Chapter 1

Classical GR, Einstein's hole argument and Physical Geometry

A warning is given to the reader that the material covered in this chaprter is not usually taught in GR courses or appear in many books. Do not skip this chapter as it is essential before progressing to loop quantum gravity.

Gravity and spacetime are the same entity. Spacetime is best described by a metric field. The field equations related to the energy-momentum tensor. It turns out to require even more sweeping revisions of our spacetime concepts. Wont have been taught in lecture courses in GR and many may not be awear of its . This amazing property hidden in GR has not been fully absorbed by the (theoretical) pysics community. The significance of this final step may be unlikely to be familiar to the reader and he/she may be shocked by the outcome.

The distinction between a vector and its vector components with respect to a coordinate induced basis vectors. It is importantant that in the following discourse, the two terms should not be confused.

spacetime is still understood as a *non-dynamical* entity which provides an arena for the laws of physics but does not itself take part.

Rubber-Sheet Analogy of Curved Space Time

One way to think of General Relativity is to use the idea of a "rubber sheet geometry" where in the absence of gravitation the sheet is flat, but a central massive body curves up the sheet in its vicinity so that a free body (which would otherwise have moved in a straight line) is forced to orbit the central body curved space is simila in nature to the curved surface of a rubber sheet. Helpfull in proving some insight into the idea of a curved spacetime.

Claims that GR can be If the marble doesn't weight too much it will move in a straight line when progected across the sheet. This

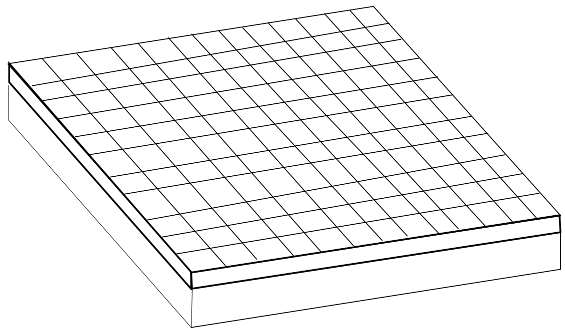

Figure 1.1: Rubbersheet simulation of geodesic moton in special relativity.

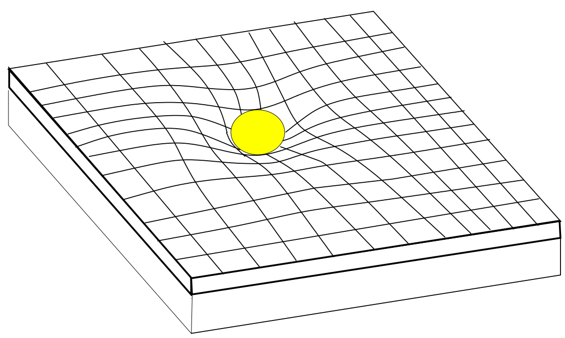

Figure 1.2: rubbersheet. It doesnot matter that the coordinates are time-dependent - it still serves as a physical refference system.

In general relativity any change in mass of the central sourse will spread out like a ripple in the rubber-sheet geometry

Geometry, Coordinates and Metrics

Before we go any further we need to clarify a few things.

The components of the field in the basis induced by the coordinate system.

In the general theory of relativity, the square of the interval is given by:

$$ds^2 = g_{ab}\, dx^a\, dx^b, \tag{1.1}$$

where g_{ab} are the components of the metric tensor, and dx^a is the differentai of the coordinate x^a. his struggle to understand "the meaning of the coordinates".

A coordinate sytem in space time $\mathcal{M}$ permits us to associate, with each space-time event of $\mathcal{M}$ four real numbers, the values of x, y, z, and t. To locate a point in space time requires specification of these four numbers.

The description will depend on the identificatoin of each point - the choice of coordinate system, and also on the directions chosen for the coordinate axes.

Let us denote the distance between two spacetime points P and Q as described by the metrics $g_{ab}(x)$ as $d_g(P, Q)$.

$$d\tau^2 = g_{ab}(x)\, dx^a\, dx^b \tag{1.2}$$

We have

$$d_g(P, Q) = \int_P^Q d\tau = \int_P^Q \left(g_{ab}(x) dx^a dx^b\right)^{1/2} \tag{1.3}$$

The description of a particular spacetime geometry will vary with the choice of coordinate system, however the distance between two spacetime points should be the same in every coordinate system. The interval

$$d\tau^2 = g_{ab}(x)\, dx^a\, dx^b = g'_{ab}(y)\, dy^a\, dy^b \tag{1.4}$$

is invariant. The coordinate differentials are different in the new coordinate system, so that must change in such a way that $d\tau^2$ is invariant. $g'_{ab}(y)$ will in general be *functionally different* from $g_{ab}(x)$ (i.e the components will involve different functions of its coordinates compared with g). It gives the same distance between space-time events and yeilds the same angle between the same two vectors. it represents the same physical solution, veiwed from a different coordinate system.

1.1 Einstein's Hole Argument

General Covariance

In special relativity there are special coordinate frames, called **inertial frames** i.e. moving with constant velocity (we'll not go into anymore detail with regards to this). The description in one frame x and another frame x' are related to each other via a Lorentz transformation

Maxwell's equations have the same form in all inertial frames. Written as equations whose form is invariant under Lorentz transformations. General covariance is the idea that the laws of nature must be the same in all reference frames, and hence all coordinate systems.

Since gravity distorts spacetime geometry there are no global inertial reference frames. That there should not be any privileged reference systems in which the laws of nature should be written. The laws of nature must therefore take the same form in all coordinate systems. :

> **General covariance is the idea that the laws of nature must be the same in all reference frames, and hence all coordinate systems.**

It is always possible to write laws in a coordinate-independent way. A physical system acting in a certain way doesn't know which coordinate system you are using to describe it. There is a more subtle point of the assertion of general covariance,

> ***Special coordinates should not have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates.***

This requirement is **NOT** the same as the more generally quoted 'principle of general covariance' stated above!!

which is that there should be **no** preffered coordinate system that has a **physical role** to play in the formulation of the equations of motion. Let us illustrae this with a simple analagy: what is the difference between Newtonian theory and special relativity? - Newtons equations of motion can be written in a Lorentz covariant form - do we conclude that Lorenzt covariance has no physical content? No, the point is that in Newton's theory there is a special frame that plays a **physical role** in formulating the equations of motion! The frame that is at rest with respect to this notion that we call the ether. If we were to insist on modifying the equations of motion in special relativity, that there should be a special inertial systems that play a role in formulating the EQM, then we would be reintroducing the Ether at least to some observers. Similarly in GR there should be **no preffered coordinate system** that plays a physical role to play in formulating the field equatios, for this would reintroduce gravity as a force, at least to some observers. - we would have lost the equivalence principle.

These arguments . In the next section we will by using very basic ideas of GR, introduced in the previous section and which should be familiar to anyone who knows anything about GR, together with some fairly straightforward maths to demonstrate a striking and, at first sight, rather alarming consequence of general covariance for GR.

General Covariance (Pure Gravity - No Matter)

a trival mathematical observation. Take the two differential equations (1.5) and (1.6). They both have the same mathematical form.

$$\frac{d^2 f(x)}{dx^2} + \omega^2 f(x) = 0, \tag{1.5}$$

$$\frac{d^2 g(y)}{dy^2} + \omega^2 g(y) = 0. \tag{1.6}$$

Say we find out that one solution to Eq(1.5) is

$$f(x) = \cos \omega x, \tag{1.7}$$

we immediatly know that

$$g(y) = \cos \omega y \tag{1.8}$$

solves Eq.(1.6). Eq. (1.8) is the same function as Eq (1.7), written as a function of y instead. It is always possible to write laws in a coordinate-independent way. A physical system acting in a certain way doesn't know which coordinate system you are using to describe it.

However, general covariance also implies a *distinct solution relating to the y-coordinate system*, which is pretty obvious once you think about it. single coordinate system this). We take these one metric function other metric is of its Now, general covariance say the laws of physics should be the same for all reference systems, demanding that the equations of motion have the same form in both our coordinate systems. So you have exactly *the same differential equation to solve in both these coordinate systems, execpt in one the independent variables are the x-coordinates and in the other the independent variables are the y-coordinates.*

$$R_{ab}(x) = 0 \tag{1.9}$$
$$R_{ab}(y) = 0 \tag{1.10}$$

Once you find out that a solution to the EOM in the x-coordinates (1.22) is, say

$$g_{00}(x) = \cos t, \quad g_{11} = x^2, \quad g_{11}, \quad \text{etc} \tag{1.11}$$

then one imeddiately knows that

$$g_{00}(y) = \cos t, \quad g_{11}(y) = y^2, \quad g_{11}(y), \quad \text{etc} \tag{1.12}$$

solves the EOM in the y-coordinates (1.23).

Once this observation is noted, it then becomes evident that if one of our metric functions is a solution then, at the same time, so is the other! Let us denote this metric function as $\tilde{g}_{ab}(y)$.

However, general covariance also implies a *distinct solution relating to the y-coordinate system*, which is pretty obvious once you think about it: General covariance demands that the equations of motion have the same form in all coordinate systems - so we have exactly the same diffrential equation to solve in both coordinate systems, (except, of course, in one the independent variable is x and in the other it is y). Say we have a metric function in the x-coordinates that is a solution, then at the same time, so is the metric function in the y-coordinates sytsem that has the same functional form! (The same functional form meaning it is the same function except that it is a function of y instead of x). Let us denote this metric function as $\tilde{g}_{ab}(y)$.

$$\boxed{g_{ab}(x = u) = \tilde{g}_{ab}(y = u).} \tag{1.13}$$

To see this more clearly, perform a coordinate transformation on the metric $\tilde{g}_{ab}(y)$.

$$\tilde{g}'_{ab}(x) = \Lambda^c_a \Lambda^d_b \tilde{g}_{cd} \tag{1.14}$$

Where $\Lambda^c_a = \frac{\partial x^c}{\partial y^a}$ is the (inverse) Jacobian matrix for the coordinate transformation $y^a = y^a(x^b)$. $\Lambda^c_a = \delta^c_a$.

$$d_{\tilde{g}}(P, Q) = \sum_P^Q \left(\tilde{g}_{ab}(y) dy^a dy^b\right)^{1/2} = \sum_P^Q \left(\tilde{g}'_{ab}(x) dx^a dx^b\right)^{1/2}. \tag{1.15}$$

Unless $\Lambda^c_a = \delta^c_a$.

$$\tilde{g}'_{ab}(x) \neq g_{ab}(x) \tag{1.16}$$

$$d_g(P, Q) \neq d_{\tilde{g}}(P, Q) \tag{1.17}$$

General covariance implies that, once we find one solution to the equations of motion of GR, it immediately gives rise to other **distinct solutions*****, one for each conceivable coordinate system!!!***

Illustrative examples

(a)

from blind mathematical point of view

$$\frac{d^2 f(x)}{dx^2} + \omega^2 f(x) = 0, \tag{1.18}$$

$$\frac{d^2 g(y)}{dy^2} + \omega^2 g(y) = 0. \tag{1.19}$$

Say we find out that one solution to Eq(1.18) is

$$f(x) = \cos \omega x, \tag{1.20}$$

we immediatly know that

$$g(y) = \cos \omega y \tag{1.21}$$

solves Eq.(1.19).

$$R_{ab}(x) = 0 \tag{1.22}$$

$$R_{ab}(y) = 0 \tag{1.23}$$

Once you find out that a solution to the EOM in the x-coordinates (1.22) is, say

$$g_{00}(x) = \cos t, \quad g_{11} = x^2, \quad g_{11}, \quad \text{etc} \tag{1.24}$$

then one imeddiately knows that

$$g_{00}(y) = \cos t, \quad g_{11}(y) = y^2, \quad g_{11}(y), \quad \text{etc} \tag{1.25}$$

solves the EOM in the y-coordinates (1.23).

I should include what was explained in [400]: Higher-dimensional Algebra and Topological Quantum Field Theory

Intuitive Explanation

The rubber sheet sits in space - this ambiant space that asigns a length to the curve. Also, there is absolute time which clocks exemplify - Newton's laws tell you the position of the rubber sheet in the ambient space at such and such a time.

However, in GR spacetime is not embeded in some further container - hence GR does not predict the proper time. The action principle describing the dynamics of the rubber sheet refers to things external to the dynamical system the theory describes i.e. absolute time and the ambient space, whereas the Einstein-Hilbert action does not.

The information contained in the solution is the value of the matter field at the place where the gravitational field takes such and such a value, for every spacetime point. This information is preserved if we actively drag the gravitational field simultaniously over the spacetime manifold. Seeing as no localization over spacetime should be preffered the two configurations are both solutions.

More subtle point - is that Pre-Maxwell there was a priveledged inertal frame in which the EOM take a particular simple form - the frame at rest with respect to the eather. If we were to drop the assumption that there is a favered frame would reintroduce the eather to at least some observers. Similarly, if there was natures priledged frame in which the fields takes a particularly simple form then we would introduce gravity as a force at least to some observers. Should note that this is what Einstein was attempty for 4 years between 1912 and 1915.

Penrose a leading person in the field of GR [17] (section 19.6):

> *" ... Gravitation is not to be regarded as a force; for, to an observer who is falling freely (such as our astrnaut A), there is no gravitational force to be felt. Instead, gravitation manifests itself in the form of spacetime curvature. Now, it is important, if this idea is to work, that there be no 'preffered coordinates' in the theory. For, if a certain limited class of coordinate systems were taken to be Nature's preffered choices, then these would define "natural observer systems" with respect to which the notion of a "gravitational force could be reintroduced, and the central role of the principle of equivalence would be lost."*

> *"The requirement, in the text, of 'no preffered coordinates' is not only vague, but also something that might be regarded as somewhat too strong. In flat space, for example, it could be reasonably said that the choices of 'Cartesian coordinates' (here the Minkowski coordinates (t, x, y, z) of [], for which the metric takes the particularly simple form $ds^2 = dt^2 - dx^2 - dy^2 - dz^2$) are 'preffered' over all other coordinate systems, and cosmological models also have special coordinate systems in which the metric form looks particularly simple (). The point is, rather, the more subtle one that such special coordinates should not have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates."*

> ***Special coordinates should not have a physical role to play, and that the equations of the theory should be such that their most natural expression does not depend on any particular choice of coordinates.***

Active Diffeomorphisms

There is a simple geometric view of how these solutions are related.

A **congruence** is a space-filling family of curves such that every point of $\mathcal{M}$ lies on just one curve.

The reader may find it helpful to note that if two functions have the same functional form is equivalent to

$$\boxed{\phi(x = u) = \tilde{\phi}(y = u), \quad \text{for all values of } u. \qquad (1.26)}$$

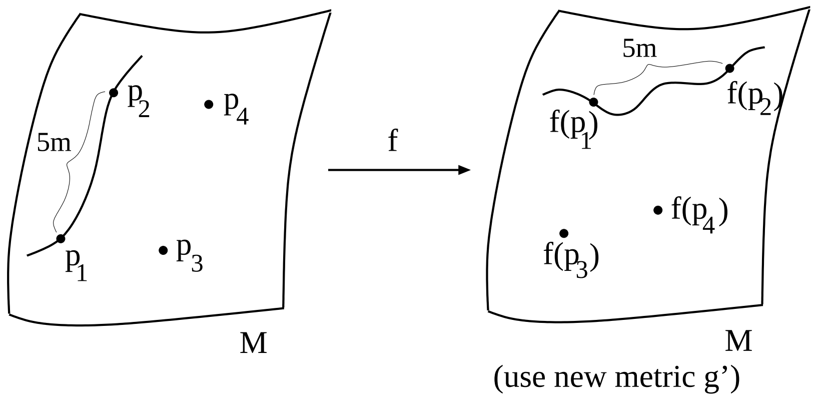

Figure 1.3: Passive spatial difeomorphism $f : M \to M$ refers to invariance under change of coordinates. The same object in a different coordinate system. Any theory of nature is invariant under passive difeomorphisms.

