Appendix Q

Consistent Discrete Classical and Quantum General relativity

Q.1 Introduction

Given data on an initial time slice which fulfill the constaint equations, will the constraint equations hold on later time slices? That is, do the evolution equations preserve the constraint equations?

The usual starting point of numerical general relativity are: the six evolution equations for h_{pq} and K_{pq}

$$\dot{h}_{pq} = 2NK_{pq} + \mathcal{L}_{N^p} h_{pq} \tag{Q.1}$$

$$\dot{K}_{pq} = N_{|pq} - N(R_{pq} + KK_{pq} - 2K^r{}_p K_{qr}) + \mathcal{L}_{N^p} K_{pq}. \tag{Q.1}$$

and the four constraint equations which put conditions on the initial data:

$$\begin{aligned} C &= R + K^2 - K^{ab}K_{ab} = 0 \\ C_m &= \nabla^a (K_{am} - Kq_{am}) = 0. \end{aligned} \tag{Q.1}$$

The discrete formulations of constrained systems are often *inconsistent*. The discrete equations you get cannot be solved simultaneously. If you solve the constraint equations at the beginning they will fail to be solved when you evolve according to the discrete evolution equations, so the discretized evolution equations produce solutions in the future that do not satisfy the discretized constraints. This is a well known problem in numerical relativity. The discrete theory is also inconsistent with regards to the Poisson bracket algebra, the discrete versions of the constraints fail to close as an algebra.

Pullin et al developed a genral technique allowing to define consistent discrete theroies. One can define a consistent discrete theory for general relativity. What you do is you discretize the **action** of the theory and then you work out a canonical theory for the discrete action. Instead of just taking the EQMs and disretizing them discretize the action. Since derrived from an action, they are going to be consistent!

The lapse and shift are not free but are determined by imposing the preservation of constraints.

The resulting theory is different from GR, yet it will generically include solutions that approximate continuum general relativity very well.

For cosmological models the generic behaviour far from the big bang approximates very well the continuum.

$$L(n,n+1) \equiv L(q_n, q_{n+1}) \equiv \epsilon \hat{L}(q,\dot{q}) \tag{Q.1}$$

$$q + q_0 \quad \text{and} \quad \dot{q} \equiv \frac{q_{n+1} - q_n}{\epsilon}. \tag{Q.1}$$

$$S = \sum_{n=0}^{N} L(q_n, q_{n+1}) \tag{Q.1}$$

$$\frac{\partial S}{\partial q_n} = \frac{\partial L(q_{n-1}, q_n)}{\partial q_n} + \frac{\partial L(q_n, q_{n+!})}{\partial q_n} = 0. \tag{Q.1}$$

$$p_{n+1} = \frac{\partial L(q_n, q_{n+!})}{\partial q_{n+1}}, \quad p_n = -\frac{\partial L(q_n, q_{n+!})}{\partial q_n}. \tag{Q.1}$$

Relations that define a type I canonical transformation

$$L(Q_n, q_{n+1}) = m\frac{(q_{n+1} - q_n)^2}{2\epsilon} - V(q_n)\epsilon \tag{Q.1}$$

$$
\begin{aligned}
p_{n+1} &= m\frac{(q_{n+1}-q_n)}{\epsilon} \\
p_n &= m\frac{(q_{n+1}-q_n)}{\epsilon} + V'(q_n)\epsilon \\
q_{n+1} &= q_n + \frac{p_n}{m}\epsilon - V'(q_n)\frac{\epsilon}{m} \\
p_{n+1} &= p_n - V'(q_n)\epsilon. \qquad \text{(Q.-1)}
\end{aligned}
$$

$$
U = \exp\left(i\frac{V(q_{i-1})\epsilon}{\hbar}\right)\exp\left(i\frac{p_{i-1}^2\epsilon}{2m\hbar}\right) = \exp H. \qquad \text{(Q.-1)}
$$

$$
\mathcal{H} = \epsilon\left(\frac{p^2}{2m} + \frac{1}{2m\omega^2 q^2} - \frac{1}{2\epsilon p q\omega^2}\right) f(\epsilon) \qquad \text{(Q.-1)}
$$

