Chapter 10

Towards Background-Independent Scattering Amplitudes

10.1 Introduction

Two masses should attract each other in the way that Newton's law specifies. In the quantum field theory this attraction comes about from an exchange of virtual gravitons. It would indicate that at low energies that loop quantum gravity has gravitons.

Why has such a calculation not been done despite the impressive advances made so far in loop quantum gravity? The transition amplitude $W(x, y)$ for a particle starting at the spacetime point x to the spacetime point y. Under an active diffeomorphism α the transition amplitude transforms as

$$W(x, y) \mapsto W(\alpha(x), \alpha(y))$$

Hence, for this transition amplitude to be background independence $W(x, y)$ would have to be constant everywhere. Of course, the reason why it this definition is of little use in a generally covariant context is because the arguments of the transition amplitude $W(x, y)$ are space-time points x, y which have no operational meaning at all in a background independent theory.[1] One obvious strategy would then be to formulate scattering amplitudes operationally, as they are measured in actual laboratory experiments.

To describe local experiments, we will be interested in such observables which (approximately) localize in some spacetime region, and the corresponding operators will need to

[1]Some may argue that $W(x, y)$ can't even be defined because the spacetime points are subject to quantum fluctuations in a quantum theory of spacetime, however, one should be careful not to make arguments based on quantum fluctuations in quantities that have no independent physical reality such as spacetime points.

include physical degrees of freedom which specify this region. Distance and time separation must be extracted from the dynamical variables. In a scattering experiment we measure incoming and outgoing particles as well as distances between instruments and elapsed time. The former are the matter field variables, the latter are gravitational field variables. It is natural to use the apparatus itself to specify the spacetime regime in which the the interaction occurs.

Scattering amplitudes of a background-independent theory can be defined as a function of the mean boundary geometry, instead of a function of the background metric, evaluated in terms of the mean value of the quantum gravitational field itself on a box encircling the interaction region. The mean geometry of the boundary are partial observables whose relation to the readings on the laboratory instruments is determined by the theory.

In the context of curved spacetime there is no Poincare invariant background spacetime... In a background independent context the notion of a global particle is not available. On the viability of the article notion in a finite region.

10.2 Difficulties in Formulating Scattering Amplitudes for BI Theories

Rovelli *et al*, [310], [311], [313].

In the standard formulism on associates A Hilbert space of states with each time-slice of a global foliation of space-time. An evolution takes place between two such time-slices and is represented by a unitary operator. Associated with states in the two such time-slices is a transition amplitude, whose modulus square determines the probability of finding the final state given that the initial one was prepared.

$$\mathcal{A}_{12} =< \psi_{int}|\hat{U}(t_1,t_2)|\psi_{fin} >, \quad \text{where } < \psi_{int}| \in \mathcal{H}^* \text{ and } |\psi_{fin} > \in \mathcal{H}. \tag{10.1}$$

$$\mathcal{A}_{12} = \int d^3x \int d^3y \; \psi^*_{int}(x,t_{int}) W(x,y) \psi_{fin}(y,t_{fin}) \tag{10.2}$$

$$W(x_1,\ldots,x_n) = Z^{-1} \int \mathcal{D}\phi \; \phi(x_1)\ldots\phi(x_n) e^{-iS[\phi]} \tag{10.3}$$

• In a background-independent theory $W(x,y) = \text{const}$.

• Conventional spacelike states don't impose any constraint on the proper time lapsed between the initial and final states. As a result, the transition amplitude will be given by a linear superposition of processes whose duration may range from microscopic to cosmic time scales.

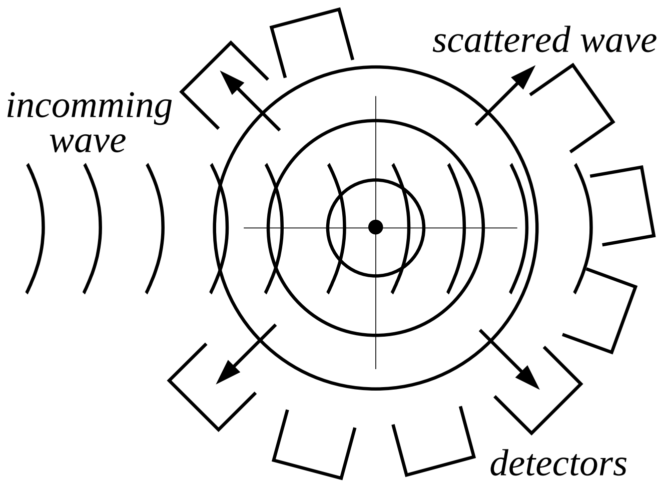


Figure 10.1: scatConven. scattering

• How do we define particles in background-independent quantum gravity where Fock states are not available?

$W[\varphi, \Sigma]$ depends on the field boundary value φ and the 3d surface Σ that bounds $\mathcal{R}$. Say we perform an active diffeomorphism, that is, keep the fields "where they are" but move around the points of the spacetime manifold. This results in a new boundary, Σ'; the field boundary value at this new surface will of course, in general, be different. Whence, if we wish to have a background-independent formulation, $W[\varphi, \Sigma]$ must not depend on Σ, ie. $W[\varphi, \Sigma] = W[\varphi]$.

Fock States and Particles

$$\hat{n}|1, 2, \ldots, N >= N|1, 2, \ldots, N > \tag{10.4}$$

10.2.1 Preparation - Measurement

Measuring Scattering amplitudes

A particle scattering in an electromagnetic field. in (a) the particle arrives at spacetime point y. We perform an active diffeomorphism on (a) and obtain new (b). In new system the particle does not arrive at spacetime point y.

Just as GR doesn't determine the distance between spacetime points, it doesn't determine this probability; the only way to preserve general covariance is if $W(x, y)$ is constant.

The detector and source must be part of the system under study. Their motion and rate are directly effected by the gravitational field. Scattering probability is defined internally.

whether the construction leads, in a first approximation, to the general relativity scattering tree amplitudes.

10.3 Conventional Scattering Theory

The scattering matrix allows the to calculate experimentally observable scattering cross sections.

10.3.1 The Propagator in Conventional Scattering Theory

the notion of a propagator $W[q_2, t_2; q_1, t_1]$. Given a wave function $\psi(q_1, t_1)$ at a time t_1, the propagator gives the corresponding wave function at a later time t_2:

$$\psi(q_2, t_2) = \int W[q_2, t_2; q_1, t_1]\psi(q_1, t_1)d^n q_1. \tag{10.5}$$

$\psi(q_2, t_2)$ is the probability amplitude that the particle is at the point q_2 at the time t_2, so $W[q_2, t_2; q_1, t_1]$ is the probability amplitude for a particle at q_1 at time t_1 to arrive at the point q_2 at time t_2. The probability that it is observed at q_2 at time t_2 is

$$P(q_2, t_2; q_1, t_1) = |W[q_2, t_2; q_1, t_1]|^2. \tag{10.6}$$

Composition

$$W[q_3, t_3; q_1, t_1] = \int d^n q_2 W[q_3, t_3; q_2, t_2]W[q_2, t_2; q_1, t_1] \tag{10.7}$$

$\psi(q_2, t_2)$ obeys Schrödinger's equation

$$\frac{\hbar^2}{2m}\nabla^2_{x_2}\psi(q_2, t_2) + i\hbar\frac{\partial\psi(q_2, t_2)}{\partial t_2} = V(q_2, t_2)\psi(q_2, t_2). \tag{10.8}$$

if

$$\psi(q_2, t_2) = \phi(x_2, t_2) - \frac{i}{\hbar}\int W_0[q_2, t_2; q_1, t_1]V(q, t)\psi(q, t) \tag{10.9}$$

where $\phi(x_2, t_2)$ obeys the free-particle equation (i.e., $V = 0$) and if W_0 obeys

$$\frac{\hbar^2}{2m}\nabla^2_{x_2} W_0[q_2, t_2; q_1, t_1] + i\hbar\frac{\partial}{\partial t_2} W_0[q_2, t_2; q_1, t_1] = i\hbar\delta^3(q_2 - q_1)\delta(t_2 - t_1). \qquad (10.10)$$

the incoming wavefunction $\phi(\vec{x}, t)$ describes a particle from the distant past. We want the wavefunction that results from the interaction with the potential $V(\vec{x}, t)$ in the distant future. We assume an idealization that there is no interaction in the $t \to -\infty$. The initial wavefunction ϕ is therefore a solution of Schrödinger's equation for free particles. The exact wavefunction $\psi(\vec{x}, t)$ then

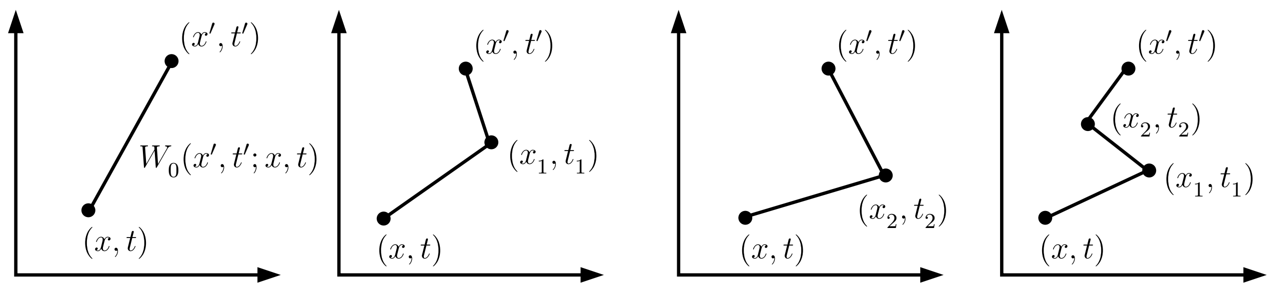


Figure 10.2: scattGraph.

$$\phi_f(\vec{x}', t') = \frac{1}{\sqrt{2\pi\hbar^3}} \exp[i(\vec{k}_f \cdot \vec{x} - \omega_f t')] \qquad (10.11)$$

Scattering matrix S

$$S_{ij} = \lim_{t \to -\infty} \langle \psi_f(\vec{x}, t) | \phi_i(\vec{x}, t) \rangle \qquad (10.12)$$

Creation Annihilation Operator Formulism

Unitarity of the $S-$Matrix

In background-dependent theory a property of the S matrix is its unitarity, when the Hamiltonian is hermitian. However, in background independent theories there is no "true time" with respect to which everything evolves. There is the time coordinate but this is an unphysical parameter in GR - observables do not depend on it - and there is no reason why evolution with respect to an unphysical parameter should be hermitian.

