Chapter 3

Formal Developments

- **Spin networks: how to generate independent Wilson loops**
- **Well defined operators: areas and volumes.**
- **Functional integration and functional calculus.**
- **Uniqueness of kinematic representation.**
- **Spatial Invariant Hilbert Space.**

In the previous talk we reviewed how a new canonical formulation of general relativity appeared to offer attractive possibilities, in particular

The phase space was identical to that of an SO(3) Yang-Mills theory.

One could solve the Gauss law using Wilson loops.

One can introduce a representation (the loop representation) where the diffeomorphism constraint can be naturally handled through knot invariants.

Promising results appeared when analyzing formal versions of the quantum Hamiltonian constraint.

We however found several aspects that need sharpening:

The calculations involving the Hamiltonian were formal, unregulated ones. We need more experience regulating operators in this formalism.

The Wilson loops were an over-complete basis of functions and that meant that wave-functions in the loop representations were bound up by complicated identities.

The variables that made the Hamiltonian constraint simple were complex variables requiring us to enforce additional reality conditions to make sure we were obtaining real general relativity.

We will see how developments that happened early in the 90s helped significantly with these issues.

3.1 Spin networks: how to generate independent Wilson loops.

Spin networks Rovelli and Smolin 1994 gr-qc/9411005

But also Penrose in the 60's Witten 1991, Kauffman Lins 90's

linear combinations of products of Wilson loops. All Wilson loops will be gauge invariant. We wish to do away with the identities that bound the wavefunctions in the loop representation. If we recall, we had

$$W_\alpha[A]W_\beta[A] = W_{\alpha\circ\beta}[A] + W_{\alpha\circ\beta^{-1}}[A] \tag{3.1}$$

Or, graphically,

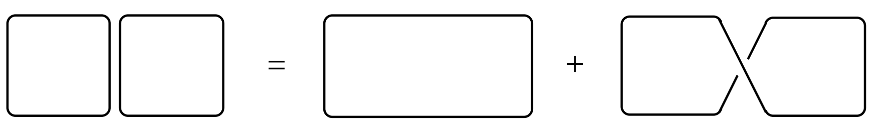

Figure 3.1: The Mandelstam identity.

Which implies,

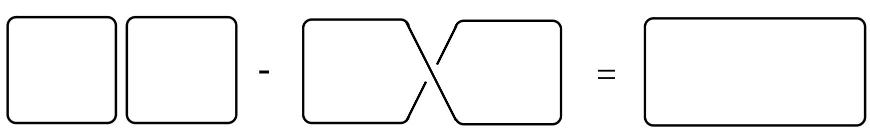

Figure 3.2: .

Let us put this a different way: suppose one has a lattice, and on this lattice we ask "which Wilson loops can I set up which are independent?".

Ignoring multiple windings, the possibilities are

And are not independent. From the previous slide we learnt that one could choose and the symmetrized combination of the other two.

Figure 3.3: .

The moral is: in the center link if we take "no loop" and "symmetrized" loops, we exhaust all independent possibilities.

The result you'll have to half-believe me is that this construction is general. that is, given any graph, you can construct independent wilson loops by choosing these two possibilities for each line.

How could this be? The idea is that in considering Wilson loops we had unnecessarily straightjacketed into considering the fundamental representation of SO(3). In general one can construct a "generalized holonomy". To do this first consider a graph embedded in 3d with intersections of any order.

Now, along each line we consider a holonomy of an SO(3) connection in the j-th representation. We can generate a gauge invariant object by contracting the holonomies at the vertices using invariant tensors for the group.

The resulting object is a generalization of the Wilson loop.

Considering higher order representation is tantamount to the "symmetrization" of the lines we discussed in the simple example.

Spin networks are linear combinations of Wilson loops. They can simply be seen as an efficient graphical device for keeping track of which combinations are independent. They are also very natural to work with.

Draw a graph in space and associate a half-integer (or irreducible representation of $SU(2)$) with each link. We denote the graph by α. For a given a connection $A_a(x)$, we parallel propagate a spinor along the link - this gives you an element of $SU(2)$. The corresponding representation gives me a matrix. We tie these matrix indices by interwiners that make the function gauge invariant. These spin networks provide a basis of states.

As put by John Baez

"...a state of quantum geometry assigns an amplitude to any system of spinning test particles tracing out paths in space, merging and splitting. These are described by spin networks: graphs with edges labelled by spins together with 'interwining operators' at vertices saying how the spins are routed. These are described using the mathematics of the group $SU(2)$."

The total Hilbert space can be written as a direct sum of finite dimensional ordinary spin-system Hilbert spaces - we all learned about in 1st year quantum mechanics courses.

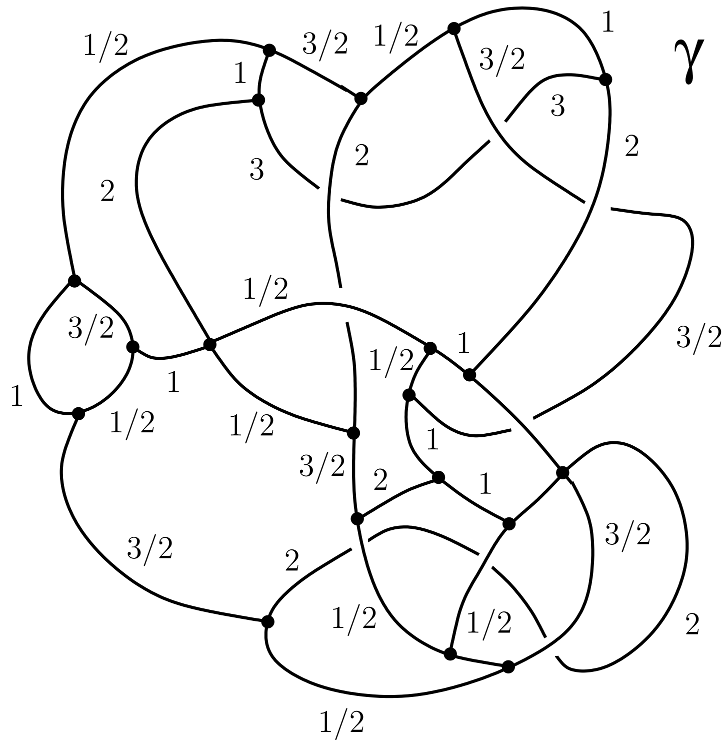

Figure 3.4: A state of quantum geometry $\Psi[\gamma]$ assigns an amplitude to any system of spinning test particles tracing out paths in space, merging and splitting. These are described by spin networks γ: graphs with edges labelled by spins together with 'interwining operators' at vertices saying how the spins are routed.

A quantum state of geometry of space assigns an amplitude to any spin network. operators corresponding to geometric quantities: lengths, areas and volumes.

The space around us is described by a huge linear combination of enormous spin networks that approximates the seemingly smooth geometry we see at distances much large than the plank length ($\sim 10^{-35}$ meters).

