Chapter 2

The early beginnings of LQG (1984-1992)

- **Motivation for Non-Perturbative (Background Independent) Quantum General Relativity (Not String Theory!)**
- **Canonical Quatization Of GR.**
- **Ashtekar's New Variables.**
- **Loops and the Loop Representation.**

2.1 Introduction

Loop quantum gravity a non-perturbative, canonical approach is mathematical rigorous, and it demands no new hypotheses each of which is especially important in quantum gravity because the lack of experimental guides.

It rests completely on the classical theory of general relativity and the core tenants of quantum mechanics.

Both General relativity and Quantum Mechanics has been successful in every test that it has encountered. overwhelmingly tested regimes of quantum theory and general relativity.

The idea of quantizing gravity.

We recall what we learned in the previous chapter: one introduces fields, electromagnetic, gravitational, etc fields over the spacetime manifold. Physical theories are still defined over spacetime, but which are invariant under active diffeomorphisms $\phi : \mathcal{M} \to \mathcal{M}$ of the spacetime manifold $\mathcal{M}$ into itself.

The Hamiltonian approach, which is very much like the approach people usually learn in a first course on quantum mechanics: if you know the wavefunction of your system at $t = 0$, the Hamiltonian tells you how it evolves in time from then on.

The canonical, background indepenent, theory was one of the first attempts based on a Hamiltonian formulation of the field equations of general relativity. Has to do with how we can reparametrize the spacial slice.

An additional degree of freedom that arises because we formulate the theory with triads instead of the metric. We can rotate the triad with a element of $SO(3)$ without changing the metric they encode.

2.2 Motivation for Non-Perturbative Quantum General Relativity (as Apposed to String Theory!)

It involves splitting the spacetime metric g_{ab} into two parts, an inert flat background η_{ab}, and another h_{ab} that measures the deviation from the background. One sets $g_{ab} = \eta_{ab} + G h_{ab}$, where G is Newton's constant. So the metric g_{ab} is the fixed metric plus small perturbations, one quantizes the latter as an interacting theory of spin-2 fields living in the flat space. The resulting theory diverges badly and is non-renormalizable, as can be shown by simple power counting arguments.

In quantum electrodynamics one is able to overcome the difficulties with divergencies by absorbing them in a redefination of finite many physical parameters. However, this procedure was not successful with quantum gravity. Even though gravitation is such a weak interaction it is not possible to treat it perturbativly. The reason for this is that transition amplitudes of n-th order in the gravitational constant diverges like a momentum integral of the general form

$$\int p^{2n-1} dp, \tag{2.1}$$

leaving us with an infinite number of ultraviolet divergent Feynman diagrams that cannot be removed by redefining finitely many physical parameters.

These arguments apply to *effective* conventional field theories, and not to quantum gravity proper!

"Man shall not separate what Einstein put together"

Renormalization was one of the guiding principles in the construction of the Standard Model, our current understanding of particle physics phenomenology. Particle physicists draw an analogy to the theory of weak interactions, where non-renormalizability of the initial "Fermi theory" forced to replace it by the renormalizable Glashow-Weinberg-Salam theory. However this analogy overlooks a crucial fact that, in the case of GR, there is a qualitatively new element. Perturbative treatments pre-supose that the space-time can be assumed to be continuous even below the scale.

take recourse to the same perturbative scheme as was done for the electromweak and strong forces. Physically interesting theories exist in their own right and perturbation methods serve as approximation techniques to extract answer to "physically interesting questions.

Perturbation treatments pre-suppose that the space-time can be assumed to be continuous even below the Plank length. This is because of a general feature of perturbation theory that to finite order in perturbation theory does not change the qualative nature of the starting point. We can illustrate this point with a particle in a SHO potential: when we calculate the energy eigenvalues we find that there is a discrete spectrum. If we wanted to treat the same problem using perturbation theory we would start with a free particle and incorporate the effects of the potential term by term. To finite order in perturbative expansion the energy spectrum is still continuous.

In string theory gravity is the exchange by gravitons with the same spacetime behaviour as photons and gluons (except for the spin). General relativity is the discovery that spacetime and the gravitational field are the same entity. Somehow the whole idea of the gravitational interaction as a result of graviton exchange on a background metric contradicts Einstein's original and fundamental idea that gravity is geometry and not a force in the usual sense. One can not begin with Minkowskian spacetime and build, say, the Scharwzchild spacetime using linear gravitational radiation. Therefore such a perturbative description of the theory is very unnatural from the outset and can have at most a semi-classical meaning when the metric fluctuations are very tiny.

Maximally breaks general covariance. Dynamical spacetime is formally equivalent to background independence; by fixing a background metric one wipes out the main point of general relativity. The machinery itself should under go suitable modifications so as to be applicable to the problem at hand. Little attention paid to this background independent approach, probably because it had yet to be backed up by concrete calculations.

" if we remove life from Einstein's beautiful theory by steam-rolling it first to flatness and linearity, then we shall learn nothing from attempting to wave the magic wand of quantum theory over resulting corpse."

The failure of standard treatments may simply be due to this grossly incorrect assumption and a non-perturbative treatment which correctly incorporates the physical microstructure of geometry may well be free of these inconsistencies.

In particular, it should be noted that what fails is the idea of perturbatively quantizing general relativity, and more precisely a given perturbative scheme fails.

It is not true that if a theory is not quantizable perturbatively it cannot be quantized. DeWitt has studied several non-linear Sigma models that do not exist pereturbatively but have been satisfactory quantum theories (constructed, for instance on a lattice)

Ashtekar

> *"The theory is not renormalizable; the perturbative quantum field theory has infinitely many undetermined parameters, rendering the theory pretty much useless as far as making physical predictions are concerned. In terms of the modern view, suggested by the theory of critical the phenomena, there is* ***new physics*** *at short distances* [here he is eluding to non-perturbative, background independent effects] *which is not captured by perturbative techniques and ot which the theory is highly sensitive. ... Renormalizabililty is a criterion of simplicity; such theories are "short-distance insensitive" so that, even in the absence of the detailed picture of the microscopic physics, one can make predictions using just a finite number of* ***effective*** *parameters which are relevant to the scale of observation. It is* ***not*** *a criterion to decide if a theory is consistent quantum mechanically.*

Einstein's theory can be stated as a variational problem: one takes a manifold (the manifold is simply a blank background upon which we place the metric to put the familiar features of space and time into it. We try all possible assignments of metrics to the manifold to find those that minnamise the Einstein-Hilbert action - these are the solutions to Einstein's equations. Nowhere in the action does there appear a fixed background metric, or any fixed geometric structure at all.

Background Independence: Experimentalists devising measurements after a while you come to realise that spacetime coordinates are a convenient mathematical device but has no physical relevance because they can always be made to fall out of the equations.

Gr is not a theory of configurations of fields over a spacetime manifold! So as you might imagine such background independent quantum field theories are very different to ordinary quantum field theories living on a fixed spacetime.

Background Independent Quantum Field Theory is Finite!

In the context of perturbation theory of quantum field theory, **regularization** is manipulating divergent integrals is not well defined, so we need to cut off the integration over d^4p. This renders each Feynmann diagram finite. Lattice regularization: Here, we assume that spacetime is actually a set of discrete points arranged. The lattice spacing then serves as the cuttoff for the spacetime integrals.

This Yang-Mills theory exists without reference to a background metric. Such a theory makes no distinction between small and large distances, as described by a background metric; take the same coordinate system but introduce two distinct metrics. According to one metric the proper distance between two points might be small and the other the proper distance between these two points might be large. The Yang-Mills theory is blind to either metric and as a result, one can argue that, such a *background independent* quantum theory will not suffer from UV divergences.

