n_D} \frac{(\alpha_1)^{n_1}\cdots(\alpha_D)^{n_D}}{\sqrt{n_1!}\cdots\sqrt{n_D!}} |n_1, n_2, \ldots, n_D > . \tag{P.7}$$

in configuration space and momentum space representation:

$$\begin{aligned} \psi_z(x) &= \sqrt{\frac{\omega}{\pi\hbar}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0) - \frac{i}{\hbar} x P_0 \right\}, & \text{(P.8)} \\ \psi_z(p) &= \sqrt{\frac{\hbar}{\pi\omega}} \exp - \left\{ \frac{\hbar}{2\omega}(p - P_0) - \frac{i}{\hbar} p X_0 \right\}. & \text{(P.9)} \end{aligned}$$

substituting $t = \hbar/\omega$ this is rewritten

$$\begin{aligned} \psi_z(x) &= \frac{1}{\sqrt{2t}} \exp - \left\{ \frac{1}{2t}(x - X_0) - \frac{i}{\hbar} x P_0 \right\}, & \text{(P.10)} \\ \psi_z(p) &= \frac{t}{\sqrt{\pi}} \exp - \left\{ \frac{t}{2}(p - P_0) - \frac{i}{\hbar} p X_0 \right\}. & \text{(P.11)} \end{aligned}$$

$$\begin{aligned} (\Delta\hat{q}_i)^2 &\equiv < \Psi_\alpha|\hat{q}_i^2|\Psi_\alpha > -[< \Psi_\alpha|\hat{q}_i|\Psi_\alpha >]^2 = \frac{1}{2}\ell_i^2, \\ (\Delta\hat{p}_i)^2 &\equiv < \Psi_\alpha|\hat{p}_i^2|\Psi_\alpha > -[< \Psi_\alpha|\hat{p}_i|\Psi_\alpha >]^2 = \frac{1}{2}\hbar^2/\ell_i^2 & \text{(P.11)} \end{aligned}$$

$$< \Psi_\beta| : F(\hat{a}_i^\dagger, \hat{a}_i) : |\Psi_\alpha >= F(\overline{\beta}_i, \alpha_j) < \Psi_\beta|\Psi_\alpha > \tag{P.11}$$

read off the most important properties of coherent states

$$< \hat{X} >_{\Psi_z} = X_0, \quad < \hat{P} >_{\Psi_z} = P_0, \tag{P.11}$$

$$e^{-t\Delta}\delta_y(x) = \frac{1}{\sqrt{4\pi t}} e^{\frac{1}{2t}(x-y)^2} \tag{P.11}$$

P.5 Coherent States of LQG

P.6 Coherent States of Compact Groups

In this section we describe the coherent state transforms (CST) for Lie groups introduced by Hall []. For simplicity, we will restrict ourselves in this section, to the case when K is simple. The general compact case can be treated in a similar way. In particular $K = U(1)^g$ will be considered in sections N.-19 and N.-19. Let K be a compact connected simple Lie group, $K_{\mathbb{C}}$ its complexification (see []) and Δ_K the Laplacian on K associated to an Ad-invariant inner product on its Lie algebra $Lie(K)$.

For each $f \in L^2(K; dx)$, where dx is the normalized Haar measure on K, the image of f by the CST, $C_t f$, is the analytic continuation to $K_{\mathbb{C}}$ of the solution of the heat equation,

$$\frac{1}{\pi}\frac{\partial u}{\partial t} = \Delta_K u, \tag{P.11}$$

in generalized coordinates the Laplacian

$$\triangle u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial q_i}\left(\sqrt{g}g^{ij}\frac{\partial u}{\partial q_j}\right) \tag{P.11}$$

with initial condition given by $u(0, x) = f(x)$.

$$\psi_g^t(h) = \sum_\pi d_\pi e^{\frac{t}{2}\gamma_\pi}\chi_\pi(gh^{-1}) \tag{P.11}$$

To obtain a more explicit description of this CST, consider the expansion of $f \in L^2(K; dx)$ given by the Peter-Weyl theorem,

$$f(x) = \sum_R tr(R(x)A_R), \tag{P.11}$$

where the sum is taken over the set of (equivalence classes of) irreducible representations of K, and $A_R 2 End V R$ is given by

$$A_R = (dim V_R)\int_K f(x)R(x)dx, \tag{P.11}$$

VR being the representation space for R. Let $X_i, i = 1, \ldots, dimK$ be an orthonormal basis for the Ad-invariant inner product on $Lie(K)$ for which the longest root has squared norm 2. Viewing $Lie(K)$ as the space of left-invariant vector fields on K, we have

$$\Delta_K = \sum_{i=1}^{dimK} X_i X_i \tag{P.11}$$

and one obtains

$$C_t f(g) = \sum_R e^{i\pi t\tau C_R} tr(R(g)A_R). \tag{P.11}$$

P.6.1 Coherent States of U(1)

Segal-Bargmann-Hall Transform

$$\Delta := \sum_{k=1}^{d} \frac{\partial^k f}{\partial x_k^2} \tag{P.11}$$

the heat equation

$$\frac{\partial u}{\partial s} = \frac{1}{2}\Delta u \tag{P.11}$$

$$B_t f(z) = \frac{1}{(2\imath t)^{-d/2}} \int_{\mathbf{R}^d} e^{-(z-x)^2/2t} f(x)dx, \quad z \in \mathbf{C}^d. \tag{P.11}$$

is just the convolution of f with a Gaussian.

Different bit

heat kernel measure:

$$\frac{d\mu}{dt} = \frac{1}{4}\Delta K_C \mu_t \tag{P.11}$$

let μ_t denote the associated heat kernel measure

$$d\mu_t(g) := \mu_t(g)dg. \tag{P.11}$$

In terms of a normalized left invariant vector field X^a on $U(1)$ as

$$\Delta = X^2. \tag{P.11}$$

$$\frac{dh(\theta,t)}{dt} = \frac{1}{2}\Delta h(\theta,t) \tag{P.11}$$

$$\frac{1}{2}\frac{\partial^2}{\partial\theta^2}h(\theta,t) = \frac{\partial}{\partial t}h(\theta,t) \tag{P.11}$$

$$\sum_m \left[-\frac{m^2}{2}a_m(t) - \dot{a}_m(t)\right] e^{im} = 0, \tag{P.11}$$

$$a_m(t) = A_m e^{-\frac{m^2}{2}t} \tag{P.11}$$

$$\begin{aligned} C_t(f)(z) &:= \frac{1}{2\pi}\int_{S^1}\left(\sum_m e^{-\frac{m^2}{2}t}x^{-m}z^m\right)\left(\sum_n a_n x^n\right)d\theta \\ &= \frac{1}{2\pi}\sum_m\sum_n a_n e^{-\frac{m^2}{2}t}z^m\int_{S^1}x^n x^{-m}d\theta \\ &= \sum_m a_m e^{-\frac{m^2}{2}t}z^m. \end{aligned} \tag{P.10}$$

$$\Delta_G = \frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial p^2}. \tag{P.10}$$

$$\frac{dh}{dt} = \frac{1}{4}\Delta_G h. \tag{P.10}$$

$$\mu_t(z) = \frac{1}{\sqrt{t\pi}}e^{\frac{p^2}{t}}\left(\sum_m e^{-\frac{m^2}{2}t}e^{im\theta}\right). \tag{P.10}$$