10.3.2 Conventional Scattering Theory in Quantum Field Theory

In the context of a Schrödinger picture of quantum field theory.

The state space at time t_1, $\mathcal{H}_{t_1}$, is a Fock space, on which the Hamiltonian is defined.

$$W[\varphi_1, t_1; \varphi_2, t_2] = \int_{varphi_1, t_1; \varphi_2, t_2} \mathcal{D}\phi \exp \frac{i}{\hbar} S[\phi], \tag{10.13}$$

or

$$W[\varphi_1, t_1; \varphi_2, t_2] =< \varphi_2 | e^{-iH(t_2 - t_1)} | \varphi_1 > \tag{10.14}$$

inserting sums of eigenstates of the energy,

$$\begin{aligned} W[\varphi_1, t_1; \varphi_2, t_2] &= < \varphi_2 | e^{-iH(t_2 - t_1)} | \varphi_1 > \\ &= \sum_{m,n} < \varphi_2 | \Psi_n >< \Psi_n | e^{-iH(t_2 - t_1)} | \Psi_m >< \Psi_m | \varphi_1 > \\ &= \sum_n \Psi_n[\varphi_2] \Psi_n^*[\varphi_1] e^{-iE_n(t_2 - t_1)} \end{aligned} \tag{10.15}$$

we know see why the kernel is a field-to-field propagator,

$$\begin{aligned} \Psi[\varphi_2, t_2] &= < \varphi_2 | \Psi > \\ &= \int D\varphi_1 < \varphi_2 | e^{-iH(t_2 - t_1)} | \varphi_1 >< \varphi_1 | \Psi > \\ &= \int D\varphi_1 W[\varphi_1, t_1; \varphi_2, t_2] \Psi[\varphi_1] \end{aligned} \tag{10.16}$$

Particles' scattering amplitudes can be derived from $W[\varphi_1, \varphi_2, T]$. For instance the 2-point function can be obtained as the analytic continuation of the Swinger function

$$S(x_1, x_2) = \lim_{T \to \infty} \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2 \, W[0, \varphi_1, T] \varphi_1(\vec{x}) W[\varphi_1, \varphi_2, (t_1 - t_2)] \varphi_2(\vec{x}) W[\varphi_2, 0, T]. \tag{10.17}$$

this can be generalised to any n-point function where the times $t_1, \ldots, t_n$ are on the $t = 0$ and the $t = T$ surfaces; these in turn, are sufficient to compute all scattering amplitudes, since time dependence of asymptotic states is trivial.

it is possible to write down creation and annihilation operators in this representation thus formalizing the particle interpretation of the theory:

$$\begin{aligned} a(\vec{k}) &= \int d^3x \; e^{i\vec{k}\cdot\vec{x}} \; \left(\omega_k \psi(\vec{x}) + \frac{\delta}{\delta\psi(\vec{x})} \right) \\ a^\dagger(\vec{k}) &= \int d^3x \; e^{-i\vec{k}\cdot\vec{x}} \; \left(\omega_k \psi(\vec{x}) - \frac{\delta}{\delta\psi(\vec{x})} \right) \end{aligned} \tag{10.18}$$

calculate the extension of the propagation kernel introduced by Feynman in the description of the quantum mechanics of a single particle, that is, the propagation kernel between field configurations defined on infinite spacial hyperplanes at fixed time.

10.4 Primer on the Graviton Propagator

Before we come on to the calculation itself we first review the graviton propagator and some of its properties, including how Newton's law can be derived from it.

to compare the low energy limit of the spin foam to quantities computed by standard QFT perturbative expansion, and the derivation of Newtonian interaction.

First discuss the photon propagator to get the idea in this simpler example.

10.4.1 Perturbation Theory of Scalar QED

The interactions of gravitons with matter can be calculated in the same way as the familiar photon case. Here we develop the quantum theory of a scalar charge coupled to the electromagnetic field.

In electromagnetism the invariance in A_μ comes about because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$

We wish to calculate the four-potential $A_\mu(x)$ produced by a source current $J^\nu(x)$ term,

$$\Box A^\nu(x) - \partial^\mu \partial_\mu A^\nu(x) = 4\pi J^\nu(x) \tag{10.19}$$

We are free to choose the most convenient gauge for the calculation intened to make.

We will choose the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0$$

In momentum space this reads

$$k^\mu A^\mu(k) = 0.$$

$$\Box A^\nu(x) = 4\pi J^\nu(x) \tag{10.20}$$

The solution of the above equation may be systematically formulated using the appropriate Green's function which we call $D_F(x-y)$, the proporgator for electromagnetism.

$$\Box D_F(x-y)(x) = 4\pi\delta^4(x-y). \tag{10.21}$$

The Fourier-transformed proporgator is defined by

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)] D_F(q) \tag{10.22}$$

Using

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp[-iq\cdot(x-y)] \tag{10.23}$$

and making comparison we get

$$D_F(q) = -\frac{4\pi}{q^2} \tag{10.24}$$

10.4.2 Gravitons

As the electromagnetic interaction can be written in terms of a vector current J^μ

the gravitational interaction can be described in terms of the coupling of the energy-momentum tensor $T_{\mu\nu}$ to the gravitational field $h^{\mu\nu}$ with coupling constant κ,

$$\mathcal{L}_{int} = -\frac{1}{2}\kappa T_{\mu\nu} h^{\mu\nu}. \tag{10.25}$$

$$h'_{ab} = h_{ab} - 2\epsilon \partial_{(a}\xi_{b)}. \tag{10.26}$$

This is the change in the metric under an infinitesimal active diffeomorphism along the vector field $\epsilon\xi^a$ while maintaining the requirement that the perturbation be small.

polarization states

The line element

$$ds^2 = dt^2 - dx^2 - [1 - \epsilon h_{23}(t-x)]d\overline{y}^2 - [1 - \epsilon h_{23}(t-x)]d\overline{z}^2. \tag{10.27}$$

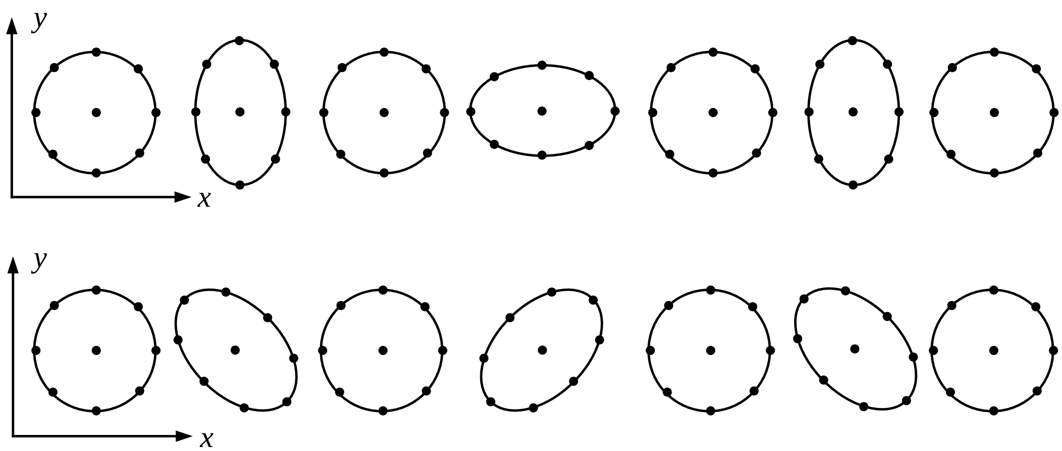


Figure 10.3: gravwavemode.

We now can proceed by perturbation theory and compute Feynmann diagrams to any order in λ.

$$g^{\mu\nu} = (\eta_{\mu\nu} + 2\kappa h_{\mu\nu})^{-1} = \eta^{\mu\nu} - 2\kappa h^{\mu\nu} + 4\kappa^2 h^{\mu}{}_{\beta} h^{\beta\nu} - 8\kappa^3 h^{\mu\beta} h_{\beta\tau} h^{\tau\nu} + \ldots \tag{10.28}$$

We need to calculate the formula for $\sqrt{-g}$

$$\begin{aligned} \sqrt{-g} &= \sqrt{-\eta}\exp\left(\frac{1}{2} tr \ln(\delta^{\beta}{}_{\nu} + 2\kappa h^{\beta}{}_{\nu})\right) \\ &= \exp\left[\frac{1}{2} tr\left(2\kappa h^{\beta}{}_{\nu} - \frac{1}{2}(2\kappa)^2 h^{\beta}{}_{\tau} h^{\tau}{}_{\beta} + \frac{1}{3}(2\kappa)^3 h^{\beta}{}_{\tau} h^{\tau}{}_{\sigma} h^{\sigma}{}_{\beta} + \ldots\right)\right] \end{aligned} \tag{10.29}$$

Plugging in these expressions for $\sqrt{-g}$ and $g^{\mu\nu}$ into the action, we get explicit forms for the coupling of matter and gravity:

$$\begin{aligned}\mathcal{L}(x) &= -\frac{\sqrt{-g}}{2\kappa^2}R + \frac{\sqrt{-g}}{2}(g^{\mu\nu}\partial_\mu\partial_\nu\varphi - m_0^2\varphi^2) \\ &= \frac{1}{2}\int d^4x(\partial^\mu\partial_\mu\varphi - m_0^2\varphi^2) - \kappa \\ +\ldots\end{aligned} \tag{10.30}$$

Any diagram which involves these vertices may now be calculated just as in electrodynamics.