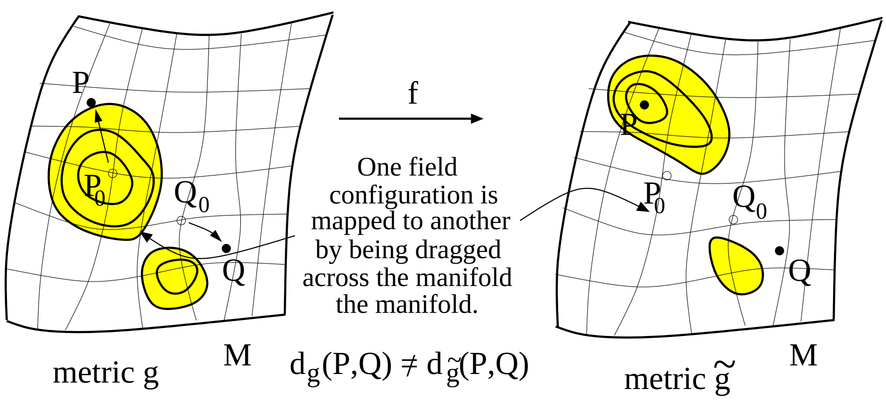

Figure 1.4: An active difeomorphism $f : M \to M$ draggs fields on the manifold while remaining in the same coordinate system. f is viewed as a map that associates one point in the manifold to another one.

If two fields $\tilde{X}(x)$ and $X(x)$ are related to each other through an active diffeomorphism then there is always a coordinate system, which we denote $y^\mu(x)$, in which $\tilde{X}(x)$ has the same functional form as $X(x)$. That is the tensor functions $X(x)$ and $\tilde{X}(y)$ are the same function but they correspond to two different coordinate systems.

$$\boxed{X^{ab...d}_{ef...g}\,(x^a = u^{(a)}) = \tilde{X}^{ab...d}_{ef...g}\,(y^a = u^{(a)}), \qquad (1.27)}$$

where $u^{(a)}$ are four numbers.

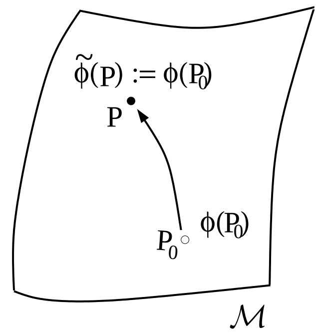

Figure 1.5: The value of $\tilde{\phi}(P)$ at P is equated to the value of $\phi(P_0)$ at P_0, i.e. $\tilde{\phi}(P) = \phi(P_0)$. Under this transformation f we indentify one point of the manifold P_0 to another point P $f : P_o \to P$.

$$g_{ab}(x^a = u^{(a)}) = \tilde{g}_{ab}(y^a = u^{(a)}), \tag{1.28}$$

see Fig(activediff5). We are dragging the tensor function at P_0 over to the point P keeping the coordinate lines "attached".

We then use a coordinate system that asigns the newly identified points the origonal coordinate values (note we are not doing a coordinate transform here - there are no Jacobian matrices involved)

As a consequence of this, if two metrics $g_{\mu\nu}(x)$ and $\tilde{g}(x)$ are related by an active diff transformation and one of them is a solution to Einstein's equations then so is the other. These are distinct metrics in that they describe different geometry.

Generalization to Gravity Coupled to Matter Fields

Expressible as tensor equations that reduce to laws consistent with special relativity in a frame in free-fall.

$$g_{ab}(x), \ \frac{\partial g_{ab}(x)}{\partial x^c}, \ \frac{\partial^2 g_{ab}(x)}{\partial x^d \partial x^c}, \text{ and } T_{ab}(x) \tag{1.29}$$

exactly the same differential equation but now involves:

$$\tilde{g}_{ab}(y), \ \frac{\partial \tilde{g}_{ab}(y)}{\partial y^c}, \ \frac{\partial^2 \tilde{g}_{ab}(y)}{\partial y^d \partial y^c}, \text{ and } \tilde{T}_{ab}(y). \tag{1.30}$$

We have an equation of motion for the matter fields. An equation of motion for the gravitational field. Each of these equations has the same mathematical form in any

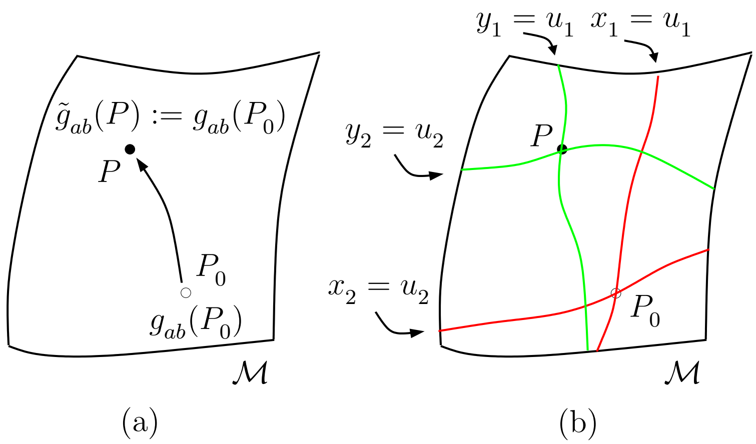

Figure 1.6: The value of the metric function $\tilde{g}_{ab}$ at P is defined by the value of the metric function g_{ab} at P_0, i.e. $\tilde{g}_{ab}(P) = g_{ab}(P_0)$. We go to a new coordinate system which assigns P the same coordinate values that P_0 has in the x-coordinates, so that $\tilde{g}_{ab}(y_1 = u_1, y_1 = u_2) = g_{ab}(x_1 = u_1, x_1 = u_2)$, compare to (1.28).

coordinate system so the arguememnts given before apply equally here. So that once we find one solution it gives rise to distinct solution, one for each concievable coordinate sytstem.

Then so are the set of fields in the y-coordinate system, that have the same functional form.

$$g_{ab}(x = u) = \tilde{g}_{ab}(y = u), \quad \text{for all values of } u. \tag{1.31}$$
$$E_a(x = u) = \tilde{E}_a(y = u), \quad \text{for all values of } u \tag{1.32}$$

General covariance implies that once we find one solution to the EOM, it immediately gives rise to other *distinct solutions*, all related to the original solution by an active diffeomorphism!

General Covariance When Spacetime is Non-Dymanical

$$\triangle A_\mu = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q_i} \left(\sqrt{g} g^{ij} \frac{\partial A_\mu}{\partial q_j} \right) \tag{1.33}$$

The new metric $g_{\mu\nu}(y)$ does not represent Minkowskian spacetime, the approach fails if spacetime is fixed and non-dynamical.

The precedure for generating new distinct solutions fails when spacetime is fixed and non-dynamical.

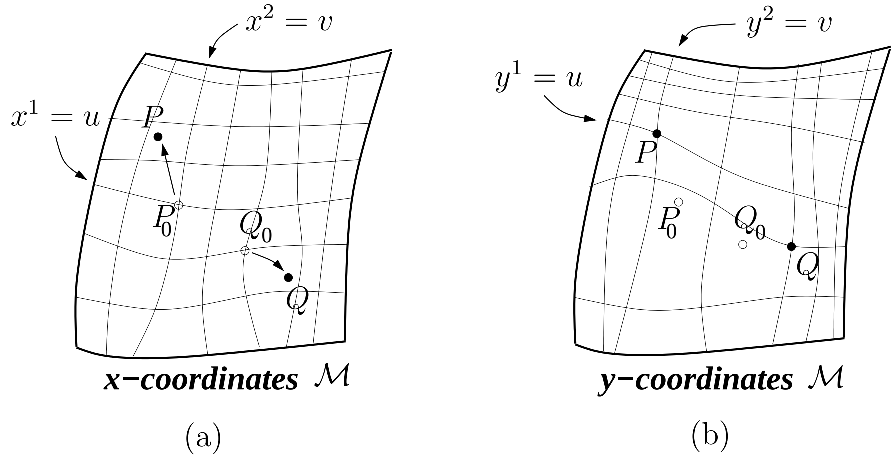

Figure 1.7: (a) An active diffeomorphism in which we indentify one point of the manifold to another point. (b) We then go to a coordinate system that asigns the newly identified points the origonal coordinate values.

It is definitly true that general covariance has no content when spacetime is non-dynamical - the physical system does not care which coordinate system we use to view it.

Einstein's Problem with General Covariance (1912)

Einstein very rapidly worked out physical implications of this that couldn't be acceptable. He demonstrates these troubles with the hole argument to which we now turn.

Consider the situation depicted by fig (1.1). Initially the universe is filled by matter but the a hole forms whcih later goes away. Now consider the effect of an active diffeomorthism that reduces to identity outside the hole.

Let us denote the distance between two spacetime points P and Q as described by the metrics $g_{\mu\nu}(x)$ and $\tilde{g}_{\mu\nu}(x)$ respectively as $d_g(P,Q)$ and $d_{\tilde{g}}(P,Q)$. They will not be equal to each other

$$\boxed{d_g(P,Q) \neq d_{\tilde{g}}(P,Q).} \tag{1.34}$$

Now the problem comes: the distance d between two distinct points P and Q which are both inside the hole although the metrics have the same initial data.

If we demand general covariance, GR doesn't determine the distance between spacetime events!

Einstein could not accept this and spent the next three years frantically, (frantically because Hilbert - maybe the best mathemation in the world at that time had also thrown

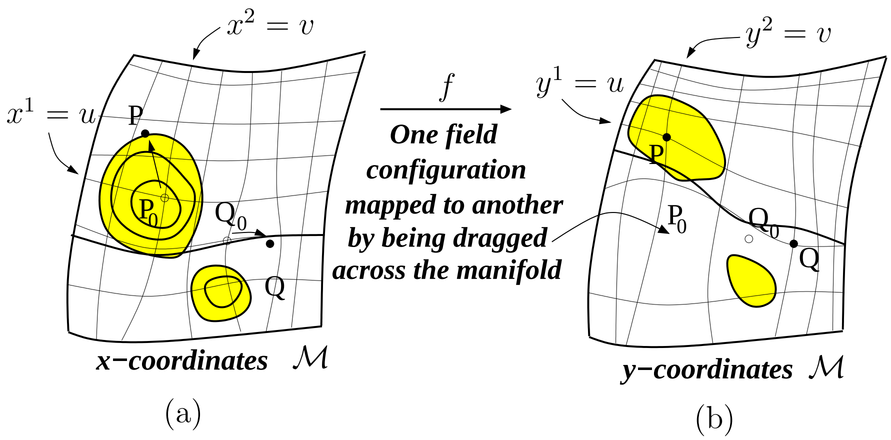

Figure 1.8: (a) An active diffeomorphism in which we actively drag the tensor function over the, in doing so indentify one point of the manifold to another. (b) We then go to a coordinate system which asigns the newly identified points their origonal coordinate values. That is to say - we carry the tensor function over the manifold, keeping the coordinate lines 'attached'.

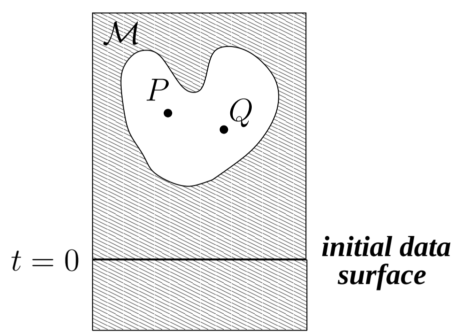

Figure 1.9: Einstein's hole argument.

himself into the problem), looking for *non-genrally covariant* field equations. If you were to look up Einstein's 1912 paper *Outline of a Theory of Gravity* you would find that he claims that the theory of gravity must be expressed in terms of the metric $g_{\mu\nu}(x,t)$, that the right-hand side must depend on the energy-momentum tensor, that the left-hand side must depend on $g_{\mu\nu}(x,t)$ and its derivatives up to second order, and that these *equations must not be generally covariant!*

The Resolution of the Hole Argument (1915)

In 1915 Einstein formulated his final field equations, however Einstein had changed his mind - they were generally covariant. What had happend? Well, he had realized that there was a mistaken assumption about the nature of spacetime and in dropping this assumption there would no longer be any incompatibility between general covariance and

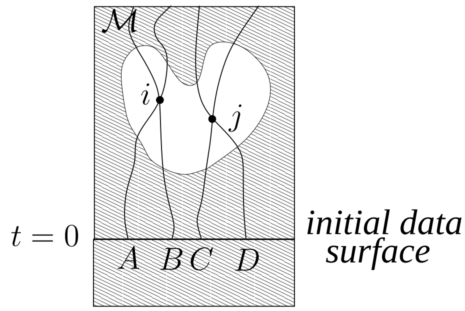

Figure 1.10: Resolution of Einstein's hole argument.

determinicity. To understand this let us see how the hole argument was resolved. The idea was to define locations using physical objects, for example particles. [1] Consider the arrangement in fig (1.1). We have four particles labeled by A, B, C and D. The particles A and B intersect in i and similarly the particles C and D intersect in j. These particles start at the initial surface and their geodesics are found by solving the equations of motion. After we have performed the active diffeomorphism we need to solve to find the geodesics for the new metric $\tilde{g}_{\alpha\beta}$. The distances between such defined locations *is deterministic*. This is because the tragectories are dragged across together with the metric by the active diff transformation. This is because we solve to find the geodesics for the transformed metric. A deterministic quantity is the distance between the two particles in fig **??**. So physical geometry is invariably define with respect to matter degrees of freedom (or in principle using degrees of freedom of the gravitational field itself).

What Einstein construed from the solution of the hole argument is that it is meaningful to refer to a location as a place where two freely falling particles intersect; however it is not meanigful to refer to a location as a point in spacetime (a spcetime event) because the distance from one such point to another is in undetermined in GR. That is spacetime points have, in themselves, no physical significance.

Remark
It is important to realize that the details of the Einstein-Hilbert action the fact that the fundemental physical theory is background invariant. You can add new terms to the Einstein-Hilbert action to change the theories high energy behaviour, but if these terms are invariant under coordinate transformations the resulting theory is also background independent. The failure to find such a theory indicates that background independence is an essetial ingredient and must be faced squarely.

[1] The lesson to be learned from the hole argument doesn't depend on whether or not the physical objects affect the gravitaion field or not. The important point is that physical objects move along geodesics. So for simpicity we consider only test particles.

1.2 Background Independence - A Farewell to Spacetime

1.2.1 Comparision of GR with the Rubber Sheet Analogy

There is another limitation of the analogy that is not so widely recognised and is of much more physic importance: the rubber sheet dynamic's is played out in space and evolves with respect to absolute time. Spacetime geometry however is not embedded in a further container and does not evolve with respect to some externaly provided time - there is no a priori given areana in which the dynamics of spacetime geometry is played out. In a sense the gravitation is its own areana! As such it is only meaningful to talk about relations between some degrees of freedom and other degrees of freedom of the gravitational field, or if matter is present, relations between matter degrees and the degrees of freedom of the gravitatoin degrees of freedom. What general relativity actually predicts is correlations between mearsurable quantities.