Coherent States of U(1)

$$\begin{aligned} h &= e^{i\theta} \\ g &= e^{i(\phi-ip)} \\ \chi_\pi(gh^{-1}) &= e^{in(\phi-\theta)}e^{np} \text{ for } \pi = n \end{aligned} \tag{P.9}$$

$$\psi_g^t(h) = \sum_{n=-\infty}^{\infty} e^{\frac{t}{i}n^2} e^{in(\phi-\theta)} e^{np}. \tag{P.9}$$

Poisson for $SU(2)$

$m, n \in \{-j, -j+1, \ldots, j-1, j\}$

$$\pi_j(g)_{mn} = \sum_{\ell} \frac{\sqrt{(j+m)!(j-m)!(j+n)!(j-n)!}}{(j-m-\ell)!(j+n-\ell)!(m-n+\ell)!\ell!} a^{j+n-\ell} d^{j-m-\ell} b^{m-n+\ell} c^{\ell} \tag{P.9}$$

$$g = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \quad a, b, c, d \in \mathbb{C}, \quad ad - bc = 1. \tag{P.9}$$

$\det(g) = \lambda_1 \lambda_2$ and $tr(g) = a + d = \lambda_1 + \lambda_2$.

$$\lambda_1 + \frac{1}{\lambda_1} = a + d, \quad \text{and} \quad \lambda_2 + \frac{1}{\lambda_2} = a + d \tag{P.9}$$

implying the quadratic equation

$$\lambda^2 - \lambda(a+d) + 1 = 0$$

with solutions

$$\lambda_1 = x + \sqrt{x^2 - 1} \quad \text{and} \quad \lambda_2 = \lambda_1^{-1} = x - \sqrt{x^2 - 1} \quad \text{where } x = \frac{a+d}{2}.$$

If $\pi_j(g)_{mn}$ is diagonal ($b = c = 0$) then (with no summation implied),

$$\pi_j(g)_{mn} = \delta_{mn} \left(a^{j+m} d^{j-m} + \sum_{\ell \neq 0} \frac{(j+m)!(j-m)!}{(j-m-\ell)!(j+m-\ell)!\ell!} a^{j+m-\ell} d^{j-m-\ell} b^{\ell} c^{\ell} \right)$$

sum over ℓ reduces to one term $\ell = 0$ and

$$\chi_\ell(g) = \sum_{m=-\ell}^{\ell} a^{\ell+m} d^{\ell-m} \tag{P.9}$$

$$\chi_\ell(g) = \chi_\ell \left(\begin{array}{cc} \alpha & 0 \\ 0 & \overline{\alpha} \end{array} \right) \tag{P.9}$$

as $ad = \alpha\overline{\alpha} = 1$ we have $\alpha = e^{i\theta}$

$$\begin{aligned}
\chi_\ell(g) &= \sum_{n=-\ell}^{\ell} e^{i\theta(\ell+n)} e^{-i\theta(\ell-n)} = \sum_{n=-\ell}^{\ell} e^{2ni\theta} \\
&= \frac{e^{i(2\ell+1)\theta} - e^{-i(2\ell+1)\theta}}{e^{i\theta} - e^{-i\theta}} \\
&= \frac{\sin(2\ell+1)\theta}{\sin\theta}
\end{aligned} \tag{P.8}$$

$$t_j^{mn}\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} = \sum_{s=-j}^{m} (-1)^{n-s} \frac{(j+n)!}{(n-s)!(j+s)!} \frac{(j-n)!}{(j-m-n+s)!(m-s)!} \alpha^{j+s} \beta^{m-s} \overline{\beta}^{n-s} \overline{\alpha}^{j-n-m+s} \tag{P.8}$$

P.6.2 Coherent States of $SU(2)$

$$\begin{aligned}
J_+|J,m> &= \sqrt{(m+1)(2J-m)}|J,m+1>, \\
J_-|J,m> &= \sqrt{m(2J-m+1)}|J,m-1>, \\
J_3|J,m> &= (-J+m)|J,m>,
\end{aligned} \tag{P.7}$$

where $|J,0>$ is a normalized vacuum $< J_-|J,0>= 0$ and $< J,0|J,0>= 1$). We denote by I_J the unit operator on on $\mathcal{H}_J$. states $|J,m>$ are given by

$$|J,m>= \frac{(J_+)^m}{\sqrt{m!_{2J}P_m}}, \tag{P.7}$$

where ${}_{2J}P_m = (2J)(2J-1)\cdots(2J-m+1)$.

These states satisfy the orthogonality and completeness conditions

$$< J,m|J,n>= \delta_{mn}, \quad \sum_{n=0}^{2J} |J,m>< J,m| = I_J. \tag{P.7}$$

We call a state

$$|v>:=e^{vJ_+-\bar{v}J_-}|J,0> \tag{P.7}$$

the generalized coherent state for $su(2)$.

$$|v>=\frac{1}{(1+|\eta|^2)^J}e^{\eta J_+}|J,0>=:|\eta> \tag{P.7}$$

P.6.3 Expectation Values and Variation Properties of $U(1)$ and $SU(2)$ Coherent States

$$\sum_n e^{-\epsilon(n-N)^2}f(n)=\sum_n e^{-\epsilon(y-N)^2}f(y)e^{2\pi iyn} \tag{P.7}$$

Let f be in $L_1(\mathbf{R},dx)$ function such that the series

$$\phi(y)=\sum_{n=-\infty}^{\infty}f(y+ns) \tag{P.7}$$

is absolutely and uniformly convergent for $y\in[0,s], s>0$. Then

$$\phi(y)=\sum_{n=-\infty}^{\infty}f(y+ns)=\frac{2\pi}{s}\sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{s}\right) \tag{P.7}$$

where $\tilde{f}(k):=\int_{\mathbf{R}}\frac{dx}{2\pi}e^{-ikx}f(x)$.