The internal labels i, j, k are positive integers determined by the external labels a, b, c: obviously $a = i + j$, $b = j + k$ and $c = i + k$ so that

$$i = (a + c - b)/2, \qquad j = (b + c - a)/2, \qquad k = (a + b - c)/2, \tag{3.2}$$

as in quantum mechanics of adding angular momentum the labels must satisfy the triangle inequalities

$$a + b \geq c, \qquad b + c \geq a, \qquad a + c \geq b \tag{3.3}$$

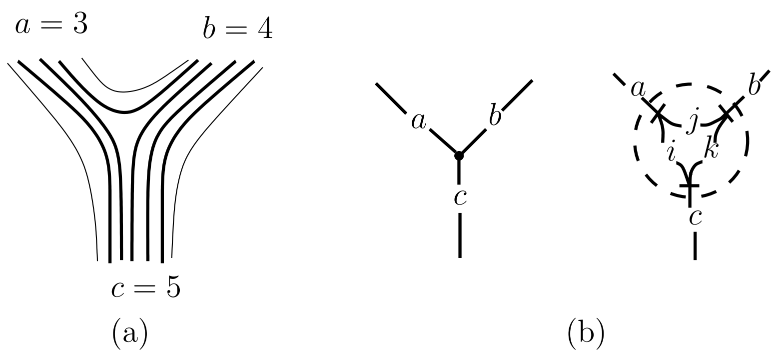

Figure 3.5: Trivalent nodes have to obey Clebsch-Gordon conditions.

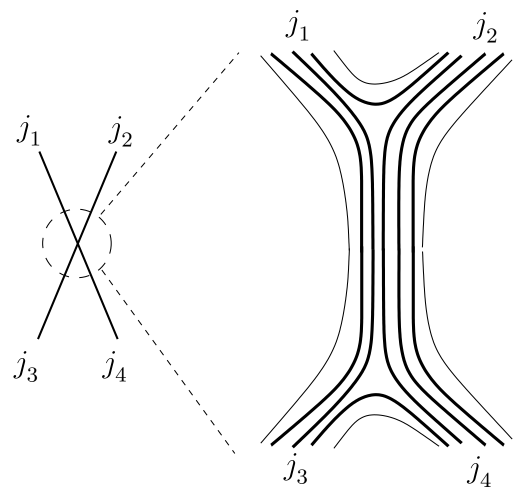

Figure 3.6: A node of valence 4 can be reduced to two trivalent nodes. The different ways of combining 4 spins

3.1.1 Some Maths of Spin Networks

mathematical - can skip this section more details given in appendix [].

$$h_e(A) \otimes h_e(A) \tag{3.4}$$

$$\underline{\underline{a}} \otimes \underline{\underline{b}} := \begin{pmatrix} a_{11}\,\underline{\underline{b}} & a_{12}\,\underline{\underline{b}} \\ a_{21}\,\underline{\underline{b}} & a_{22}\,\underline{\underline{b}} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ c_{21} & c_{22} & c_{23} & c_{24} \\ c_{31} & c_{32} & c_{33} & c_{34} \\ c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \tag{3.5}$$

$i, j, k, l = 1, 2,\ A, B = 1, 2, 3, 4$ The row and column labels of C are ***composite*** labels $a_{ij} b_{kl} = c_{AB}$

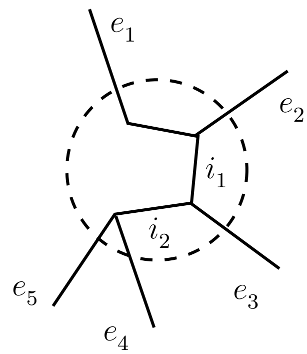

Figure 3.7: valdecomp. The decomposition of a higher valent intersection valence 4...

$$(A \otimes B)(C \otimes D) = AC \otimes BD \tag{3.6}$$

$\underline{\underline{M}}$ a unitary representation of $SU(2)$, i.e., it satisfies

$$[\underline{\underline{M}}_i, \underline{\underline{M}}_j] = \epsilon_{ijk}\, \underline{\underline{M}}_k, \quad \text{for } i, j, k = 1, 2, 3. \tag{3.7}$$

block diagonalize $\underline{\underline{M}}$, that is put it into the form:

$$\underline{\underline{M}}_i = \begin{pmatrix} \underline{\underline{A}}_i & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{B}}_i & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{C}}_i \end{pmatrix}. \tag{3.8}$$

It is obvious that the satisfy the $SU(2)$ algebra individually, $[\underline{\underline{A}}_i, \underline{\underline{A}}_j] = \epsilon_{ijk}\, \underline{\underline{A}}_k$, ect.

$$\underline{\underline{M}}_i = \underline{\underline{A}}_i \oplus \underline{\underline{B}}_i \oplus \underline{\underline{C}}_i \quad \text{for } i = 1, 2, 3. \tag{3.9}$$

$$\underline{\underline{a}} \otimes \underline{\underline{b}} = \underline{\underline{A}} \oplus \underline{\underline{B}} \oplus \underline{\underline{C}} \tag{3.10}$$

The irreducible representations of $SU(2)$ have dimensions $j(j+1)$.

$A \otimes B$

$$a_{ij} b_{kl} = c_{ik;jl}. \tag{3.11}$$

the row and column labels of the matrix elements of C are *composite* labels: the row label ik, is obtained from the row labels of the matrix elements of A and B and the column label, jl is obtained from the corresponding column labels.

Class function

$$f(x, y, \ldots, z) = f(g^{-1}xg, g^{-1}yg, g^{-1}\ldots zg) \tag{3.12}$$

The character of an irreducible representation $\chi_i(x) := \sum_\alpha M_{\alpha\alpha}(x)$. The obvious property that $\chi_i(x) = \mathrm{Tr}(\mathbf{U}^{-1}\mathbf{M}\mathbf{U}) = \mathrm{Tr}\mathbf{M}$. Class functions can be expanded (**Peter-Weyl theorem**)

$$f(x) = \sum_i a_i\, \chi_i(x) \tag{3.13}$$

where summation is over the irreducible representations. This is easily generalized to case of more than one arguement

$$f(x, y, \ldots, z) = \sum_{i,j,k} a_{ij\ldots k}\, \chi_i(x)\chi_j(y)\ldots\chi_k(z) \tag{3.14}$$

The *Peter-Weyl Theorem* applied to $U(1)$ gives the Fourier series theory:

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \frac{e^{in\theta}}{\sqrt{2\pi}}, \tag{3.15}$$

where $f(\theta) \in L^2(U(1))$.