Spacetime has no character independent of observation - the way we experiance the world is only through observation and measurment. When the dynamics of gravity can be ignored the gravitational field gives rise to , through the observations that we make,the appearance a background spacetime.

How can field theory not be defined on a spacetime. Spacetime may be curved and change with time but in GR things still move on spacetime; fields and particles have dynamics on a curved spacetime. Physics on a curved spacetime is not GR! A dynamical theory of spacetime which is also generaly covariant is background independent! (the reader should note this conculsion of background independence that in the argument we didn't need to specify the exact form of the field equations).

Not taught so as to not overburden or confuse the student with conceptual difficulties raised by this - or teachers lecturers are not aware of this .

1.2.2 The View of the World that Emerges

The View of the World That Emerges

It is meaningless to talk of the geometry of spactime (in the absence of dynamical entities) as if it were an entity having independent existence. When one makes a measurement of *physical "geometry"* one is making a measurement of a certain aspect of the relationships that exists between physical objects that live in the world - there is no "geometry" without matter! This idea is not new and goes back to the times of Aristotle and Decartes; they advocated that space is an abstraction of the fact that some parts of matter can be in touch with others (see Descartes [81]). Of course, Aristotle and Decartes were knew nothing of relativity and the bringing together of space and time.

Einstein's modern day version is not motivated by mere philosophical considerations but lies at the basis of a physical theory, for which there has been obtained spectacular empirical support - binary pulsar's period decay due to gravitational radiation, discovery of black holes in the sky, the existence of an exanding universe.

Einstein's modern day version goes a step further; what the reader might find more disturbing is that, GR is also telling us that there is no time either! Which begs the question how do we describe dynamics without reference to time or evolution!?

This picture of spacetime is fundementally different from Newton's (and Minkowski's spacetime) notion. Newton introduced the idea of physical space as an idependent entity because he needed it for his dynamical theory. The Newtonian picture of the world is a background space on which on which matter moves. Points exist irrespective of whether there is matter present or not.

Not just point particles. It generalizes to fields on the spacetime manifold. Any two sytems related by an active diffeomorphism are physically indistinguishable. A physical system is not described by a field configuration (or by the location of the particles), but rather by the equivalence class of field configurations (and particle locations), related by all diffeomorphisms.

In GR a point in spacetime becomes an abstract notion which in itself has no physical meaning. This is the resolution of Einstein's struggle to understand "the meaning of the coordinates".

When spacetime is dynamical, general covariance is formally equivalent to background-independence.

How the field is localized over the spacetime manifold has no physical significance. In general relativistic physics, the "location" of physical objects and physical fields is not determined with respect to a pre-existing space. Physical are only "located" with respect to each other. Physical meaning lies in the relationships between fields. The spacetime manifold has turned out to be a conveinent mathematical device, devoid of any physical meaning.

A spacetime point and the electromagnetic field at Andromeda and drag it to the room your are in - amazingly it should represent the same physical situation!! No longer can we think of a point of the spacetime manifold as a place "where" things happen.

It is meaningless to talk about one spacetime point being causally related to another!! No background causal structure.

It was only at this point, in 1915, that GR was born. It was in making this final step that Einstein remarked ***"beyond my wildest expectations"***. So as the reader doesn't think this is all being made up we quote Einstein's own words

"All our spacetime verifications invariably amount to a determination of spacetime coincidences. If, for example, events consisted merely in the motion of material points, then ultimately nothing would be observable but the meeting of two or more of these points."

Spacetime geometry has no meaning independent of observations.

Spacetime measutements and gravitational experiments are made by using objects, matter fields or particles and their mutal relationships.

mutal relationships between gravitational field and objects, matter fields or particles are preserved under active diffeomorphisms.

quantum mechanics must undergo the same deep transformation that classical special relatvistic mechanics had to undergo in jumping to generel relativistic physics: In general realtivisic physics, the "location" of physical objects and physical fields is not determined with respect to a preexisting space.

Einstein's Hole Argument: Compact Form

One point gets associated with another point. Go to a coordinate system which asigns this newly identified point the same coordinate values that the old point had in the x-coordinates. That way $\tilde{g}$ in the y-coordinates, i.e. $\tilde{g}_{ab}(y)$, is the same function g is in the x-coordinates , i.e. $g_{ab}(x)$. That is $g_{\mu\nu}(x) = \tilde{g}_{\mu\nu}(x)$. Now, general covariance demands that the EOM be the same in both coordinae systems. So we have exactly the same differentiation equation in both coordinate systems. Therefore, if one of the metric functions is a solution then so is the other. Now consider the situation in fig() with initial data and metric solution $g_{ab}(x)$. Let us perform an active diff transformation which reduces to identity outside the hole. We obtain another distinct solution $\tilde{g}_{ab}(x)$ with the same initial conditions. The upshot is if we demand physical theories to be general covariant, GR does not determine the distance between to spacetime points.

The resolution to the

"Geometry up to Diffeomorphisms"

– – – – – – – – – –

1.2.3 Common Misunderstandings

Often the general relativist will use terms which have a different meaning to many people in the rest of the physics community, leading to much confusion.

When a general relatvist referes to diffeomorphisms they are most likely referring to active diffeommorphisms and not passive diffeomrophisms (if they are using the coordinate-free geometry formulism then the only diffeomorphisms are active diffeomorphisms!)

When it is said that GR is invariant under diffeomorphisms, it is meant that the theory is invaraiant under active diffeomorphisms. These are the gauge transformations of GR and they should not be confused with the freedom of chosing coordinates on the space-time M. Invariance under coordinate transformations is not a special feature of GR, all physical theories are invaraint under coordinate transformations!

It is sometimes stated that an active diffeomorphism is just a coordinate transformation viewed differently. This is misleading, consider a non-uniform translation in Minkowski spacetime. Under a passive transformation the resulting spacetime is, of course, still Minkowski but under the active transformation the resulting spacetime is no longer Minkowski. (Under a uniform translation the active transformation results in Minkowski spacetime but this is only because of the homogeneity of Minkowski spacetime).

People should be aware of the differing use of the term general covariance. The principle is defined as the condition that the equations of motion should take the same form in all coordinate systems. However, when a general relatvist says that GR is a generally covariant theory they are not emphasing that it is invaraint under general coordinate transformations but rather that the theory is background independent as a direct consequence of coordinate invariance.

Even though the full content of GR is that it discards the very notion of space-time, a general relatvist may continue to use the terms "space" and "time" but it should be understood that they do so only to indicate certain aspects of the gravitational field.

1.2.4 The Blessing of background independence - Non-Peturbative Quantum Gravity Finite and Requires No Renormalization!

The reason is fairly simple:

There is *no* background metric to provide scales so there can be no UV-divergences in observable quantities arising from indefinitely small distances as happens in ordinary, *background-dependent* field theories!!!

or put another way

The absence of the divergences that usually plague interacting field theories in a Minkowskian background spacetime can be understood intuitively from the diffeomorphism invariance of the theory "short and long distances are gauge equivalent".

quote a more precise argument

???? built from the product of a pair of field operators evaluated at a single point, it is not well-defined. In this scheme, one introduces an artificial separation of the single point to a pair of closely separated points and . The problematic terms involving field products such as $\hat{\phi}^2(x)$ becomes $\hat{\phi}(x)\hat{\phi}(x')$, whose expectation value is well defined. If one is interested in the low energy behavior captured by the point-defined quantum field theory one takes the coincidence limit. Once the divergences present are identified, they may be removed (regularization) or moved (by renormalizing the coupling constants), to produce a well-defined, finite stress tensor at a single point.????

" A background independent operator must always be finite. This is because the regulator scale and the background metric are always introduced together in the regularization procedure. This is necessary, because the scale that the regularization parameter refers to must be described in terms of a background metric or coordinate chart introduced in the construction of the regulated operator. Because of this the dependence of the regulated operator on the cuttoff, or regulator parameter, is related to its dependence on the background metric. When one takes the limit of the regulator parameter going to zero one isolates the non-vanishing terms. If these have any dependence on the regulator parameter (which would be the case if the term is blowing up) then it must also have dependence on the background metric. Conversely, if the terms that are nonvanishing in the limit the regulator is removed have no dependence on the background metric, it must be finite. "

The field is specified against a backgroand spacetime, assigning for the components of the electric and magnetic fields, E_i, B_i. The uncertainity relations apply to physical, observable quantities, such as the position and momentum of a particle, or the values of the magnetic and electric fields.

The proper length between two abstractly defined spacetime points, and nor are areas of a surface or the volume of a region specified using coordinates.

The idea of quantum fluctuations of spacetime points, a naive yet popular conception of what a quantum theory of gravity might entail.

When the dynamics of gravitational field can be neglected, we recover the background dependent matter field theories of particle physics.

The conventional mathemaical formulism of quantum field theory relies very much on the existence of a background space time. take general relativity seriously reconstruct quantum field theory from scratch in a form that does not require background space.

Observables of Quantum Gravity are Finite

The regularization parameter can often be derictly interpreted as the distance between spacetime points.

Einstein's theory can be stated as a variational problem: one takes a manifold (the manifold is simply a blank background upon which we place the metric to put the familiar features of space and time into it. We try all possible assignments of metrics to the manifold to find those that minmise the Einstein-Hilbert action - these are the solutions to

Einstein's equations. Nowhere in the action does there appear a fixed background metric, or any fixed geometric structute at all.

Einstein's theory can be recast as a Yang-Mills theory. This Yang-Mills theory exists without reference to a background metric. Such a theory makes no disticntion between small and large distances, as described by a background metric; take the same coordinate system but introduce two distinct metrics. According to one metric the proper distance betwen two points might be small and the other the proper distance between these two points might be large. The Yang-Mills theory is blind to either metric and as a result, one can argue that, such a *background independent* quantum theory will not suffer from UV divergences.

By introducing a point-splitting regularization. If the end result has no memory of which particular metric we used we wont have broken spatial diff invariance. In General relativity the proper distance between two abstractly defined points, has in itself, no physical meaning and so we should not expect UV-divergences to occur in physical quantities as we remove the regulator by sending $\epsilon \to 0$.

By replacing point particles with strings, a graviton interactions are "smeared" such that there is now no localized interaction point anymore. However this simple argument there is no localized interaction point for the simple reason that (according to classical GR) spacetime points have no independent physical reality! Say we have some object at some position in the spacial manifold. According to GR its position can have no physical meaning. Background independent theories smeare themselves! Rather singular functions 1.2.4

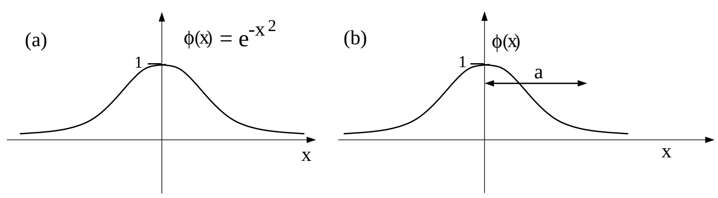

Figure 1.11: Illustration of smearing. operator valued distributions.

(operator counterparts of distributions reffered to as **operator valued distributions**, but we wont get into the details of this here, see apendix ??). The location of this object in space has no physical meaning and we can remove this dependency be averaging the position of the object over the whole spacial manifold; in doing so no position is favoured over anyother. This has the effect of smearing the oject over the whole spacial manifold - the resulting mathematical object is much more regular and does not give rise to UV-divergencies. These vague argunents have been made precise in [**?**] and our hopes confermed to be true??. We do not need the smearing affects of strings over point particles, back-ground independent field theories *smear themselves*! In [] the UV-divergencies come back when background dependence is reintroduced.

It is a "topological field theory " but with "local degrees of freedom" and as such knows nothing about rulers or clocks.

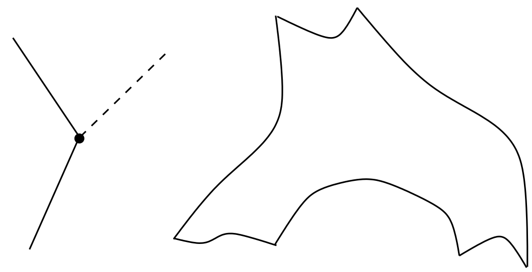

Figure 1.12: Regime where gravity is very strong so that the non-perturbative and background independence of GR must be taken into account. That spacetime points have no independent physical reality casts doubt on the hand-wavey argument I gave above.

There is no longer a clear cut distinction between physical objects that form the reference system (laboratory walls) and physical objects whose dynamics we are describing.

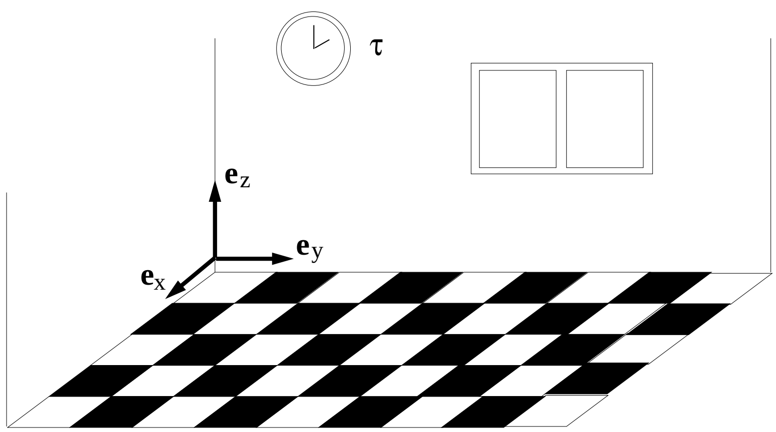

Figure 1.13: Labortary walls exemplify Newton's absolute space and the clock absolute time. We can define positions relative to the wall.

1.3 Physical Geometry

The kinematic phase space expresses all potential outcomes of measuments of partial observables. Dynamics is a restriction on on these configurations and expresses the existence of correlations among measurements of partial observables.

[279]

> To measure t we use clocks. A clock is a system with a variable, for instance the position of a hand, which has a simple behaviour in t. In this

paper, we shall denote a clock variable (the position of the hand) as T; we shall denote varaibles of different clocks as $T, T', T'', \ldots$. Good clocks may have, for instance, linear behaviour in t:

$$T(t) = \alpha t. \tag{1.35}$$

It is an elementary-physics-course observation that we never really measure t; rather, we always measure $T's$. The value of a physical quantity Q, measured at time t, denoted $Q(t)$. Since time is determined by measuring a clock varaible T, what is actually measured is not $Q(t)$, but only the combined quantity $Q(T)$. Thus, t does not ever appear in laboratory measurements.

What we learn from GR is that coordinates do not have any meaning independent of observations, invloving clocks and light signals. According to GR, a coordinate system is defined only by explicity carrying out space-time measurements.