Canonical formulation for constrained discrete dynamical systems

$$
L(n,n+1) = p_n(q_{n+1}-q_n) - \epsilon\mathcal{H}(q_n,p_n) - \lambda_{nB}\phi^B(q_n,p_n) \qquad \text{(Q.-1)}
$$

$$
\begin{aligned}
P^q_{n+1} &= \frac{\partial L(n,n+1)}{\partial q_{n+1}} = p_n, \quad P^q_n = -\frac{\partial L(n,n+1)}{\partial q_n} = p_n + \epsilon\frac{\partial\mathcal{H}(q_n,p_n)}{\partial q_n} + \lambda_{nB}\frac{\partial\phi^B(q_n,p_n)}{\partial q_n} \\
P^p_{n+1} &= \frac{\partial L(n,n+1)}{\partial p_{n+1}} = 0, \quad P^p_n = -\frac{\partial L(n,n+1)}{\partial p_n} = -(q_{n+1}-q_n) + \epsilon\frac{\partial\mathcal{H}(q_n,p_n)}{\partial p_n} + \lambda_{nB}\frac{\partial\phi^B(q_n,p_n)}{\partial p_n} \\
P^{\lambda_B}_{n+1} &= \frac{\partial L(n,n+1)}{\partial\lambda_{(n+1)}B} = 0, \quad P^{\lambda_B}_n = \phi^B(q_n,p_n).
\end{aligned}
\qquad \text{(Q.-3)}
$$

$$
\phi^B(q_n, P^q_{n+1}) = 0. \qquad \text{(Q.-3)}
$$

$$
\phi^B(q_{n+1}, P^q_{n+1}, \lambda_{nB}) = 0, \qquad \text{(Q.-3)}
$$

$$
\lambda_{nB} = \lambda_{nB}(q_{n+1}, P^q_{n+1}, v^\alpha) \qquad \text{(Q.-3)}
$$

The final evolution equations are obtained by substituting the Lagrangian multipliers.

Notice that here the Lagrange multipliers were determined without imposing any gauge fixing. Notice that more precisely what has been determined is "$\lambda \times \epsilon$".

For a completely parametrized theory is no explicit dependence on ϵ, which may be fixed arbitrarily. Once the time interval (or the lattice spacing) is chosen, lapse is determined.

Q.1.1 "Dirac's" canonical approach to general discrete systems

They have recently developed a "Dirac's" canonical approach to general discrete systems.

$$
\begin{aligned}
L(n, n+1) &\equiv L(q_n, q_{n+1}) \\
{P^q}_{n+1}{}^a &= \frac{\partial L(n, n+1)}{\partial {q^a}_{n+1}} \\
{P^q}_n{}^a &= -\frac{\partial L(n, n+1)}{\partial {q^a}_n}
\end{aligned}
\tag{Q.-4}
$$

$$
\left| \frac{\partial^2 L(n, n+1)}{\partial {q^b}_{n+1} \partial {q^a}_n} \right| = 0. \tag{Q.-4}
$$

Primary constraints

$$
\Phi_A({q^a}_n, {P^q}_n{}^a) = 0 \tag{Q.-4}
$$

$$
q^a_{n+1} = f^a({q^b}_n, {P^q}_n{}^b, V^A, U^A) \tag{Q.-4}
$$

V and U arbitrary functions. Consistency:

$$
\Phi^A({q^a}_{n+1}, {P^q}_{n+1}{}^a) = \Phi_A({f^a}_n, \frac{\partial L(q_n, f^a)}{\partial {q^a}_{n+1}}) = 0 \tag{Q.-4}
$$

$$
C(q_n, P^q_n) \qquad V^A = V^A(q_n, P^q_n, v^\alpha, u^\rho) \tag{Q.-4}
$$

$$
q^a_{n+1} = f^a({q^b}_n, {P^q}_n{}^b, V^A(q, P^q_n, v, u), U^A(q, P^q_n, v, u)) = \tilde{f}^a(q^b_n, P^{qb}_n, v^\alpha, u^\rho) \tag{Q.-4}
$$

$$
{P^q}_{n+1}^b = g^a({q^b}_n, {P^q}_n{}^b, v^\alpha, u^\rho). \tag{Q.-4}
$$

By using a Type II, II, or IV transformation one can show that this evolution is canonical, preserves the Poisson brackets and the canstraint surface. This is equivalent to what happens in the continuum, consistency may be achieved by determing the complete constraint surface and a total Hamiltonian that preserves the Poisson structure.