$$\begin{aligned} h_e^{\mathbf{C}} &:= g_e=\sum_{n=0}^{\infty}\frac{i^n}{n!}\{h_e,C\}_{(n)} \\ &= \sum_{n=0}^{\infty}\frac{i^n}{n!}(-{}_j^e\frac{\tau_j}{2})^n\ h_e \\ &= e^{-i\tau_j p_j^e/2}h_e \end{aligned} \tag{P.6}$$

polar decomposition

P.7 Wigner Function

Quantum analog of the classical phase space probability distribution. For a particle travelling along one dimension, the Wigner function in terms of the wavefunction $\psi(x)$ through the formula:

$$W(q,p) = \frac{1}{2\pi\hbar} \int dx \psi^*(q + x/2) e^{ix/\hbar} \psi(q - x/2), \tag{P.6}$$

and it completely characterizes the quantum state. The integral of $W(q,p)$ over q and p is one, which follows from he normalization of the wavefunction. Expectation values of the quantum observables can be obtained from the Wigner function by integrating it with appropriate

Classical states statistics are described by the function $f(q,p)$, which is the probability function in the phase space, i.e.,

$$f(q,p) \geq 0, \quad \int f(q,p)\, dp = P(p), \quad \int f(q,p)\, dq = \tilde{P}(p), \tag{P.6}$$

with $P(p)$ and $\tilde{P}(p)$ probability distributions for position and momentum, respectively.

We consider an observable $X(q,p)$ which is a function on the phase space of the system under consideration. The characteristic function for the observable $X(q,p)$

$$\chi(k) =< e^{ikX} > \tag{P.6}$$

is given by the relation

$$\chi(k) = \int e^{ikX(q,p)} f(q,p)\, dq\, dp. \tag{P.6}$$

$$w(X) = \frac{1}{2\pi} \int \chi(k) e^{-ikX}\, dk \tag{P.6}$$

is a real nonnegative function which is normalized

$$\int w(X)\, dX = 1. \tag{P.6}$$

$$w(X) = \int f(q,p) \delta(X(q,p) - X)\, dq\, dp. \tag{P.6}$$

P.8 Noiseless Subsystems and Quantum Gravity

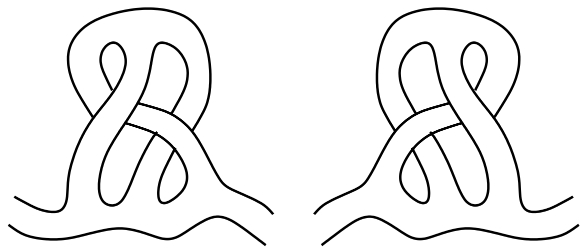

Figure P.1: ConElecNeutrF. The effect of charge conjugation on an electron-neutrino.

P.8.1 Propagating Modes vs Background Independence

"Normally when a particle is in interaction with an environment, information about its state dissipates into the environmnet - we say that it decoheres, It's difficult to prevent this decoherence from happening ... whihc depoendends on the efficacy on a particle's being in a pure quantum state. ... Markopoulou that their insights applied to teh problem of how a quantim particle could emerge from a quantum spacetime ... " not finished here

The NS method is background independent in the sense that it does **not** rely on a particular graph/state, there is a genuine superposition of spinfoams / states inside the boundary and is defined via the dynamics.

BUT:

The NS is a subsystem of a subsystem $= \mathcal{H}_A^{SBH}$ in ?? It finds the first if you give the second.

It does need the boundary / interaction dynamics:

P.8.2 NS and Locality - Microscopic vs Macroscopic Locality

One can try to assign geometric / local / causal properties to the subsystems / subgraphs. But they are not eigenstates of the dynamics.

We **have** to take into account the quantum sum over geometries.

Locality in an NS example:

$$\mathcal{H}^S \simeq \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$$

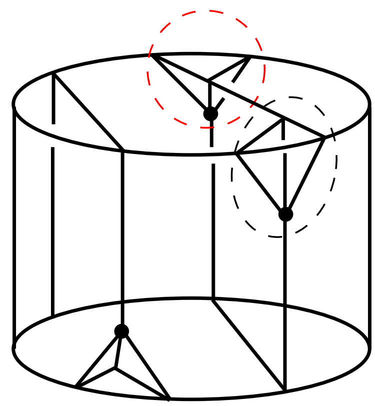

Figure P.2: NSlocalSpin.

with Z_3 .. in $A^{int'}$.

$$SOOO \otimes E$$

$\rightarrow \mathcal{H}^{NS} \simeq \mathbb{C}^2$, but it's **none** of the origin of Perez.

NS Suggests:

• give boundary (interaction) dynamics

• look at appropriate symmetries and extract NS

• assign locality / geometric / causal properties to the effective NS's.

P.8.3 Outlook

- Black hole case

  • excitations in $\mathcal{H}^{QSBH}$ that correspond classically to ..

  • Rotating black hole

  • black hole in spin foams / other models

  • $G_q \rightarrow SO(3)$

- Propagating modes in quantum gravity with boundary.

- Weaves (Dynamical) from translation symmetry in A^{int}.

- Separate scales: environment $\leftrightarrow$ small-scale ... system $\leftrightarrow$ course-grained weave.

- Theory of noiseless subsystems:
  - approx / emergent definition
  - Poincare.

P.9 The Standard Model from Loop Qunatum Gravity?

With the new decomposition, it is straightforward to check that operators in $\mathcal{A}_{evol}$ can only affect the $\mathcal{H}_{\Gamma}^{T}$ and that $\mathcal{H}_{\Gamma}^{B}$. Check this explicitly by showing that the actions of braiding and twisting of the edges of the graph and the evolution moves commute.

Proof:

$\square$

P.10 Summary

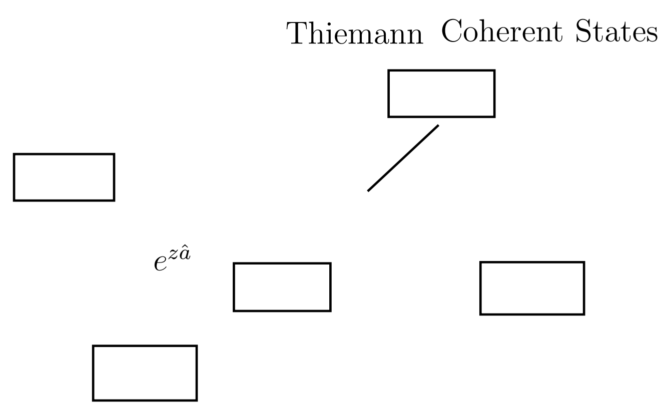

Figure P.3: u

P.11 Biblioliographical notes

In this chapter I have relied on the following reffferences:

P.12 Worked Exercises and Details

□

Poisson's formula and some inequalities.