In the theoretical analysis of an experiment, one (arbitrary) coordinate system $\vec{x}, x^o$, and then equations of motion, as (M.-19) with the data in the following way. First we have to locally solve the coordinates $\vec{x}, x^o$ with respect to quantities $f_1 \ldots f_4$ that represent the physical objects used as clocks and as spatial reference system

$$f_1(\vec{x}, x^o) \ldots f_4(\vec{x}, x^o) \rightarrow \vec{x}(f_1, \ldots \ldots, f_4), x^o(f_1, \ldots, f_4) \tag{1.36}$$

and then express the rest of the remaining fields (f_i $i = 5 \ldots N$) as functions of $f_1 \ldots f_4$

$$f_i(f_1, \ldots, f_4) = f_i(\vec{x}(f_1, \ldots, f_4), x^o(f_1, \ldots, f_4)). \tag{1.37}$$

If, for instance, $F(\vec{x}, t)$ is a scalar, then for every quadruplet of numbers $f_1 \ldots f_4$, the quantity $F_{(f_1 \ldots f_4)} = F(f_1, \ldots, f_4)$ can be compared with experimental data. This procedure is routinely performed in any analysis of experimental gravitational data - the physical time f_4 representing the the reading of the laboratory

1.3.1 Physical GPS Coordinates

It is difficult to discuus observables without some pressuposed notion of time; we pressume, the undeniable, existense of clocks. We must at some point verify such variables emerge from the timeless, fundemental description in the suitable physical regime.

$$\begin{aligned} ct_P &= \frac{1}{2}[c(t_A + t_B) + (x_B - x_A)], \\ x_P &= \frac{1}{2}[c(t_A - t_B) + (x_B + x_A)] \end{aligned} \tag{1.38}$$

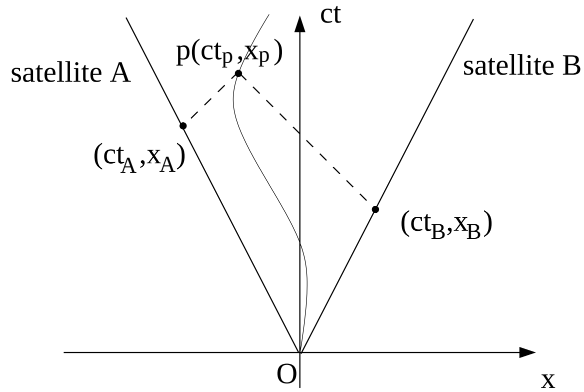

Figure 1.14: GPS.

1.3.2 Physical Area

Geometric quantities are the observables of the gravitational field. When the gravitational field is quantized, geometric observables will be represented by operators. We then try to compute their spectrum and eigenstates.

Naive formular:

$$
\begin{aligned}
\| d\xi^1 \times d\xi^2 \| &= \| d\xi^1 \| \, \| d\xi^2 \| \sin\theta \\
&= \sqrt{\| d\xi^1 \|^2 \| d\xi^2 \|^2 (1 - \cos^2\theta)} \\
&= \sqrt{\| d\xi^1 \|^2 \| d\xi^2 \|^2 - (\| d\xi^1 \cdot d\xi^2 \|)^2} \\
&= \sqrt{|q_{11} q_{22} - (q_{12})^2|} \, d\sigma_1 \, d\sigma_2 \\
&= \sqrt{\det q} \, d^2\sigma
\end{aligned}
\tag{1.39}
$$

$$
A(S) = \int d^2\sigma \sqrt{\det q} \tag{1.40}
$$

$$
q^S_{uv} = \frac{\partial x^\alpha}{\partial \sigma^u} \frac{\partial x^\beta}{\partial \sigma^v} q_{\alpha\beta} \tag{1.41}
$$

$$
n_a = \frac{1}{2} \epsilon^{uv} \epsilon_{\alpha\beta\gamma} \frac{\partial x^\beta}{\partial \sigma^u} \frac{\partial x^\gamma}{\partial \sigma^v} \tag{1.42}
$$

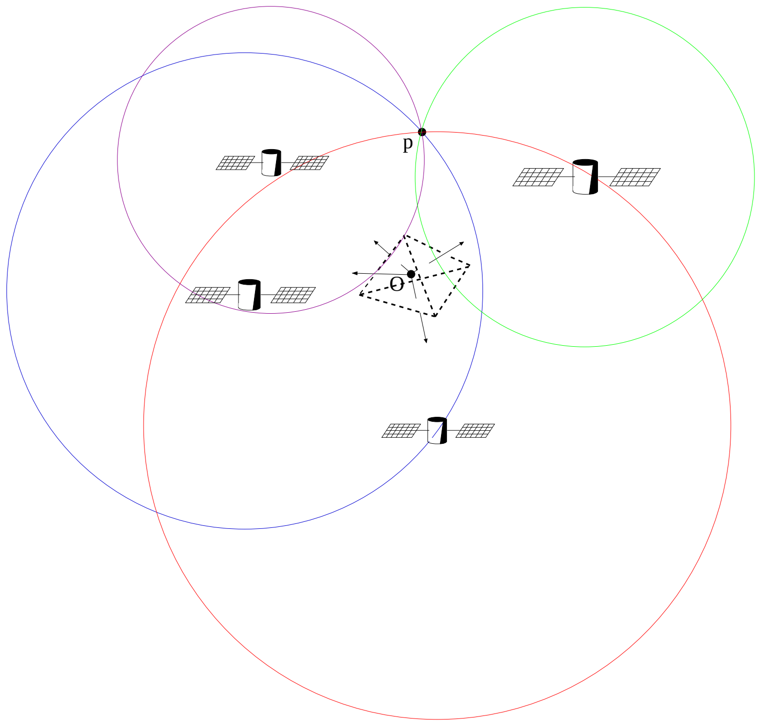

Figure 1.15: GPS3D A spacetime point in Minkoski spacetime can be expressed as a relation amogst 4? measurable variables. This definition of spacetime location retains meaning in the jump to GR.

$$
\begin{aligned}
A[S] &= \int_S d^2\sigma \sqrt{\det q} = \sqrt{\frac{1}{2}\epsilon^{uu'}\epsilon^{vv'}q^S_{uv}q^S_{u'v'}} \\
&= \int_S d^2\sigma \sqrt{n_\alpha n_\beta \tilde{E}^{\alpha i}\tilde{E}^{\beta j}}
\end{aligned}
\tag{1.43}
$$

1.3.3 Description of a Measurement of Area

i. the gravitational field,

ii. two particles,

iii. a two dimensional surface (the "table").

The simultaneity surface Σ is the set of points in $\mathcal{M}$ whose light cone intersects X in two points at the same proper time distance along X from P.

We also assume other physical objects exist light pulses travelling along geodesics, apparatus that detect the arrival of light pulses, clocks that measure proper time along world

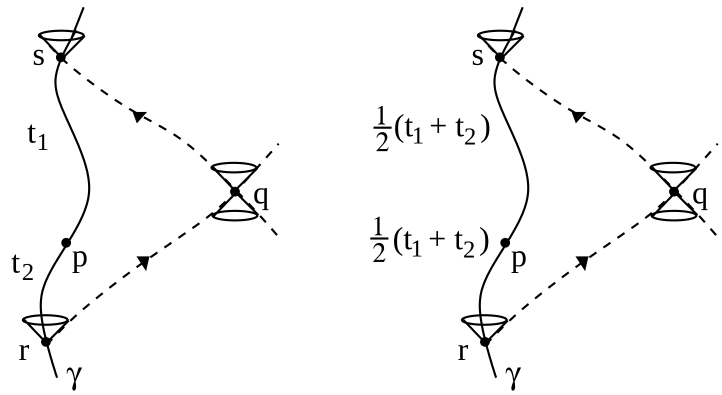

Figure 1.16: clock time.

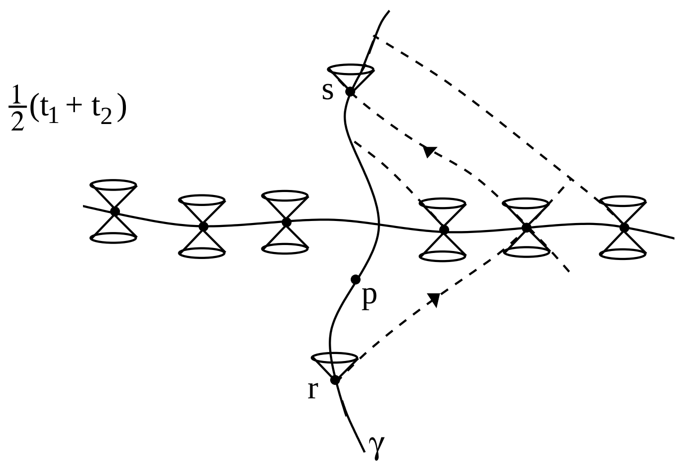

Figure 1.17:

lines, recording devises and so on. We do not consider these other physical objects as part of the dynamical system observed

it is invaraint under coordinate transformations that preserve the coordinate choice made??? what about acttive diffs??

The intersection between the surface of simultanity Σ of the observer and the table world history S is a two dimensional surface $S = \Sigma \cap T$.

A Coordinate choice

In our preffered coordinates :

(i) P is the origin,

(ii) The 3-surface T is defined by $-1 < x^1 < +1$, $-1 < x^1 < +1$ and $x^3 = 0$.

(iii) The world line X is defined by $x^1 = x^2 = x^3 = 0$

No physical quantities will not depend upon which coordinates we choose, just as in electrodynamics where the electric $E_i(x)$ and magnetic $B_i(x)$ fields dont depend on which particular gauge potential $A_\mu(x)$ we choose to calculate $E_i(x)$ and $B_i(x)$. Let us on this

1.3.4 Einstein's Field Equations

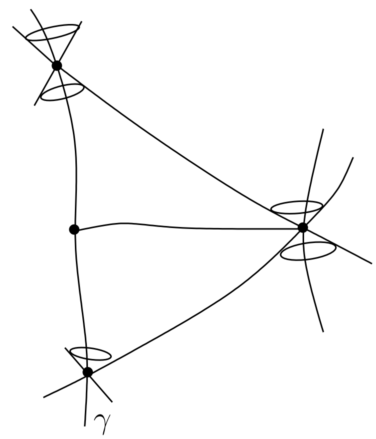

Figure 1.18: measLocation.

Measurement of Relative Velocities

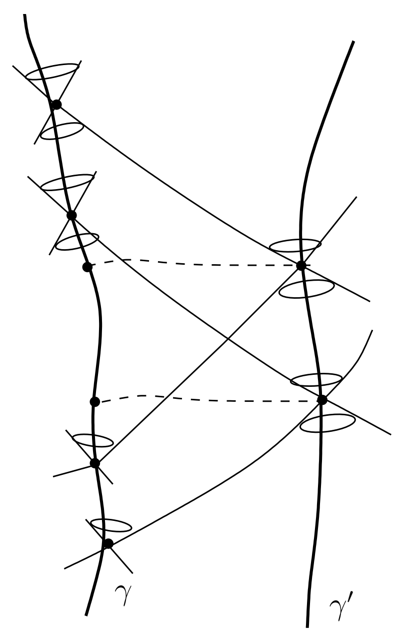

Figure 1.19: measVelocity.

1.3.5 The velocity composition law

1.3.6 Dust as Matter Reference System

1.4 Some conceptual issues

There are suprising features of classical and quantum general relativity that take some getting used to. Here we present a some for the reader to ponder over. On a first reading the reader may want to skip this section and move straight to Connections verses Metrics. The talks are more focussed on techniqual issues - although of course the conceptual and techiqual interplay.

Since there is no background space-time metric, what does "time evolution" mean? the theory does not have a unique inert notion of time, fundemental ingredient in quantum mechanics. The predictons of classical general relativity do not depend explicitly on the coordinate time t. What the theory predicts are correlations between physical variables, not the way physical variables evolve with respect to a preffered time coordinate t.

This is a general property of generally covariant theories. $\mathcal{H} = 0$ - has to do with the fact that there is no physical meaning physical notion of time in GR. We illustrate ths with a simply example: we have a non-relativistic particle. We will put in a gauge symmetry by hand

$$\begin{aligned} S[q(t+\epsilon)] - S[q(t)] &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t}\epsilon + \frac{\partial \mathcal{L}}{\partial q^i}\dot{q}^i \epsilon + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\frac{d}{dt}(\dot{q}^i \epsilon)\right] dt \\ &= \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial t}\epsilon + \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)\frac{d\epsilon}{dt}\right] dt \\ &= [\mathcal{L}\epsilon]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \mathcal{L}\right)\frac{d\epsilon}{dt}\, dt \end{aligned} \tag{1.44}$$

Parameterization-invariance means that the integral must zanish for arbitrary $d\epsilon/dt$, so that we have

$$\boxed{\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\dot{q}^i - \mathcal{L} = 0.} \tag{1.45}$$

It is a general result that the Hamiltonian zanishes for invariant under parameterization-invariance.

There is an extensive literature on this subject. We will just settle by refering to C.Isham's paper "The problem of time in conanical quantum Gravity" and by quoting Rovelli, [36] gr-qc/0306059

"...In nonrelativistic physics, time and position are defined with respect to a system of reference bodies and clocks that are implicititly assumed to exist and not to intetract with the physical system studied. In gravatational physics, one discovers that no body or clock exists which does not interact with the gravitationalfield: the gravitaional field affects directly the motion and rate of any reference body or clock. Thereforeone cannot separate reference bodies and clocks from the dynamical variables of the system. General relativity - in fact, any general covariaant theory -is always a theory of interacting variables that necessary include the physial bodies and clocks used as references to characterize spacetime points. "

Our notion of time is meaningless in a generic situation. The features of time is not present at the fundemental level, rather it "emerges" as features of the semi-classical limit, (same sort of idea of how the notion of temperature is an emerges in the thermodynamical limit, but is meaningless at the fundemental level of molecules). How does this come about? Well, what the theory does is it describes relative location and relative evolution of dynamical objects. Pick observable degrees of freedom that act as a "clock" and find its correlations to some other set of measurable degrees of freedom. Then by changing the reading of the clock and observing the changes in the other measurable quantities, we can, *to some approximation*, recover our usual notion of mechanics defined by evolution in fixed background time. The notion is meanigless in a generic situation.

There is no longer a clear cut distinction between physical objects that form the reference system (laboratory walls) and physical objects whose dynamics we are describing.

An active spatial diffeomorphism $f : M \to M$ relates different objects in M in the same coordinate syatem. f is viewed as a map that associates one point in the manifold to another one. GR is the only theory of nature that is invariant under active difeomorphisms. It is this invariance that allows for the possibility of quantum gravity to be finite.

1.5 Relational Mechanics

1.5.1 Covariant Hamiltonian Formulation

dynamics fixes relations among vaiables, so that knowing some of them we can predict the others. The absence of any preffered time vairable. The relation aspect of evolution.

learn

rephrase it in the language of the presymplectic formulation of a conventional system, and from here, extend it to presympectic systems that do not correspond to a conventional system.

Parametrized Harmonic Oscillator

In its usual presentation, classical mechanics appears to give time a special role. However, mechanics can be formulated so as to treat the time variable on the same footing as the other variables in an extended phase space. Phase space variables being on the same footing.