It should be noted that we are not considering I'm not talking about *passive diffeomorphisms*, which is when you view the same physical system in a different coordinate system; the invariance under passive transformations is a property of any theory of nature. GR is the only theory that is invariant under *active dffeomorphism transformations*.

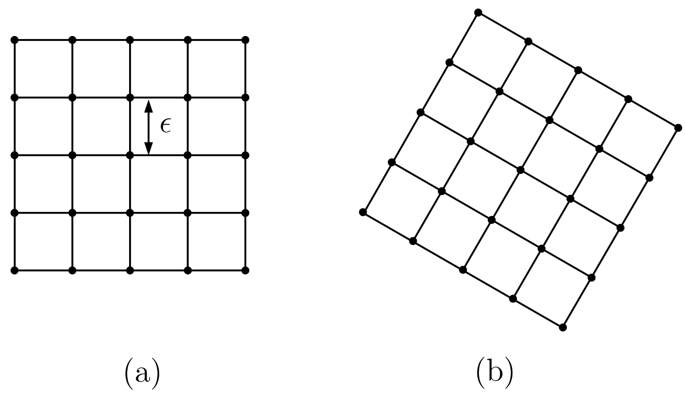

Figure 2.1: The regularized operator depends on the orientation of the lattice.

The lattice spacing serves as the cutoff for the spacetime integrals. a lattice spacing. The regularized operator depends on the choice of orientation of the lattice.

The techniques have to regularize a field theory uses the background metric, be it Minkowski or curved spacetime metric.

because they all depend on the presence of a background metric. new regularization procedures which may be applied to field theories constructed without a background metric. Additional background structure is introduced in the definition of the regularized operator. We introduce a fidicual metric. For given numerical value of the regularization parameter ϵ, if we change the fidicual metric we will get a different regularized operator. Its dependency on the metric is related to its dependency on the regularization parameter ϵ.

This argument can be made more precise more techniqual:

smolin

" A background independent operator must always be finite. This is because the regulator scale and the background metric are always introduced together in the regularization procedure. This is necessary, because the scale that the regularization parameter refers to must be described in terms of a background metric or coordinate chart introduced in the construction of the regulated operator. Because of this the dependence of the regulated operator on the cuttoff, or regulator parameter, is related to its dependence on the background metric. When one takes the limit of the regulator parameter going to zero one isolates the non-vanishing terms. If these have any dependence on the regulator parameter (which would be the case if the term is blowing up) then it must also have dependence on the background metric. Conversely, if the terms that are nonvanishing in the limit the regulator is removed have no dependence on the background metric, it must be finite. "

One must face this problem non-perturbatively. The canonical approach is well suited for the task. In the sixties a non-perturbative quantization programme was formulated within this approach. The idea was to first represent quantum states as functionals of 3-metrics and then select physical states by demanding that they be annihilated by the quantum constraint operators. Unfortunately, the task turned out to be too difficult in the quantum theory and not a single solution could be obtained. In fact, it was not even possible to make sense of the quantum constraint equations.

The approach aims at unifying quantum mechanics and general relativity by developing new non-perturbative techniques from the outset and by staying as close as possible to conventional quantum theory and experimentally tested GR. Superstring theory, based on the simple idea of flat space-time, has always been able to exploit the tools of standard quantum theory. In contrast, the connection approach has had to start from scratch.

short scale structure of Loop quantum gravity introduces a physical cutoff. very high momentum integrations that originate the ultraviolet divergences.

introduced by quantum gravity cure the ultraviolet difficulties of conventional quantum field theory.

2.3 Connections verses Metrics

On its own, the manifold is simply a blank background; the whole point of the metric is to put the familiar features of space and time into it. The metric is all about distances between points. To avoid making assumptions about the space-time metric it must abandon it. time".

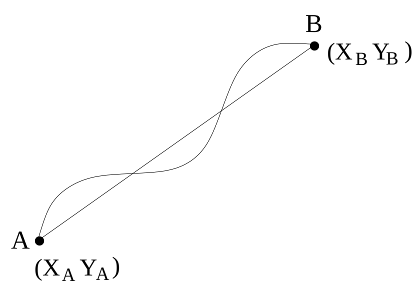

Figure 2.2: With the metric approach, the shortest distance from A to B is defined by coordinates, and a Phythagors-like metric equation for S.

We wish to use variables that are in accordance with with our wish to construct a background independent quantum theory??

While the metric is all about distances between points, a connection captures the notion of parallelism along curves. Connections are powerful, and allow physicists to do detailed calculations but without having to make any assumptions about the nature of space-time they are dealing with. Many mathematical concepts can also be freed from their reliance on specific assumptions of space and time by switching from metrics to connections.

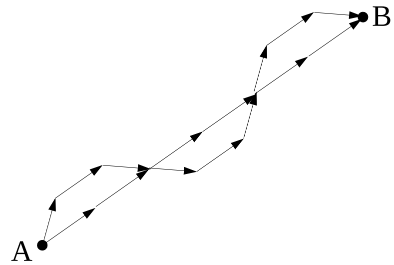

Figure 2.3: With connections, the shortest distance is defined as the route along which every tangent vector is parallel to its neighbour so there is no need for coordinates X and Y.

A key difference is that now there are no background fields whatsoever. Most of the techniques used in standard quantum field theory are deeply rooted in the availability of a flat background metric.

2.4 Canonical Quatization Of GR.

Rovelli [279]

"In nonrelativistic physics, time and spacial position are defined with respect to a system of reference bodies and clocks that are always implicitly assumed to exist and not to interact with the system studied. In gravitational physics, one learns that no body or clock exists which does not interact with the gravitational field: the gravitational field affects directly the motion and rate of any reference body or clock. ..."

In passing from the Lagranegian to Hamiltonian formalism. It is this stage by the use of a particular "time variable" plays such an important role in

the physical states describe only those aspects that are independent of any choice of coordinate

Saying spacetime is 'dynamic' does not mean that it is 'changes' with respect to any given external time. Time is within, not external to spacetime. Accordingly, solutions to Einstein's equations, which are whole spacetimes, do not describe anything evolving. In order to take such an evolutionary form, which is, for example, necessary to formulate an intial value problem, we have to re-introduce a notion of 'time' with reference to which we may speak of 'evolution'. This is done by introducing a structure the somehow allows us to spilt spacetime into space and time.

2.4.1 Sketch of canonical quantization:

1) Pick a Poisson algebra of classical quantities.
2) Represent these quantities as quantum operators acting on a space of quantum states.
3) Implement any constraint you may have as a quantum operator equation and solve to find physical states.
4) Construct an inner product on physical states.
5) Develop a semiclassical approximation and compute expectation values of physical quantities.

Now, the general scheme outlined above has been arund ever since the work of DeWitt

Quantum theory of gravity, I-III by Bryce S. DeWitt, Phys. Rev. 160 (1967), 1113-1148, 162 (1967) 1195-1239, 1239-1256.

However, the problem has always been making the scheme mathematically rigorous, or else to do some kind of calculations that shed some light on the meaning of it all.

see appendix ...

So the first thing you need is a canonical formulation of your theory, in this case general relativity. This was first worked out by Dirac and Bergmann in the late 60's.

This split is necessary to got to phase space and the Hamiltonian formalism, which we need to do in order to formulate a quantum theory.

What is the "configuration space" which is analogous to the set of particle positions in classical mechanics.