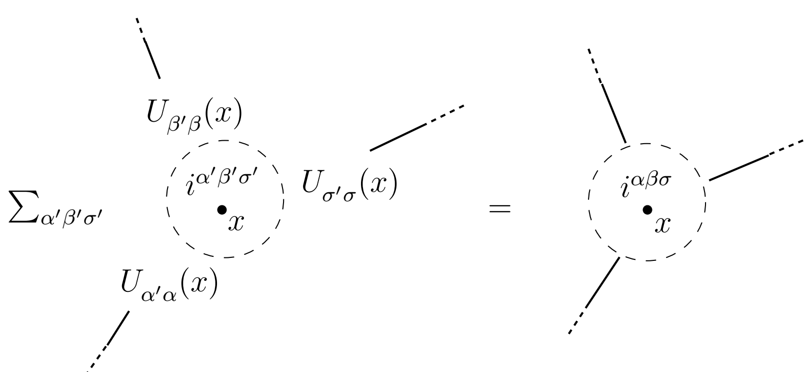

Figure 3.8: $i^{\alpha\beta\sigma}$. no free ends if diagrams closed will be Gauss gauge invariant.

$$h_e(A) \mapsto U_{\alpha\alpha'}(x)h_e(A)U_{\alpha\alpha'}(y) \tag{3.16}$$

where x is the starting point of the link and y the end point.

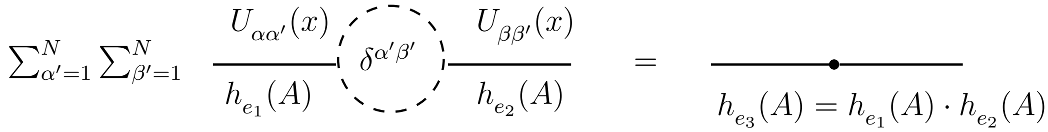

Figure 3.9: A node of valence 2 has $i^{\alpha\beta} = \delta^{\alpha\beta}$ as the trival interwiner.

the interwiners are constants $i^{\alpha\beta\sigma}$ $(\alpha, \beta, \sigma = 1, 2)$ such that

$$i^{\alpha'\beta'\sigma'} = i^{\alpha\beta\sigma}\, U_{\alpha\alpha'} U_{\beta\beta'} U_{\sigma\sigma'}. \tag{3.17}$$

In the case where the interwiners are unabiguisly defined, but for greater than 3 there are different choices. If we draw analogy with composition of angular momentum. This reflects the fact that there is more than one way to add 3 angular momentun.

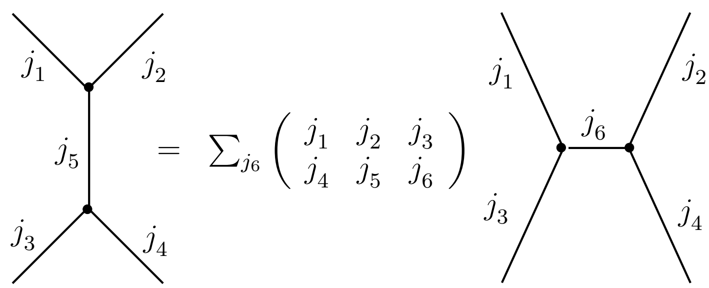

Figure 3.10: .

$$[\rho_{j_1}(H_{e_1}(A))]^{\alpha_1}_{\beta_1} \cdots [\rho_{j_n}(H_{e_1}(A))]^{\alpha_n}_{\beta_n}\, v^{\beta_1 \ldots \beta_n} = v^{\alpha_1 \ldots \alpha_n} \tag{3.18}$$

Case (a) ψ_{AB}:

$$\epsilon_{AB}\epsilon^{CD} = \delta^C_A \delta^D_B - \delta^D_A \delta^C_B. \tag{3.19}$$

This identity can be incorpriated

$$\begin{aligned} \psi_{AB} &= \psi_{[AB]} + \psi_{(AB)} \\ &= \psi_{(AB)} + \frac{1}{2}\psi_{[CD]}(\delta^C_A \delta^D_B - \delta^D_A \delta^C_B) \\ &= \psi_{(AB)} + \left(\frac{1}{2}\psi_{[CD]}\epsilon^{CD}\right)\epsilon_{AB} \\ &\equiv \psi_{1AB} + \psi_0 \epsilon_{AB} \end{aligned} \tag{3.20}$$

where

$$\psi_0 = \frac{1}{2}\psi_{CD}\epsilon^{CD} \tag{3.21}$$

Both $\psi_0\epsilon_{AB}$ and ψ_{1AB} are separately idependent of the binor identity. Any spinor can be decomposed in this way incorpriating the binor identitiy. Let us consider a three component spinor ψ_{ABC}.

Case (b) ψ_{ABC}:

$$\begin{aligned} 3\psi_{(ABC)} &= \psi_{A(BC)} + \psi_{B(AC)} + \psi_{C(AB)} \\ &= 3\psi_{A(BC)} - (\psi_{A(BC)} - \psi_{B(AC)}) - (\psi_{A(BC)} - \psi_{C(AB)}) \\ &= 3\psi_{A(BC)} - \epsilon_{AB}\sigma_C - \epsilon_{AC}\sigma_B \end{aligned} \tag{3.22}$$

where $\sigma_C = \epsilon^{AB}(\psi_{A(BC)} - \psi_{B(AC)})$. This rearanged gives

$$\psi_{A(BC)} = \psi_{(ABC)} + \frac{1}{3}\epsilon_{AB}\sigma_C + \frac{1}{3}\epsilon_{AC}\sigma_B. \tag{3.23}$$

$$\psi_{A(BC)} = \psi_{ABC} - \frac{1}{2}\epsilon_{BC}\psi_{AD}^{\ \ \ D} \tag{3.24}$$

Using (3.23) in (3.24) we arrive at for ψ_{ABC}

$$\psi_{ABC} = \psi_{(ABC)} + \frac{1}{2}\epsilon_{BC}\psi_{AD}^{\ \ \ D} + \frac{1}{3}\epsilon_{AB}\ \sigma_C + \frac{1}{3}\epsilon_{AC}\ \sigma_B. \tag{3.25}$$

obtaining the desired expansion in which all terms are separately independent of the binor identity.

Case (c) $\psi_{A\ldots F}$: Proof is by induction. More generally (3.23) implies

$$\psi_{A(BC\ldots F)} = \psi_{(ABC\ldots F)} + \frac{1}{n}\epsilon_{AB}\rho_{(C\ldots F)} + \cdots + \frac{1}{n}\epsilon_{AF}\rho_{(B\ldots F)} \tag{3.26}$$

where

$$\rho_{(C\ldots F)} = \epsilon^{AB}(\psi_{A(BC\ldots F)} - \psi_{B(AC\ldots F)}) \tag{3.27}$$

Spinor decomposition

Any decomposition involves

$$\psi_A, \psi_{(AB)}, \psi_{(ABC)}, \ldots \text{ and products of } \epsilon_{AB} \tag{3.28}$$

$$\hat{\sigma}^2 = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k \hat{\sigma}^k}{4} \right) \otimes \cdots \otimes 1 \tag{3.29}$$

$$\hat{\sigma}^2 \psi_{(AB\ldots F)} = n(n+1)\psi_{(AB\ldots F)} \tag{3.30}$$

A representation of a group G in a vector space V over k is defined by a homomorphism

$$\pi : G \to GL(V).$$

The degree of the representation is the dimension of the vector space.