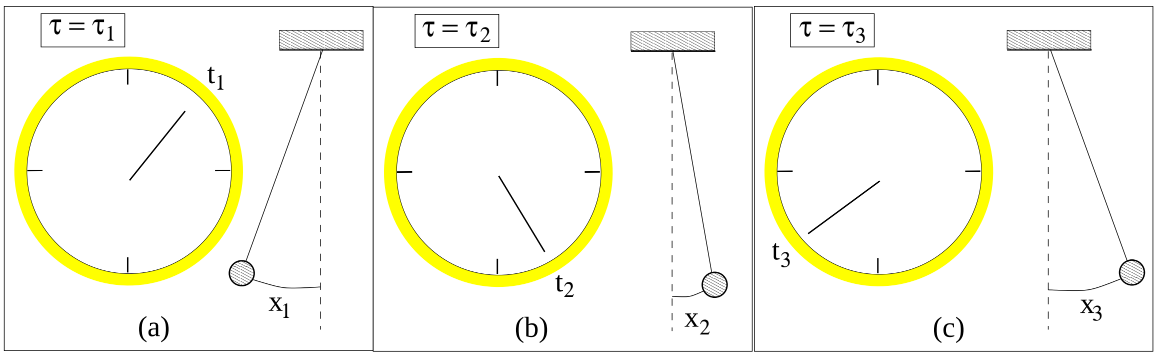

Figure 1.20: partialobs. τ is an unphysical parammeter labelling different possible correlations between the time reading t of the clock and the elongation x of the pendulm.

$$x(\tau), \qquad t(\tau) \tag{1.46}$$

The role of the coordinates (and in particular of the time coordinate) can be clarified by means of this analogy. The coordinates have the same physical status as the arbitrary parameter τ that we use in order to label and distinguish the set of relations between the reading on the clock and the elongation of the pendulm. The 'time coordinate' in GR has no physical meaning and serves only to label and distinguish points of space-time and is a mere matter of convenience.

[82]

$$S = \int d\tau \left[\frac{dx}{d\tau} p + \frac{dt}{d\tau} p_t - \lambda \left(p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \right) \right] \tag{1.47}$$

$$\begin{aligned} \frac{dp}{d\tau} &= -\lambda m\omega^2 x, & \frac{dx}{d\tau} &= \lambda \frac{p}{m}, \\ \frac{dp_t}{d\tau} &= 0, & \frac{dt}{d\tau} &= \lambda. \end{aligned} \tag{1.48}$$

variation with respect to λ gives the first class constraint. These constraints form a first-class system, which means that the Poisson bracket of two of them is again a linear combination (generally with phase-space dependent coeffcients) of the, any other constraints are called second class, see Apendix F for more on constrained Hamiltonian system.

$$\mathcal{C} = p_t + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = 0 \tag{1.49}$$

$$\frac{dx}{dt} = \frac{dx}{d\tau} \Big/ \frac{dt}{d\tau} = \frac{\lambda p/m}{\lambda} = \frac{p}{m} \tag{1.50}$$

$$\frac{dp}{dt} = \frac{dp}{d\tau} \Big/ \frac{dt}{d\tau} = \frac{-\lambda m\omega^2 x}{\lambda} = -\omega^2 x \tag{1.51}$$

or

$$p = m\frac{dx}{dt}, \quad m\frac{d^2x}{dt^2} = -m\omega^2 x. \tag{1.52}$$

What we have done is promoted the time to a dynamical variable. This has conjugate momentum which we denote p_t. The original degrees of freedom and time are left as functions of some physically irrelevant parameter τ (physically meaningles as it does not correspond to a measurable quantity - just like the time coordinate in GR). Time t can be considered independent of the other degrees of freedom when a Lagrangian multiplier term is added to the action. Invariance of this extended Hamiltonian under variations of τ implies that it is constrained to vanish, (just as in the case of GR)).

have

$$\begin{aligned} x(\tau) &= A\cos(\omega\tau) + B\sin(\omega\tau), \\ t(\tau) &= \tau, \\ p(\tau) &= -m\omega A\sin(\omega\tau) + m\omega B\cos(\omega\tau), \\ p_t(\tau) &= -\frac{1}{2}m\omega^2(A^2 + B^2), \end{aligned} \tag{1.53}$$

as a solution, where (A,B) are the physical observables coordinatize the *physical* phase space, $\Re$

$$\begin{aligned} A &= \cos(\omega t)x - \frac{1}{m\omega}\sin(\omega t)p, \\ B &= \frac{1}{m\omega}\cos(\omega t)p + \sin(\omega t)x. \end{aligned} \tag{1.54}$$

Note $A = x(t = 0) \equiv x_0$, and $B = \frac{p(t=0)}{m\omega} \equiv \frac{p_0}{m\omega}$, i.e. the position x_0 of the harmonic oscillator when the internal clock measures $t = 0$ physical observables.

The space-time arguments x and t are *not observables* exactly like the τ of the parametrized description of a pendulm and clock.

1.5.2 Depamerizable Mechanics: Identification of a "time" variable

If $q^a = (t, q^i)$ and

$$\mathcal{C} = p_t + \mathcal{C}_0(p^i, q^i) \tag{1.55}$$

then

$$S = \int d\tau \left(p_t \frac{dt}{d\tau} + p_i \frac{dq^i}{d\tau} + N(p_t + \mathcal{C}_0(p_i, q^i)) \right) \tag{1.56}$$

varying N:

$$p_t = -\mathcal{C}_0 \tag{1.57}$$

substituting this into the action

$$S = \int d\tau \left(p_i \frac{dq^i}{d\tau} - \mathcal{C}_0 \frac{dt}{d\tau} \right) \tag{1.58}$$

$$S = \int d\tau \frac{dt}{d\tau} \left(p_i \frac{dq^i}{dt} - \mathcal{C}_0 \right) \tag{1.59}$$

$$S = \int dt \left(p_i \frac{dq^i}{dt} - \mathcal{C}_0(p^i, q^i) \right) \tag{1.60}$$

it must be true that a canonical transformation can be made on the phse space such that the Hamiltonian constraintcan be written in the form (1.55).

GR cannot be *deparamitized*! This is because there is no unique notion of time

The absence of any preffered time vairable

1.5.3 Fully Constrained Hamiltonian Systems

true observables are composed of correspondences between partial observables, one of which is the reading of a clock.

One parameter families of that are given by the same partial observers at different clock feadings.

Presympletic phase space

The space of partial observables is the extended configuration space $\mathcal{C}$, and the dynamics is governed by a vanishing Hamiltonian $\mathcal{H}$ on the phase space associatedd with $\mathcal{C}$ (denoted $T^*\mathcal{C}$).

$$S[q^i, p_i, \lambda^m] = \int_{\tau 1}^{\tau 2} d\tau \left\{ \frac{dq^i}{d\tau} p_i - \lambda^m \mathcal{C}_m(q^i, p_i) \right\}, \tag{1.61}$$

which is invariant arbitrary reparametrizations of the parameter τ. The parameter is unphysical and unobservable, like the time coordinate in general relativity. The unreduced, or extended phase-space γ_{ex} is coordinatized by the canonical pairs (q^i, p_i); $i = 1, 2, \ldots, N$.

The variation of the action with respect to the canonical coordinates q^i, p_i gives the equations of motion

$$\frac{dq^i}{d\tau} = \lambda^m \frac{\partial \mathcal{C}_m(q^i, p_i)}{\partial p_i}, \tag{1.62}$$

$$\frac{dp_i}{d\tau} = \lambda^m \frac{\partial \mathcal{C}_m(q^i, p_i)}{\partial q^i}. \tag{1.63}$$

while variation with respect to the Lagrange multipliers λ^m (where the index m labels the M constraints) gives us the constraint equations

$$\mathcal{C}_m = \mathcal{C}_m(q^i, p_i) = 0, \quad m = 1, 2, \ldots, M \tag{1.64}$$

In the example above $q^1 = x$, $p_1 = p$, $\lambda^1 = \lambda$ and $\mathcal{C}_1 = \mathcal{C} = p_t + \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$

The canonical 2-form on γ_{ex}

$$X_{d\mathcal{C}_m} = -\frac{\partial \mathcal{C}_m}{\partial p_i}\frac{\partial}{\partial q^i} + \frac{\partial \mathcal{C}_m}{\partial q^i}\frac{\partial}{\partial p_i}. \tag{1.65}$$

The pair $(\Gamma_{ex}, \omega_{ex})$ forms a symplectic space.

The first class constraints satisfy, in general, a "non-Lie" algebra

$$\{\mathcal{C}_m, \mathcal{C}_n\} = C^l_{mn}(q^i, p_i)\mathcal{C}_l, \tag{1.66}$$

and the number of independent physical degrees of freedom of the theory is $D = N - M$. The constraint surface γ of γ_{ex} defined by the constraint equations (1.64) is a $(2D + M)-$dimensional manifold. The restriction ω of ω_{ex} to the consraint surface γ is of rank $2D$. The M null directions of ω are the infinitesimal transformations generated by the constraints. They define the gauge orbits on γ.

The space (γ, ω) is a presymplectic space, which contains the full dynamical information about the system. Hence dynamical systems in this form are also called "presymplectic systems". γ can be parameterized by the set of independent coordinates $(\tilde{q}^a, \tilde{p}_a, t^m)$, where $(\tilde{q}^a, \tilde{p}_a)$, $a = 1, 2, \ldots, D$ are canonical variables that coordinatize the physical phase space γ_{ph}, and t_m, $m = 1, 2, \ldots, M$ coordinatize the orbits. In general this coordinatization can hold only locally, and different charts may be needed to cover the entire space.

where t is the coordinate in $\mathbb{R}$, and corresponds to the external time variable. The difference between the conventional formulation and the presymplectic formulation is only in the fact that this time variable is treated on the same footing as the other variables. As a concrete example, we may imagine that H is the harmonic oscillator Hamiltonian describing the small oscillations of a pendulum, while t is the reading of a physical clock. Then the presymplectic system (??) describes how two equal-footing physical variables (the pendulum amplitude and the clock reading) evolve with respect to one another. In general covariant systems, such as any general relativistic system, this 'equal footing' status between all physical variables is an essential feature of the theory. It expresses the major physical discovery of general relativity: the complete relativity of spacetime localization.

Note that the canonical coordinates $\tilde{q}^a$, and $\tilde{p}_a$ are the physical observables of the system. They are gauge-invariant. They satisfy $\{\tilde{q}^a, \tilde{p}_b\} = \delta^a_b$ on the physical phase space. In these coordinates, the physical symplectic form on ω_{ph} is $\omega_{ph} = d\tilde{p}_a \wedge \tilde{q}^a$. The p q general solution of the equations of motion is simply given by the embedding equations of the orbits in γ_{ex}, that is

$$q^i = q^i(t^m; \tilde{q}^a, \tilde{p}_a), \tag{1.67}$$
$$p_i = p_i(t^m; \tilde{q}^a, \tilde{p}_a). \tag{1.68}$$

Each set $(\tilde{q}^a, \tilde{p}_a)$ determines a solution; along each solution, the quantities (q^i, p_i) depend on the M parameters t^m (instead than just on a single time variable) because of the gauge freedom in the evolution. The inverse relations of (??)-(??) give the dependence of the physical observables $\tilde{q}^a$, and $\tilde{p}_a$ from the original coordinates

$$\tilde{q}^a = \tilde{q}^a(q^i, p_i), \tag{1.69}$$
$$\tilde{p}_a = \tilde{p}_a(q^i, p_i), \tag{1.70}$$

as well as the orbit coordinates t^m

$$t^m = t^m(q^i, p_i). \tag{1.71}$$

The quantities (??) and (??) commute with all the constraints, and provide a complete set (in the sense of Dirac) of gauge-invariant observables. Every other physical observable can be obtained from them.

In general, $2N - M$ of these equations are independent. For each physical state of the system, determined by the value of $(\tilde{q}^a, \tilde{p}_a)$, these equations define an M dimensional subspace in the phase space. Therefore each state determines a set of relations on the original phase space variables. These relations represent the dynamical information on the system; they provide the full solution of the dynamics in a gauge-invariant fashion.

$$S = \int d\tau \left(p_a \frac{dq^a}{d\tau} + N\mathcal{C}(p_a, q^a) \right) \tag{1.72}$$

$$\tilde{q}^a = \tilde{q}^a(q^i, p_i), \tag{1.73}$$
$$\tilde{p}_a = \tilde{p}_a(q^i, p_i), \tag{1.74}$$

1.6 Action Principle for General Relativity

Just as the action principle for electrodynamics sould be invariant under gauge transformations, an action principle for general relativty should be invariant under its gauge transformations which we have learnt are infintesimal active diffeomorphisms, not to be confused with coordinate transformations.

It turns out, however, that actions that are invaraint under coordinate transformations are automatically invariant under active diffeomorphisms! This we prove in the following subsection.

In the second subsection we demonstrate the first order Palatini action principle of GR. This is an action formulation in which the connection is considered as an independent variable.

1.6.1 Invariance of Integral Scalars Under Active Diffeomorphisms

With a coordinate transformation the volume element

$$d^4x = \frac{1}{4!}\epsilon_{\mu\nu\sigma\rho}dx^\mu dx^\nu dx^\sigma dx^\rho$$

transforms as

$$d^4x' = d^4xJ \tag{1.75}$$

where

$$J = \det\left(\frac{\partial x^{\mu'}}{\partial x^\alpha}\right).$$

Now

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x^{\mu'}} g_{\alpha\beta}(x) \frac{\partial x^\beta}{\partial x^{\nu'}}$$

We look on the right-hand side as the product of three matrices, and take the determinant of both sides giving

$$g' = J^{-2}g.$$

Thus

$$\sqrt{-g} = J\sqrt{-g'}. \tag{1.76}$$

Therefore

$$\sqrt{-g}d^4x = \sqrt{-g'}d^4x'$$

and so the quantity

$$\sqrt{-g}d^4x$$

is invariant under coordinate transformations. Suppose F is a scalar field, $F = F'$. Then

$$\int F\sqrt{-g}d^4x = \int F\sqrt{-g'}Jd^4x = \int F'\sqrt{-g'}d^4x'$$

if the region of integration for x' corresponds to that for x. Therefore

$$\int F\sqrt{-g}d^4x$$

is invariant under coordinate transformations. We refer to as an integral scalar.

Now let us investigate how an integral scalar transforms under an active diffeomorphism. We start with seeing how $\sqrt{-g}d^4x$ transforms under active diffeomorphisms.

Consider the "small" volume depictured in fig (1.21). Under an active diffeomorphism the "corner" points of the infintesimal volume are changed but then evaluated at the original values of the coordinates, therefore

$$d^4x = d^4y.$$

and the quantity

$$d^4x$$

is invariant under active diffeomorphisms.

We see that in integrals over spacetime volume, the volume element d^4x does not change under an active diffeomorphism, while it does change under a coordinate transformation. However, the volume element $\sqrt{-g}\ d^4x$ is invariant under a coordinate transformation but not under an active diffeomorphism.

We will now show that any scalar integral is invariant under an active diffeomorphism that vanishes at the end points of integration. Let us find the effect of an active diffeomoprhism generated by the vector field ξ_ν on the metric $g_{\mu\nu}$.

$$y^\mu = x^\mu + \epsilon\xi^\mu(x)$$

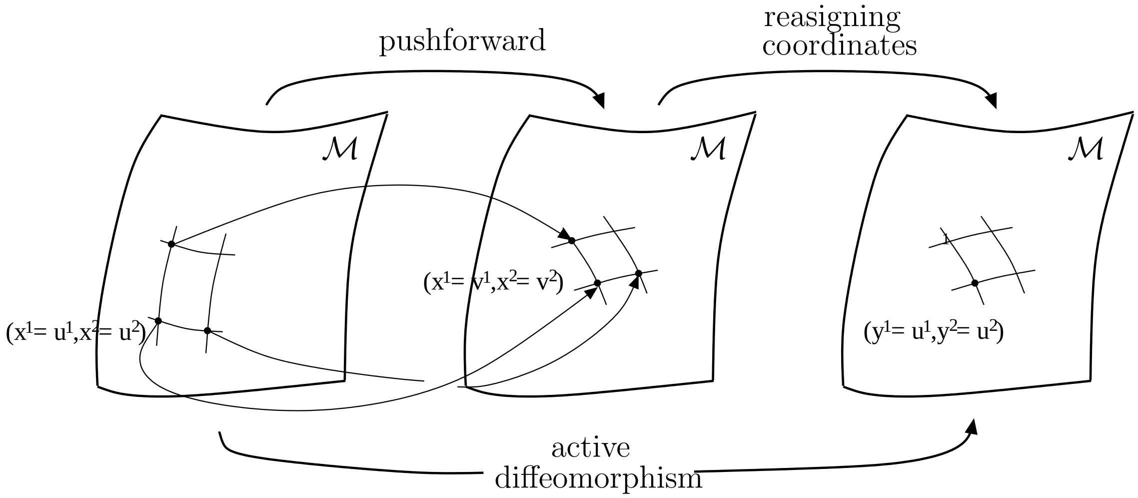

Figure 1.21: We first do a pushforward mapping the "corners" of the infintesimal volume to new points. We then do a coordinate transformation to assign the new points the original coordinate values.