Poisson's formula

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \varphi(2\pi n + x) &= \sum_{p=-\infty}^{\infty} a_n e^{ixp} \\
&= \sum_{p=-\infty}^{\infty} e^{ixp} \left\{ \frac{1}{2\pi} \int_0^{2\pi} e^{-ipy} \sum_{n=-\infty}^{\infty} \varphi(2\pi n + y) dy \right\} \\
&= \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{ixp} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} \varphi(2\pi n + y) e^{-ipy} dy \qquad \text{(P.5)}
\end{aligned}
$$

The sum over n in the last expression can be rewriten as follows:

$$
\begin{aligned}
\sum_{n=-\infty}^{\infty} \int_0^{2\pi} \varphi(2\pi n + y) e^{-ipy} dy &= \sum_{n=-\infty}^{\infty} \int_{2\pi(n+1)}^{2\pi n} \varphi(y) e^{-ipy} dy \\
&= \int_{-\infty}^{\infty} \varphi(y) e^{-ipy} dy. \qquad \text{(P.5)}
\end{aligned}
$$

Therefore

$$
\sum_{n=-\infty}^{\infty} \varphi(2\pi n + x) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} e^{-ipy} \int_{-\infty}^{\infty} \varphi(y) e^{-ipy} dy \qquad \text{(P.5)}
$$

we set $x = 0$, we obtain

$$
\sum_{n=-\infty}^{\infty} \varphi(2\pi n) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(y) e^{-ipy} dy. \qquad \text{(P.5)}
$$

Coherent equations.

(b) $A = za^\dagger$ and $B = -\bar{z}a$

$$[A, B] = -|z|^2[a^\dagger, a] = |z|^2 \tag{P.5}$$

$$e^B|0>= e^{-\bar{z}a}|0>= (I - \bar{z}a + \frac{1}{2}a^2 - \dots)|0>= |0> \tag{P.5}$$

$$e^{za^\dagger - \bar{z}a}|0>= e^{-|z|^2/2}e^{za^\dagger}|0>= e^{-|z|^2/2}|z> . \tag{P.5}$$

$\hat{a}_i|\Psi_\alpha >= \alpha_i|\Psi_\alpha >$

$$< \Psi_\beta|\hat{a}_i|\Psi_\alpha >= \alpha_i < \Psi_\beta|\Psi_\alpha > \tag{P.5}$$

$< \Psi_\beta|\hat{a}_j^\dagger = (\hat{a}_j|\Psi_\beta >)^* = \overline{\beta}_j < \Psi_\beta|$

$$< \Psi_\beta|\hat{a}_i^\dagger|\Psi_\alpha >= \overline{\beta}_i < \Psi_\beta|\Psi_\alpha > \tag{P.5}$$

$$\begin{aligned} < \Psi_\beta|(\hat{a}_i^\dagger)^m(\hat{a}_j)^n|\Psi_\alpha > &= \\ &= (\overline{\beta}_i)^m(\alpha_j)^n < \Psi_\beta|\Psi_\alpha > \end{aligned} \tag{P.5}$$

===================

$$< \Psi_\beta|\Psi_\alpha >= e^{-(|\alpha|^2+|\beta|^2)/2}e^{\overline{\beta}\alpha}$$

$$< \Psi_{phy}|\Phi_{phy} >= \frac{1}{\Lambda}\int_0^\Lambda d\lambda < e^{-i\lambda\hat{C}}\Psi|\Phi >$$

$$\hat{U}(\lambda)\Psi_\alpha := e^{-iC(\alpha)}\Psi_{\alpha(\lambda)}$$

where $\alpha_j(\lambda) = \alpha_j - i\kappa_j$

$$< e^{-iC(\alpha)}\Psi_{\alpha(\lambda)}|\Psi_\alpha >= e^{iC(\alpha)} < \Psi_{\alpha(\lambda)}|\Psi_\alpha >= e^{iC(\alpha)}e^{-(|\alpha|^2+|\alpha_j-i\kappa_j|^2)/2}e^{(\overline{\alpha}_j-i\overline{\kappa}_j)\alpha}$$

$$\|\Psi_\alpha^{phy}\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \tag{P.6}$$

======

$$
\begin{aligned}
&< \Psi_\alpha^{phy}|\Psi_\alpha^{phy} >= \frac{1}{2\pi}\int_0^{2\pi} d\lambda < \hat{U}(\lambda)\Psi_\alpha|\Psi_\alpha > \\
= &\ \frac{1}{2\pi}\int_0^{2\pi} d\lambda e^{-i\lambda\Delta}\left[e^{-|\alpha|^2/2}\sum_{n_1,\ldots,n_D}\frac{(e^{-i\kappa_1}\alpha_1)^{n_1}\ldots(e^{-i\kappa_D}\alpha_D)^{n_D}}{\sqrt{n_1!}\ldots\sqrt{n_D!}}|n_1,\ldots,n_D>\right]^\dagger \\
&\qquad e^{-|\alpha|^2/2}\sum_{\tilde{n}_1,\ldots,\tilde{n}_D}\frac{\alpha_1^{n_1}\ldots\alpha_D^{n_D}}{\sqrt{\tilde{n}_i!}\ldots\sqrt{\tilde{n}_D!}}|\tilde{n}_1,\ldots,\tilde{n}_D> \\
= &\ \frac{e^{-|\alpha|^2}}{2\pi}\int_0^{2\pi} d\lambda e^{-i\lambda\Delta}\sum_{n_1,\ldots,n_D}\sum_{\tilde{n}_1,\ldots,\tilde{n}_D} e^{i\sum_j n_j\kappa_j} < n_1,\ldots,n_D|\frac{(\alpha_D^\dagger)^{n_D}\ldots(\alpha_1^\dagger)^{n_1}\alpha_1^{\tilde{n}_1}\ldots\alpha_D^{\tilde{n}_D}}{\sqrt{n_1!}\ldots\sqrt{n_D!}\sqrt{\tilde{n}_i!}\ldots\sqrt{\tilde{n}_D!}}|\tilde{n}_1,\ldots,\tilde{n}_D> \\
= &\ e^{-|\alpha|^2}\sum_{n_1,\ldots,n_D}\frac{|\alpha_1|^{2n_1}\ldots|\alpha_D|^{2n_D}}{n_1!\ldots n_D!}\frac{1}{2\pi}\int_0^{2\pi} d\lambda e^{-i\lambda\Delta}e^{i\sum_j n_j\kappa_j} \\
= &\ e^{-|\alpha|^2}\sum_{n_1,\ldots,n_D}\frac{|\alpha_1|^{2n_1}\ldots|\alpha_D|^{2n_D}}{n_1!\ldots n_D!}\delta_{\sum_j n_j\kappa_j,\Delta}\ ,
\end{aligned}
\tag{P.1}
$$

Baker-Campbell-Hausdorff formula.