Direct Products and Clebsch-Gordan Coefficients

$$\tau^{\mathbf{i}}_{(\mathbf{j})} = \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2} \right) \otimes \cdots \otimes 1 \tag{3.31}$$

We wish to calculate $\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})}$. The terms below for $k \neq k'$ wont contribute to the commutator as the order of multiplication for $k = k'$ is irrelevant,

$$\sum_{k \neq k'} \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^k}{2} \right) \otimes \cdots \otimes 1 \right) \left(1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^{k'}}{2} \right) \otimes \cdots \otimes 1 \right) - (k \leftrightarrow k') = 0 \tag{3.32}$$

$$\begin{aligned}
\tau^{\mathbf{i}}_{(\mathbf{j})}\tau^{\mathbf{j}}_{(\mathbf{j})} - \tau^{\mathbf{j}}_{(\mathbf{j})}\tau^{\mathbf{i}}_{(\mathbf{j})} &= \\
&= \sum_{k=1}^{2s+1} \left[1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i}{2} \right) \left(\frac{\hat{\sigma}^j}{2} \right) \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^j}{2} \right) \left(\frac{\hat{\sigma}^i}{2} \right) \otimes \cdots \otimes 1 \right] \\
&= \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \left(\frac{\hat{\sigma}^i \hat{\sigma}^j}{4} - \frac{\hat{\sigma}^j \hat{\sigma}^i}{4} \right) \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk} \sum_{k=1}^{2s+1} 1 \otimes \cdots \otimes \frac{\hat{\sigma}^k}{2} \otimes \cdots \otimes 1 \\
&= \epsilon_{ijk} \tau^{\mathbf{k}}_{(\mathbf{j})}
\end{aligned} \tag{3.33}$$

We say the vector space V carries a representation π of $SU(2)$.

$$\begin{aligned}\frac{1}{2}\hbar\hat{\sigma}^3_{AA'BB'CC'} &:= \frac{1}{2}\hbar\hat{\sigma}^3_{AA'}\delta_{BB'}\delta_{CC'} \\ &\vdots \\ &\frac{1}{2}\hbar\hat{\sigma}^3_{CC'}\delta_{AA'}\delta_{BB'}\delta_{EE'} \\ &(AA';BB';CC'=0,1).\end{aligned} \tag{3.34}$$

Summarising

A spin network function is labelled by a graph γ, a set of non-trivial representations $\{\pi_e\}$ one for each edge of γ, and a set of contraction matrices $\{M_v\}$, one for each vertex of γ, which contract indices of the tensor product $\otimes_{e\in E(\gamma)}\pi_e(h_e)$ in such a way that the resulting function is gauge invariant.

One can show that these functions are linearly independent.

$\rightarrow$ The interwiner operators associated a vertex of a spin network can be understood as specifying how we could connect the ropes of the edges meet at the same vertex. In the language of representaion of representation theory of groups, this corresponds to the fact that tensor products of irreducible represenations can be completely decomposed into the direct sum of irreducible representations. Hence they are invariant tensors of irreducible representations of groups and given by the stanadrd Clebsch-Gordon theory. In the case of $SU(2)$ spin networks, when the vertex is tri-valent, the decomposition of the product is unique. For $n > 3$, an n-valent vertex can be divided into tri-valanetvertices by making use of interwiner operators. At the same time, the colors asssociated with edges which meet at a vertx must satify consistent conditions. $\leftarrow$

3.1.2 A Note on Spatial Diffeomorphism Invariance

The next step in the construction of the theory is to factor away diffeomorphism invariance. Diffeomorphism invariance identifies two spin-networks that can be deformed into each other as gauge equivalent; in the same way that two solutions of the Einstein equationsthat are related by a coordinate transformation. An s-knot is an equivalence classe of spin networks related by diffeomorphisms.

This is a key step for two reasons. First of all, $\mathcal{H}$ is a "huge" non-separable space (). Prevents Wilson loops are distributional . However, gravity has a cure for this; to factor out spatial diffeomorphism heuristically, we are averaging the loop over the position of the loop and so in some sense the loop is smeared over the whole manifold. The only remaining information contained in the loop is its knotting which are indexed by a countable index set (it isn't quite as straightforward as this; for networks with nodes of valence 5 and greater in factoring away smoothe diffeomorphisms are still labelled by continuous parameters, (see Appendix M)).

One can show that the space of square integrable functionals, i.e.$\int |\Psi[A]|^2 \leq \infty$, has a basis of "spin networks". In the nexted section we will look at rigorously quantized geometrically interesting obseravables area of surfaces and volumes of space, which have been obtained operators on $L^2(\mathcal{A})$. The matrix elements of these operators can be explicitly computed in the spin network basis

3.2 Constructing Well Defined Operators: Areas and Volumes

3.2.1 Area.

[94], [97]

Rovelli and Smolin 94;Ashtekar, Lewandowski et al 95

Given a surface, we want to compute its area quantum mechanically.

$$A = \int_S d^2\sigma \sqrt{E^{ai}E^{bi}n_a n_b} \tag{3.35}$$

Conceptual consideration What are the true observables as these are subject to Heisenberg's indeterminacy principle. The details of this are presented in appendix L - Physical geometery and geometric operators. Although the expression for area is invariant under a change of coordinates it fails to be invariant under a diff transformation as a consquence of the fact that the area of an abstract surface defined in terms of coordinates is not invariant under active diff transformations. In fact, physically mearsurable areas in general relativity corresponed to surfaces defined by physical degrees of freedom, for instance matter (the area of a table). However, it is reasonable to expect that the fully gauge invariant operator corresponding to a physically defined area (say defined with matter) has precisely the same mathematical form as the gauge invariant operator studied here. The reason is that one can use the matter degres of freedom to gauge fix the diffeomorphisms - so that a non-diff-invariant quantity in pure gravity correspondes to a diff-invariant quantity in a gravity+matter theory.)

To promote this quantity to an operator, we need to handle the product of triads and also the square root. To do this, we start by partitioning the surface in small elements of area and notice that since the triads are functional derivatives quantum mechanically, one only gets contributions from the small elements of area pierced by a line of the spin net.