Finally one can recognise the second class constraints and impose them strongly. While some symmetries of the continuum are broken the discretization others are preserved.

The procedures may be extended to quantum field theories and reproduces, for standard gauge results obtained by using tranfer matrix techniques on a lattice.

$$S = -\frac{a_0 a^3}{2} \sum_P \frac{S_P}{A_P^2} \mathrm{Tr}(U_P) \tag{Q.-4}$$

It also provides a very simple descrition of BF theories on a lattice

$$\begin{aligned} L(n, n+1) &= \sum_v \mathrm{Tr}\Big[B^0_{n,v} h^1_{n,v} h^2_{n,v+e_1} (h^1_{n,v+e_2})^\dagger (h^2_{n,v})^\dagger + B^2_{n,v} V_{n,v} h^1_{n+1,v} (V_{n,v+e_1})^\dagger \\ &- B^1_{n,v} V_{n,v} h^2_{n+1,v} (V_{n,v+e_2})^\dagger (h^2_{n,v})^\dagger + \mu_{n,v}(V_{n,v} V^\dagger_{n,v} - I) + \sum_{k=1}^{2} \lambda^k_{n,v} (h^k_{n,v} (h^k_{n,v})^\dagger - I)\Big]. \end{aligned} \tag{Q.-5}$$

For standard gauge theories as Y-M and BF the symmeties may be preserved under the discretization and there is not any need of determining Lagrange multipliers.

Q.1.2 Canonical discrete quantization of general relativity

The Lagrangiean of Euclidean general relativity in terms of the Ashtekar variables.

$$L = \int_\Sigma E^{ai} F^i_{a0} + \epsilon_{abc} [E^{bi} E^{cj} \epsilon^{ijk} N + N^b E^{ck}] F^k_{de} \epsilon^{ade}, \tag{Q.-5}$$

This Lagrangian can easily be discretized as follows:

$$\begin{aligned} L(n, n+1) &= \sum_v \mathrm{Tr}\Big[E^a_{n,v} h^a_{n,v} V_{n,v+e_1} (h^a_{n+1,v})^\dagger (V_{n,v})^\dagger \\ &+ K_{1,n,v} h^2_{n,v} h^3_{n,v+e_2} (h^3_{n,v+e_3})^\dagger (h^3_{n,v})^\dagger + cyc. \\ &+ \alpha_{a,n,v} \left(h^a_{n,v} (h^a_{n,v})^\dagger - 1\right) + \beta_{n,v}(V_{n,v} V^\dagger_{n,v} - 1)\Big] \end{aligned} \tag{Q.-6}$$

Where h and V are $SU(2)$ holonomies

$$K_{a,n,v} = \epsilon_{abc}[E^b_{n,v}E^c_{n,v}N_{n,v} + N^b_{n,v}E^c_{n,v}]. \tag{Q.-6}$$

The canonical quantization is now straightforward. Each of the action variables

$$E^a_{n,v},\ h^a_{n,v},\ V_{n,v},\ \alpha^a_{n,v},\ \beta^b_{n,v},\ N_{n,v},\ N^b_{n,v} \tag{Q.-6}$$

will have canonical momenta and evolution equations given by conanical transformations concerning level n and $n+1$.

The $SU(2)$ gauge symmetry is exactly preserved by the diffeomorhism and Hhhhamiltonian constraints are solved determining the multipliers N, N^a as we discussed above.

The final degrees of freedom are the electric field and the $SU(2)$ holonomy and as the Gauss invariance is exactly reserved the final quantum theory may be in principle treated in terms of loops.