10.4.3 Newton's Law from the Graviton Propagator

Electromagnetism

In electromagnetism the vector potential and current are related by

$$\Box A_\mu(x) = j_\mu(x)$$

or in momentum space

$$A_\mu(k) = -\frac{1}{k^2}j_\mu(k) \tag{10.31}$$

The calculation of scattering amplitudes is in conventional QED is done with the help of propagators connecting currents - Feynman diagrams!

The dynamics of electromagnetism is contained in specification of the interaction between a current and the field by

$$j^\mu A_\mu.$$

or in terms of sources this becomes two currents:

$$-j'_\mu\frac{1}{k^2}j^\mu. \tag{10.32}$$

For a particular choice of coordinates, the vector k_μ may be expressed as

$$k^\mu = (\omega, \mathbf{k}, 0, 0) \tag{10.33}$$

Then the current-current interaction when the exchanged particle has four-momentum k_μ is given by

$$-j'_\mu \frac{1}{k^2} j^\mu = -\frac{1}{\omega^2 - \mathbf{k}^2}(j'_0 j^0 - j'_1 j^1 - j'_2 j^2 - j'_3 j^3). \tag{10.34}$$

The conservation of charge

$$\partial_\mu j^\mu(x) = 0$$

in momentum space becomes the restriction

$$k_\mu j^\mu = 0. \tag{10.35}$$

In the particular coordinate systme we have choosen, this restriction connects the third and fourth components of the currents by

$$j^3 = \frac{\omega}{\mathbf{k}} j^4. \tag{10.36}$$

Substituting this into the amplitude (10.34), we find that

$$-j'_\mu \frac{1}{k^2} j^\mu = \frac{j'_4 j^4}{\mathbf{k}^2} + \frac{1}{\omega^2 - \mathbf{k}^2}(j'_1 j^1 + j'_2 j^2) \tag{10.37}$$

We would wish to take the inverse Fourier transform to convert this to a space-interaction. The first term is independent of the frequency and the inverse Fourier transform of

$$\frac{j'_4 j_4}{\mathbf{k}^2}$$

is

$$\frac{e^2}{4\pi r}\delta(t - t') \tag{10.38}$$

which represents an instantaneous acting Coulomb potential.

Gravity

$$E_0 = -\frac{1}{32\pi}\int d^3x \int d^3y \frac{\rho(\vec{x})\rho(\vec{y})}{|\vec{x}-\vec{y}|}, \qquad m = \int \rho(\vec{x})\; d^3x. \tag{10.39}$$

10.4.4 Physics From Spinfoam Kernel

The spinfoam graviton:

$$W_{\mu\nu\rho\sigma}(x,y) = \frac{1}{\mathcal{N}}\sum_s < s|h_{\mu\nu}|s >< s|h_{\rho\sigma}|s > \Psi_q[s]K[s]. \tag{10.40}$$

Consider the normalized projections

$$W_{ab} := \frac{1}{n_a^2 n_b^2} n_a^\mu(x) n_b^\rho(x) n_b^\sigma(y) W_{\mu\nu\rho\sigma}(x,y). \tag{10.41}$$

In the linearised continuum theory,

$$W_{ab} = f_\zeta(\varphi_{ab})\frac{1}{|x-y|^2}. \tag{10.42}$$

where ζ is a gauge-fixing parameter. This result should be reproduced by the spinfoam gravition in the semiclassical limit.

10.5 Background-Independent Strategy

General Boundaries

In the non-general relativity case - A metric is induced on Σ from the background metric g_{ab} of the manifold $\mathcal{M}$. (b) In the general relativistic case the metric of the surface Σ is actively dragged across with the surface Σ.

Observables get defined on a closed finite boundary. Classical dynamics expresses a set of relations between them. The boundary observables are partial obsevables - they are quantities whose measurement can be operationally defined in principle - in scattering experiments the geometry

we recapitulate

In the theoretical analysis of an experiment, one (arbitrary) coordinate system $\vec{x}, x^o$, and then equations of motion, as (M.-19) with the data in the following way. First we have to locally solve the coordinates $\vec{x}, x^o$ with respect to quantities $f_1 \dots f_4$ that represent the physical objects used as clocks and as spatial reference system

$$f_1(\vec{x}, x^o) \dots f_4(\vec{x}, x^o) \rightarrow \vec{x}(f_1, \dots \dots, f_4), x^o(f_1, \dots, f_4) \tag{10.43}$$

and then express the rest of the remaining fields (f_i $i = 5 \dots N$) as functions of $f_1 \dots f_4$

$$f_i(f_1, \dots, f_4) = f_i(\vec{x}(f_1, \dots, f_4), x^o(f_1, \dots, f_4)). \tag{10.44}$$

If, for instance, $F(\vec{x}, t)$ is a scalar, then for every quadruplet of numbers $f_1 \dots f_4$, the quantity $F_{(f_1 \dots f_4)} = F(f_1, \dots, f_4)$ can be compared with experimental data. This procedure is routinely performed in any analysis of experimental gravitational data - the physical time f_4 representing the the reading of the laboratory clock.

Distance and time separation must be extracted from the dynamical variables. In a scattering experiment we measure incoming and outgoing particles as well as distances between instruments and elapsed time. The former are the matter field variables, the latter are gravitational field variables.

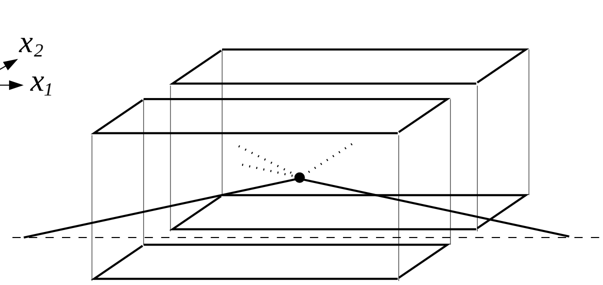


Figure 10.4: scatteringP.

$$W(x, y; \Sigma, \Psi) = \int \mathcal{D}\varphi \; \varphi(x) \; \varphi(y) \; W(\Sigma, \Psi) \; \Psi[\varphi] \tag{10.45}$$

$$W[\varphi] = \int_{\phi|_\Sigma} \mathcal{D}\phi e^{-\frac{i}{\hbar}S[\phi]}. \tag{10.46}$$

The boundary value of the gravitational field determine the geometry of the boundary surface Σ.

the boundary values of the gravitational field on the boundary surface Σ determines the the geometry of Σ, and therefore encodes the relative distance of the measuring devices and the proper time elapsed from the bottom to the top of the spacetime region considered.

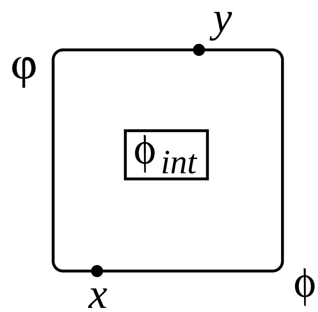


Figure 10.5: Boundfuncxy. Here x and y are points in the 3d metric manifold comprising a boundary with the topology of a 3-sphere S_3 and metric q, i.e. (S_3, q). Metric relations between x and y are determined by q. Under an active diffeomorphism the points x and y change together with q, leaving the metric relations between x and y invariant.

the metric of Σ is not coded in the location of Σ on a manifold: it is coded in the boundary value of the gravitational field on Σ.

Scattering probabilities are determined internally. Scattering amplitude and the spacetime geometry are both encoded in the quantum state. Reflects the fact that there are no *external* reference bodies.

Transition amplitudes are associated with regions of spacetime and states are associated with their boundaries.

But in GR the information on the geometry of a surface is not in Σ. It is in the state of the (gravitational) field on the surface!

Hence, choose $\Psi[\varphi]$ to be a state peaked on a given geometry q of Σ!

Distance and time separations between x and y are now well defined with respect to the mean boundary geometry q.

There is a region, R, of spacetime where the scattering experiment is performed.

expresses the amplitude of having a certain set of initial and final fields, as well as boundary fields, measured by apparatus that are located in spacetime, in the manner described by (the geometry) the surface Σ, this geometry being determined by φ itself.

$$W(x_1, \ldots, x_n) = Z^{-1} \int \mathcal{D}\varphi \; \varphi(x_1) \ldots \varphi(x_n) W[\varphi, \overline{\Sigma}] W[\varphi, \Sigma] \tag{10.47}$$

$$W_0[\varphi, \overline{\Sigma}] = \int_{\phi|_\Sigma = \varphi} \mathcal{D}\phi_{\overline{R}} e^{-S^{(0)}_{\overline{R}}[\phi]} \equiv \Psi_\Sigma[\varphi]. \tag{10.48}$$

we may expect an expression of the form

$$W(x_1, \ldots, x_n; q) = Z^{-1} \int_{\phi|_\Sigma = \varphi} \mathcal{D}\varphi \; \varphi(x_1) \ldots \varphi(x_n) \; \Psi_q[\varphi] \; W[\varphi] \tag{10.49}$$

the boundary can be viewed as expressing initial, final as well as boundary values of ϕ and $W[\phi]$ expresses the corresponding amplitude. The quantum dynamics give probabilities for ensembles of boundary measurements.