So we have $\hat{A} = \lim n \to \infty \sum_I \sqrt{A^2}_I$,

We need the action of the triad on a spin network state, which is very similar to on a loop state,

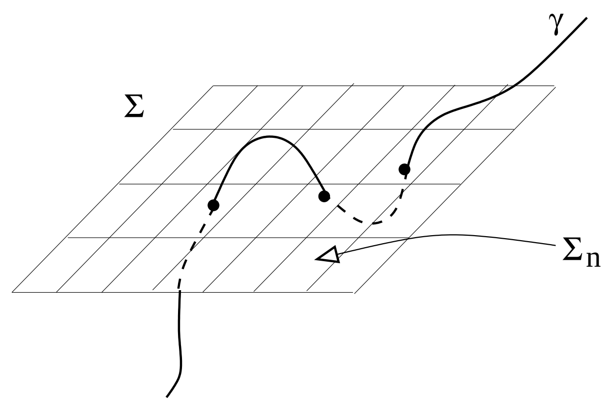

Figure 3.11: A partition of Σ.

Where X_i^j is a generator of SO(3) in the J level representation and s_X is a spin network that is opened at the point x and the generator X is inserted in that place. Since these quantities are distributional, we need to regularize. We will regularize by: a) smearing the E's along the small surface and point-splitting the product. The result is

$$A \tag{3.36}$$

Notice that we have six one dimensional Dirac deltas. All these "cancel each other" and we are left with a simple expression givenby the square of the SO(3) generator in the J-th spin representation. From angular momentum theory, we know that the value of such square is $j(j+1)$, so the end result for the area operator is,

$$A = 8\pi\hbar G\gamma \sum_i \sqrt{j_i(j_i+1)} \tag{3.37}$$

So we see that the area has a well defined action, and although we used background structures to regularize, the final result is topological and background independent. The spectrum of the operator is discrete, and admits a simple interpretation in which the spin of the lines of a spin network can be viewed as "quanta of area".

Ashtekar and Lewandowski have done a complete analysis that includes the possibility of lines being parallel to the surface and produced the complete spectrum of the operator. gr-qc/9602046,9711031

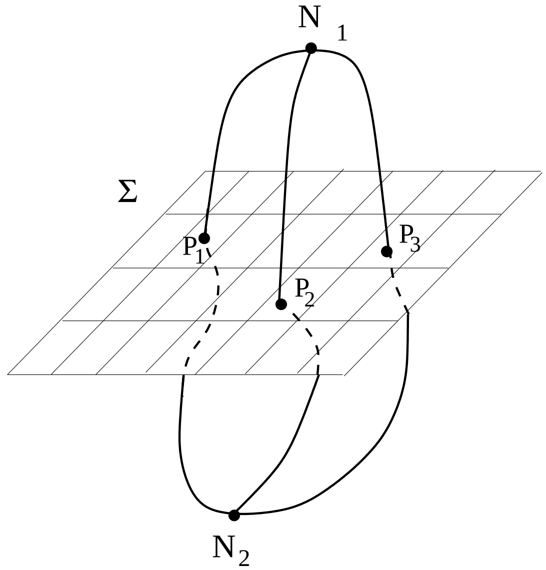

Figure 3.12: A simple spin network S intersecting the surface Σ.

While the quantum operator is well-defined onj states which are cylindrical with respect to any one graph, at the end we have to ensure that the resulting family of operators is consistent. This is a non-trivial issue.

The prediction of the spectrum is very specific and has immediate consequences. One naively would have assumed that the spectrum of the area would go as,

However, it doesn't. With the prediction of

The spacing of the eigenvalues diminishes rapidly for large values of the area. We will se that this has consequences at the time of computing the entropy of black holes.

[97] - The difference ΔA between are eigenvalues as $A \to \infty$ is

$$\Delta A \leq 4\pi\beta l_P^2 \frac{\sqrt{8\pi\beta}}{\sqrt{A}} + \mathcal{O}\left(\frac{l_P^2}{A}\right) l_P^2. \tag{3.38}$$

the area spectrum is used in the statisticalmechanical caculation of black-hole entropy.

A last note. Some objections have ben raised obout the last point... Some objections are based on the intuition that the position of the matter defining the surface could be subjected to quantum fluctuations, preventing the possibility of defining a sharp surface. This objection is incorect. Neither the position of the matter, nor the area of the surface, have physical independent reality. It is only the gravitaional field in the location determined by matter, or, the other way round, the location of the matter in the gravitaional field, that have physical reality. The two do not form independent sets of degrees of freedom subjected to independent quantum fluctuataions. See ...".

3.2.2 Volume operator:

[94], [98]

The volume operator

$$V = \int \sqrt{q} = \int \sqrt{\frac{1}{3!}\epsilon^{ijk}\epsilon_{abc}\tilde{E}_i^a\tilde{E}_j^b\tilde{E}_k^c} \tag{3.39}$$

I will omit the details since the calculation goes very much as for the area operator: one first breaks the integral into a sum over little cubic regions. In each of these regions. In each of regions. In each of these regions one smears the E operators with two dimensional integrals. Because of the epsilons the quantity is only non-vanishing at a place where there is a vetex of the spin network. One is left with three two dimensional integrals, three one dimensional integrals and three-dimensional Dirac deltas so the result is finite.

The group factor is a bit more difficult to compute than before, but corresponds to three traced generators contracted with epsilon. It can be computed. The end result is that the volume is finite, has a discrete spectrum, and the non-vanishing contributions come from **four-valent** intersections or higher. The eigenvalues depends on the value of the valences that enter the intersections.

[94], [95]

From [27] Smolin says, *The area and volume operators can be promoted to genuine physical observables, by gauge fixing the time gauge so that at least locally time is measured by a physical field* [The papers smolin references are here [91], [92]]. *The discrete spectra remain for such physical observables, hence the spectra of area and volume constitute genuine physical predictions of hte quantum theory of gravity.*

3.2.3 Quantum Geometry

So a picture of quantum geometry emerges in which the lines of flux are associated with "quanta of area" and the intersections of the lines quanta of volume. A spin-network is not in space it is space. To ask where is the spin-network is like asking where is a solution to Einsein's equation.

In classical geometry the volumes of regions and areas of the surfaces depend on the values of the gravitaional fields. They are coded in the metric tensor. On the other hand in the quantum picture the geometry is is coded in the choice of spin network. These spin networks correspond to the classical description, one can find a spin network which describes, to some level of approximation, the same geometry.

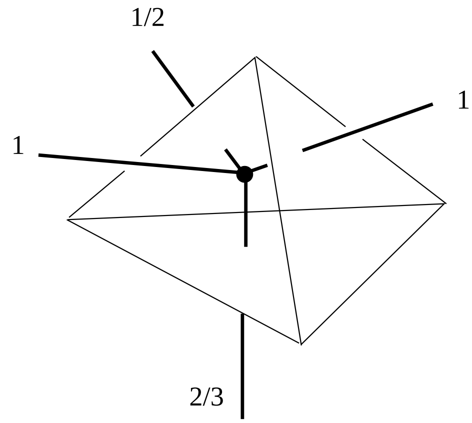

Figure 3.13: A node of a spin network (in bold) and its dual 3-cell (here a tetrahedron). The colouring of the node determines the quantized volume of the tetrahedron. The colouring of the edges determines the quantized area of faces via equation ().