We are proving a reduced version of the Baker-Campbell-Hausdorff formula: the following

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp(-[\hat{A},\hat{B}]/2) \tag{P.1}$$

holds when $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$.

Proof:

□

First we prove

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{A} + \alpha[\hat{A},\hat{B}] + \frac{\alpha^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{\alpha^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \cdots \tag{P.1}$$

To derive this we write $f(\alpha) = e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}}$, take derivatives of $f(\alpha)$ and then putting $\alpha = 0$. First

$$f(0) = \hat{B}.$$

Taking the first derivative

$$\frac{d}{d\alpha}e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = e^{\alpha\hat{A}}[\hat{A},\hat{B}]e^{-\alpha\hat{A}} \tag{P.1}$$

and setting $\alpha = 0$ gives $[\hat{A},\hat{B}]$. Taking the derivative of (P.12) we obtain

$$\frac{d^2}{d\alpha^2}e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = e^{\alpha\hat{A}}[\hat{A},[\hat{A},\hat{B}]]e^{-\alpha\hat{A}}$$

and setting $\alpha = 0$ gives $[\hat{A},[\hat{A},\hat{B}]]$ and so on.

As $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$ the RHS of (P.12) reduces to $\hat{B} + \alpha[\hat{A},\hat{B}]$ so that

$$e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha[\hat{A},\hat{B}].$$

We now introduce the function $g(\alpha) = e^{\alpha\hat{A}}e^{\alpha\hat{B}}$, then

$$\begin{aligned}\frac{dg}{d\alpha} &= \hat{A}e^{\alpha\hat{A}}e^{\alpha\hat{B}} + e^{\alpha\hat{A}}\hat{B}e^{\alpha\hat{B}} \\ &= \left(\hat{A} + e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}}\right)e^{\alpha\hat{A}}e^{\alpha\hat{B}} \\ &= \left(\hat{A} + e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}}\right)g(\alpha).\end{aligned} \tag{P.0}$$

We have derived the differential equation

$$\frac{dg}{d\alpha} = \left(\hat{A} + \hat{B} + \alpha[\hat{A},\hat{B}]\right)g(\alpha) \tag{P.0}$$

whose solution is

$$g(\alpha) = e^{\alpha\hat{A}+\hat{B}+\frac{\alpha^2}{2}[\hat{A},\hat{B}]}. \tag{P.0}$$

So that

$$e^{\alpha\hat{A}}e^{\alpha\hat{B}} = e^{\alpha\hat{A}+\hat{B}+\frac{\alpha^2}{2}[\hat{A},\hat{B}]}. \tag{P.0}$$

Putting $\alpha = 1$ gives

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}. \tag{P.0}$$

As $[\hat{A},\hat{B}]$ commutes with both $\hat{A}$ and $\hat{B}$ we can write

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{A}+\hat{B})\exp([\hat{A},\hat{B}]/2). \tag{P.0}$$

"Complexifier" form

$$\Psi_0(A) = \mathcal{N}\exp(-\hat{C})\delta(A) \tag{P.0}$$

where

$$\hat{C} := \frac{1}{4\hbar\kappa}\int d^3x \int d^3y\; W_\Lambda(\underline{x},\underline{y})(\hat{\tilde{g}}^{ab}_\Lambda(\underline{x}) - \delta^{ab}) \tag{P.0}$$

and

$$\hat{\tilde{g}}^{ab}_\Lambda(\underline{x}) := \int d^3x' \hat{E}^a_i(\underline{x})\delta_\Lambda(\underline{x}-\underline{x}')\hat{E}^b_i(\underline{x}') \tag{P.0}$$

Metric operator:

$$\begin{aligned} &\int d^3x' \hat{E}^a_i(\underline{x})\delta(\underline{x}-\underline{x}')\hat{E}^b_i(\underline{x}')\tilde{S} \\ = &\sum_v \sum_{edges e_1,e_2 of v} \int d^3x' F^a_{v,e_1}(\underline{x})\delta(\underline{x}-\underline{x}')F^a_{v,e_2}(\underline{x}) \end{aligned} \tag{P.0}$$

Irreducible representations of $SU(2)$.

$$P \in \mathbb{C}[z_1,z_2], \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad z = (z_1,z_2), \tag{P.0}$$

and

$$zg = (az_1 + cz_2, bz_1 + dz_2). \tag{P.0}$$

$$P_k(z_1,z_2) = z_1^k z_2^{n-k}, \quad 0 \geq k \geq n, \tag{P.0}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z_1^k z_2^{n-k} \quad = \quad (az_1+cz_2)^r(bz_1+dz_2)^{n-r} \tag{P.1}$$

Could take the basis of mononials of degree n in the order

$$z_1^n, z_1^{n-1}z_2, \ldots, z_1 z_2^{n-1}, z_2^n. \tag{P.1}$$

OR

$$P_\ell(z_1, z_2) = z_1^{\ell+j} z_2^{\ell-j}, \quad -\ell \geq j \geq \ell, \quad \ell = 0, \frac{1}{2}, 1, \ldots \tag{P.1}$$

where ℓ is an integer or half integer

$$\frac{z_1^{2\ell}}{\sqrt{(2\ell)!}}, \frac{z_1^{2\ell-1}z_2}{\sqrt{(2\ell-1)!(1)!}}, \frac{z_1^{2\ell-2}z_2^2}{\sqrt{(2\ell-2)!(2)!}}, \ldots, \frac{z_1 z_2^{2\ell-1}}{\sqrt{(1)!(2\ell-1)!}}, \frac{z_2^{2\ell}}{\sqrt{(2\ell)!}}. \tag{P.1}$$

These are the basis vector of a vector space we denote V_ℓ. We easily see that it is closed under the linear transformation

$$z_1^{\ell+j} z_2^{\ell-j} \to (az_1 + bz_2)^{\ell+j}(-cz_1 + dz_2)^{\ell-j}$$

as

$$\begin{aligned}
& (az_1 + bz_2)^{\ell+j}(-cz_1 + dz_2)^{\ell-j} \\
= \ & \sum_{r=0}^{\ell+j}\sum_{q=0}^{\ell-j} (az_1)^{\ell+j-r}(bz_2)^r(-cz_1)^q(dz_2)^{\ell-j-q} \\
= \ & \sum_{r=0}^{\ell+j}\sum_{q=0}^{\ell-j} \left(\begin{array}{c} \ell+j \\ r \end{array} \right) \left(\begin{array}{c} \ell-j \\ q \end{array} \right) \left(a^{\ell+j-r}b^r(-c)^q d^{\ell-j-q}\right) \cdot z_1^{\ell+(j-r+q)} z_2^{\ell-(j-r+q)}
\end{aligned} \tag{P.0}$$

Does $j - r + q$ vary between $-\ell$ and ℓ? The minimum value $j - r + q$ can take is when $r = \ell + j$ and $q = 0$, and hence is $-\ell$. The maximum value $j - r + q$ can take is when $r = 0$ and $q = \ell - j$, and hence is ℓ.