Q.1.3 Discrete Quantum Gravity Applied to Cosmology

A very simple example

$$L = E\dot{A} + \pi\dot{\phi} - NE^2(-A^2 + (\Lambda + m^2\phi^2)|E|) \tag{Q.-6}$$

The system has four hase space variables and one constraint. Therefore it has two independent observables ($\{\mathcal{O}, C\}$):

$$\mathcal{O}_1 = \phi, \quad \mathcal{O}_2 = \pi + \frac{2}{3}\frac{m^2\phi}{\Lambda + m^2\phi^2}AE. \tag{Q.-6}$$

$$L(n, n+1) = E_n(A_{n+1} - A_n) + \pi_n(\phi_{n+1} - \phi_n) - N_n E_n^2(-A_n^2 + (\Lambda + m^2\phi_n^2)|E_n|) \tag{Q.-6}$$

The evolution is given by the canonical transformation generated by the by L, the Lagrange multilier at each step is determined by the preservation of the constraint. The final discrete evolution equations are:

$$\Theta = \Lambda + m^2\phi^2, \tag{Q.-6}$$

$$
\begin{aligned}
N_n &= \frac{[-P^A_n\Theta + A^2_n]\Theta}{2A^5_n}. \\
P^A_{n+1} &= A^2_n\Theta^{-1} \\
A_{n+1} &= A_n + \frac{A^2_n - P^A_n\Theta}{2A_n} \\
\phi_{n+1} &= \phi_n \\
P^\phi_{n+1} &= P^\phi_n - (A^2_n - P^A_n\Theta)A_n m^2\phi_n\Theta^{-2}.
\end{aligned} \tag{Q.-9}
$$

$A^2_n - P^A_n\Theta$ is a measure of the step of the discretization.

There are two constants of motion, that are the discrete counterpart of the observables

$$
O_n = \phi_n \qquad O'_n = P^\phi_n + \frac{2}{3}\frac{m^2\phi_n}{\Theta}A_n P^A_n, \quad O_{n+1} = O_n. \tag{Q.-9}
$$

Notice that the discrete theory has four phase space degrees of freedom instead of the two of the continuum theory. The additional degrees of freedom charactrize the step of the discretization and encode remnants of the gauge invariance in the discrete theory.

Although the graphs suggest that the triad goes to zero at $n = 0$ and therefore one has a singularity this is not the case.

We here show the approach to the singularity in the discrete and continuum case. The discrete theory has a small but non-vanishing triad a $n = 0$. R. Gambini and J. Pullin gr-qc/0212033

The rate of contraction/expansion changes when going through the big crunch/bang. Question: is that a remnent of the reparametrization invariance or does it have physical consequences?

The answer to this question is related with the existence of more degrees of freedom and therefore more constants of motion. In fact, the discrete canonical transformation is singuluar for $A = 0$. If one tries to introduce a generator of this evolution:

$$
A_{n+1} = A_n + \{A_n, \mathcal{H}_n\} + \frac{1}{2!}\{\{A_n, \mathcal{H}_n\}, \mathcal{H}_n\} + \cdots \tag{Q.-9}
$$

$$
\mathcal{H}_n = \frac{\mathcal{C}^2_n}{4\Theta A_n}\left[1 + \sum_{k=1}^{\infty} a_k \left(\frac{\mathcal{C}_n}{A^2_n}\right)^k\right] \tag{Q.-9}
$$

H_n diverges for $\left(\frac{\mathcal{C}_n}{A^2_n}\right)^k > 2$. This happens for $n = 0$ when the system goes through the singularity.

H_n, which is constant on each region characterizes the spacing of the discretization in an invariant way, and in that sense suggest a procedure for taking the continuum limit.

It may provide a mechanism for changing the values of the fundamental constants?

Quantization

$$\psi_{n+1}[A, \phi] \tag{Q.-9}$$

$$< A_1, \phi_1|U|A_2, \phi_2 >= \sqrt{\frac{2|A_2|}{\Theta}} \exp(-iA_2^2(A_1 - A_2)sgA_2) \tag{Q.-9}$$

It is very easy to obtain the unitary evolution operator from the discrete action. The inner product is also trivially defined in this space. The elimiation of the constraints simplify all the quantization process.