It is extremely convenient that these spin-network states are eigenvectors of geometric operators. An example of this which we will come to is the microscopic source of the entropy of a black hole.

What are the knotting and linking of loops and graphs to do with?

It has been shown that the diffeomorphism invariant states were characterized by the knotting and linking of loops and graphs. What features of geometry do the knotting and linking measure? Observables sufficient to label the degrees of freedom of quantum geometry were identified in the area and volume operators, which measure combinatorial and labeling information, but which are insensitive to the topology of the embedding.

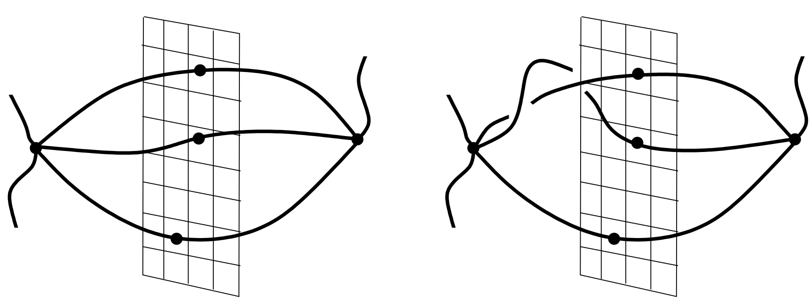

Figure 3.14: knotAresVolF. Area and volume operators are insensitive to the topology of the embedding.

Recently, results we describe in [275] show that some of the information in the embedding may have nothing to do with geometry, but instead describes emergent particle states. We will say more about this at the end of chapter 7.

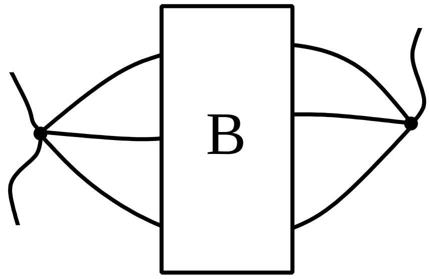

Figure 3.15: knotAresVolF2. Infinte degeneracy.

Other applications of spin network states

They are also employed in a different context; they employed in a technique for constructed a measure of integration called *clyndrical measure thoery*. In the next section They will appear as a natural candidates for *cylindrical functions* used in this construction.

Similarly in Ashtekar's approach A is like the "position" and E is like the canonically conjugate "momentum". Spin network states are eigenstates of E, so they are like "momentum" eigenstates.

out of place

3.3 Algebraic Quantization

The approach is conservative in the sence that one is following a non radical extension of the usual procedure of (algebraic) quantization.

Choose a subset, $\mathcal{S}$, of the set of all complex valued functions on phase space Γ such that

(1) $\mathcal{S}$ is large enough to allow any sufficiently regular function on Γ to be expressed as a sun of their products of its elements;

(2) $\mathcal{S}$ is closed under Poisson brackets and also closed under complex conjugation;

(3) elememnts of $\mathcal{S}$ are to have unambuguous quantum analogs.

In short, the elements of $\mathcal{S}$ should play the role played by the q's and p's when the phase space is $\mathbb{R}^{2n}$.

3.3.1 The GNS Construction

• A state F on $\mathcal{A}$ is a positive linear functional ('expectation-value'of the operators in $\mathcal{A}$): For any $a \in \mathcal{A}$, $F(a)$ is a complex number such that:

$$F(a + \lambda b) = F(a) + \lambda F(b), \qquad \lambda \in \mathbb{C}; \quad F(I) = 1; \quad F(a^* a) \geq 0.$$

• Given any F, the GNS construction provides a Hilbert space $\mathcal{H}$ and a representation of $\mathcal{A}$ by operators on $\mathcal{H}$ such that

i) the repressentation is cyclic; i.e. there exists a vector Ψ_F in $\mathcal{H}$ such that $\{\mathcal{A} \cdot \Psi_F\}$ is dense in $\mathcal{H}$; and

ii) $F(a) = (\Psi_F, a\Psi_F)$ for all $a \in \mathcal{A}$.

• Very general procedure. e.g., Every ireducible representation of $\mathcal{A}$ is cyclic.

If θ is an automorphism on $\mathcal{A}$ (i.e. a structure preserving map from $\mathcal{A}$ to otself), and if $F[\theta(a)] = F[a]$ then θ is unitarily implemented on $\mathcal{H}$; There exists a unitary operator U_θ on $\mathcal{H}$ such that

$$(\theta(a))\Psi = (U_\theta^{-1} a U_\theta)\Psi,$$

for all $\Psi \in \mathcal{H}$ and $U_\theta \Psi_F = \Psi_F$.

A powerful and economic way to ensure that (gauge-)symmetries are unitarily implemented. In Minkowski field theories, $F(a) =< 0|a|0 >$ is Poincare invariant.

3.3.2 C^*-algebras

Representations

Stuff discussed here will be proven in detail in appendix O.

Definition of a C^*-algebra.

first intro

An algebraic structure called a C^*-algebra. A concrete C^*-algebra is a linear space $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$, that is, a bunch of operators closed under addition, multiplication, scalar multiplication, and taking adjoints which is also complete with respect to the operator norm.

A C^*-algebra can be defined abstractly without any reference to linear operators acting on a Hilbert space. An abstract C^*-algebra is given by a set on which addition, multiplication, adjoint conjugation, and a norm are defined, satisfying the same algebraic relations as their concrete counterparts.

second intro

The elements of $\mathcal{O}$ are abstract mathematical entities, completely unrestricted except for the above conditions. A representation of the C^*-algebra $\mathcal{O}$ is a set of particular objects that satisfy the conditions of a C^*-algebra.

A representation of a C^*-algebra $\mathcal{O}$ consists of $(\mathcal{H}, \pi)$, where $\mathcal{H}$ is a complex Hilbert space and π is a morphism of $\mathcal{O}$ to the C^*-algebra $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$. Clearly, the conditions of the definition of a C^*-algebra $\mathcal{O}$ are satisfied.

Cyclic representations and states

The Hydrogen atom. We generate a complete set of eigenstates by acting on the ground state by ladder operators.

A cyclic representation of $\mathcal{O}$ is a triple $(\mathcal{H}, \pi, \Omega)$, where $\Omega \in \mathcal{H}$ such that $\|\Omega\| = 1$ and $\pi(\mathcal{O})$ is dense in $\mathcal{H}$.