Differentiating, we get

$$\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu + \epsilon \partial_\nu \xi^\mu. \tag{1.77}$$

$$\begin{aligned} g^{\mu\nu}(y) &= \frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma} g^{\rho\sigma}(x) \\ &= (\delta^\mu_\rho + \epsilon\partial_\rho \xi^\mu)(\delta^\nu_\sigma + \epsilon\partial_\sigma\xi^\nu) g^{\rho\sigma}(x) \\ &= g^{\mu\nu}(x) + \epsilon g^{\rho\nu}(x)\partial_\rho\xi^\mu + \epsilon g^{\mu\rho}(x)\partial_\rho \xi^\nu \end{aligned} \tag{1.78}$$

$$g^{\mu\nu}(y) = g^{\mu\nu}(x^\rho + \epsilon\xi^\rho) = g^{\mu\nu}(x) + \epsilon\xi^\rho \partial_\rho g^{\mu\nu}(x) \tag{1.79}$$

$$\frac{\delta g^{\mu\nu}}{\epsilon} = \frac{g^{\mu\nu}(y) - g'^{\mu\nu}(y)}{\epsilon} \tag{1.80}$$

$$\begin{aligned} \delta g^{\mu\nu}/\epsilon &= \xi^\rho\partial_\rho g^{\mu\nu} - g^{\mu\rho}\partial_\rho\xi^\nu - g^{\rho\nu}\partial_\rho\xi^\mu \\ &= \xi^\rho\partial_\rho g^{\mu\nu} - g^{\mu\rho}(\nabla_\rho\xi^\nu - \Gamma^\nu_{\sigma\rho}\xi^\sigma) - g^{\rho\nu}(\nabla_\rho\xi^\mu - \Gamma^\mu_{\sigma\rho}\xi^\sigma) \\ &= \xi^\rho(\partial_\rho g^{\mu\nu} + g^{\mu\sigma}\Gamma^\nu_{\rho\sigma} + g^{\sigma\nu}\Gamma^\mu_{\rho\sigma}) - g^{\mu\rho}(\nabla_\rho\xi^\nu) - g^{\rho\nu}(\nabla_\rho\xi^\mu) \\ &= \xi^\rho\nabla_\rho g^{\mu\nu} - \nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu \\ &= -\nabla^\mu\xi^\nu - \nabla^\nu\xi^\mu \end{aligned} \tag{1.81}$$

We have under a diffeomorphism

$$\delta g^{\mu\nu} = -\epsilon \nabla^\mu \xi^\nu - \epsilon \nabla^\nu \xi^\mu.$$

Then from

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

we have

$$\delta\sqrt{-g} = \epsilon(\nabla_\alpha \xi^\alpha)\sqrt{-g}. \tag{1.82}$$

Putting this together and as the Lie derivative of a scalar is the directional derivative

$$\begin{aligned}
\delta S_{\epsilon\xi} &= \int \mathcal{L}_{\epsilon\xi}(F\sqrt{-g}) d^4x \\
&= \epsilon \int (\xi^\mu \nabla_\mu F + F \nabla_\mu \xi^\mu)\sqrt{-g} d^4x \\
&= \epsilon \int \nabla_\mu (F\xi^\mu)\sqrt{-g} d^4x \\
&= \epsilon \int F\xi^\mu d^3\Sigma_\mu
\end{aligned} \tag{1.83}$$

where we have used Gauuss' law. For variations with $\xi^\mu = 0$ at the bounadries, $\delta S_{\epsilon\xi} = 0$.

What is the physical implication of this? Note if the metric was a nondynamical object it could not be varied without changing the original action principle. Since we are assuming the metric is not a nondynamical object that would otherwise break the background independence of the theory - diffeomorphism invariance is formally equivalent to general covariance, namely the invariance of the field equations under arbitrary changes of the spacetime coordinates $\vec{x}$ and t. Therefore if the action principle is based on an integral scalar **it will have active diffeomorphisms as a gauge transformation**.

1.6.2 Palatini's first order formalism of Einstein's equation.

This is a special connection built out of the metric and its derivatives is the so-called metric connection uniquely determined by the relation

$$\nabla_\alpha g_{\mu\nu} = 0$$

which follows from the requirement that a vector parallely transported around an infintesimal loop retains the same length.

If the action is to be a scalar, the Lagrangian for the spacetime metric cannot depend on the first derivatives $\partial_\alpha g_{\mu\nu}$, because $\nabla_\alpha g_{\mu\nu} = 0$ and the first derivatives can all be transformed to zero at a point. Thus for the action to be an integral scalar, one is forced to include second derivatives of the metric. The Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ is the simplest scalar that can be formed from second derivatives of the metric.

$$\mathcal{L}_G = (-g)^{1/2} R \tag{1.84}$$

This as a functional of g_{ab} and its first and second derivatives, namely,

$$\mathcal{L}_G = \mathcal{L}_G(g_{ab}, g_{ab,c}, g_{ab,cd}), \tag{1.85}$$

The Euler-Lagrange equations include an extra term because of the dependence on $g_{ab,cd}$,

$$\frac{\partial \mathcal{L}}{\partial g_{ij}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{g}_{ij}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial \mathcal{L}}{\partial \ddot{g}_{ij}}\right) = 0 \tag{1.86}$$

The direct calculation is horrendous and there is a more efficient method which we present in appendix D. But here we detail a particular approach which uses a choice of basic variables considered more suitable to for quantization, i.e. in which connections are considered as independent variables.

We wish to consider the connection as a dynamical degree of freedom in its own right, with no dependency on the metric. How can we relinquish the connection's relation to the metric but still have a theory with the same physical content?

Inspiration comes from single particle mechanics. The variables $x(t)$ and $p(t)$ become two independently adjustable functions in a new variational principle [**?**],

$$I = \int [p\dot{x} - H(p, x, t)] dt \tag{1.87}$$

$$\delta I = p\delta x|_{x',t'}^{x'',t''} + \int_{x',t'}^{x'',t''} \left[\left(\dot{x} - \frac{\partial H}{\partial p}\right)\delta p + \left(-\dot{p} - \frac{\partial H}{\partial x}\right)\delta x\right] dt. \tag{1.88}$$

Out of the extrematization with respect to the $p(t)$, one recovers the standard formula for the momentum in terms of the velocity,

$$\dot{x} = \frac{\partial H(p, x, t)}{\partial p}. \tag{1.89}$$

Extremisation with respect to the $x(t)$ give the equation of motion

$$\dot{p} = -\frac{\partial H(p, x, t)}{\partial x}. \tag{1.90}$$

Palatini's first order formalism of Einstein's equation.

$$\mathcal{L}_G = \mathcal{L}_G(g^{ab}, \Gamma^a_{bc}, \Gamma^a_{bc,d}) \tag{1.91}$$

$$S = \int_\Sigma (-g)^{1/2} g^{ab} R_{ab}(\Gamma^a_{bc}, \Gamma^a_{bc,d}) d^3x. \tag{1.92}$$

However, we consider the

$$\mathcal{L}_G = \mathcal{L}_G(\tilde{g}^{ab}, \Gamma^a_{bc}, \Gamma^a_{bc,d}) \tag{1.93}$$

$$S = \int_\Sigma \tilde{g}^{ab} R_{ab}(\Gamma^a_{bc}, \Gamma^a_{bc,d}) d^3x. \tag{1.94}$$

with respect to $\tilde{g}^{\mu\nu}$ only, and the principle of stationary action gives immediately the vacuum field equations

$$R_{ab}(\Gamma^a_{bc}) = 0. \tag{1.95}$$

The more difficult part is showing that independent variation of the connection Γ^a_{bc} in (1.94) implies the connection is the metric connection: $\Gamma^c_{ab} = g^{cd}(\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})/2$, (or rather the condition $\nabla_c g_{ab} = 0$, which is equivalent). We do this now. In a freely falling frame the connection vanishes

$$\begin{aligned} \delta R^\alpha{}_{\beta\mu\nu} &\overset{*}{=} \partial_\mu(\delta\Gamma^\alpha_{\beta\nu}) - \partial_\nu(\delta\Gamma^\alpha_{\beta\mu}) \\ &\overset{*}{=} \nabla_\mu(\delta\Gamma^\alpha_{\beta\nu}) - \nabla_\nu(\delta\Gamma^\alpha_{\beta\mu}) \end{aligned} \tag{1.96}$$

If a tensor equation holds in one coordinate system it holds in all. The Palatini equation

$$\delta R^{\alpha}{}_{\beta\mu\nu} = \nabla_{\mu}(\delta\Gamma^{\alpha}_{\beta\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\beta\mu}) \tag{1.97}$$

Contracting on α and μ we get the useful formula

$$\delta R_{\beta\nu} = \nabla_{\alpha}(\delta\Gamma^{\alpha}_{\beta\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\beta\alpha}) \tag{1.98}$$

$$\begin{aligned} \delta S &= \int_V (\sqrt{-g}g^{\mu\nu})\delta R_{\mu\nu} d^4x \\ &= \int_V (\sqrt{-g}g^{\mu\nu})[\nabla_{\alpha}(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_{\nu}(\delta\Gamma^{\alpha}_{\mu\alpha})]d^4x \end{aligned} \tag{1.99}$$

Integrating by parts

$$\begin{aligned} \delta S &= \int_V [\nabla_{\nu}(\sqrt{-g}g^{\mu\nu})\delta\Gamma^{\alpha}_{\mu\alpha} - \nabla_{\alpha}(\sqrt{-g}g^{\mu\nu})\delta\Gamma^{\alpha}_{\mu\nu}]d^4x \\ &= \int_V [(\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu}))\delta\Gamma^{\rho}_{\mu\nu}]d^4x \end{aligned} \tag{1.100}$$

The integrand vanishes:

$$\sum_{\mu,\nu}[\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu})]\delta\Gamma^{\rho}_{\mu\nu} = 0 \tag{1.101}$$

where we have reinstated the summation for clarity. The variations in $\Gamma^{\rho}_{\mu\nu}$ are aribtrary, but symmetric in μ and ν, so the above is equivalent to

$$(2\sum_{\mu<\nu} + \sum_{\mu=\nu})\ [\frac{1}{2}\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) + \frac{1}{2}\delta^{\mu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\nu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu})]\delta\Gamma^{\rho}_{\mu\nu} = 0$$

where the expression in the brakets of (1.101) has been replaced by its symmetric part. In this summation all the $\delta\Gamma^{\rho}_{\mu\nu}$ are independent of each other and so we have

$$\frac{1}{2}\delta^{\nu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\mu\sigma}) + \frac{1}{2}\delta^{\mu}_{\rho}\nabla_{\sigma}(\sqrt{-g}g^{\nu\sigma}) - \nabla_{\rho}(\sqrt{-g}g^{\mu\nu}) = 0 \tag{1.102}$$

We now show in turn that the covariant derivatives of $(\sqrt{-g}g^{\mu\nu})$, $\sqrt{-g}$, $g^{\mu\nu}$, and $g_{\mu\nu}$ vanish. Setting $\nu = \rho$ in (1.102) and summing over ν implies

$$\nabla_\sigma(\sqrt{-g}g^{\mu\sigma}) = 0$$

Using this in (1.102) we get

$$\nabla_\rho(\sqrt{-g}g^{\mu\sigma}) = 0 \tag{1.103}$$

Taking the determinant of

$$g^{\mu\sigma}\nabla_\rho\sqrt{-g} + \sqrt{-g}\nabla_\rho g^{\mu\sigma} = 0$$

considered as a matrix with indices μ, σ implies

$$\nabla_\rho\sqrt{-g} = 0. \tag{1.104}$$

Which then implies

$$\nabla_\rho g^{\mu\nu} = 0. \tag{1.105}$$

Which we use in the following

$$\begin{aligned} 0 = g_{\mu\alpha}\nabla_\rho(\delta^\alpha_\nu) &= g_{\mu\alpha}\nabla_\rho(g^{\alpha\sigma}g_{\sigma\nu}) \\ &= g_{\mu\alpha}g^{\alpha\sigma}\nabla_\rho g_{\sigma\nu} \\ &= \delta^\sigma_\mu\nabla_\rho g_{\sigma\nu} \\ &= \nabla_\rho g_{\mu\nu}. \end{aligned} \tag{1.106}$$

This completes the proof.

1.7 Hamiltonian Formulation

1.7.1 Gauge Transformations in Phase Space

A dynamic system is said to be constrained if its physical phase space is a submanifold $\overline{\Gamma}$ of the original phase space Γ, called the constraint surface. The constraint surface is defined by the vanishing of a set of functions $C_k(q,p)$ called the constraints:

$$\overline{\Gamma} := \{p \in \Gamma \mid C_k = 0, \quad k = 1, \ldots, K\}. \tag{1.107}$$

Consistency resquires that the constraints are preserved under time evolution, this may imply further constraints, which in turn have to be checked for consistency, leading to the possibly of other constraints that go by the names of secondary, tertiary e.t.c. Hopefully at some point this iteration (the Dirac-Bergman algorithm) terminates and one arrives at a finite, total number of constraints

$$C_k = 0, \quad k = 1, \ldots, K, \tag{1.108}$$

that are consistent with evolution.

To understand the phase space of a constrained system, one needs to distinguish between first class and second class constraints. First class constraints are defined by the property that their Poisson brackets with all other constraints vanish on the constraint surface. All other constraints, i.e. those that are not first class, are called second class.

First class constraints play two roles: as well as specifying the constraint surface, along with the other constraints, they also generate flows in the constraint surface.

A constrained system is said to be of first class if for all covectors n_α normal to $\overline{\Gamma}$, $\Omega^{\alpha\beta} n_\alpha$ is tangent to $\overline{\Gamma}$.