Since the monomial set is closed under the linear transformation g, it will provide a $(2\ell + 1) \times (2\ell + 1)$ matrix representation.

For any $g \in SU(2)$, let $U_\ell(g)$ be the linear transformation of $\mathcal{H}_\ell$ given by

$$(U_\ell(g)f)(v) = f(g^{-1}v) \tag{P.0}$$

for all $f \in \mathcal{H}_\ell$ and $v \in \mathbb{C}^2$. This is a representation: $U_\ell(1)$ is the identity, and for any $g, h \in SU(2)$ we have

$$
\begin{aligned}
(U_\ell(g)U_\ell(h)f)(v) &= (U_\ell(h)f)(g^{-1}v) \\
&= f(h^{-1}g^{-1}v) \\
&= f((gh)^{-1}v) \\
&= (U_\ell(gh)f)(v) \qquad \text{(P.-2)}
\end{aligned}
$$

for all $f \in \mathcal{H}_\ell$, $v \in \mathbb{C}^2$.

Matrix elements

We derive the symmetric version, based on the basis vectors (P.12),

$$\pi_\ell(g)_{mn} = \alpha^{m+n}(-\overline{\beta})^{\ell-m}\beta^{\ell-n}\sum_{s=0}^{\ell-n}\frac{\sqrt{(\ell+m)!(\ell-m)!(\ell+n)!(\ell-n)!}}{(m+n+s)!(\ell-m-s)!(\ell-n-s)!s!}\left(-\frac{|\alpha|}{|\beta|}\right)^s \tag{P.-2}$$

Setting

$$\phi_k^\ell(z_1,z_2) = \frac{z_1^{\ell+k}z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}} \tag{P.-2}$$

The matrix elements are defined by

$$U_\ell(g)\phi_n^\ell = \sum_{k=\ell}^{-\ell}\phi_k^\ell(z_1,z_2)\ \pi_\ell(g)_{kn} \tag{P.-2}$$

From (P.-2)

$$\begin{aligned}
U_\ell(g)U_\ell(h)\phi_n^\ell &= U_\ell(g)\sum_{j=-\ell}^{\ell}\phi_j^\ell\ \pi_\ell(h)_{jn} \\
&= \sum_{j=-\ell}^{\ell}\sum_{k=-\ell}^{\ell}\phi_k^\ell\ \pi_\ell(g)_{kj}\pi_\ell(h)_{jn} \\
&= U_\ell(gh)\phi_n^\ell \\
&= \sum_{k=-\ell}^{\ell}\phi_k^\ell\ \pi_\ell(h)_{kn}(gh)
\end{aligned} \tag{P.-4}$$

that is

$$\sum_{k=-\ell}^{\ell}\frac{z_1^{\ell+k}z_2^{\ell-k}}{\sqrt{(\ell+k)!(\ell-k)!}}\left(\sum_{j=-\ell}^{\ell}\pi_\ell(g)_{kj}\pi_\ell(h)_{jn} - \pi_\ell(h)_{kn}\right) \tag{P.-4}$$

Setting $z_2 = 1$ and multiplying both sides by $z_1^{-\ell-m-1}$, integrating z_1 around the unit circle about the origin of the z_1-plane we obtain

$$\sum_{j=-\ell}^{\ell}\pi_\ell(g)_{mj}\pi_\ell(h)_{jn} = \pi_\ell(h)_{mn} \tag{P.-4}$$

i.e. $\pi_\ell(g)_{kj}$ is a matrix representation of $SU(2)$.

We defined the representation $\pi_\ell(g)_{mn}$ by

$$
\begin{aligned}
U_\ell(g)\phi_n^\ell(z_1,z_2) &= \frac{1}{\sqrt{(\ell+n)!(\ell-n)!}}U_\ell(g)z_1^{\ell+n}z_2^{\ell-n} \\
&= \frac{(\alpha z_1-\overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n}}{\sqrt{(\ell+n)!(\ell-n)!}} \\
&= \sum_{m=\ell}^{-\ell}\phi_m^\ell(z_1,z_2)\;\pi_\ell(g)_{mn} \\
&= \sum_{m=\ell}^{-\ell}\frac{z_1^{\ell+m}z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}\;\pi_\ell(g)_{mn} \qquad \text{(P.-6)}
\end{aligned}
$$

To arrive at (P.12), first we expand into the binomial expansion

$$
\begin{aligned}
&(\alpha z_1-\overline{\beta}z_2)^{\ell+n}(\beta z_1+\overline{\alpha}z_2)^{\ell-n} \\
=&\left(\sum_{t=0}^{\ell+n}(-1)^{\ell+n-t}\begin{pmatrix}\ell+n\\ t\end{pmatrix}\alpha^t\overline{\beta}^{\ell+n-t}z_1^tz_2^{\ell+n-t}\right)\left(\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\beta^{\ell-n-k}\overline{\alpha}^kz_1^{\ell-n-k}z_2^k\right) \\
=&\sum_{t=0}^{\ell+n}\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ t\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\alpha^t(-\overline{\beta})^{\ell+n-t}\beta^{\ell-n-k}\overline{\alpha}^k\cdot z_1^{\ell+t-n-k}z_1^{\ell-t+n+k}
\end{aligned}
\qquad \text{(P.-8)}
$$