If $(\mathcal{H}, \pi, \Omega)$ is a cyclic representation of a $*-$algebra, then

$$A \to \omega(A) :=< \Omega|\pi(A)\Omega > \tag{3.40}$$

defines a linear functional on $\mathcal{O}$. Remember a functional is something that acts on a vector and gives you back a complex number. The linear functional on $\mathcal{O}$ is usualy called, slightly confusingly at first, a *state*: A state is an assignment of an expectation value to each member of a collection of 'observables' (the elements of a C^*-algebra), if we know the expectation value of a complete set of commuting "observables" we know the state $|\Psi>$ of the system. For example, if we have the expectation value of the enery, z-component and the total angular momentum of an electron in a Hydrogen atom then we know the quntum state $\psi_{E_n, J^z_m, J}(x)$ of the electron.

Now the converse of the above statement is also true, and is known as the GNS construction: Given one has a state ω one can construct a cyclic rerpresentation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$.

However there is a proviso, a property you want the inner product to have is that it be positive definite, i.e. $(A, A) = 0$ implies that A is the null vector. A representation π is said to be faithful if, for $A \in \mathcal{O}$, $\pi(A) = 0$ implies $A = 0$. If the representation is not faithul the C^*-algebra would contain elements of $\mathcal{O}$ for which $\omega(A^*A) = 0$ does not imply $A = 0$. A representation can be forced to be faithful as is discussed below.

Given one has a faithul representation and a state ω on a $C\star-$algebra, one can construct a cyclic rerpresentation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$.

3.3.3 Null vectors

If an inner product $(A, A) = 0$ does not imply that A is null it is said to be only positive semidefinite.

The presence of a null ideal $\mathcal{N}$ requires to construct the Hilbert space by transferring our attention from the $C\star-$algebra $\mathcal{O}$ ... to consideration of equivalence classes ... and the induced multiplication, addition and scalar multiplification between and of classes. This forces the positive semidefinite inner product to become positive definite.

Must establish if induced multiplication, addition between classes, scalar multiplication and inner product is independent of the particular representative of the equivalence class.

Let

$$\mathcal{N} = \{A \in \mathcal{O} \ : \ \omega(A^* A) = 0\}.$$

Using the Cauchy-Schwarz inequality

$$|\omega(A^* B)|^2 \leq \omega(B^* B)\omega(A^* A),$$

we find that

$$\omega(N^* A) = \omega(A^* N) = \omega(N^* N) = 0 \tag{3.41}$$

whenever $N \in \mathcal{N}$. We consider each operation of a C^*-algebra in turn and then its inner product.

Conjugation $[A^*] = [A]^*$?

$[A^*]$ is comprised of those elements $A^* + N$ where N runs through the elements belonging to $\mathcal{N}$. Let N be any element in $\mathcal{N}$, we easily have

$$\omega((N^*)^* N^*) = \omega(N N^*) = \omega(N^* N) = 0$$

hence $N^* \in \mathcal{N}$. It follows from

$$(A + N)^* = A^* + N^*$$

that $A^* + N^*$ is a representative of $[A]^*$.

Multiplication $[A][B] = [AB]$?

Let N, N' be any two elements in $\mathcal{N}$,

$$(A + N)(B + N') = AB + (AN' + NB + NN') \equiv AB + N'' \tag{3.42}$$

Eq (3.41) implies $N'' \in \mathcal{N}$ so $AB + N''$ is a representative of $[AB]$.

Addition $[A] + [B] = [A + B]$?

Let N, N' be any two elements in $\mathcal{N}$, then obviously

$$\omega((N + N')^*(N + N')) = 0$$

so that $N + N' \in \mathcal{N}$. It then follows from

$$A + N + B + N' = A + B + (N + N')$$

that $A + B + (N + N')$ is a representative of $[A + B]$.

Scalar multiplication $\alpha[A] = [\alpha A]$?

As ω is linear

$$\omega((\alpha N)^* \alpha N) = \alpha^* \alpha \, \omega(N^* N) = 0$$

hence $\alpha N \in \mathcal{N}$. It then follows from

$$\alpha(A + N) = \alpha A + (\alpha N),$$

that $\alpha A + \alpha N$ is a representative of $[\alpha A]$.

Inner product $\omega([A]^*[B]) = \omega(A^* B)$?

Now we prove $(A + N, B + N')_\omega$ is independent of the representatives.

$$(A, B)_\omega = \omega(A^* B) \tag{3.43}$$

For any two elements A and B, $A \rightarrow \overline{A} = A + N$ and $B \rightarrow \overline{B} = B + N'$, where N and N' are elements of $\mathcal{N}$, we have

$$\begin{aligned}
\omega(\overline{A}^*\overline{B}) &= \omega((A^*+N^*)(B+N')) \\
&= \omega(A^*B)+\omega(N^*B)+\omega(A^*N')+\omega(N^*N') \\
&= \omega(A^*B) \qquad (3.44)
\end{aligned}$$

$\mathcal{N}$ is what is called a *two sided ideal* in $\mathcal{O}$.

Define the scalar product on $\mathcal{O}/\mathcal{N}$ by

$$([A],[B])_\omega = \omega(A^*B) \qquad (3.45)$$

Completion of $\mathcal{O}/\mathcal{N}$ with respect to this norm is the Hilbert space $\mathcal{H}_\omega$.

3.3.4 GNS Constrution

Given a state ω over an abstract C^*-algebra $\mathcal{A}$, the Gelfand-Naimark-Segal construction provides us with a Hilbert space $\mathcal{H}_\omega$ with a preferred state Ω_ω, and a representation π_ω of $\mathcal{A}$ as a concrete algebra of bounded operators on $\mathcal{H}_\omega$, such that

$$\omega(A) = <\Omega_\omega|\pi_\omega(A)\Omega_\omega> . \qquad (3.46)$$

Now one invokes the Gelfand-Naimark theorem which asserts that:

> Every C^*-algebra with identity is isomorphic to the C^*-algebra of all continous, bounded, complex functions on a compact, Hausdorff space.

Compact means finite in size. A Hausdorff space is a space with a certain separibility condition: for two distinct points there exists a neighbourhood for each point which are disjoint.

3.4 The Holonomy-Flux Algebra

3.4.1 Differentiability Classes of Manifolds and Loops

A function is called real analytic at a point if it possesses derivatives of all orders and given by a convergent power series locally. For example, a function on the real line $\mathbb{R}$

is analytic at the point p if there exists an interval (a, b) containing p such that in this interval the function can be expanded as a convergent series

$$f(x) = a_0 + a_1(x-p) + a_2(x-p)^2 + a_3(x-p)^3 + \dots, \quad (3.47)$$

where

$$a_0 = f(p),\ a_1 = f'(p),\ a_2 = \frac{f''(p)}{2!},\ a_3 = \frac{f'''(p)}{3!}, \dots \quad (3.48)$$

A function is analytic if it is analytic at each point in its whole domain. The set of all analytic functions is contained in the set of smooth functions. Analytic functions are also referred to as C^ω-smooth functions.