One starts by considering the Hilbert actions $S = \int L dt = \int d^4x \sqrt{-g} R$

And considers a foiliation of space-time into space and time,

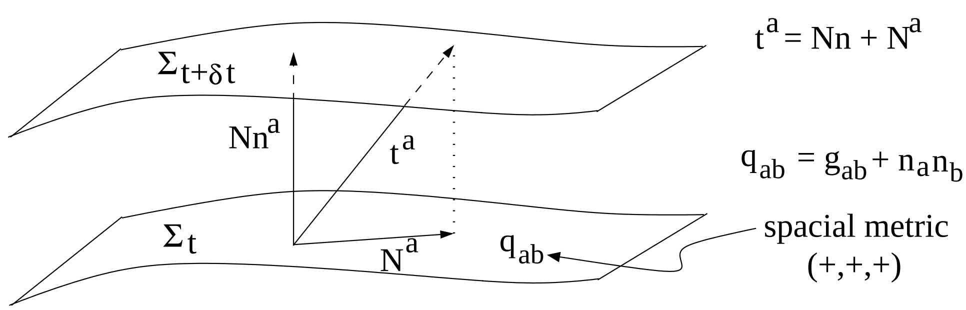

Figure 2.4: A spacetime diagram illustrating the definition of the lapse N and shift vector N^a

$$ds^2 = N^2 dt^2 + q_{ab}(dx^a + N^a dt)(dx^b + N^b dt) \tag{2.2}$$

The quantity $N(t, x^a)$ is called the *lapse* function - it measures the difference between the coordinate time t, and the proper time, τ, on curves normal to the hyper-surfaces Σ_t, the normal being $n_a = (-N, 0, 0, 0)$ in the above coordinates. The quantity $N_a(t, x^a)$ is called the *shift vector* - it measures the difference between a spacial coordinates are comoving if $N_a(t, x^a)$

One can also define an *extrinsic curvature* which describes how a spacial hypersurfaces Σ_t curve with respect to the 4-dimensional spacetime manifold.

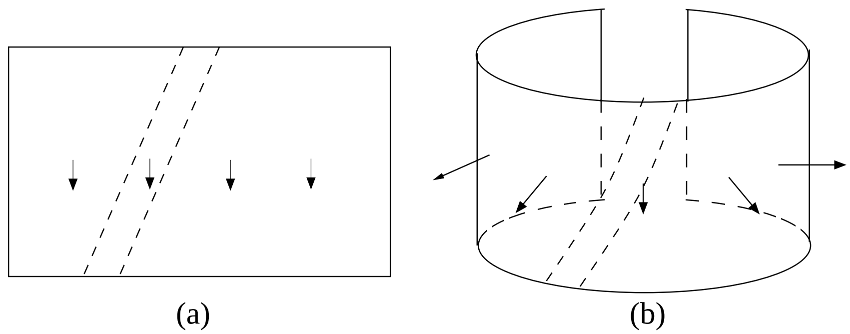

Figure 2.5: Both have the same intrinsic geometry, i.e. both have a induced metric that is flat. However, the plane does not curve with respect to the 3-dimensional space manifold and the spilt-cylinder does. The spilt-cylinder is said to have non-zero extrinsic curvature.

The canonically conjugate variables to q_{ab} is called π^{ab}. We could have chosen a different foliation, and used the corresponding induced metric and its conjugate momenta as our dynamical variables. The foliation used is specified by the shift and lapse functions; these are freely specifiable and do not appear as dynamical variables in the Hamiltonian, instead they appears Lagrangian multipliers imposing constraints...

Canonical momenta are densities. I will denote this from here with a tilde (this turns out to have some importance later).

If one examines the Hilbert action, one notices that there are no time derivatives of the lapse N or shift N^a. This is understood since N merely represents a particular paramatrisation of time-like curves and so is not a dynamical degree of freedom. The shift N^a

As their canonically conjugate momenta vanish. These are conditions that have to be satisfied on any spatial surface. They are called constraints. There are four of them, three associated with the shift are called "momentum" or "diffeomorphism" constraints. They are either denoted as a co-vector (density) C_a or some times, for convenience they are integrated against a fixed test vector and denoted $C(N) = \int d^3x N^a(x) C_a(x)$

The one associated with the lapse is called the "Hamiltonian" constraint and is a (doubly densitized) scalar, which sometimes it is presented integrated against an arbitrary density of weight -1,

$C(N) = \int d^3x N(x) C(x)$

With the variables we are using, the explicit form of the constraints is,

It is a general result that the Hamiltonian vanishes for invariant under parameterization-invariance. We illustrate with a simple example,

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \dot{q}^i - \mathcal{L} = 0. \tag{2.3}$$

Where/what is physical "time" in Quantum Gravity? Note: x0 is not time . Theory is reparametrization invariant. H does NOT generate time translation

$$\exp\left(\frac{-i x_0 H}{\hbar}\right) \Psi[\mathcal{G}] = \Psi[\mathcal{G}] \tag{2.4}$$

The vector or diffeomorphism constraint has a simple geometric interpretation. If one considers its Poisson bracket with a function of the canonical variables, one finds,

That is, the constraint is in the canonical language, the infinitesimal generator of diffeomorphisms on the spacial surface.

I will use q^{ab} to represent the spacial metric.

In particular, the constraints themselves are covariant,

The diffeomorphism algebra is a "Lie" algebra, but the algebra involving the Hamiltonian constraint has non-constant structure functions.

If one performs a Legendre transform, one finds that the theory has a "Hamiltonian" that is a combination of the constraints.

$$H = N^a C_a + NC \tag{2.5}$$

If one considers the Poisson bracket of the "Hamiltonian" with a function of the canonical variables, one gets the "time" evolution along the vector t^a we introduced, interpreted as a "shift" given by the Hamiltonian constraint.

Notice that we are dealing with an unusual theory in the sense that the Hamiltonian vanishes. This is in line with the fact that we introduced a fiducial notion of the time by picking a foliation. The choice of a preferred Lorentz frame specifies a preferred time variable t. However, the choice of inertial frame should have no bearing on physical predictions. The choice of time slicing of space time specifies a preferred coordinate time variable t. The choice of time coordinate t should not explicitly appear.

How would one go about quantizing this theory? One could start by picking as the classical Poisson algebra just the three-metric and its canonically conjugate momenta. One would then consider functions, for instance, of the three metric and represent,

$$\hat{q}_{ab}(q) = q_{ab}(q); \qquad \hat{e}^{ab} = \frac{\delta}{\delta q_{ab}}(q) \tag{2.6}$$

One would then require that the wavefunctions be annihilated by the constraints, written as quantum operator equations. Geometrically this means that the wavefunctions be invariant under the symmetries of the theory.

Here one runs into trouble. The Hamiltonian constraint is a complicated, non-polynomial expression that needs to be regularized. Usual regularization procedure do not preserve the covariance of the theory or require external background structures.

Moreover, we have almost no experience on handling these kinds of wavefunctions. We know very little about the functional space we are dealing with and in particular do not have a measure of integration that would yield a physically valid inner product.

These difficulties stalled the progress on canonical quantization since the 60's.

At the beginning we claimed that the observables of the theory must be finite - that was the easy bit - we must then face the issue of what to compute with a quantum theory like this. One should only consider of physical interest quantities that have vanishing Poisson brackets with all the constraints. No such quantities are known for general relativity (in a compact manifold), We will see in chapter 6, that if the Master constraint programme works out, that they will be able to construct such quantities.