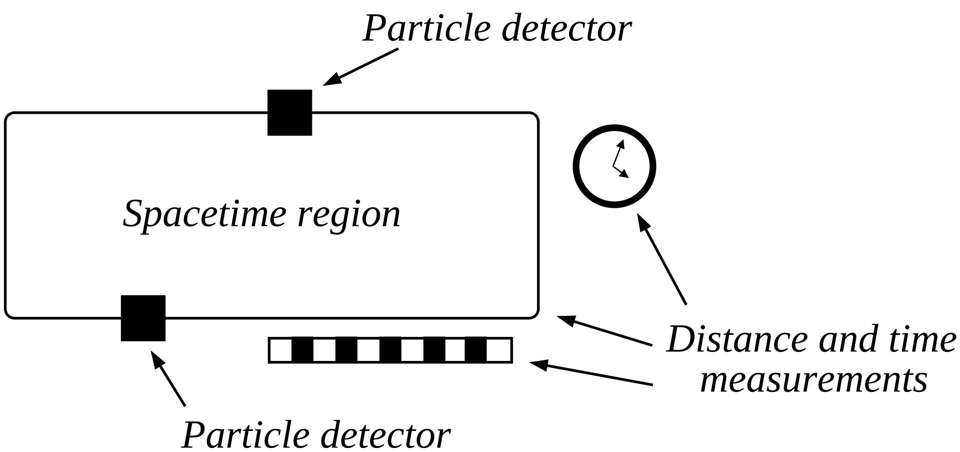


Figure 10.6: RegionPartDec. The boundary geometry is measured by the physical apparatus that surronds a potential interaction region.

In GR distance and time measurements are field measurements like the other ones: they determine the boundary data of the problem.

10.6 Elementary Quantum Mechanics for Comparison Purposes

Consider the two-point function of a single harmonic oscillator with mass m angular frequency ω. This is given by

$$G_0(t_1, t_2) =< 0|x(t_1)x(t_2)|0 >=< 0|xe^{-\frac{i}{\hbar}H(t_1-t_2)}x|0 > \tag{10.50}$$

In the Schrödinger picture, the R.H.S. of (M.-19) reads

$$G_0(t_1, t_2) = \int dx_1 dx_2 \overline{\psi_0(x_1)} x_1 W(x_1, t_1; x_2, t_2) x_2 \psi_0(x_2) \tag{10.51}$$

The following fact is not difficult to prove. $W(x_1, t_1; x_2, t_2)$ is the amplitude to propagate from one point to another in a given time interval. If a particle is initially at position x_2

and we consider W as a function of the final position and time, it is none other than the wave function for a particle with the specific initial condition.

As such, the propagator satisfies the Schrödinger equation at its final point and is the wave function after the elapse of a time $t_1 - t_2$.

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t) e^{i\int_{-\infty}^{t_1} \mathcal{L}(x,dx/dt)} \tag{10.52}$$

is the functional integral restricted to the interval $(-\infty, t_1)$. As is well known from Euclidean theory this gives the vacuum state.

$$t_2 - t_1 \to -iT. \tag{10.53}$$

$$1 = \sum_n^\infty |n >< n|, \qquad H|n >= |n > E_n, \tag{10.54}$$

$$\begin{aligned} < x_1|e^{-HT}|x_2 > &= \sum_n < x_1|e^{-HT}|n >< n|x_2 > \\ &= \sum_n e^{-E_nT} < x_1|n >< n|x_2 > \\ &= \sum_n e^{-E_nT}\psi_n(x_1)\psi_n(x_2)^* \end{aligned} \tag{10.55}$$

$$\psi_0(x) = \lim_{T\to\infty} W[x, 0, T] \tag{10.56}$$

$$\psi_0(x) = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t)\ e^{i\int_{-\infty}^{t_1} \mathcal{L}(x,\dot{x})} \tag{10.57}$$

$$1 = \int_{x(-\infty)=0}^{x(t_1)=x} \mathcal{D}x(t)\ e^{i\int_{-\infty}^{\infty} \mathcal{L}(x,\dot{x})} \tag{10.58}$$

Similarly:

$$\overline{\psi_0(x)} = \int_{x(t_2)=x}^{x(+\infty)=0} \mathcal{D}x(t) e^{i\int_{t_2}^{+\infty} \mathcal{L}(x,dx/dt)} \tag{10.59}$$

Consider an harmonic oscillator, with mass m compute the 2-point function $G(t_1, t_2) =< 0|x(t_1)x(t_2)|0 >$ in canonical formulism

we get

$$< 0|x(t_1)x(t_2)|0 >=< 0|xe^{iH(t_2-t_1)}x|0 >= \frac{1}{2\omega}e^{i\frac{3}{2}\omega T} \tag{10.60}$$

with $T := t_2 - t_1$.

Compute it from the propagator kernel $W[x_1, x_2, T]$, as in (M.-19).

We have

$$G(t_1, t_2) = \frac{1}{\mathcal{N}}\int dx_1 dx_2\, W[x_1, x_2, T]\ \Psi_0[x_1]x_1\Psi_0[x_2]x_2, \tag{10.61}$$

where the normalisation is $\mathcal{N} = \int dx_1 dx_2 W[x_1, x_2, T]\Psi_0[x_1]\Psi_0[x_2]$, and $\Psi_0[x]$ is the vacuum state. Using expressions (M.-19)

$$\begin{aligned} \Psi_0[x] &= \left(\frac{\omega}{\pi}\right)^{1/4} \exp -\frac{1}{2}\frac{\omega}{\hbar}x^2, \\ W[x_1, x_2, T] &= \sqrt{\frac{\omega}{2\pi i \sin\omega T}} \exp\left(-i\frac{\omega}{2}\frac{(x_1^2 + x_2^2)\cos\omega T - 2x_1 x_2}{\sin\omega T}\right) \end{aligned} \tag{10.62}$$

Substituting (10.61) into this, we obtain

$$G(t_1, t_2) = \frac{1}{\mathcal{N}}\int dx_1 dx_2\ x_1 x_2 e^{-\frac{1}{2}x_i A_{ij} x_j} \tag{10.63}$$

with

$$A_{ij} = \omega \begin{pmatrix} 1 + i\cot\omega T & -i/\sin\omega T \\ -i/\sin\omega T & 1 + i\cot\omega T \end{pmatrix}$$

This is easily evaluated, we introduce the term $j_1 x_1 + j_2 x_2$ in the exponent

$$\frac{1}{\mathcal{N}}\int dx_1 dx_2\ x_1 x_2 \exp\left(-\frac{1}{2}x_i A_{ij} x_j + j_1 x_1 + j_2 x_2\right)$$

$G(t_1, t_2)$ is then

$$G(t_1,t_2) = \frac{\partial^2}{\partial j_2 \partial j_1}\left(\frac{1}{\mathcal{N}}\int dx_1 dx_2\; e^{-\frac{1}{2}x_i A_{ij} x_j + j_1 x_1 + j_2 x_2}\right)\Big|_{j=0}$$

This Gaussian integral in the brackets is evaluated by completing the square (see appendix O), the answer is

$$\frac{\pi}{\det A^{1/2}}\exp(j_i A^{-1}_{ij} j_j).$$

Finally, we get

$$G(t_1,t_2) = A^{-1}_{12} = \frac{1}{2\omega}e^{i\omega T}.$$

Up to the vacuum energy contribution $e^{i\frac{1}{2}\omega T}$, the result coincides with the canonical evaluation.

$$G_0(t_1,t_2) = \int \mathcal{D}x(t)\; x(t_1)x(x_2)e^{i\int_{-\infty}^{\infty}\mathcal{L}(x,dx/dt)} \tag{10.64}$$

Lets break it up into

$$G_0(t_1,t_2) = \int dx_1 dx_2\; \overline{\psi_0(x_1)}x_1\; W[x_1,x_2;t_1,t_2]\; x_2\psi_0(x_2) \tag{10.65}$$

where

$$W[x_1,x_2;t_1,t_2] = \int_{x(t_2)=x_2}^{x(t_1)=x_1} \mathcal{D}x(t)\; e^{i\int_{t_2}^{t_1}\mathcal{L}(x,dx/dt)} \tag{10.66}$$

is the path integral restricted to the open interval (t_1,t_2) integrated over the paths that start at x_2 and end at x_1

The normalisation of the measure in (M.-19) is determined by

$$1 = \int \mathcal{D}x(t)\; e^{i\int_{-\infty}^{\infty}\mathcal{L}(x,dx/dt)} \tag{10.67}$$

Breaking this integral in the same manner as we did above one gets

$$1 = \int dx_1 dx_2\; \overline{\psi_0(x_1)}\; W[x_1,x_2;t_1,t_2]\; \psi_0(x_2) \tag{10.68}$$

or equivalently

$$1 =< 0|e^{-\frac{i}{\hbar}H(t_1-t_2)}|0 > \tag{10.69}$$

10.6.1 Covariant Interpretation

$$G_0(t_1, t_2) =< W_{t_1,t_2} \mid \hat{x}_1\hat{x}_2\Psi_0 >, \tag{10.70}$$

in terms of states and operators in the Hilbert space $\mathcal{K}_{t_1,t_2} = \mathcal{H}^*_{t_1} \otimes \mathcal{H}_{t_2}$

(i) "the boundary state"

$$\Psi_0(x_1, x_2) = \overline{\psi_0(x_1)}\psi_0(x_2)$$

represents the joint boundary configuration of the system at the two times t_1 and t_2 if no excitation of the oscillator is present; it describes the joint outcome of a measurement at t_1 and measurement t_2, both detecting no excitations.

(ii) The two operators $\hat{x}_1$ and $\hat{x}_2$ create an ("incoming") excitation at t_2 and an ("outgoing") excitation at t_1; thus the state $\hat{x}_1\hat{x}_2\Psi_0$ can be interpreted as a boundary state representing the joint outcome of a measurement at t_1 and a measurement at t_2, both of them detecting a single excitation.

(iii) The bra $W_{t_1,t_2}(x_1, x_2) = W[x_1, x_2; t_1, t_2]$ is the linear functional coding the dynamics, whose action is on the two-excitation states associates it an amplitude, which can be compared with other similar amplitudes. For instance, observe that

$$< W_{t_1,t_2} \mid \hat{x}_2\Psi_{t_1,t_2} >= 0; \tag{10.71}$$

that is, the probability amplitude of measuring a single excitation at t_2 and no excitation at t_1 is zero.