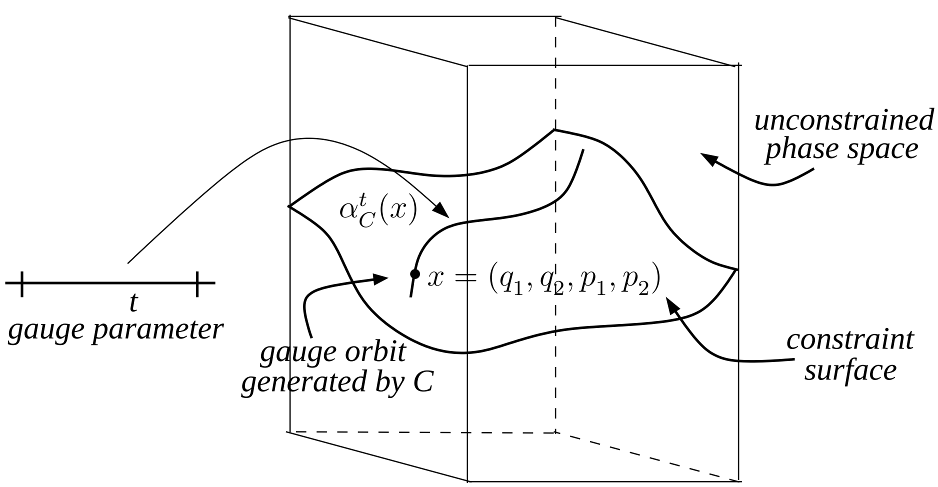

Figure 1.22: partComptDitt1.

The Hamiltonian constraint of the theory is a first class constraint and should therefore be viewed as a gauge transformation. However, since the constraint is responsible for generating time evolution is the unphysical unfolding of a gauge transformation!

The definition of an observable (invariant quanitity under gauge transformations) in a constrained system is a variable that weakly (on the constraint surface) commutes with all the first class constraints . However, since one of these constraints is the generator of time evolution (the Hamiltonian constraint), the observables are constants of motion.

These functions are called *structure functions*. If they are constants, they are called structure constants and the constraint functions C_k are generators of a Lie algebra of the set of functions on Γ.

Due to non-trivial structure *functions* in the commutators involving the hamiltonian constriant of GR makes the situation outside the range of standard techniques. A way out of this is to use an alternative set of constraints whose commutators only involve structure constants - for example, the so-called Master constriant.

1.7.2 Second Class Constraints and Gauge Fixing

E.g. The Lorentz gauge constraint

$$\partial^\mu A_\mu(x) = 0. \tag{1.109}$$

These appear in the constrained BF theory equivalent to the first order tetrad formulation of GR. How to promote these second class constraints is of cruical importance in formulating improved version spin foams (quantum spacetime) capable of reproducing the correct semiclassical limit (see chapter 4).

1.8 Partial, Complete and Dirac Observables

A partial observable is

A complete observable is

Dirac Observables

Different choices of clocks can be interpreted as different setups for a physical measurent.

$$\mathbf{x}(t) = \mathbf{x} - \frac{\mathbf{p}}{p_0}(t - x^0) \tag{1.110}$$

a true observable formed by two partial observables.

Has vanishing Poisson brackets with the constraint $\mathbf{p}^2 - m^2$:

$$\{x_1(t), p_1^2 + p_2^2 + p_3^2 - m^2\} = \{x_1(t), p_1^2\} \tag{1.111}$$

For systems with one constraint the idea works in the following way: Assume that the system is totally constrained so that the constraint generates the time evolution (which is then considered as a gauge transformation). Use a phase space function T, which is not a Dirac observable, as a clock which "measures" the time flow, denoted $T(\tau)$ i.e. the gauge transformation. Consider another phase space function f and calculate the value of f "at the time" at which T assumes the value τ, that is, the combined quantity $f(T)$ when $T = \tau$. Since the value of f at a fixed time τ does not change with time, the result will be time independent, i.e. a Dirac observable. Moreover varying τ gives a one-parameter family of Dirac observables. Following Rovelli, T and f partial observables and the one-parameter family of Dirac observables complete observables.

Dittrich generalizes these ideas to an arbitary number of gauge degrees of freedom [300].

Briefly discuss the notion.

In a familiar setting say we want the position or momentum some time later, what would do this for you is

$$q(t+\delta t) = q(t) + \delta t \, \frac{\partial q}{\partial t} = q(t) + \delta t \, \{\mathcal{H}, q(t)\}.$$

Now if we want to know what the position is at $t_0 + 2\delta t$ we have

$$\begin{aligned} q(t+2\delta t) &= q(t+\delta t) + \delta t \, \{\mathcal{H}, q(t+\delta t)\} \\ &= q(t) + 2\delta t \, \{\mathcal{H}, q(t)\} + \delta t^2 \, \{\mathcal{H}\{\mathcal{H}, q(t)\}\}. \end{aligned} \tag{1.112}$$

Similarly

$$\begin{aligned} q(t+3\delta t) &= q(t+\delta t) + \delta t\{\mathcal{H}, q(t+\delta t)\} \\ &= q(t) + 3\delta t\{\mathcal{H}, q(t)\} + 3 \cdot 2\delta t^2\{\mathcal{H}\{\mathcal{H}, q(t)\}\} + \delta t^3 \, \{\mathcal{H}\{\mathcal{H}\{\mathcal{H}, q(t)\}\}\}. \end{aligned} \tag{1.113}$$

Let us introduce the shorthand notation

$$\{\mathcal{H}, q(t_0)\}_{(2)} \equiv \{\mathcal{H}\{\mathcal{H}, q(t_0)\}\}, \quad \{\mathcal{H}, q(t_0)\}_{(3)} \equiv \{\mathcal{H}\{\mathcal{H}\{\mathcal{H}, q(t_0)\}\}\}, \quad \ldots$$

It is not too difficult find

$$q(t_0+t) = q(t_0+N\delta t) \;=\; q(t) + t\,\{\mathcal{H}, q(t_0)\} + \frac{t^2}{2!}\{\mathcal{H}, q(t_0)\}_{(2)} + \cdots + \\ + \ldots \frac{t^n}{n!}\{\mathcal{H}, q(t_0)\}_{(n)} + \ldots \\ =: \;\exp t\{\mathcal{H},\}q(t_0). \quad (1.114)$$

$$\alpha^t_{\mathcal{H}}(q)(x) := \exp t\{\mathcal{H},\}q(t).$$

$$\alpha^t_C(x)$$

$$\alpha^t_C(f)(x) := f(\alpha^t_C(x)) \quad (1.115)$$

If $\alpha^s_C(x)$ is the flow generated by the constraint C starting from the point x, then the value of the function $\alpha^s_C(f)$ at the point x is given by

$$\alpha^s_C(f)(x) := f(\alpha^s_C(x)). \quad (1.116)$$

It can be calculated with the series

$$\alpha^s_C(f(x)) = \sum_{k=0}^{\infty} \frac{s^k}{k!}\{C, f(x)\}_k. \quad (1.117)$$

Partial observables

“a physical quantity to which we can associate a (measuring) procedure leading to a number”, [51]. If we assume that one can associate to an arbitrary phase space function such a measuring procedure, then any phase space function is a partial observable. Partial observables need not be a Dirac observable.

T is a kind of clock, whose values .

$$\alpha^s_C(T)(x) = \tau \quad (1.118)$$

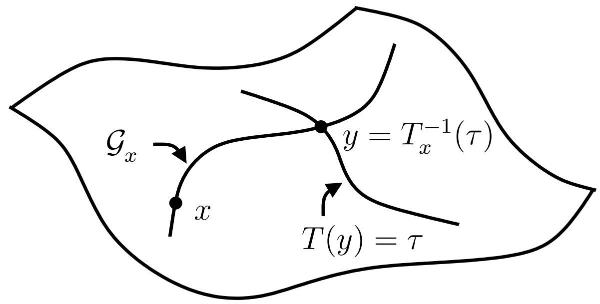

Figure 1.23: comptasGIE.

Complete observables

A complete observable is "a quantity whose value can be predicted by the theory in classical theory)".

Now given another phase space function f and a phase space point x one can calculate the value the function takes at the point of the gauge orbit through x at which T assumes the value τ. The complete observable predicts the value of f for the "time" τ. We denote this predicted value

$$F_{[f;T]}(\tau, x). \tag{1.119}$$

More formally, the condition defining $F_{[f;T]}(\tau, x)$ is

$$F_{[f,T]}(\tau, x) := \alpha_C^s(f(x))_{|\alpha_C^s(T(x))=\tau}. \tag{1.120}$$

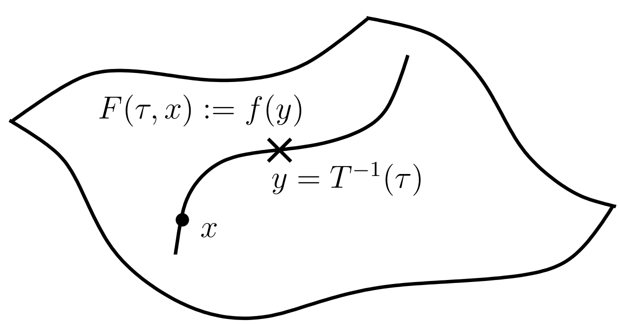

Figure 1.24: partComptDitt3a. $F_{[f,T]}(\tau, x)$ is defined as the value of f evaluated at the point y on the gauge orbit where $T(y) = \tau$. Solving for y we can write $F_{[f,T]}(\tau, x) = f(y) = f(T^{-1}(\tau))$. Note that $F_{[f,T]}(\tau, x)$ only depends on τ and not on the location of x along the gauge orbit.

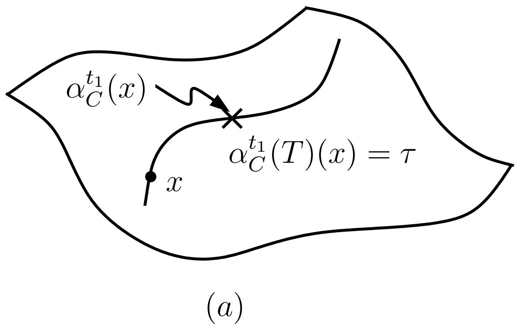

Figure 1.25: partComptDitt3. (a) $t = t_1$ when the clock function $T(\alpha_C^t(x))$ assumes the value τ. (b) The function $F_{[f,T]}(\tau, x)$ gives the value that the function $f(\alpha_C^t(x))$ assumes if the function $T(\alpha_C^t(x))$ assumes the value τ. $F_{[f,T]}(\tau, x)$ is a complete observable generated from the partial observables $T(x)$ and $f(x)$.

Clock variables

The gauge orbits are one-dimensional. Hence, we can parametrize a gauge orbit with the values of a phase space function T, if this phase space function changes strictly monotonously along the orbit, there should be only one point on each gauge orbit where the function T assumes the value τ.

1.8.1 A Complete Observable is a Dirac Observable

Since the definition of $F_{[f;T]}(\tau, x)$ depends on the phase space point, we have actually constructed a phase space function

$$F_{[f;T]}(\tau, \cdot).$$

However, by definition, $F_{[f,T]}(\tau, x)$ only depends on the gauge orbit through x and not on where x is located on the orbit. Hence, if the phase space points x and y both lie on the same gauge orbit we have

$$F_{[f,T]}(\tau, x) = F_{[f,T]}(\tau, y), \tag{1.121}$$

i.e. $F_{[f;T]}(\tau, \cdot)$ is constant along gauge orbits (see fig.(1.26)).

No matter which point on a particular gauge orbit we start at, f always assumes the same value where T assumes the value τ. Hence the complete observable $F_{[f;T]}(\tau, \cdot)$ is a Dirac observable.

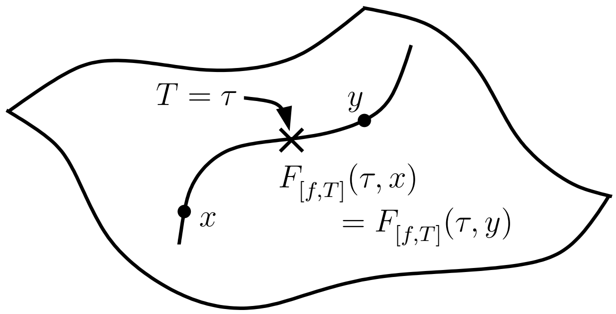

Figure 1.26: partComptDitt5. If we started at another phase space point y on the same orbit, we would still evaluate f at the same point on that gauge orbit as before, namely at that point, at which T assumes the value τ.

To every evolution orbit it assigns one value, namely the prediction of f for that moment at which T is equal to τ. But this value will change if we change τ, i.e. predict f for another value of the clock varaible T. In this sence we can 'evolve' $F_{[f;T]}(\tau, \cdot)$ through all values of the clock variable T.

We now consider some examples.

Examples

(i) Parameterized Harmonic oscillator

phase space point: $x = (q, p, t, p_t)$

$$C = p_t + \frac{p^2}{2m_1} + \frac{1}{2}m\omega^2 q^2 \tag{1.122}$$

where p_t is the conjugated momentum to the time variable t and p are conjugated to the position variables q. A natural choice for a clock variable is t and we can ask for the position of the first particle at that moment at which t assumes the value τ. We will denote this observable by $F_{[q;t]}(\tau)$ and it can be easily calculated to be

$$F_{[q;t]}(\tau) = q \cos\omega(\tau - t) + \frac{p}{m\omega}\sin\omega(\tau - t). \tag{1.123}$$

$$\begin{aligned}
\{C, q(x)\} &= \{p_t + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2, q\} \\
&= -\frac{p}{m}
\end{aligned}$$

$$\begin{aligned}\{C,\{C,q(x)\}\} &= -\frac{1}{m}\{C,p(x)\}\\ &= -\frac{1}{m}\{p_t+\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2,p\}\\ &= -\omega^2 q\end{aligned}$$

$$\begin{aligned}\alpha^s_C(q) &= q(\alpha^s(x))\\ &= \exp s\{C,\}q(x)\\ &= q(1-\frac{s^2}{2!}\omega^2+\frac{s^4}{4!}\omega^4-\dots)-\frac{p}{m\omega}(s\omega-\frac{s^3\omega^3}{3!}+\dots)\\ &= q\cos\omega s-\frac{p}{m\omega}\sin\omega s\end{aligned}$$

$$\begin{aligned}\{C,t\} &= \{p_t+\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2,t\}\\ &= -1\end{aligned}$$

$$\begin{aligned}\alpha^s_C(t) &= t(\alpha^s(x))\\ &= \exp s\{C,\}t(x)\\ &= t-s\end{aligned}$$

$$\begin{aligned}F_{[q;t]}(\tau)(x=\{q,p,t,p_t\}) &= q(\alpha^s_C)_{|t(\alpha^s_C)=\tau}\\ &= q\cos\omega(\tau-t)+\frac{p}{m\omega}\sin\omega(\tau-t) \qquad (1.124)\end{aligned}$$

(ii)

phase space point: $x=(q_1,p_1,q_2,p_2,t,p_t)$

$$C=p_t+\frac{p_1^2}{2m_1}+\frac{p_2^2}{2m_2} \qquad (1.125)$$

where p_t is the conjugated momentum to the time variable t and p_1, p_2 are conjugated to the two position variables q_1, q_2. A natural choice for a clock variable is t and we can ask for the position of the first particle at that moment at which t assumes the value τ. We will denote this observable by $F_{[q_1;t]}(\tau)$ and it can be easily calculated to be

$$F_{[q_1;t]}(\tau) = q_1 + \frac{p_1}{m_1}(\tau - t). \tag{1.126}$$

$$\begin{aligned} \{C, q_1(x)\} &= \{p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}, q_1\} \\ &= -\frac{p_1}{m_1} \end{aligned}$$

$$\begin{aligned} \{C, \{C, q_1(x)\}\} &= \frac{1}{m_1}\{C, p_1(x)\} \\ &= -\frac{1}{m_1}\{p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}, p_1\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \alpha_C^s(q_1) &= q_1(\alpha^s(x)) \\ &= \exp s\{C, \}q_1(x) \\ &= q_1 - \frac{p_1}{m_1}s \end{aligned} \tag{1.127}$$

$$\begin{aligned} \{C, t(x)\} &= \{p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}, t\} \\ &= 1 \end{aligned} \tag{1.128}$$

$$\begin{aligned} \alpha_C^s(t) &= t(\alpha^s(x)) \\ &= \exp s\{C, \}t(x) \\ &= t + s \end{aligned} \tag{1.129}$$

$$\begin{aligned} F_{[q_1;t]}(\tau)(x = \{q_1, p_1, t, p_t\}) &= q_1(\alpha_C^s)_{|t(\alpha_C^s)=\tau} \\ &= q_1 + \frac{p_1}{m_1}(\tau - t) \end{aligned} \tag{1.130}$$