So we must think of this as a functional of $g^{\alpha\beta}$ and its first and second derivatives, namely,

$$\mathcal{L}_G = \mathcal{L}_G(g_{\alpha\beta}, g_{\alpha\beta,\gamma}, g_{\alpha\beta,\gamma\delta}) \tag{2.7}$$

$$\left[\int_\Sigma N\mathcal{H}, \int_\Sigma M\mathcal{H}\right] = i\frac{16\pi}{M_p^2}\int_\Sigma (N\partial_i M - M\partial_i N)h^{ij}\mathcal{H}_j, \tag{2.8}$$

$$\left[\int_\Sigma N^i\mathcal{H}_i, \int_\Sigma M^j\mathcal{H}_j\right] = i\frac{16\pi}{M_p^2}\int_\Sigma [\vec{N},\vec{M}]^k\mathcal{H}_k, \tag{2.9}$$

$$\left[\int_\Sigma N\mathcal{H}, \int_\Sigma M^j\mathcal{H}_j\right] = -i\frac{16\pi}{M_p^2}\int_\Sigma \mathcal{L}_{\vec{M}}N\mathcal{H}, \tag{2.10}$$

where $\mathcal{L}_{\vec{M}}$ denotes the Lie derivative along the vector field M^j and $[\vec{N},\vec{M}]$ denotes the commutator of the two vector fields. The operators $\int_\Sigma N^i\mathcal{H}_i$ generate diffeomorphisms of

Σ and, on classical solutions, the operators $\int_\Sigma N\mathcal{H}$ generate displacements of the hypersurface Σ along the vector field Nn^a, where n^a is the future-pointing spacetime normal to Σ.

These constraints form a first-class system, meaning that the Poisson bracket of them is again a linear combination (generally with phase-space dependent coefficients) of the constraints.

(2.9) says that the vector constraints form a Lie subalgebra which, however, according to (2.10), is not an ideal. This means that the flows generated by the scalar constraints are not tangential to the constraint-hypersurface that is determined by the vanishing of the vector constraints, except for the points where the constraint hypersurfaces for the scalar and vector constraints intersect. This means that generally one cannot reduce the constraints, simply because the scalar constraints do not act on the solution space of the vector constraints. This difficulty persists in the implementation of the constraint equations of operator constraints in canonical quantum gravity and is in fact the source of much difficulty in understanding the dynamics and semiclassical linit.

2.5 Ashtekar's New Variables.

As we have already discussed, a major obstacle to progress in the canonical approach had been the complicated nature of the field equations in the traditional variables, (q_{ab}, p^{ab}). This obstacle was removed in 1984 with the proposal by Ashtekar of a new set of variables for studying canonical quantum gravity. In terms of these, all the constraint become polynomial, in fact, at worst quartic.

with the introduction of Ashtekar's new canonical variables.

I The first step consists of using triads (which with the benefit of hindsight will be denoted $\tilde{E}_i^\alpha(x)$ to encode the metric information,

$$(\det q(x))q^{\alpha\beta}(x) = \tilde{E}_i^\alpha \tilde{E}_i^\beta \tag{2.11}$$

$\tilde{E}_i^\alpha(x)$ represents three spacial vectors. Here i is an index that distinguishes the three vectors and should not be confused with the coordinate basis indices which we denote with greek letters. It turns out that the use of such framefields brings out a different point of view on the connection and curvature, one in which GR has a strong resemblance to particle physics field theories.

The basis system $\tilde{E}_i^\alpha(x)$ fulfilling (2.11) can be chosen somewhat arbitrary. We have the freedom to choose a different basis i.e. $\tilde{E}_i^{\prime\alpha}(x) = M_i^j(x)\tilde{E}_i^\alpha(x)$. The metric $q_{ab}(x)$ is left unchanged by local $SO(3)$ transformations such that

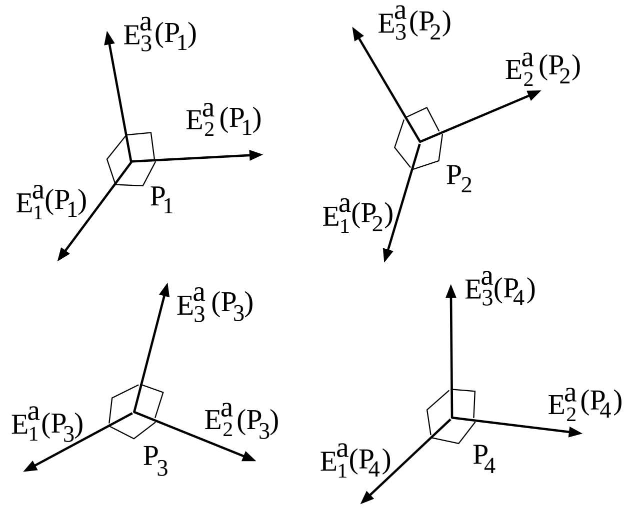

Figure 2.6: Triad field.

$$E_i^a(x^a) \to E'^a_{\ i}(x^a) = O^j_{\ i}(x^a) E_j^a(x^a), \tag{2.12}$$

where $O^j_{\ i}(x^a)$ is a matrix in $SO(3)$ which depends on position in space. When "Physical quantities" are left invariant, such transformations are known as *gauge transformations*, and theories invariant under them are called *gauge theories.*

Now we have introduced these frame fields we now need to know how to compare vectors in frames at different points.

$\nabla_a V_b = \partial_a V_b + \Gamma^c_{ab} V_c$. The same remedy is applied to $\partial_a V^i(x^a)$ and we introduce a connection

$$\omega_{a\ j}^{\ i}(x) \tag{2.13}$$

with two tetrad indicies and one spacetime index.

$$\mathcal{D}_a V^i(x) = \partial_a V^i(x) + \omega_{a\ j}^{\ i} V^j(x) \tag{2.14}$$

The same information contained in $\omega_{a\ j}^{\ i}(x)$ is also contained in the field

$$\Gamma^i_a = \epsilon^i_{\ jk} \omega_a^{\ jk}, \tag{2.15}$$

and it is in this field that is used as a basic variable in the canonical theory.

The canonically conjugate variable is related to the extrinsic curvature, $K_a^i = K_{ab}E^{bi}$ *This presentation follows Barbero gr-qc/9410014*

The constraints become,

$$\epsilon_{ijk} K_a^j E^{ak} = 0 \tag{2.16}$$

Where $\zeta = -1$ for Lorentzian and $\zeta = 1$ for Euclidean space-time

$$\mathcal{S}[e^I, \omega^{IJ}] = \frac{1}{4\kappa} \int_{\mathcal{M}} d^4x \epsilon^{abcd} \epsilon_{IJKL} e_a^I e_b^J \left(R_{cd}^{KL} - \frac{\Lambda}{6} e_c^K e_d^L \right) \tag{2.17}$$

Formulating general relativity with triads instead of metrics was not new, it had been tried before and the problems are similar to using the metric formulation when one tries to build a quantum theory.

Ashtekars's new insight was to introduce a new variable canonically conjugate to the triad, via the (complex) canonical transformation,

$$A_a^i = \Gamma_a^i + iK_a^i \tag{2.18}$$

The first constraint becomes a Gauss law, stating that the theory is invariant under frame rotations (Euclidean group in 3 dimensions, that is, SO(3)).

$$D_a E_i^a = 0 \tag{2.19}$$

The second constraint, which we know correspond to diffeomorphism invariance, are written simply as the vanishing of the Poynting vector.

$$F_{ab}^i E_i^b \equiv \vec{\mathbf{B_i}} \times \vec{\mathbf{E^i}} = 0 \tag{2.20}$$

where $\vec{\mathbf{B_i}}$ is the "magnetic" field constructed from A_{ai}.

The complicated Hamiltonian constraint can be made a simple expression quadratic in the momenta,

$$\epsilon^{ijk} \tilde{E}_i^a \tilde{E}_j^b F_{abk} \equiv \epsilon^{ijk} \vec{\mathbf{B_i}} \cdot \vec{\mathbf{E_j}} \times \vec{\mathbf{E_k}} = 0 \tag{2.21}$$

Another interesting aspect of this reformulation is that we can think of general relativity as an unusual type of SO(3) Yang-Mills theory. In particular its (unconstrained) phase-spaces are the same. General relativity can be viewed as Yang-Mills theory with some extra constraints (and a different Hamiltonian).