Finally the

$$1 =< W_{t_1,t_2} |\Psi_0 >; \tag{10.72}$$

which requires that the boundary state Ψ_0 is a solution of the dynamics, in the sense

10.6.2 Regarding Phases of the Boundary State and of the Propagator

There is a phenomenon regarding phase of the boundary state $\Psi_{q_1,p_1,q_2,p_2}(x_1, x_2)$ and the propagator $W_{t_1,t_2}(x_1, x_2)$ that will be important to notice in ascessing the viability of the scattering amplitude calculation in the next section.

We know this from elementary quatum mechanics the phase of the semiclassical state determines where the state is peaked in theconjugate variables (see (7.47)).

$$S[q'] = S[q+\eta] = S[q] + \int dt\eta(t)\frac{\delta S[q]}{\delta q(t)} + \mathcal{O}(\eta^2). \tag{10.73}$$

The two paths contribute exp$iS[q]/\hbar$ and exp$iS[q']/\hbar$ to the PI; the combined contribution is

$$A \approx e^{iS[q]/\hbar}\left(1 + \exp\frac{i}{\hbar}\int dt\eta(t)\frac{\delta S[q]}{\delta q(t)}\right), \tag{10.74}$$

where we have neglected corrections of order η^2. We see that the difference in phase between the two paths, which determines the interference between the two contributions, is

$$\hbar 1 R dt(t) S[q]/q(t).$$

We see that the smaller the value of h, the larger the phase difference between two given paths. So even if the paths are very close together, so that the difference in actions is extremely small, for sufficiently small $\hbar$ the phase difference will still be large, and on average destructive interference occurs.

The exceptional path which extremizes the action, i.e., the classical path, $q_c(t)$. For this path, $S[q_c + \eta] = S[q_c] + \mathcal{O}(\eta^2)$. Thus the classical path and a very close neighbour will have actions which differ by much less than two randomly-chosen but equally close paths (Figure M.-19). This means that for fixed closeness of two paths and for fixed $\hbar$, paths near the classical path will on average interfere constructively (small phase difference) whereas for random paths the interference will be on average destructive.

Thus heuristically, we conclude that if the problem is classical (action $S \gg \hbar$), the most important contribution to the PI comes from the region around the path which extremizes the PI. In other words, the particle's motion is governed by the principle that the action is stationary. This, of course, the principle of least action from which the Euler-Lagrange equations of classical mechanics are derived.

In the classical limit the propagator is approximated by the exponential of a solution to the Hamilton-Jacobi system

Expanding the Hamilton function around q_1 and q_2 to first order

$$S_{t_1,t_2}(x_1, x_2) = S_{t_1,t_2}(q_1, q_2) + \frac{\partial S}{\partial x_1}(x_1 - q_1) + \frac{\partial S}{\partial x_2}(x_2 - q_2) \tag{10.75}$$

but

$$\frac{\partial S}{\partial x_1} = p_1 \quad \text{and} \quad \frac{\partial S}{\partial x_2} = -p_2. \tag{10.76}$$

10.7 3d "Nutshell" Model

Implementation in the full 4d quantum gravity is difficult because of the technical complexity of the theory. It is useful to test and illustrate it in a simple context.

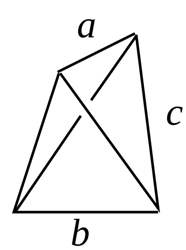


Figure 10.7: equiltetra.

10.7.1 Elementary geometry of equilateral tetrahedra

Consider a tetradhedron embedded in euclidean three-dimension space. Let a be the length of one of the edges (we call it the "top" edge) and b the length of the opposite ("bottom") edge, namely the edge disjoint from the top edge.

Elementary geometry gives (worked examples)

$$\sin\frac{\theta_a}{2} = \frac{b}{\sqrt{4c^2 - a^2}}, \qquad \sin\frac{\theta_b}{2} = \frac{a}{\sqrt{4c^2 - b^2}}, \qquad \cos\theta_c = \frac{ab}{\sqrt{(4c^2 - a^2)(4c^2 - b^2)}} \tag{10.77}$$

It follows that

$$\cos\theta_c = \sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}. \tag{10.78}$$

For later purpose, we consider also the case in which $c \gg a, b$. In this case, we have, to the first relevant order,

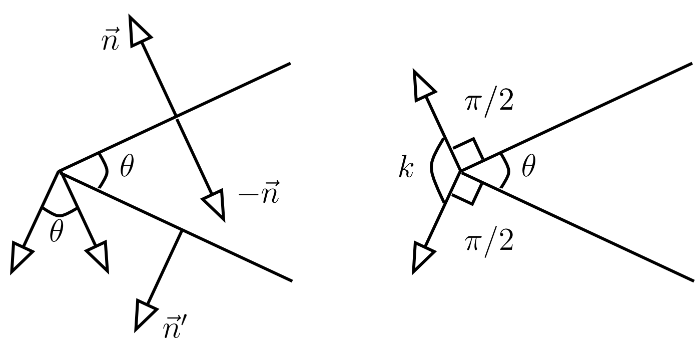
$$\theta_a = \frac{b}{c}, \qquad \theta_b = \frac{a}{c}, \qquad \theta_c = \frac{\pi}{2} - \frac{ab}{4c^2} \tag{10.79}$$

and

$$\theta_a = \frac{\pi}{2} - \frac{\theta_a\theta_b}{4} \tag{10.80}$$

We consider also the three external angles at the edges

$$k_a(a,b,c) = \pi - \theta_a(a,b,c), \qquad k_b(a,b,c) = \pi - \theta_b(a,b,c), \qquad k_c(a,b,c) = \pi - \theta_c(a,b,c). \tag{10.81}$$



(a) (b)

Figure 10.8: (a) The internal angle θ is obtained from $\cos\theta = (-\vec{n})\cdot\vec{n}'$. (b) The external angle k is obtained from $\cos k = \vec{n}\cdot\vec{n}'$, and as can be seen from the diagram $k = \pi - \theta$.

Notice that these are the discretized (discretized due to triangulation) extrinsic curvature of the surface of the tetrahedron. We have denoted them with the letter k, as it is often used for the extrinsic curvature. Using (10.77) and the relation (10.81), the relation between the edge lengths a, b, c and the externl angles k_a, k_b, k_c can be written in the form

$$
\begin{aligned}
a &= \sqrt{4c^2 - b^2}\cos\frac{k_b}{2}, \\
b &= \sqrt{4c^2 - a^2}\cos\frac{k_a}{2}, \\
ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)}\cos k_c\ ;
\end{aligned}
\tag{10.82}
$$

(where we used $\sin(\pi/2 + x) = \cos x$ and $\cos(\pi + x) = -\cos x$) while (10.78) reads

$$
\cos k_c = -\cos\frac{k_a}{2}\cos\frac{k_b}{2.}
\tag{10.83}
$$

10.7.2 Classical theory

Regge action

Consider the action of general relativity, in the case of a simply connected finite region of spacetime $\mathcal{R}$. In the presence of a boundary $\Sigma = \partial\mathcal{R}$ we have to add a boundary term to the Einstein-Hilbert action, in order to have well defined equations of motion. The full action reads

$$
S_{GR}[g] = \int_{\mathcal{R}} d^n x\sqrt{\det g}R + \int_{\Sigma} d^{n-1}x\sqrt{\det q}\ k.
\tag{10.84}
$$

Since the bulk action vanishes on a vacuum solution of the equations of motion, the Hamilton function of GR reads

$$
S[q] = \int_{\Sigma} d^{n-1}x\sqrt{\det q}\ k[q]
\tag{10.85}
$$

Let i be the index labelling the links of the triangulation and denote the length of the link i by l_i. In three dimensions, the bulk Regge action is

$$
S_{Regge} = \sum_i l_i\left(2\pi - \sum_t \theta_{i,t}(l)\right),
\tag{10.86}
$$

where $\theta_{i,t}(l)$ is the dihedral angle of the tetrahedron t at the link i, and the angle in the paranthesis is therefore the deficit angles at i.

We choose the minimalist triagulation formed by a single tetrahedron, and furthermore, consider the case in which the tetrahedron is equilaterial. Then there are no internal links, the Regge action is the same as the Regge Hamiltonian function, and is given by

$$S(a,b,c) = -ak_a(a,b,c) - bk_b(a,b,c) - ck_c(a,b,c). \tag{10.87}$$

The dynamical model and its physical meaning

Define the momenta

$$p_a(a,b,c) = \frac{\partial S(a,b,c)}{\partial a}, \qquad p_b(a,b,c) = \frac{\partial S(a,b,c)}{\partial b}, \qquad p_c(a,b,c) = \frac{\partial S(a,b,c)}{\partial c} \tag{10.88}$$

and equate them to constants

$$p_a(a,b,c) = p_a, \qquad p_b(a,b,c) = p_b, \qquad p_c(a,b,c) = p_c. \tag{10.89}$$

The action $S(a,b,c)$ is a homogeneous function of degree one meaning

$$S(\rho a, \rho b, \rho c) = \rho S(a,b,c). \tag{10.90}$$

Define $a' = \rho$, $b' = \rho$, $c' = \rho c$. Then by differentiating both sides of (10.90) with respect to ρ we find

$$\begin{aligned} S(a,b,c) &= \frac{\partial S}{\partial a'}\frac{\partial a'}{\partial \rho} + \frac{\partial S}{\partial b'}\frac{\partial b'}{\partial \rho} + \frac{\partial S}{\partial a'}\frac{\partial c'}{\partial \rho} \\ &= a\frac{\partial S}{\partial a'} + b\frac{\partial S}{\partial b'} + c\frac{\partial S}{\partial c'} \\ &= a\frac{\partial S}{\partial(\rho a)} + b\frac{\partial S}{\partial(\rho b)} + c\frac{\partial S}{\partial(\rho c)}. \end{aligned}$$

Setting $\rho = 1$ gives

$$S(a,b,c) = a\frac{\partial S(a,b,c)}{\partial a} + b\frac{\partial S(a,b,c)}{\partial b} + c\frac{\partial S(a,b,c)}{\partial a}. \tag{10.91}$$