We will show that this is equal to

$$
\sum_{m=-\ell}^{\ell}\alpha^{m+n}(-\beta^*)^{\ell-m}\beta^{\ell-n}\left(\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ m+n+k\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}\left(-\left|\frac{\alpha}{\beta}\right|\right)^k\right)z_1^{\ell+m}z_1^{\ell-m} \qquad \text{(P.-8)}
$$

equivalently

$$
\sum_{m=-\ell}^{\ell}\sum_{k=0}^{\ell-n}\begin{pmatrix}\ell+n\\ m+n+k\end{pmatrix}\begin{pmatrix}\ell-n\\ k\end{pmatrix}Q_{m,n,k} \qquad \text{(P.-8)}
$$

where $Q_{m,n,k}=\alpha^{m+n+k}(-\beta^*)^{\ell-m-k}(\alpha^*)^k\beta^{\ell-n-k}\cdot z_1^{\ell+m}z_1^{\ell-m}$. We do this by working backwards. First write

$$
\begin{aligned}
B \quad := \quad & \sum_{m=-\ell}^{\ell} \sum_{k=0}^{\ell-n} \left(\begin{array}{c} \ell+n \\ m+n+k \end{array} \right) \left(\begin{array}{c} \ell-n \\ k \end{array} \right) Q_{m,n,k} \\
= \quad & \sum_{k=0}^{\ell-n} \sum_{m=-\ell}^{\ell} \left(\begin{array}{c} \ell+n \\ m+n+k \end{array} \right) \left(\begin{array}{c} \ell-n \\ k \end{array} \right) Q_{m,n,k} \\
= \quad & \sum_{k=0}^{\ell-n} \sum_{t=n+k-\ell}^{\ell+n+k} \left(\begin{array}{c} \ell+n \\ t \end{array} \right) \left(\begin{array}{c} \ell-n \\ k \end{array} \right) Q_{t-n-k,n,k}
\end{aligned}
\tag{P.-10}
$$

but note that

$$
\left(\begin{array}{c} \ell+n \\ \ell+n+k \end{array} \right) = 0 \quad \text{if} \quad k > 0,
$$

because $1/(-N)! = 0$ for positive integer N. As such the upper limit on the t summation will always be $\ell+n$. We now explicitly write out each term in the k summation. In doing so we will use

$$
\left(\begin{array}{c} \ell+n \\ n+k-\ell \end{array} \right) = 0 \quad \text{if} \quad k < \ell-n
$$

which again follows from $1/(-N)! = 0$ for positive integer N. Hence,

$$
\begin{aligned}
B \quad = \quad & \sum_{t=n-\ell}^{\ell+n} \left(\begin{array}{c} \ell+n \\ t \end{array} \right) \left(\begin{array}{c} \ell-n \\ 0 \end{array} \right) Q_{t-n,n,0} + \sum_{t=n+1-\ell}^{\ell+n} \left(\begin{array}{c} \ell+n \\ t \end{array} \right) \left(\begin{array}{c} \ell-n \\ 1 \end{array} \right) Q_{t-n-1,n,1} + \dots \\
\dots \quad & + \sum_{t=0}^{\ell+n} \left(\begin{array}{c} \ell+n \\ t \end{array} \right) \left(\begin{array}{c} \ell-n \\ \ell-n \end{array} \right) Q_{t-\ell,n,\ell-n} \\
= \quad & \sum_{t=0}^{\ell+n} \sum_{k=0}^{\ell-n} \left(\begin{array}{c} \ell+n \\ t \end{array} \right) \left(\begin{array}{c} \ell-n \\ k \end{array} \right) Q_{t-n-k,n,k}
\end{aligned}
\tag{P.-11}
$$

Now $Q_{t-n-k,n,k} = \alpha^t \beta^{\ell-n-k} \overline{\alpha}^k \overline{\beta}^{\ell+n-t} \cdot z_1^{\ell+t-n-k} z_1^{\ell-t+n+k}$ and we have proven (P.-8).

We divide the last line above by $[(\ell+n)!(\ell-n)!]^{1/2}$ and get $(\alpha z_1 - \overline{\beta} z_2)^{\ell+n} (\beta z_1 + \overline{\alpha} z_2)^{\ell-n} / [(\ell+n)!(\ell-n)!]^{1/2}$. From (P.-6)

$$
\sum_{m=-\ell}^{\ell} \alpha^{m+n}(-\beta^*)^{\ell-m}\beta^{\ell-n}\left(\sum_{k=0}^{\ell-n}\left(\begin{array}{c}\ell+n\\ m+n+k\end{array}\right)\left(\begin{array}{c}\ell-n\\ k\end{array}\right)\left(-\left|\frac{\alpha}{\beta}\right|\right)^k\right)\frac{z_1^{\ell+m}z_1^{\ell-m}}{\sqrt{(\ell+n)!(\ell-n)!}}
$$

$$
= \sum_{m=\ell}^{-\ell}\frac{z_1^{\ell+m}z_2^{\ell-m}}{\sqrt{(\ell+m)!(\ell-m)!}}\,\pi_\ell(g)_{mn} \tag{P.-11}
$$

We can then read off the matrix elements (P.12) .

$\square$

Examples

$\ell = 0, 1/2, 1, 3/2, \ldots$

For $2\ell =$ to an even integer, the representatioon is the $2\ell+1$ dimensional tensorial representation of $SO(3)$. For $2\ell =$ to an odd integer, π_ℓ is a spinor representation. For matter, 1/2 describe elementary particles of half-integer spin.

(1) $\ell = 1/2$

$$
\begin{aligned}
\pi_{1/2}(g)_{\frac{1}{2}\frac{1}{2}} &= \alpha\sum_{k=0}^{0} = \alpha \\
\pi_{1/2}(g)_{-\frac{1}{2}-\frac{1}{2}} &= \alpha^{-1}(-\overline{\beta})\beta\sum_{k=0}^{1}\frac{1}{(k-1)!(1-k)!(1-k)!k!}\left(-\left|\frac{\alpha}{\beta}\right|\right)^k = \overline{\alpha} \\
\pi_{1/2}(g)_{-\frac{1}{2}\frac{1}{2}} &= = \beta
\end{aligned} \tag{P.-12}
$$

$$
\pi_{1/2}\left(\begin{array}{cc}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{array}\right) = \left(\begin{array}{cc}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{array}\right) \tag{P.-12}
$$

(2) $\ell = 1$:

$$
\pi_1\left(\begin{array}{cc}\alpha & \beta\\ -\overline{\beta} & \overline{\alpha}\end{array}\right) = \left(\begin{array}{ccc}\alpha^2 & \sqrt{2}\alpha\beta & \beta^2\\ -\sqrt{2}\alpha\overline{\beta} & |\alpha|^2-|\beta|^2 & \sqrt{2}\overline{\alpha}\beta\\ \overline{\beta}^2 & -\sqrt{2}\overline{\alpha}\overline{\beta} & \overline{\alpha}^2\end{array}\right) \tag{P.-12}
$$

Unitarity of the representation

the transpose $g \to g^T$

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \to \pi_\ell(g^T)_{mn} = \pi_\ell(g)_{nm} = (\pi_\ell(g)^T)_{mn} \tag{P.-12}$$

the complex conjugate $g \to \overline{g}$

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \to \begin{pmatrix} \overline{\alpha} & \overline{\beta} \\ -\beta & \alpha \end{pmatrix} \quad \Rightarrow \quad \pi_\ell(g)_{mn} \to \pi_\ell(\overline{g})_{mn} = (\overline{\pi_\ell(g)})_{mn} \tag{P.-12}$$

Combining both operations $g \to g^\dagger$, this induces $\pi_\ell(g^\dagger) = \pi_\ell(g)^\dagger$ and so we can write

$$\pi_\ell(g)^\dagger = \pi_\ell(g^\dagger) = \pi_\ell(g^{-1}) = \pi_\ell(g)^{-1}. \tag{P.-12}$$

Therefore the representation is unitary.