We therefore can think of attempting the quantization of general relativity as we would do for a Yang-Mills theory. In electromagnitism, A is like the "position" variable and E is like the "momentum", because the equation

$$\frac{dA}{dt} = E$$

(in the temporal gauge) is like

$$\frac{dq}{dt} = p.$$

Similarly, in Ashtekar's approach A is the configuration variable and E is the canonically conjugate "momentum". Therefore, it is natural to consider a polarization in which one considers wavefunctions of the connection $\Psi[A]$. Notice that this significantly different from what one would have done in the metric representation.

$$\hat{x} = x, \quad -i\hbar\frac{d}{dx} \tag{2.22}$$

$$\hat{A}_i^a = A_i^a, \quad \hat{E}_a^i = -i\hbar\frac{\partial}{\partial A_i^a} \tag{2.23}$$

acting on the functional $\Psi[A]$, such that

$$\hat{C}_i\Psi = 0, \quad \hat{C}_a\Psi = 0, \quad \hat{C}\Psi = 0, \tag{2.24}$$

the **connection representation**.

However, let us instead consider the more general canonical transformation,

$$A_a^i = \Gamma_a^i + \beta K_a^i \tag{2.25}$$

where β is an arbitrary complex number called the Immirzi parameter. Then the constraints become,

$$\tilde{G}_i \equiv \nabla_a \tilde{E}_i^a = 0 \tag{2.26}$$

$$\tilde{V}_a \equiv F_{ab}^i \tilde{E}_i^b = 0 \tag{2.27}$$

$$\tilde{\tilde{S}} \equiv -\zeta\epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{abk} + \frac{2(\beta^2\zeta - 1)}{\beta^2}\tilde{E}_{[i}^a\tilde{E}_{j]}^b(A_a^i - \Gamma_a^i)(A_b^j - \Gamma_b^j) = 0 \tag{2.28}$$

So we see that if we choose the Immirzi parameter to be the imaginary unit (in the Lorentzian case) or one (in the Euclidean case), the constraints become polynomial functions of the fundamental variables.

2.6 Loops and the Loop Representation.

In the of quantization we choose certain functions of the basic variables to be fundamental operators, in terms of which all other operators are constructed. In the case of ordinary text-book quantum mechanics the standard choice we take the basic variables position and momentum themselves to be the fundamental operators. Then using these we construct the Hamiltonian and other operators representing physical observables as made up . A seemingly sensible choice would be to choose the connection representation, (2.23), as it parallels Schödinger representation. We contemplate a different choice - the loop representation.

There is an alternative to thinking about geometry in terms of the curvature fields at each point in space is to instead think about the holonomy around loops in space.

The idea is to take any path that starts at one point and comes back to the same point (this is called a loop), and consider a vector which is carried along the path always keeping the new vector parallel to the previous vector. In curved space the intial and final vectors will be related by a rotation transformation. This rotation transformation is called the holonony of the loop. It can be calculated for any loop, so the holonomy of a curved space is an assignment of rotations to all loops in the space.

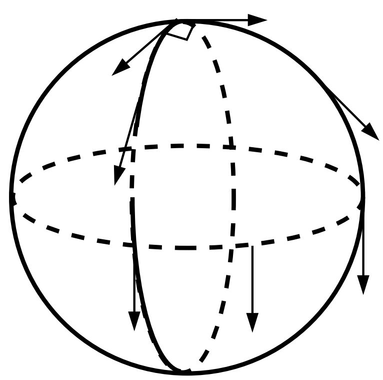

Figure 2.7: SphcurvEx. This is a manifestation of the curvature of the sphere.

In the loop representation the fundamental operator is the **Wilson loop** rather than the connection field operator $\hat{A}_i^a$.

In such a representation, the first constraint just requires that the wavefunctions be invariant under SO(3) gauge invariant, just like wavefunctions of Yang-Mills theory.

An example of wavefunction that is invariant under SO(3) gauge transformation is the trace of the holonomy of the connection along a loop, also called "Wilson loop",

$$h_\gamma[A] = \exp\left(\oint_\gamma A_a(x)\dot{\gamma}^a(x)ds\right) \tag{2.29}$$

where $\dot{\gamma}^a(x) = dx^a/ds$. Let us call $\Delta\phi = \int_\gamma A_a(x)\dot{\gamma}^a(x)ds$. Recalling Stokes's law

$$\oint_\gamma \mathbf{V}\cdot d\mathbf{l} = \int_S \nabla\times\mathbf{V}\cdot d\mathbf{S} \tag{2.30}$$

$$\begin{aligned}\Delta\phi &= \frac{e}{\hbar c}\int_S \nabla\times\mathbf{A}\cdot d\mathbf{S} \\ &= \frac{e\Phi}{\hbar c}\end{aligned} \tag{2.31}$$

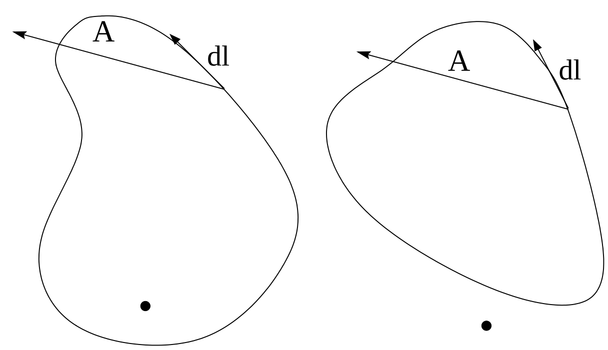

Figure 2.8: loop.

$\Phi_B = \int \mathbf{B}\cdot d\mathbf{S}$ is the magnetic flux, (a gauge invariant quantity), through the surface. Hence, the holonomy is a gauge invariant quantity and so solves the Gauss constraint for Electro-magnetism.

Tells us how we are to compare basis at different points; we are to think of them as related to each other by a rotation.

A short digression into the use of two-component spinors as a mathematical object, first introduced as a mathematical tool to study aspects of calssical GR by Penrose.

Vectors and tensors are defined by their transformation rules, spinors are too..

Going to introduce spinors, but only as a mathematical object - they anything quantum mechanical. We are familiar with the rotation matrix of a vector, from quantum mechanics courses, we know the rotation matrix of a spinor. When one parallel transports a vector around a closed loop, the final direction is related to the initial direction by a rotation matrix. We can read off the angles and then plug them into the rotation matrix of the spinor - this relates the initial direction of the spinor to the final one, (mod some subtleties 1-2 instead of 1-1 - remember when you rotate a spinor through angle 2π this changes the sign of the spinor, one has to rotate through 4π to return to the original value of the spinor). So we have two equivalent descriptions - we can, instead of using vectors, we can encode information about the curvature of space in terms of the rotation of a spinor under parallel propagation around a closed loops. It is more conveinent to use the $su(2)$ valued connection.

so if we transport a spinor along the entire closed curve γ, it will return multiplied by a group element.