This allows us to identify immediately

$$p_a(a,b,c) = -k_a(a,b,c), \qquad p_b = -k_b(a,b,c), \qquad p_c = -4k_c(a,b,c). \tag{10.92}$$

Inserting the explicit form (10.77) of the angles, we obtain the evolution equation

$$\begin{aligned} a &= \sqrt{4c^2 - b^2}\cos\frac{p_b}{2}, \\ b &= \sqrt{4c^2 - a^2}\cos\frac{p_a}{2}, \\ ab &= -\sqrt{(4c^2 - a^2)(4c^2 - b^2)}\cos\frac{p_c}{4} \end{aligned} \tag{10.93}$$

10.7.3 Time evolution

10.7.4 Quantum theory

10.7.5 Time evolution in the quantum theory

10.8 Geometry of a 3-Simplex and a 4-Simplex

10.8.1 3-Simplex

10.8.2 4-Simplex

Let us consider five 4d unit vectors $\hat{N}_I \in \mathcal{S}^3$, $I = 1\ldots5$, and introduce the ten angles defined by their scalar products, $\cos\phi_{IJ} = \hat{N}_I \cdot \hat{N}_J$, with the convention $\phi_{II} = 0$. Finally, we define the 5×5 Gram matrix, $G_{IJ} = \cos\phi_{IJ}$. These five vectors are not linearly independent, so we can find $v_I \in \mathbb{R}^5$ such that:

$$\sum_{I=1} v_I \hat{N}_I = 0. \tag{10.94}$$

This means that the 5-vector v_I is a null vector for the Gram matrix G_{IJ} . In particular, we get a constraint on the angles ϕ_{IJ}:

$$\sum_{I=1} v_I \hat{N}_I \cdot \hat{N}_J = 0 \quad \Rightarrow \quad \text{for all } J. \tag{10.95}$$

This constraint can be interpreted geometrically as follows. The five unit vectors define a unique geometric 4- simplex (embedded in $\mathbb{R}^4$) up to a global scale (4-volume of the simplex). They are the (outward) normals to the five tetrahedra of the 4-simplex. The closure condition of the 4-simplex reads exactly as (53) with the v_I being the (oriented) 3-volumes of the tetrahedra. Furthermore, we can differentiate the equation (54) and contract it with the null vector. This gives:

$$\sum_{I,J} v_I v_J \sin\phi_{IJ}\, d\phi_{IJ} \tag{10.96}$$

10.9 Quantum Gravity - The Four Ingredients

Giving meaning to the expression

$$W(x,y;q) = \int \mathcal{D}\varphi\; \varphi(x)\varphi(y)\; W[\varphi]\; \Psi_q[\varphi] \tag{10.97}$$

(i) $\int \mathcal{D}\phi \rightarrow \sum_{s-knots}$;

(ii) $W[\phi] \rightarrow W[s]$ defined by GFT spinfoam model;

(iii) $\Psi_q \rightarrow$ a suitable coherent state on the geometry q;

(iv) $\phi(x) \rightarrow$ graviton field operator from LQG.

(i) a proper definition of the space of 3d fields φ integrated over and a well posed definition of the integration measure.

(ii) An explicit expression for the boundary propagator $W[\varphi]$.

(iii) An explicit expression for the boundary state $\Psi_q[\varphi]$.

(iv) A definition of the field operator $\varphi(x)$.

10.9.1 Space of 3d Fields

$$W(x_1,\ldots,x_n;q) = Z^{-1}\sum_s\; c(s)\; \varphi(x_1)\ldots\varphi(x_n)\; \Psi_q[s]\; W[s] \tag{10.98}$$

10.9.2 The Boundary Propagator $W[\varphi]$ from Group-Field-Theory

A spin foam model can be recast in the form of a rather peculiar field theory over the cartesian product of a group [].

$$W[s] = \int \mathcal{D}\Phi \; f_s[\Phi] \; e^{\int \Phi^2 - \lambda \int \Phi^5}. \tag{10.99}$$

$$W[s] = \sum_{\partial\sigma} \prod_{faces} A_{faces} \prod_{verties} A_{vertex} \tag{10.100}$$

which has a nice interpretation as a discretization of the Misner-Hawking sum over geometries

$$W(^3g) = \int_{\partial g = ^3 g} \mathcal{D}g \; e^{iS_{Eintein-Hilbert}[g]} \tag{10.101}$$

with background triangulations summed over as well.

To first order in λ the only nonvanishing connected term in $W[s]$ is for

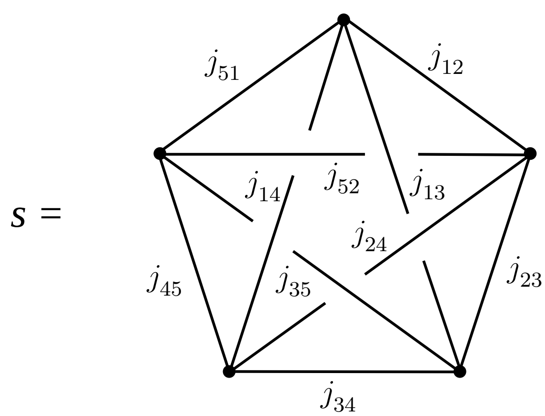


Figure 10.9: tenjsymbRov.

And the dominate contribution for large j is given by the spinfoam σ dual to a *single* 4-simplex. This is

$$W[s] = \frac{\lambda}{5!} \left(\prod_{n<m} dim(j_{nm}) \right) \; A_{vertex}(j_{nm}) \tag{10.102}$$

Regge action

Consider the action of general relativity, in the case of a simply connected finite region of spacetime $\mathcal{R}$. In the presence of a boundary $\Sigma = \partial\mathcal{R}$ we have to add a boundary term to the Einstein-Hilbert action, in order to have well defined equations of motion. The full action reads

$$S_{GR}[g] = \int_{\mathcal{R}} d^n x \sqrt{\det g} R + \int_{\Sigma} d^{n-1} x \sqrt{\det q}\; k. \qquad (10.103)$$

Since the bulk action vanishes on a vacuum solution of the equations of motion, the Hamilton function of GR reads

$$S[q] = \int_{\Sigma} d^{n-1} x \sqrt{\det q}\; k[q] \qquad (10.104)$$

$$S_{Regge} = \qquad (10.105)$$

10.9.3 The Boundary State $\Psi_q[\varphi]$

Linearized quantum gravity gives us a crucial hint, and provides us with a straightforward way to *interpret* semiclassical boundary states. Indeed, consider linearized quantum gravity, namely the well defined theory of a noninteracting spin-2 graviton field $h_{\mu\nu}(x)$ on a flat spacetime with background metric $g^0_{\mu\nu}$. This theory has a preferred vacuum state $|0>$.

can be obtained from the analysis of the coherent states in LQG. Here q is the intrinsic and extrinsic (a coherent state depends on both classical position and momentum) geometry of the closed 3d surface

Choose a boundary geometry q: Let q be the geometry of the 3d boundary (Σ, q) of a spherical 4d ball, with linear size $L \gg \sqrt{\hbar} G$.

Interpret s as the (dual) of a triangulation of this geometry. Choose a *regular* triangulation of (Σ, q); interpret the spins as the areas of the corresponding triangles, using the standard LQG interpretation of spin networks.

This determines the "background" spins $j^{(0)}_{nm} = j_L$. $\Psi_q(s)$ must be peaked on these values. Choose a Gaussian state around these values with α, to be determined.

A Gaussian can have an arbitrary *phase*:

$$\Psi_q[s] = \exp\left\{ -\frac{\alpha}{2} \sum_{n<m} (j_{nm} - j^{(0)}_{nm})^2 + i \sum_{n<m} \Phi^{(0)}_{nm} j_{nm} \right\}. \qquad (10.106)$$

must be a coherent state, determined by coordinate and *momentum*, namely by the intrinsic 3-geometry and the *extrinsic* 3-geometry q!!

The $\Phi^{(0)}_{nm} = \phi$ are the background **dihedral angles**.

The phase factor in this state is important. As we know from elementary quantum mechanics (see equation (7.47)), it determines where the state is peaked in the variables conjugate to the spins j_{mn}. The form of the Regge action is $S_{Regge} = \sum_{n<m} \Phi_{nm}(j_{mn}) j_{nm}$, where $\Phi_{nm}(j_{mn})$ are dihedral angles at the triangles, which are function of the areas themselves and that $\partial S_{Regge}/\partial j_{nm} = \Phi_{nm}$. It is then easy to see that these dihedral angles are the variables conjugate to the spins.

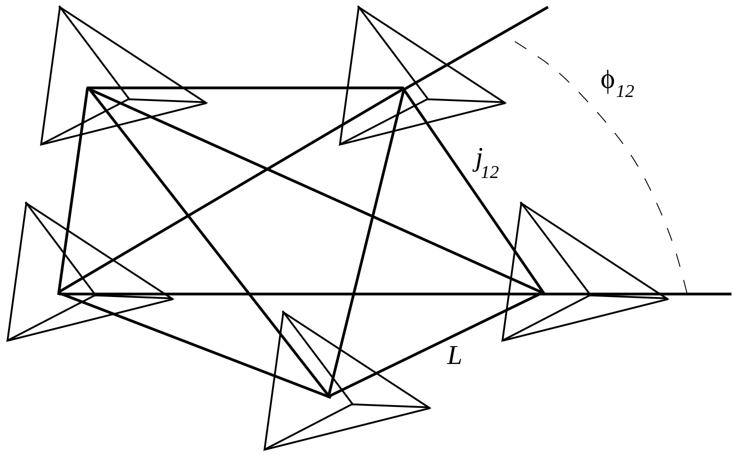


Figure 10.10: DihedAngRov.