It Poisson commutes with the constraint and is therefore a Dirac observable.

$$\begin{aligned}\{F_{[q_1;t]}, C\} &= \{q_1 + \frac{p_1}{m}(\tau - t)\ ,\ p_t + \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2}\} \\ &= -p_1\{p_t, t\}\frac{1}{m_1} + \{q_1, p_1^2\}\frac{1}{2m_1} \\ &= 0 \end{aligned} \tag{1.131}$$

The Poisson bracket of two observables $F[q_1;t](\tau_1)$ and $F[q_1;t](\tau_2)$ at two different clock values τ_1 and τ_2 is phase space independent

$$\begin{aligned}\{F[q_1;t](\tau_1), F[q_1;t](\tau_2)\} &= \{q_1 + \frac{p_1}{m_1}(\tau_1 - t)\ ,\ q_1 + \frac{p_1}{m_1}(\tau_2 - t)\} \\ &= \frac{1}{m_1}(\tau_2 - \tau_1) \end{aligned} \tag{1.132}$$

Now one could also choose the position of the second particle as a clock variable and ask for the position of the first particle at that moment at which the second particle has position τ_2. The corresponding observable is

$$F_{[q_1;q_2]}(\tau') = q_1 + \frac{p_1}{m_1}\frac{m_2}{p_2}(\tau' - q_2). \tag{1.133}$$

$$\alpha_C^s(q_1) = q_1 + \frac{p_1}{m_1}s, \qquad \alpha_C^s(q_2) = q_2 + \frac{p_2}{m_2}s$$

we rewrite the second equation as

$$s = \frac{m_2}{p_2}(\alpha_C^s(q_2) - q_2)$$

to use it to replace s

$$\begin{aligned}F_{[q_1;q_2]}(\tau')(x = \{q_1, p_1, t, p_t\}) &= q_1(\alpha_C^s)_{|q_2(\alpha^s)=\tau} \\ &= \left(q_1 + \frac{p_1}{m_1}s\right)_{|q_2(\alpha_C^s)=\tau'} \\ &= q_1 + \frac{p_1}{m_1}\frac{m_2}{p_2}(\tau' - q_2) \end{aligned} \tag{1.134}$$

If one ignores that τ and τ' refer to different clocks, (1.133) looks of course quite different from (1.126). However if one takes into account that the value τ is reached at that moment at which

$$\tau' = F_{[q_2;t]} = q_2 + \frac{p_2}{m_2}(\tau - t) \tag{1.135}$$

and uses this to replace τ' in (1.133) one will get back to (1.126). In so far both choices of clock variables give us the same time evolution if one takes into account the "translation" (1.126) between the clock readings τ' and τ'.

(iii) FRW-cosmology with massless scalar field

phase space point: $x = (a, P_a, \phi, P_\phi)$

with constraint:

$$C = \frac{1}{2}(-\frac{P_a^2}{a} + \frac{P_\phi^2}{a^3})$$

Partial observables: clock

$$T = \phi \tag{1.136}$$

$$f = a \tag{1.137}$$

Clock observable:

$$F^{\tau}_{[f;T]}(x) = f(x') \tag{1.138}$$

where

$x' \sim x$ and $T(x') = \tau$

The complete observable for this example $\tau = 0$ is

$$F^{\tau=0}_{[a;\phi]}(x) = a \exp(-\mathrm{sgn}(P_a P_\phi)(0 - \phi)) \tag{1.139}$$

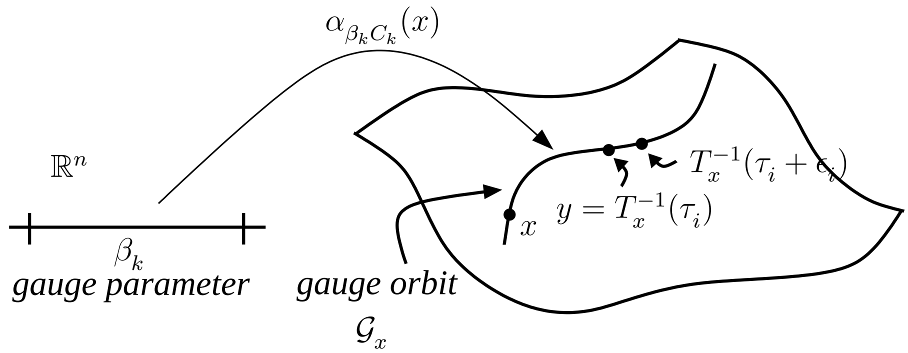

Figure 1.27: partComptDitt4.

1.8.2 Systems with Several Constraints

$$F_{[f;T^K]}(\tau) = \sum_{r=0}^{\infty} \frac{1}{r!} \{\cdots\{f, \tilde{C}_{K_1}, \cdots\}, \tilde{C}_{K_r}\}\, (\tau^{K_1} - T^{K_1}) \cdots (\tau^{K_r} - T^{K_r}). \qquad (1.140)$$

1.8.3 Infinitely Many Constraints

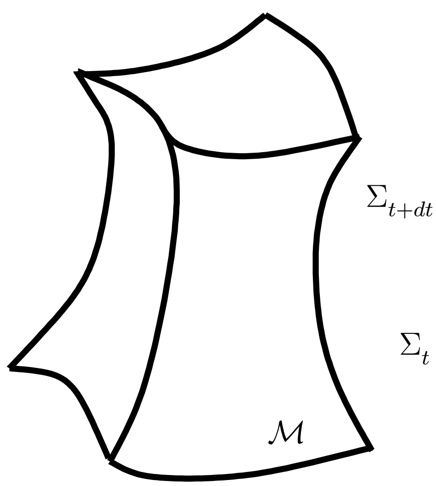

Figure 1.28: FolofSpacetime. Foliation of spacetime $\mathcal{M}$ by a family of spacelike hpersurfaces $(\Sigma_t)_{t\in\mathbb{R}}$.

1.8.4 Observables for Canonical General Relativity

Discuss this in more detail in Appendix E and also in the chapter on the master constraint.

True observables of general relativity. To explicitly find them would this would amount to solving Einstein's equations. This is a problem of the classical theory, unrelated to the problem of its quantization.

For the case of gravity in four space time dimensions and for asymptotically flat boundary conditions there are 10 gauge invariant phase space functions known. These are the ADM charges [305] given by the generators of the Poincare transformations at spatial infinty. For gravity coupled to matter, in some cases gauge invariant functions describing matter are known but in general no phase space functions which describe the gravitational degrees of freedom (with the exception of the ADM charges). Yet there are infinitely many gauge invariant degrees of freedom.

Longstanding open problem in classical GR. Partial, complete observables provide a method for calculating Dirac observables, [302].

defines complete observables for systems with several, including infinite, number of constraints.

can construct explicit expression for complete observerable in canonical general relativity which is pameratized by one parameter insted of infinite - could develope approximation schemes.

1.8.5 Approximate Observables for Canonical General Relativity

• Approximate Dirac observables with a dynamical interpretation can be calculated explicitly to an arbitrary order

• Precise inderstanding of linearized theory and (quantum) field theory on a fixed background as approximations to full general relativity

• Formulism can be used to address construction and interpretation of Dirac observables

• Can be generalized to expansion around symmetry reduced/cosmological sectors $\Rightarrow$ use knowledge on symmetry reduced sectors to construct approximate Dirac observables for full theory

• Need a better understanding of the (quantum) interpretation of complete observables, in particular role of clock variables.

1.9 Recovering Time

The hypothesis that the fundamental theory of nature can be formulated in a timeless language [287], and that temporal phenomena could be emergent [288], [289], [290].

see week 41 beaz

Rovelli wants to use thermodynamics to **define** what we call time as we usualy mean. does this as follows. Given a classical statistical state ρ, find some Hamiltonian H such that ρ is the Gibbs state $\exp(-H/kT)$. In lots of cases this isn't hard; it basically amounts to

$$H_0 = -kT \ln \rho_0 \tag{1.141}$$

Therefore, in a statistical context we have in principle an operational procedure for determining which one is the time variable: Measure ρ_0; compute H_0 from (1.141); compute the hamiltonian flow $s(t)$ of H_0 on Σ: the time variable t is the parameter of this flow. The multiplicative constant in front of H_0 just sets the unit in which time is measured.

Of course, H will depend on T, but this is really is just saying that fixing your temperature fixes your units of time!

1.10 Why Quantise Gravity

Combine general relativity and quantum theory into a single theory that can claim to be the complete theory of nature.

General relativity has the problem with infiniteis as inside a black hole the density of matter and the strength of the gravitational field quickly become infinite. That also the case in the very early in the history of the universe.

Infinities in qunatum mechanics occur whenever you attempt to apply it to fields, such as the electromagnetic field. The problem is that the electric and magnetic fields have values at every point in space so an infinite number of variables. In quantum theory, there are uncontrollable fluctuations in the values of every quantum variable. An infinite number of variables, fluctuating uncontrollably, can lead to equations that get out of hand and predict infinite numbers.

It has been a long held hope that when gravity is taken into account, the fluctuations will be tamed and will be finite, and that the infinities of classical general relativity will be brought under control by quantum theory.

1.11 The Problem of Quantising Gravity

Say you didn't know Schrodinger's equation and wanted to formulate it for a general potential. It goes without saying that it is sensible for this equation to not depend on some particular classical tragectory! Of course the equation you arrive at will have solutions that approximate some particular trajectory, but this is a different matter. This here is what you need to appreicate in order to understand Ashtekar's statement!

Now say you want to formulate the equations of quantum gravity. As with Shcrodinger's equation for an electron in a general potential, the general equation for QG should not depend on some particular classical tragectory, that is, it should not depend on some particular classical spacetime!! As above, the equation's of QG have solutions that approximate some classical spacetime but this is something else.

Physics in the absence of spacetime. How can you do that? For example, if you want to (and this is what is mostly done in LQG) you can introduce a classical spacetime as a mathematical device on which to formulate the theory, but you do it in a way that the end result doesn't depend on the fidicual spacetime you used. What you end up with is anything, quite the opposite. Turns out that LQG is essentially unique (at least at the kinematic level).

If someone tells you that you're crazy, you'll believe anything! you can ask should we take Schrodinger's equation for a general system, that appears in all these physics texts books, and replace it with something that depends on some particular classical tragectory?

Obviously, there are interpretative issues that such a candidate for a theory of QG need to be addressed to be taken seriously which are just as important as the tequiniqual side! Significant progress in formulating an interpretative framework for all such theories - For example Rovelli has some very nice simple, intersting and natural ideas.

1.11.1 The Problem of Time

absolute time: Information brought in from outside.

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L} = 0. \tag{1.142}$$

The Schrödinger equation $\frac{i\hbar^2}{2m} \frac{\partial}{\partial t} |\psi >= \hat{\mathcal{H}} |\psi >$

$$\hat{\mathcal{H}} |\psi >= 0 \tag{1.143}$$

There are prblems with the presence of this constraint.

That the Shrödinger equation, the varaibles "x" and "t" play very different roles. In contrast t is assumed to be a continous external parameter. expected to have a clock that behaves perfectly classically and is completely external to the system under study. As we have seen, for a generally corvariant theory, any reference system must be part of the dynamical system under study; in light of GR there is no such thing as a perfect clock.

How is one to do quantum mechanics in this new conceptual setting? As the previous discussion of the classical situation would suggest the answer: "relationally". One could envision promoting *all* variables of a system to quantum operators, and choosing one of them to play the role of a "clock". Say we call such a variable t (it could for instance be, the angular position of a real clock, or maybe the elongation of a pendulumn). One could compute conditional probabilities for the other variables to take certain values x_0 when the "clock" variable takes the value t_0. If the "clock" we use does correspond to a variable that is behaving classicaly as a clock, then the conditional probabilities will approximate well the probabilities computed in ordinary Schrödinger theory.

the conditional probabilities are still well defined, but they do not approximate Schrödinger theory. If there is no variable that can be considered as a good classical clock, Shrödinger's quantum mechanics does not make any sense and the relational quantum mechanics is therefore a generalization of Shrödinger's quantum mechanics.

"... Conventional mechanics describes evolution in "background time": the independent time variable t is assumed to to represent a mearsurable but non-dynamical physical quantitiy. This is true in the classical as well as in the quantum theory, where evolution in background time is given by unitary transformations. With general relativity, however, we have learned that there is no non-dynamical measurable time in nature: there is no background time in particular. Therefore, at the fundemantal level physics cannot describe evolution in time. It can only describe relations, or correlatins, between mearsurable quantities. It is therefore neccessary to extend the formulation of classical and quantum mechanics to such background independen context. [[37]] ... "

Rovelli has the clearest head with conceptual issues. What is observable in classical and quantum gravity [278].

"...Precisely as classical mechanics, the conventional formulation of QM describes evolution of states and observables in time. Precicsely as classical mechanics, this is not sufficient to deal with general relativistic systems, because these systems do not describe evolution in time; they describe correlations between observables. Therefore a formulation of /qm slightly more general than the conventional one - or quantum version of the relativistic mechanics... ...The possibility of such a formulation is discussed in the first part of this chapter [chapter 5 [?]]. In the second part, I discuss the physical interpretation of QM.

relational QM. No general notion of time at the fundemental level. No good clocks at the Plank scale.

The gravitational interactions become huge so we need a quantum theory of gravity that preserves the non-perturbative and background independence of the classical theory. The big bang - creation of the universe The notion of time is meaniglese in a generic situation. Asking what happens before the big bang in quantum gravity is unlikely to make any sense because the classical notion of realtional time breeaks down near to the singularity.

There are schools of thought in relation to the "problem of time" in generally covariant theories. One trys to single out the “correct” time variable out of all the variables out of the covariant theory. Another approach takes general covariance more seriously, and keeps all variables on an equal footing.

May not need a radical upheavle of the present formulation of quantum mechanics. various approaches:

Important

The intemaitly related problem of quantum cosmology.

1.12 The Problem of Quantum Cosmology

Quantum cosmology is the application of quantum theory to the universe as a whole. In quantum cosmology there is no external clock or observer to the system of interest.there is no a priori classical world. The Copenhagen interpretation assumes from the outset the existence of a classical world with respect to which the results of a quantum measurements are interpreted.

the quantum constraint equations define quantum states that are supposed to describe the whole universe.

we need an interpretaion of the states of a quantum theory that allows the observer to be part of the quantum system.

Many worlds: This interpretation disposes of the notion of collapse of the wave function altogether. Instead asserts that everything that can happen *does* happen. Not only is there a world in which Schödinger’s cat lives, there is also a disconnected world in which Schödinger’s cat dies! The many-worlds approach is an option to use when we describing the universe as a whole and there is no external large system...

Laws of physics give rise to many different solutions.