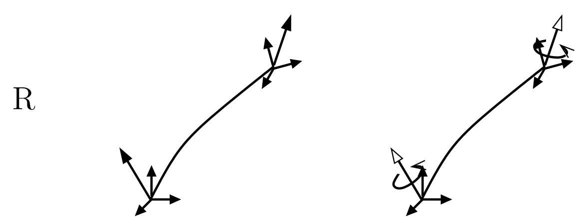

Figure 2.9: introduce spinors.

$$\hat{R} = \exp\left(i\mathbf{a} \cdot \mathbf{J}\right) \tag{2.32}$$

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a} \cdot \sigma\right) \tag{2.33}$$

we specify a path and perform a series of infintesimal parallel transports along that path. If the points $x_0, x_1, x_2, \ldots, x_n$ form a series of infintesimal separated points along a path γ, parallel transport is

$$\begin{aligned} U(x_0, x_n) &= (1 + A_a^i(x_0)(x_1 - x_0)^a \tau_i)(1 + A_a^i(x_1)(x_2 - x_1)^a \tau_i) \ldots \\ &\cdots \times \ (1 + A_a^i(x_{n-1})(x_n - x_{n-1})^a \tau_i) \end{aligned} \tag{2.34}$$

Let s be a parameter of the path γ, running from 0 at x_0 to 1 at x_n. Then define the Wilson line as the power-series expansion of the exponential, with matrices in each term ordered so that higher values of s stand to the left. This prescription is called *path-ordering* and is denoted by the symbol $\mathcal{P}$.

$$\mathbf{H}_\gamma[A] = \mathcal{P} \exp\left(\int_\gamma A_a^i(x)\tau_i \frac{dx^a}{ds} ds\right) \tag{2.35}$$

This transforms as

$$\mathbf{g}(x_0)\mathbf{H}_\gamma[A]\mathbf{g}^{-1}(x_n) \tag{2.36}$$

Consider when γ is a closed path. This transforms as

$$\mathbf{g}(x_0)\mathbf{H}_\gamma[A]\mathbf{g}^{-1}(x_0) \tag{2.37}$$

$$W_\gamma[A] = \mathrm{Tr} W_\gamma[A] \tag{2.38}$$

$A_a^i(x) = A_a^{i\,\prime}(x) + \tau_i \partial_a \phi(x)$

$$W_\gamma[A] = \mathrm{Tr}\mathbf{H}_\gamma[A] = \mathrm{Tr}(g(x_0)\mathbf{H}_\gamma[A']g^{-1}(x_0)) = W_\gamma[A'] \tag{2.39}$$

Wilson loops are gauge invariant:

$$W_\gamma[A] = W_\gamma[A'] \tag{2.40}$$

will use Wilson loops[1] are basis variables instead of the connection. We are replacing 3-d fields with 1-d objects - aren't they going to be rather singular? Yes in QCD. Gravity has a cure for this problem: factorize out this dependencey by averaging over the position of the loop, thus in affect smearing the Wilson loop over the whole od space.

Supplementary: Going from vectors to spinors.

the use of two-component spinors as a mathematical object, first introduced into GR by Penrose.

$$\hat{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.41}$$

$$\begin{aligned} & a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 = \mathbf{a}\cdot\sigma \\ & = \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \mathbf{A} \end{aligned} \tag{2.42}$$

The determinate of $\mathbf{A}$ is obviously $-(a_1^2 + a_2^2 + a_3^2) \equiv -\mathbf{a}\cdot\mathbf{a}$. Consider the effect of a similarity transformation on $\mathbf{A}$,

[1] holonomies are defined without the use of a background metric and so are in accordance with with our wish to construct a background independent quantum theory

$$\mathbf{A}' := \mathbf{U}\mathbf{A}\mathbf{U}^{-1} \tag{2.43}$$

$\det \mathbf{A}' = \det \mathbf{U} \det \mathbf{A} \det \mathbf{U}^{-1} = \det \mathbf{A}$, which means

$$\mathbf{a}' \cdot \mathbf{a}' = \mathbf{a} \cdot \mathbf{a} \tag{2.44}$$

so the similarity transformation induces a transformation on a_1, a_2, a_3 - that preserves the length of the vector $\mathbf{a}$ i.e. a rotation. Let us write the matrix $\mathbf{A}$ as

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \times \begin{pmatrix} \xi_1 & \xi_2 \end{pmatrix} \equiv \begin{pmatrix} \eta_1\xi_1 & \eta_1\xi_2 \\ \eta_2\xi_1 & \eta_1\xi_2 \end{pmatrix} \equiv \begin{pmatrix} a_3 & a_1 - ia_2 \\ a_1 + ia_2 & -a_3 \end{pmatrix} \tag{2.45}$$

then ξ transforms as

$$\xi' = \mathbf{U}\xi \tag{2.46}$$

If one checks one finds that the transformation is precisely that of the rotation transformation on a spinor!

$$\hat{U} = \exp\left(\frac{i}{2}\mathbf{a} \cdot \sigma\right) \tag{2.47}$$

$$\Psi \to \Psi' = \hat{U}\Psi(x) = \exp\left(i\mathbf{a}(x) \cdot \hat{\mathbf{T}}\right)\Psi(x) \tag{2.48}$$

$$\tau \cdot a \to \tau \cdot a' = \exp\left(\frac{i}{2}\mathbf{a} \cdot \sigma\right)(\tau \cdot a)\exp\left(\frac{i}{2}\mathbf{a} \cdot \sigma\right) \tag{2.49}$$

effects a rotation of the vector field r around the axis $\mathbf{n} = \mathbf{r}/a$ by an angle $|a|$.

There are many combinations of loops one can consider. In Figure (2.6) is an example of interconnected loops. We can also envisage Wilson loops that intersect themselves or other Wilson loops, see Figure (2.6), (these are usually called networks rather than loops).

Wilson loops are known to be annihilated by the Gauss law, but not by the diffeomorphism constraint. The constraint has a simple geometric action, though, stating that the constraint shifts the loop,

$$C(\tilde{N})W_\gamma[A] = W_\gamma[A] \tag{2.50}$$

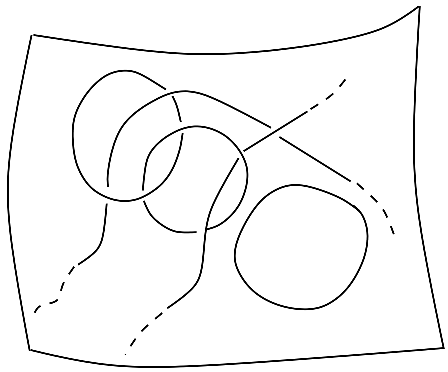

Figure 2.10: There are many combinations of loops one can consider.

Wilson loops are known to be an overcomplete (more later) basis for all gauge invariant functions (*Giles PRD24, 2160, (1981))*

With the Wilson loops we solve the Gauss law constraint. One is left with the diffeomorphism Hamiltonian constraint. It is clear that trying to solve the difeomorphism constraint in terms of the connection field, $A_i^a(x)$, will not be too effective.

The way to handle this is by changing representation to the loop representation - this is illustrated in Figure (2.6).

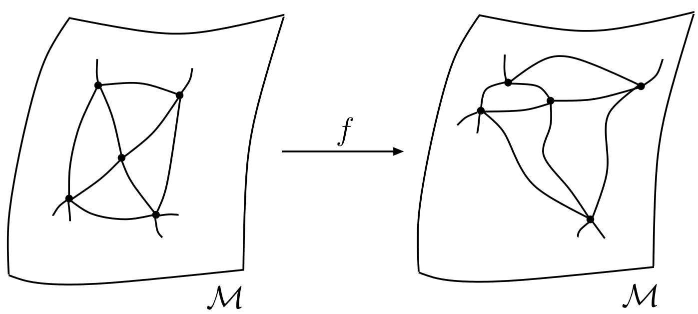

Figure 2.11: Discrete objects are ideal for dealing with the requirement of spatial difeomorphism invariance. Physically relevant information is represented by abstract combinatorics.