10.9.4 Definition of the Field Operator $\varphi(x)$

$$G_q^{abcd}(x,y) = \frac{\sum_s W[s]\ \hat{h}^{ab}(x)\hat{h}^{cd}(y)\ \Psi_q[s]}{\sum_s W[s]\ \Psi_q[s]} \tag{10.107}$$

the fluctuation of the metric over the flat metric

$$h_s^{ab}(x) = (q_{S(s)}^{ab}(x) - \delta^{ab}) = E^{ai}(x)E^{bi}(x) - \delta^{ab} \tag{10.108}$$

We know the action of the operator $\hat{E}^{ai}$ on a spin network state $|s>$,

$$E^{Ii}(n)E_i^I(n)|s>= (8\pi\hbar G)^2\ j_I(j_I+1)|s>, \tag{10.109}$$

whether or not the action is diagonal depends on the orientation of the surface associated to the area operator.

Define

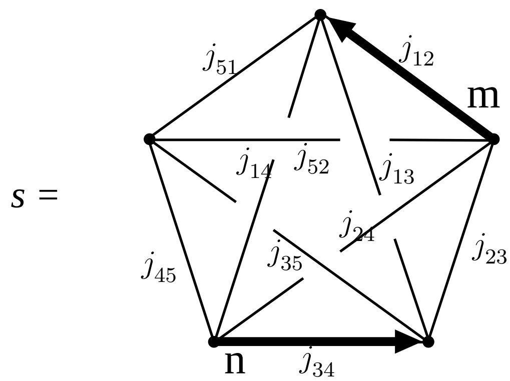


Figure 10.11: tenjsymbRov2.

$$W^{abcd}(x,y;q) = \mathcal{N}\sum_{ss'} W[s'] < s'|h^{ab}(x)h^{cd}(y)|s > \Psi_q[s]. \tag{10.110}$$

$$W(L) = W^{abcd}(x,y;q)n_a n_b n_c n_d \tag{10.111}$$

Standard perturbation theory gives

$$W(L) = i\frac{8\pi}{4\pi^2}\frac{1}{|x-y|_q^2} = i\frac{8\pi\hbar G}{4\pi^2}\frac{1}{L^2} \tag{10.112}$$

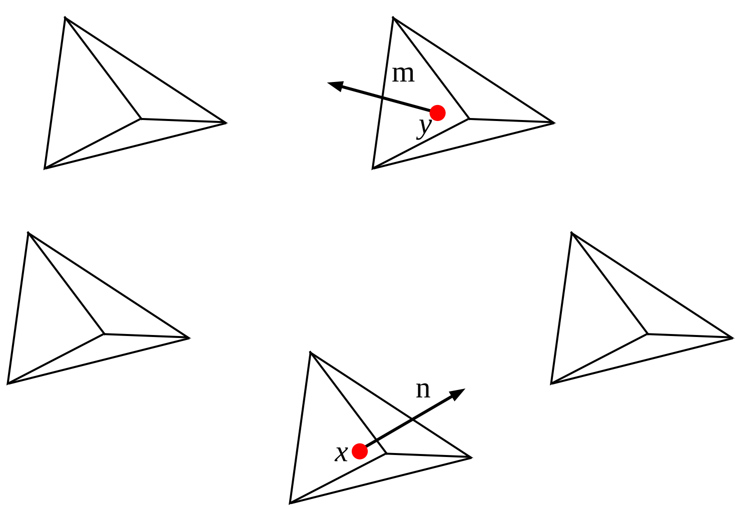


Figure 10.12: DihedAngRov2.

$$\begin{aligned} W(L) &= W^{abcd}(x,y;q)n_a n_b n_c n_d = \\ &= N\frac{\lambda(\hbar G)^4}{5!}\sum_{j_{nm}}(j_{12}(j_{12}+1)-j_L^2)\,(j_{34}(j_{34}+1)-j_L^2) \\ &\quad A_{vertex}(j_{nm})\, e^{-\frac{\alpha}{2}\sum_{n,m}(j_{nm}-j_l)^2-i\Phi\sum_{n,m}j_{nm}} \end{aligned} \tag{10.113}$$

$$A_{vertex} \sim e^{iS_{Regge}} + e^{-iS_{Regge}} + D \tag{10.114}$$

$e^{-i\Phi\sum_{n,m}j_{nm}}$ is a rapidly oscillating phase.

But since

$$S_{Regge}(j_{nm}) = \sum_{n<m}\Phi_{nm}(j)j_{nm} \tag{10.115}$$

and

$$S_{Regge}(j_{nm}) \sim \Phi\sum_{nm}j_{nm} + \frac{1}{2}G_{(nm)(kl)}\delta j_{nm}\delta j_{kl} \tag{10.116}$$

only the "good" component of A_{vertex} survives!

This is the "forward propagating" component of A_{vertex}.

The Gaussian "integration" gives finally

$$W(L) = \frac{4i}{\alpha^2}G_{(12)(34)} \tag{10.117}$$

where $G_{(nm)(kl)}$ is the ("discrete") derivative of the dihedral angle, with respect to the area (the spin').

$$G_{(nm)(kl)} = \frac{\partial\Phi_{mn}(j_{ij})}{\partial j_{kl}}\Big|_{j_{ij}=j_L} \tag{10.118}$$

It can be computed from geometry, giving $G_{(12)(34)} = \frac{8\pi\hbar Gk}{L^2}$, where k is a numerical factor ~ 1

Cfr the "nutshell" dynamics in 3d gravity

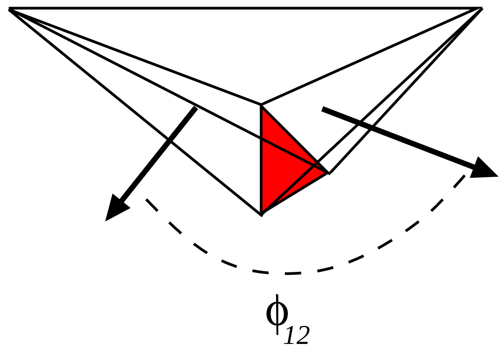


Figure 10.13: TetphiRov.

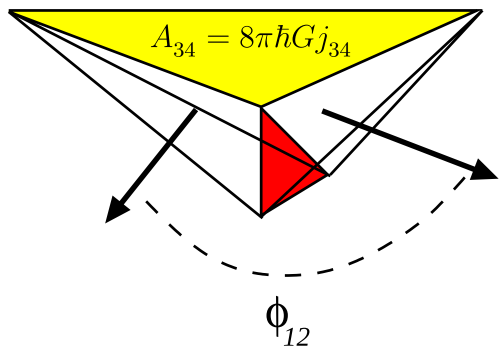


Figure 10.14: TetphiRov2.

Adjusting numerical factors $\alpha^2 = 16\pi^2 k$, this gives

$$W(L) = i\frac{8\pi\hbar G}{4\pi^2}\frac{1}{L^2} = i\frac{8\pi}{4\pi^2}\frac{1}{|x-y|_q^2} \tag{10.119}$$

which is the correct graviton ‘propagator-component’.

→ This is only valid for $L^2 \gg \hbar G$.

For small L, the propagator is affected by quantum gravity effects, and is given by the $10j$ symbol combinatorics.

→ This is equivalent to the Newton law.

10.9.5 Bringing it Together

$$W^{a_1 b_1 \ldots a_n b_b}(x_1, \ldots, x_n) = Z_{LT}^{-1} \sum \; c(s)\; h^{a_1 b_1}(x_1) \ldots h^{a_n b_n}(x_n) \tag{10.120}$$

$$W^{a_1 b_1 \ldots a_n b_b}(x_1, \ldots, x_n) = Z^{-1} \int \mathcal{D}g \; g^{a_1 b_1}(x_1) \ldots g^{a_n b_n}(x_n) \; e^{-S_{EH}[g]} \tag{10.121}$$

10.10 The Complete LQG Graviton Propagator

Some components of the graviton two-point function were computed in the last section using the spinfoam Barrett-Crane vertex. In [285] Rovelii *at el* complete the calculation of the remaining components. They find that, under the above assumptions, the Barrett-Crane vertex does *not* yield the correct long distance limit. They argue that the problem is general and can be traced to the intertwiner-independence of the Barrett-Crane vertex, and therefore to the well-known mismatch between the Barrett-Crane formalism and the standard canonical spin networks.

Alternatives to the Barrett-Crane model have been proposed which are more physically reasonable from which the correct graviton propagator could be recovered: Engle, Pereira and Rovelli [201], Livine and Speziale [200] as well as Freidel and Krasnov [286].

10.10.1 Problems with the Non-diagonal Matrix Elements of the Propagator

$$G^{abcd}(x, y) = < 0|h^{ab}(x) h^{cd}(y)|0 > \tag{10.122}$$

10.10.2 Graviton Propagator from the New Spin Foam Model

Now with the simplicity constraints appropriatly implemented, we need to check the new spin foam model yeilds a better behaved graviton propragator than the Barrett-Crane model does.

10.11 Which is more Fundamental: Particles or Quantum Fields

We investigate the analogy between then otion of "quanta" in the first-quantized theory of "particles" in QFT.