The elimination of the constraints simplifies the quantization and allows to treat in a simlper way old conceptual problems as the issue of time.

Notice that the evolution variables n does not have any intrinsic meaning and it is not associate with any dynamical variable. n cannot be taken as a clock variable and it is not and observable quantity.

$$S = \int \left[\dot{q} + p_0\dot{q}^0 - N(p_0 + \frac{p^2}{2m} + \lambda q) \right] d\tau, \tag{Q.-9}$$

$$L(n, n+1) = p^n(q_{n+1} - q_n) + p_0^n(q_{n+1} - q_n^0) - N_n(p_0^n + \frac{p_n^2}{2m}\lambda q_n). \tag{Q.-9}$$

the conditional robability to obtain $q = x$ given $q^0 = t$

$$(q = x|q^0 = t) = \frac{\sum_{n=-\infty}^{\infty} |\Psi(x, t, n)|^2}{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx|\Psi(x, t, n)|^2} \tag{Q.-9}$$

One can show that this relational description recovers usual quantum mechanics when the discrete approximation approaches the continuum limit and when the clock variable behaves sufficiently to a classical clock.

The procedure can be extended to situations where the simultaneity surfaces are not transverse to the classical orbits of the system.

The system has an unitary evolution in n. At t cannot br perfectly correlated with n, even in the semi-classical regime of the clock, the evolution in t is not perfectly unitary. In fact one can show that the density matrix evolves according to

$$\frac{\partial}{\partial t}\rho_2 = -i[\mathcal{H}_2, \rho_2(t)] - \sigma[\mathcal{H}_2, [\mathcal{H}_2, \rho_2(t)]]. \tag{Q.-9}$$

This equation was first proposed by Milburn based on phenomenological arguments, and is a particular tpye of non-unitary evolutions considered by Lindblad.

Their derivation allows to estimate σ that is of order of the Planck time.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \tag{Q.-9}$$

This equation doesnot violate the conservation of energy like Hawking prosal for information loss. One could exect to confirm this type of equation by studying some mesoscopic quatum systems.

Information loss problem in Black Holes

It provides a new and very effective mechanism for treating the information loss problem in Black Holes. It eliminates the puzzle as a fundamental question.

They have shown that for any Black hole bigger than 600 Plank masses the information loss induced by our equation is enough to dissipate all the black hole information before to its evaporation.

For very small black holes, Hawking's semi-classical analysisis not valid.

Bose-Einstein Experiment of Quantum Gravity Decoherence

Plank-scale-induced deviations whose detection is a function of the number of particles.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0)e^{-i\omega_{nm}t}e^{(-\sigma(\omega_{nm})^2)t} \tag{Q.-9}$$

One could exect to confirm this type of equation by studying some mesoscopic quatum systems.

Q.2 Semi-Discrete Approach

Significant departure from what everyone else was doing in LQG. They could not make use of of kinematic tools of LQG, like spin-network states, Astekar-Lewendowski diff-invariant measure, ... Thiemann paper *"One way out could be to look at constraint quantization from an entirely new point of view which proves useful also in discrete formulations of classical GR, that is, numerical GR. While being a fascinating possibility, such a procedure would be a rather drastic step in the sense that it would render most results of LQG obtained so far obsolete."*, [77].

discrete time but keep space continuous in the classical action. But would be not be a discord with GR in which space and time on same footing?

Get coupled non-linear PDEs.

Same kinematics but deal with spacial constraint in the usual way.

$$\begin{aligned} S = & \int \quad dtd^3x \left[Tr \left(\tilde{P}^a (A_a(x) - V(x) A_{n+1,a}(x) V^{-1}(x) + \partial_a (V(x)) V^{-1} \right) \right. \\ & - \quad N^a C_a - NC + \mu Tr(V(x) V^\dagger(x) - 1) \Big] \end{aligned} \qquad \text{(Q.-9)}$$

Q.3 Biblioliographical notes

In this chapter I have relied on the following refferences:

[448]

Q.4 Worked Exercies and Details

Proofs