Basing ourselves on the fact that Wilson loops are a basis we can formally expand any gauge invariant function of a connection as,

$$\Psi[A] = \text{``}\sum_{\gamma} \Psi[\gamma] W_{\gamma}[A]\text{''} \tag{2.51}$$

At the moment this is only a formal expression, we will see later that we can get better control of it. It is analogous to what one does when one goes to the momentum representation in quantum mechanics

$$\Psi[x] = \int dk \Psi[k] \exp(ikx) \tag{2.52}$$

The coefficients of the expansion are functions of loops. They are the "wavefunctions in the loop representation" given by the "inverse loop transform"

$$\Psi[\gamma] = \int dA \Psi[A] W_\gamma[A] \tag{2.53}$$

Position - Momentum representations of point particle QMs.	Connection - Loop representations of quantum gravity.
$< x\|\psi > \leftrightarrow \psi(x)$	$< A\|\psi > \leftrightarrow \psi(A)$
$< p\|\psi > \leftrightarrow \psi(p)$	$< \alpha\|\psi > \leftrightarrow \psi(\alpha)$
$< x\|p > \leftrightarrow e^{ipx}$	$< \alpha\|A > \leftrightarrow -\mathrm{Tr}\mathcal{P}e^{\oint_\alpha A \cdot dl}$
$\psi(p) = \int dx e^{ipx} \psi(x)$	$\psi(\alpha) = -\int d\mu[A] \mathrm{Tr}\mathcal{P}e^{\oint_\alpha A \cdot dl} \psi(A)$

Why would we want to go to the loop representation? Because (i) this representation it is immediate to write solutions to the diffeomorphism constraint. One simply has to consider functions of loops that are invariant when one applies a diffeomorphism to the loop. That is, they have to be what mathematicians call **knot invariants**.

We can illustrate the basic idea with rotational invariant wavefucntions in two dimensions: take any square integrable function in polar coordinates $\varphi(r, \theta)$

$$\tilde{\varphi}(r, \theta) = \int_0^{2\pi} d\alpha \varphi(r, \theta + \alpha) \tag{2.54}$$

$$\frac{d}{d\theta} \tilde{\varphi}(r, \theta) = 0 \quad \text{ i.e. } \tilde{\varphi}(r, \theta) = \tilde{\varphi}(r) \tag{2.55}$$

This opens an unexpected connection between knot theory and quantum gravity, first noted by Rovelli and Smolin (1988). It also illustrates why using these new variables opens new perspectives on the problem not available with the traditional variables.

2.6.1 Projector Technique (Group Averaging)

Note, (2.54) can be written as the formal equation

$$\tilde{\varphi}(r, \theta) = \int_0^{2\pi} d\alpha \exp\left(i\alpha \hat{p}_\theta\right) \varphi(r, \theta), \quad \text{where } \hat{p}_\theta = -i \frac{d}{d\theta}, \tag{2.56}$$

where $\exp\left(i\alpha \hat{p}_\theta\right) \varphi(r, \theta)$ has the meaning

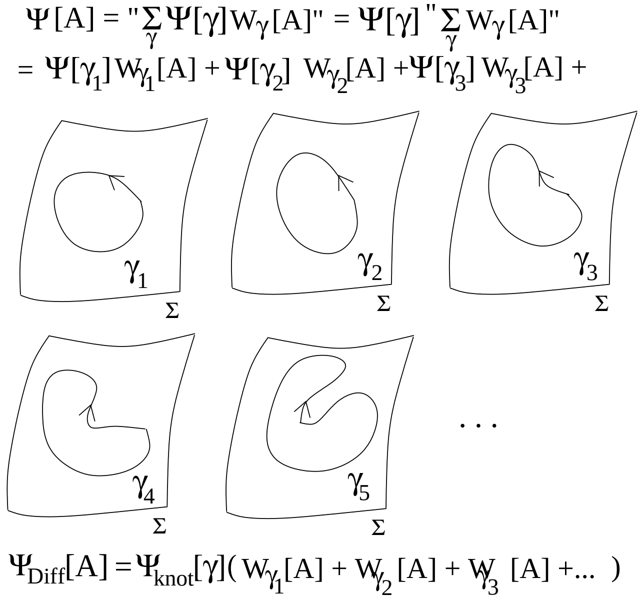

Figure 2.12: We can generate spatially diff invariant wavefunctions $\Psi_{Diff}[A]$ by averaging of the associated loop γ $\Psi[A]$. Each of the loops γ_i for $i = 1, 2, \ldots$ are topologically equivalent but geometrically inequivalent.

$$\exp\left(i\alpha \hat{p}_\theta\right) \varphi(r,\theta) = \exp\left(\alpha \frac{d}{d\theta}\right) \varphi(r,\theta) = \left(1 + \frac{d}{d\theta} + \frac{1}{2}\frac{d^2}{d\theta^2} + \ldots\right) \varphi(r,\theta) = \varphi(r, \theta + \alpha) \tag{2.57}$$

which is evidently the Taylor expansion.

see Appendix A4.1. With (2.54) we were able to generate rotationally invariant functions, $\tilde{\varphi}_{rot}(r,\theta) = \tilde{\varphi}(r)$, from any given function $\varphi(r,\theta)$. This suggests the formal solution to the spatial diffeomorphism constraint $\hat{C}_a \Psi[A] = 0$,

$$\Psi_{Diff}[A] = \int \mathcal{D}[N^a] \exp\left(\int_\Sigma d^3x i N^a(x) \hat{C}_a(x)\right) \Psi[A] \tag{2.58}$$

where $\hat{C}^a$ generates infintesimal spatial transformations and where $N^a(x)$ plays the role of the parameter α in (2.56).

Alternatively, can be viewed as a "delta functional"

$$P \sim \prod_x \delta(C_a(x)) \sim \int \mathcal{D}[N^a] \exp\left(i \int d^3x N^a C_a(x)\right). \tag{2.59}$$

analogous with the representation of the delta function as the integral of an exponential

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dp e^{-ipx}. \tag{2.60}$$

In talk 3 we apply this same technique to solving the Hamiltonian constraint. This will bring us in contact to the idea of spinfoams - the quantum geometry of spacetime.

2.7 The Hamiltonian Constraint

What about the Hamiltonian constraint on a Wilson loop,

$$HW_\gamma[A] = \epsilon^{ijk}\tilde{E}_i^a\tilde{E}_j^b F_{ab}^k W_\gamma[A] \tag{2.61}$$

The Hamiltonian involves products of operators that need to be regularized. Let us be simple minded and consider a point-splitting regularization with a fiducial background metric. Let us also consider a factor ordering in which the electric fields are to the right. We then have

$$F_{ab}^k\ \tilde{E}_i^a\tilde{E}_j^b\ W_\alpha[A] = F_{ab}^k\ \frac{\delta^2 W_\alpha[A]}{\delta A_a^i \delta A_b^j} \sim F_{ab}^k\ \dot{\gamma}^a\dot{\gamma}^b = 0 \tag{2.62}$$

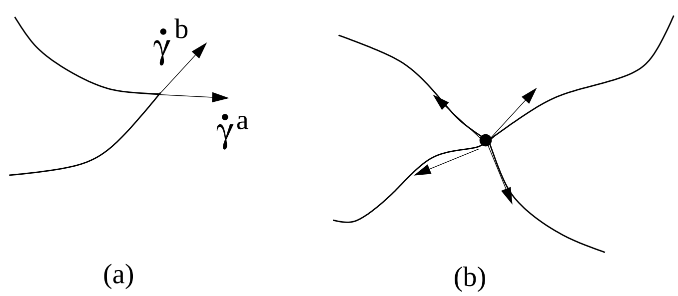

Figure 2.13: If there are kinks as in (a) or intersections as in (b) the product $\gamma^a(x_P)\gamma^b(x_P)$ is not always a symmetric a symmetric quantity, (we are ignoring the very important regularization issue!). Hence, such loops don't solve the Hamiltonian constraint, see (2.62).

[45]

"The central result obtained in the loop representaion is that one can find a large class of solutions of the *full* set of quantum constraint equations. More precisely, we find the general solution of the diffeomorphism constraint and an infinite-dimensional space of solutions of all the constraints."