Consider a quantum particle moving in one dimension, having the SHO Hamiltonian

$$\hat{H} = \frac{\hat{p}}{2m} + V(\hat{x}). \tag{10.123}$$

we introduce the operators

$$\begin{aligned}
\hat{a} &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{x} + i\frac{\hat{p}}{\sqrt{m\omega}}\right) \\
\hat{a}^\dagger &= \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\hat{x} - i\frac{\hat{p}}{\sqrt{m\omega}}\right)
\end{aligned} \tag{10.124}$$

Using the commutation relation $[\hat{x}, \hat{p}] = i$, we obtain

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

This, together with the trivial commutation relations $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$, shows that $\hat{a}^\dagger$ and $\hat{a}$ are the raising and lowering operator, respectively. As we speak of one particle, the number operator $\hat{N} = \hat{a}^\dagger\hat{a}$ now cannot be called the number of particles. Instead, we use a more general terminology (applicable to (??) as well) according to which $\hat{N}$ is the number of "quanta". But quanta of what?

$$\mathcal{L}(\phi, \partial_\alpha\phi) = \frac{1}{2}[(\partial^\mu\phi)(\partial_\mu\phi) - m^2\phi^2] \tag{10.125}$$

from the Lagrangian density (10.125) turns out to be

$$\hat{H} = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(\hat{N}_{\mathbf{k}} + \frac{1}{2}\right),$$

with $\hat{N}_{\mathbf{k}} \equiv \hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}}$, which is an analog of the first term in (93??). This analogy is related to the fact that (82??) represents a relativistic-field generalization of the harmonic oscillator. (The harmonic-oscillator Lagrangian is quadratic in $\mathbf{x}$ and its derivative, while (82??) is quadratic in ϕ and its derivatives). The Hamiltonian (97??) has a clear physical interpretation; ignoring the term 1/2 (which corresponds to an irrelevant ground-state energy $\sum_{\mathbf{k}} \omega_{\mathbf{k}} 1/2$), for each $\omega_{\mathbf{k}}$ there can be only an *integer* number $n_{\mathbf{k}}$ of quanta with energy $\omega_{\mathbf{k}}$, so that their total energy sums up to $n_{\mathbf{k}}\omega_{\mathbf{k}}$. These quanta are naturally interpreted as "particles" with energy $\omega_{\mathbf{k}}$.

10.11.1 QFT Notion of a Particle in a Background-Independent Context

definition of quantities that can be interpreted as particle transition amplitudes.

Particle physicists who hold that QFT is fundamentally a formalism for describing processes involving particles, such as scattering or decays.

In fact, it is well known that the Poincare group plays a central role in the particle interpretation of the states of the field: Wigner's celebrated analysis [312] has shown that the particle states are the irreducible representations of the Poincare group in the QFT state space. The defining properties of the particles, mass and spin (or helicity), are indeed the invariants of the Poincare group.

One of the main problems regarding quantum field theory in curved space-time is how to introduce the concept of particles.

The problem becomes even more severe when we go to full quantum gravity as there is no background spacetime.

However, particles are what we observe in experiments.

particle states that can be defined in background independent formulation and the usual particle states of quantum field theory, defined on infinite spacelike region.

Global and Local Particles [311].

particles described by the n-particle Fock states are idealizations that do not correspond to the real objects detected in the detectors. There is no reason for interpreting the Fock basis as "more physical" than any other basis in the state space of QFT.

10.11.2 Normal Modes and Global Particles

$$H_0 = H_1 + H_2 + V = \frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) + \lambda q_1 q_2, \tag{10.126}$$

where p_1, p_2 are the momenta conjugate to q_1, q_2 and $\lambda \ll \omega^2$.

$$q_a = \frac{q_1 + q_2}{\sqrt{2}}, \qquad q_b = \frac{q_1 - q_2}{\sqrt{2}}, \tag{10.127}$$

with eigenvalues

$$\omega_a^2 = \omega^2 + \lambda, \qquad \omega_b^2 = \omega_2^2 - \lambda$$

In terms of these

$$H = H_a + H_b = \frac{1}{2}(p_a^2 + \omega^2 q_a^2) + \frac{1}{2}(p_b^2 + \omega^2 q_b^2)$$

10.11.3 Local Particles

We can define an orthonormal basis in this Hilbert space by diagonalizing a complete set of commuting self-adjoint operators.[2] Let us choose the set formed by H_1 and H_2 where

$$H_1|n_1, n_2 >= E_1^{(n_1)}|n_1, n_2 >, \qquad H_2|n_1, n_2 >= E_2^{(n_2)}|n_1, n_2 >$$

10.11.4 Open Issues

• Is it just chance?

• Other components? Full tensorial structure? (Modesto, Speziale)

• Do higher order terms in λ change the result? (Modesto)

• Other models GFT/C seems to give the same result (Modesto)

• $n-$point functions? Computing the undetermined constants of the non-renormalizable perturbative QFT?

• ...

10.11.5 Inclusion of Matter - The Standard Model

10.11.6 Conclusion

i. **Low energy limit.** 'One component of' the graviton propagator or the Newton's law appears to be correct, to 1st order in λ.

ii. **Barret-Crane vertex.** Only the 'good' component of the $10j$ symbol survives, the others are suppressed by the rapidly oscillating phase in the vacuum state that peaks the state on its correct extrinsic geometry. The BC vertex works.

iii. **Scattering amplitudes.** A technique to compute n-point functions within a background formalism exists.

[2]Just as in the hydrogen atome where we use the self-adjoint operators J, j_z to form an orthonormal basis.

10.12 Scattering Amplitudes from the Master Constraint Programme

10.13 Biblioliographical notes

In this chapter I have relied on the following references:

D. Colosi: *On some aspects of canonical and covariant approaches to quantum gravity.*

H. Nikolic: *Quantum mechanics: Myths and facts.*

10.14 Worked Exercises

Given the equilateral tetrahedron,

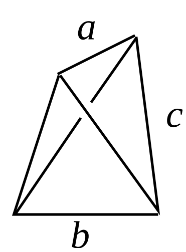


Figure 10.15: The equilateral tetrahedron.

Prove

(a)

$$\sin \frac{\theta_a}{2} = \frac{b}{\sqrt{4c^2 - a^2}}$$

(b)

$$\sin \frac{\theta_b}{2} = \frac{a}{\sqrt{4c^2 - b^2}}$$

(c)

$$\cos \theta_c = \frac{ab}{\sqrt{(4c^2 - a^2)(4c^2 - b^2)}}$$

Proof:

(a) We take the bottom left-hand corner as the origin. Denote the distance from the origin to the midpoint of the top edge by x, then

$$x^2 = c^2 - \frac{a^2}{4}$$

and

$$\frac{1}{2}\sqrt{4c^2 - a^2}\sin\theta_a = \frac{1}{2}b.$$

(b)

By symmetry $a \leftrightarrow b$

$$\frac{1}{2}\sqrt{4c^2 - b^2}\sin\theta_b = \frac{1}{2}a.$$

(c)

We obtain the last expression for the "internal" angle from the scalar product of the normals to two adjacent triangles, by working in the orthonormal basis determined by the top and bottom edges.

The bottom edge is taken to be the $x-$axis, the top edge parallel to be the z-axis. A normal to these $x-$ and $z-$axis, pointing vertical, is taken to be the $y-$axis.

The bottom edge is described by $b\hat{e}_x$. The vertical height, h, from the mid-point of the bottom edge to the midpoint of the top edge is first obtained by calculating the distance from the origin to the midpoint of the top edge:

$$\sqrt{c^2 - \frac{a^2}{4}} = \frac{1}{2}\sqrt{4c^2 - a^2}$$

then

$$h = \sqrt{\left(\frac{1}{2}\sqrt{4c^2 - a^2}\right)^2 - \frac{b^2}{4}} = \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}$$

We define $\vec{c}$ as the vector from the origin to the "out of page" end of the top edge, and $\vec{c}'$ as the vector from the origin to the "in of page" end of the top edge.

$$\begin{aligned}\vec{c} &= \frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y + \frac{a}{2}\hat{e}_z \\ \vec{c}' &= \frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y - \frac{a}{2}\hat{e}_z\end{aligned} \tag{10.128}$$

The normal to the triangle formed between the origin and two ends of the top edge is proportional to

$$\begin{aligned}\vec{c} \times \vec{c}' &= \left(\left\{\frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y\right\} + \frac{a}{2}\hat{e}_z\right) \times \left(\left\{\frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y\right\} - \frac{a}{2}\hat{e}_z\right) \\ &= a\hat{e}_z \times \left(\frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y\right) \\ &= -\frac{a}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_x + \frac{ab}{2}\hat{e}_y.\end{aligned} \tag{10.129}$$

This has norm:

$$\begin{aligned}|\vec{c} \times \vec{c}'| &= \sqrt{\frac{a^2}{4}(4c^2 - a^2 - b^2) + \frac{a^2b^2}{4}} \\ &= \frac{a}{2}\sqrt{4c^2 - a^2}\end{aligned} \tag{10.130}$$

The normal to the triangle formed between the origin and “out of page” end of the top edge and the right-hand end on the bottom edge is proportional to

$$\begin{aligned}\vec{b} \times \vec{c} &= b\hat{e}_x \times \left(\frac{b}{2}\hat{e}_x + \frac{1}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_y + \frac{a}{2}\hat{e}_z\right) \\ &= -\frac{ab}{2}\hat{e}_y + \frac{b}{2}\sqrt{4c^2 - a^2 - b^2}\hat{e}_z\end{aligned} \tag{10.131}$$

This has norm:

$$\begin{aligned}|\vec{b} \times \vec{c}| &= \sqrt{\frac{b^2}{4}(4c^2 - a^2 - b^2) + \frac{a^2b^2}{4}} \\ &= \frac{b}{2}\sqrt{4c^2 - b^2}\end{aligned} \tag{10.132}$$

We then obtain an expresion for the dihedral angle θ_c,

$$
\begin{aligned}
\cos\theta_c &= \left(-\frac{\vec{c}\times\vec{c}'}{|\vec{c}\times\vec{c}'|}\right)\cdot\frac{\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|} \\
&= \frac{1}{\frac{a}{2}\sqrt{4c^2-a^2}}\,\frac{1}{\frac{b}{2}\sqrt{4c^2-b^2}}\times \\
&\quad \left(\frac{a}{2}\sqrt{4c^2-a^2-b^2}\,\hat{e}_x-\frac{ab}{2}\hat{e}_y\right)\cdot\left(-\frac{ab}{2}\hat{e}_y+\frac{b}{2}\sqrt{4c^2-a^2-b^2}\,\hat{e}_z\right) \\
&= \frac{1}{\frac{a}{2}\sqrt{4c^2-a^2}}\,\frac{1}{\frac{b}{2}\sqrt{4c^2-b^2}}\,\frac{a^2b^2}{4} \\
&= \frac{ab}{\sqrt{(4c^2-a^2)(4c^2-b^2)}}. \qquad (10.133)
\end{aligned}
$$

□