$\square$

According to Euler's theorem, every rotation R in $\mathbb{R}^3$ can be written as $R = R_3(\phi)R_2(\theta)R_3(\psi)$, see fig (P.4).

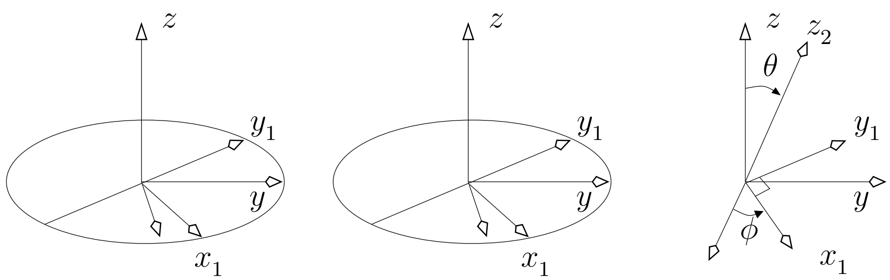

Figure P.4: EulerTherm.

$$\begin{aligned} \pi_{\frac{1}{2}}(\phi,\theta,\psi) &= \exp\left(-i\frac{\phi}{2}\sigma_3\right)\exp\left(-i\frac{\theta}{2}\sigma_2\right)\exp\left(-i\frac{\psi}{2}\sigma_3\right) \\ &= \begin{pmatrix} \exp(-i\frac{\phi}{2}) & 0 \\ 0 & \exp(i\frac{\phi}{2}) \end{pmatrix} \begin{pmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{pmatrix} \begin{pmatrix} \exp(-i\frac{\psi}{2}) & 0 \\ 0 & \exp(i\frac{\psi}{2}) \end{pmatrix} \end{aligned} \tag{P.-13}$$

Together

$$\begin{pmatrix} \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & -\exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \\ \exp(-i\frac{\phi}{2})\sin(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) & \exp(-i\frac{\phi}{2})\cos(\frac{\theta}{2})\exp(-i\frac{\psi}{2}) \end{pmatrix} \tag{P.-13}$$

Irreducibility

According to Schur's lemma, the representation is irreducible if the matrix which commutes with all the elements of the representation is a constant matrix. This we will use to prove the representations (labelled by ℓ) are irreducible.

Consider special case in (P.12) $\alpha = e^{-im\theta/2}$ $\beta = 0$

$$\pi_\ell(0, 0, R_3(\phi))_{mn} = \delta_{mn} e^{-im\phi} \tag{P.-13}$$

where $m, n = \ell, \ell - 1, \cdots - \ell + 1, -\ell,$

$$(e^{-in\theta} - e^{-im\theta}) M_{nm} = 0. \tag{P.-13}$$

implying that M is diagonal

Thus any matrix M that commutes with the representation is a constant matrix.

$\square$

Completeness

i.e. any function $f(g)_{mn}$ can be expanded in terms of elements $\pi_\ell(g)_{mn}$ of the irreducible elements with summation over labelled by ℓ,

$$f(g)_{mn} = \sum_\ell c_\ell \, \pi_\ell(g)_{mn} \tag{P.-13}$$

analagous to a Fourier transform corresponding to the abelean group $U(1)$ whose irreducible representations are $e^{i\ell\phi}$

$$f(\phi) = \sum_{\ell=-\infty}^{\infty} c_\ell \, e^{i\ell\phi} \tag{P.-13}$$

where ℓ is an integer labelling the irreducible representations.

character $\chi^{(\ell)}(\theta)$

$$\chi^{(\ell)}(\theta) = \sum_{m=-\ell}^{\ell} = \frac{\sin\left(j + \frac{1}{2}\right)\theta}{\sin\left(\frac{\theta}{2}\right)} \tag{P.-13}$$

$\square$

Clebsch-Gordon coefficients.

Vector addition

$$\chi^{(j_1)}\chi^{(j_2)} = \sum_{m_2=-j_2}^{j_2} e^{-im_2\theta} \sum_{m_1=-j_1}^{j_1} e^{-im_1\theta} \tag{P.-12}$$

Set $m = m_1 + m_2$, and assume $j_1 \geq j_2$ without loss of generality. Then

$$\begin{aligned}
\chi^{(j_1)}\chi^{(j_2)} &= (e^{+j_2\theta} + \cdots + e^{-j_2\theta}) \left(\frac{e^{-i(j_1+1)\theta} - e^{ij_1\theta}}{e^{-i\theta} - 1} \right) \\
&= \frac{e^{-i(j_1+j_2+1)\theta} - e^{i(j_1+j_2)\theta}}{e^{-i\theta} - 1} + \cdots + \frac{e^{-i(j_2-j_1+1)\theta} - e^{i(j_2-j_1)\theta}}{e^{-i\theta} - 1} \\
&= \chi^{(j_1+j_2)} + \chi^{(j_1+j_2-1)} + \cdots + \chi^{(j_1-j_2)}
\end{aligned} \tag{P.-13}$$

Thus

$$\pi_{j_1} \otimes \pi_{j_2} = \sum_{j=|j_1-j_2|}^{j_1+j_2} \pi_j \tag{P.-13}$$

For example

$$\pi_{1/2} \otimes \pi_{1/2} = \pi_0 \oplus \pi_1 \tag{P.-12}$$

The composition of two spin haaalf particle is the direct sum of a scalar (singlet) and a spin one (doublet). Or two $j = 1/2$ edges of a spin network shares a tri-valent vertex with either a $j = 0$ or $j = 1$ edge.

$$\pi_1 \otimes \pi_1 = \pi_0 \oplus \pi_1 \oplus \pi_2 \tag{P.-11}$$

$$\begin{aligned}
\mathbf{J}^2\psi(j,m) &= j(j+1)\psi(j,m) \\
J_z\psi(j,m) &= m\psi(j,m)
\end{aligned} \tag{P.-11}$$