2.8 Need for Intersecting Loops

That means that not any wavefunction of the gravitational field. In particular, it implies that a function that is non-vanishing on smooth loops only" is not a good candidate.

As we see, just using functions of loops is a problem.

The other point we need to consider the expression of the (doubly densitized) metric in terms of the new variables

$$q^{ab} = \tilde{E}^a_i \tilde{E}^b_i \tag{2.63}$$

And think of the corresponding quantum operator, we see that the metric is, as a matrix, degenerate everywhere on loop states except at intersections; the metric has only one non-vanishing component, along the loop.

We include the cosmological constant term in the Hamiltonian constraint

$$H_\Lambda = H + \Lambda \det q \tag{2.64}$$

where Λ is the cosmological constant and $\det q$ the determinant of the three metric. This term is replaced by tetrad variables

$$H = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} + \frac{\Lambda}{6} \epsilon^{ijk} \epsilon_{abc} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k. \tag{2.65}$$

The cosmological term acting on a Wilson loop

$$\frac{\Lambda}{6} \epsilon^{ijk} \epsilon_{abc} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k W_\gamma[A] \sim \epsilon^{ijk} \epsilon_{abc} \gamma^a \gamma^b \gamma^c \tag{2.66}$$

which is obviously zero for smooth Wilson loops. General relativity with and without a cosmological constant are very different, this suggests these solutions don't have physical relevance.

The volume operator

$$V = \int \sqrt{q} = \int \sqrt{\frac{1}{3!} \epsilon^{ijk} \epsilon_{abc} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k} \tag{2.67}$$

is zero everywhere except at intersections.

For all these reasons we therefore must consider such loops.

2.8.1 Intersecting Loops

The Hamiltonian constraint has a simple action in the same representation in which the diffeomorphism constraint can be solved.

Acting on states the actin of the Hamiltonian constraint is concentrated at the intersection of loops. As a result of this one can find an infinite dimensional space of exact solutions to the Hamiltonian constraint.[64] [65].

These include an infinite space of solutions which consists of all the states which have support on loop that have no intersections. JacobSmolin????

There are, in addition, a large number of states with support on intersecting loops [66]

Supplementary: Solutions for all the constraints that have intersections.

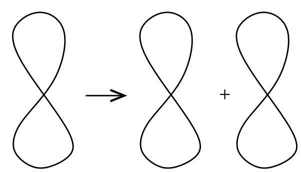

Figure 2.14: Action of the Hamiltonian on a four-valent spin network.

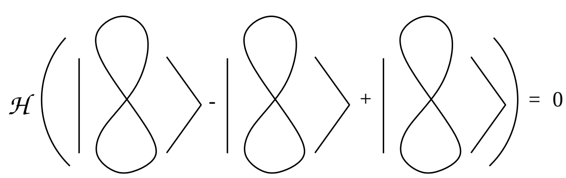

Figure 2.15: Solution of the Hamiltonian of four-valent spin network.

2.9 Difficulties

Many suggestive facts about the possibility of making breakthroughs in solving the quantum constraint equations of canonical quantized general relativity were discovered. But most of the results were formal and loaded with difficulties.

2.9.1 Over-Completeness of Wilson Loops

In the Loop representation quantum states are represented by functions of sets of loops, of the form $\Psi[\alpha, \beta, \gamma]$. We wish to use the space of functions of single loops - in doing so, however, we find difficulty we glossed over in the presentation. We said that the Wilson loops were an "overcomplete" basis of gauge invariant states.

Before I explain what that means, I first introduce some notation for the Wilson loops that we will be needing below. If γ_i represent segments of curves then $\gamma_i \circ \gamma_j$ says that the beginning of γ_j is joined to the end of γ_i. The segment γ is necessarily an open curve. Also the inverse γ^{-1} is the segment with opposite orientation. With this, the Wilson loop satisfies the following properties:

$$\begin{aligned} W[0] &= 1 \\ W[\gamma] &= W[\gamma^{-1}] \\ W[\gamma] &= W[\gamma \circ \lambda \circ \lambda^{-1}] \end{aligned} \tag{2.68}$$

In the fundamental representation of $SU(2)$ (Pauli) matrices are not generic 2×2 matrices, but they actually satisfy certain relationships, like for instance,

$$\mathrm{T}rA\ \mathrm{T}rB = \mathrm{T}r(AB) + \mathrm{T}r(AB^{-1}). \tag{2.69}$$

or in terms of Wilson loops,

$$W_\alpha[A] W_\beta[A] = W_{\alpha\circ\beta}[A] + W_{\alpha\circ\beta^{-1}}[A] \tag{2.70}$$

It means wavefunctions in the loop representation are not free quantities. We are talking about the coefficients in the basis of the overcomplete vectors. It would be good to define a set of vectors that are free of such linear identities. In the above simple network it is easy to do (Talk 2). However, for more complicated networks the number of such identities sky-rockets.

2.9.2 Complex General Relativity and Reality Conditions

Finally, we saw that the canonical transformation defining the Ashtekar connection had the form

$$A_a^i = \Gamma_a^i + iK_a^I \tag{2.71}$$

And the “Immirzi” parameter had to be the imaginary unit for the Hamiltonian to have the simple form we have been considering. This implies that one is dealing with a theory of complex variables. To recover real Lorentzian GR we must impose *reality conditions* at the end of any calculation. However, this did not appear to be easy.

2.9.3 Hamiltonian Constraint and Spacial Diffromorphism

A less immediately obvious but more serious problem is that the Hamiltonian constraint, used by Ashtekar, is a scalar density of weight two, (i.e. it transforms as $\tilde{\tilde{S}} \rightarrow \tilde{\tilde{S}}' = J^2 \tilde{\tilde{S}}$ where J is the Jabcobian of the coordinate transformation); such objects cannot be promoted to quantum operators without breaking spatial diffeomorphism invariance.

2.9.4 Formulism Based on Surfaces of Simultaneity

absolute time provided a natural foliation of spacetime into spacial cross sections.

In special relativity because of relativity of simultaneity they do not agree on a unique time slicing

General covariance tells us that observables of quantum gravity do not depend on the time coordinate. So should we really formulate the quantum theory from a Hamiltonian mechanics based on the space of fields at fixed “time”. We discuss this further in the last chapter.

The notion of a special spacelike surface over which initial data are fixed conflicts with diffeomorphism invariance. A generally covariant notion of observable at a given time, make very little physical sense. A consistent definition of state and observable in a generally covariant context cannot explicitly involve time.

2.9.5 Observables

We may have an infinite number of solutions but have no idea of the physical significance - need interpretation of operators acting on physical states. We need operators corresponding to classical observables. The solutions are useless until a way to interpret them is found.

2.9.6 Locality

Locality is a tricky issue in background independent quantum theories of gravity because there is no background metric with which to measure distances or intervals. It is non-trivial to construct diffeomoerphism invariant observables that that measure local properties of fields.

2.10 Summary

In loop quantum gravity, the gravitational field is described in terms of its effect on the parallel transport of spin-1/2 test particles, leading to a formalism in which the quantum geometry of space is described using Wilson loops. This theory makes specific predictions concerning the discreteness of geometrical observables such as area and volume.

Amazingly, the Hamiltonian constraint has a simple action in the same representation in which the diffeomorphism constraint can be solved. Acting on states the action of the Hamiltonian constraint is concentrated at the intersection of loops. As a result of this one can find an infinite dimensional space of exact solutions to the Hamiltonian constraint.

Many suggestive facts about the possibility of making breakthroughs in solving the quantum constraint equations of canonical quantized general relativity were discovered. But most of the results were formal and loaded with difficulties.