When we write parametric equations for a curve in the Euclidean space, say, $(x^1 = x^1(t), \dots, x^n = x^n(t))$, for (a, b) we expect the curve to be continuous (as a function from (a, b) to $\mathbb{R}^2$) provided that each of the 'coordinate functions' $x^1(t), \dots, x^{n-1}(t)$ and $x^n(t)$ are continuous (as a function from (a, b) to $\mathbb{R}$). A curve in Euclidean space $\mathbb{R}^n$ is analytic if it can be expanded as a Taylor series locally.

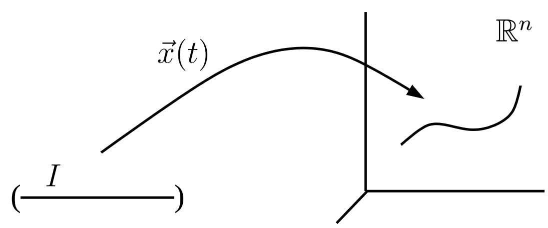

Figure 3.16: DiffCurveFig. The map $\lambda(t)$ from the open interval $I = (a, b)$ of the real line to the coordinates on $\mathbb{R}^n$ characterizes the differentiability class of the curve.

A curve in a manifold $\mathcal{M}$ is analytic if and only if its image under a chart is an analytic curve in $\mathbb{R}^n$, that is, if the map $\phi \circ \lambda$ from an open interval $I = (a, b)$ to $\mathbb{R}^n$ in Fig.(C.7.1) is an analytic map. A curve is piecewise analytic if it is made up of a finite number of pieces, each of which is analytic.

Analytic and smooth diffeomorphims from the manifold to itself.

This is possible [??] but then the technical discussion becomes much more complicated because, e.g., two smooth curves can intersect one another at an infinite number of points.

3.4.2 Integration on a Manifold

One can not sum over vectors and tensors because the result will be ambiguous - at different points the vectors and tensors transform differently. Unlike tensors in general,

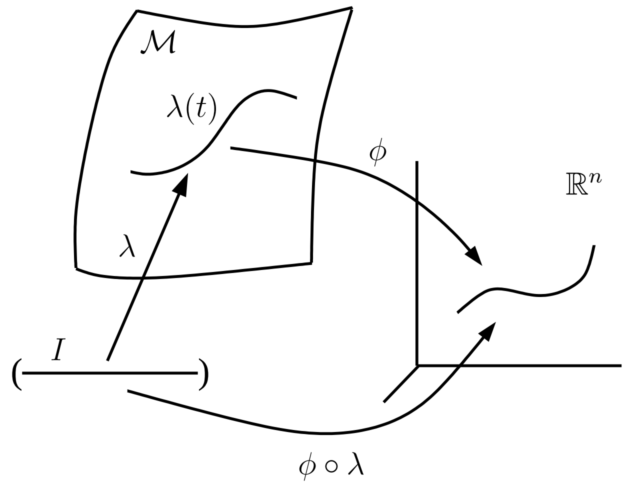

Figure 3.17: tangvectorForm. The map $\phi \circ \lambda$ from the open interval $I = (a, b)$ of the real line to the coordinates characterizes the differentiability class of the curve $\lambda(t)$.

we can add a scalar field field evaluated at two different points, x_1 and x_2 say, and the resulting quantity is still a scalr, since under a coordinate transformation, the sum transforms as

$$\phi'(x_1') + \phi'(x_2') = \phi(x_1) + \phi(x_2)$$

Summation must be over over scalars! As the volume element $d\Omega$ is a scalar density of weight -1 it follows that we can only integrate scalar density Φ of weight $+1$ over a region Ω, $\int_\Omega \Phi d\Omega$ since at each point $\Phi d\Omega$ is a scalar and can be added together. There are similar statements about which can be made about about integrations over curves and surfaces. It is natural to integrate a one-form X^a on a curve γ

$$\int_\gamma X_a dx^a \tag{3.49}$$

and a two form Y_{ab} over a surface S

$$\int_S Y_{ab} dS^{ab} \tag{3.50}$$

An *embeding* of one manifold into another.

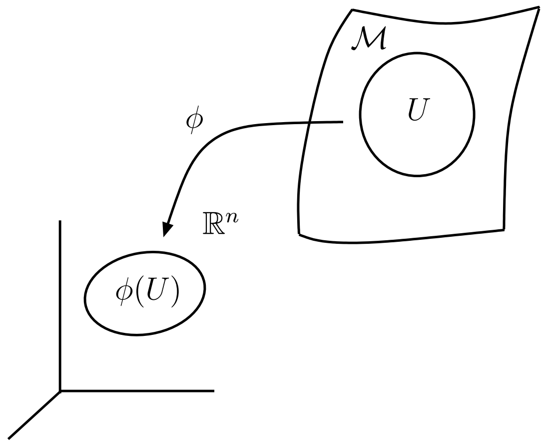

Figure 3.18: DiffClass0. A chart on $\mathcal{M}$ comprise an open set U of $\mathcal{M}$, called a coordinate patch, and a map $\phi : U \to \mathbb{R}^n$.

3.4.3 The Holonomy-Flux Algebra

There is no metric to raise and lower indices so we need to be honest about the type of tensor the variables of the theory are. There is the connection one-form

$$A_a^i(x)$$

the Electric field

$$E_i^a(x)$$

we also have the totally-antisymmteric tensor density

$$\epsilon_{abc}$$

From this tensor we can form the two-vector density

$$\tilde{E}_{ab}(x) := \epsilon_{abc} E_i^c(x)$$

Wilson loop functions are the obvious candidates for configuration variables. These will be associated with piecewise analytic loops on Σ, i.e., with piecewise analytic maps $\alpha : S^1 \to \Sigma$. (Thus, the loops do not have a preferred parameterization, although in the intermediate stages of calculations, it is often convenient to choose one.) The Wilson loop variables $T\alpha(A)$ are given by:

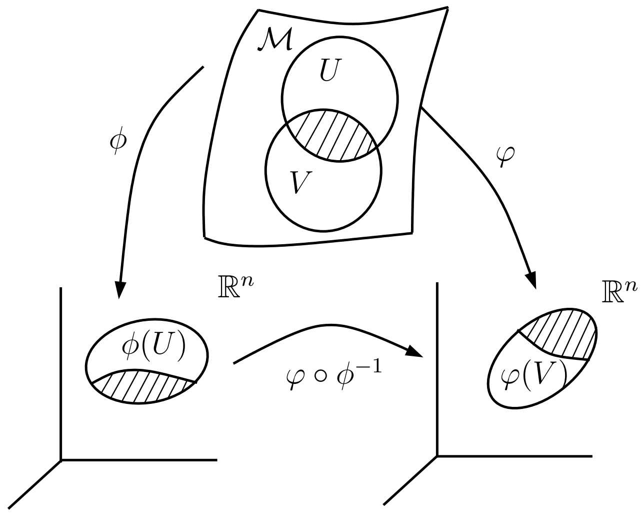

Figure 3.19: DiffClass2. The neighbourhoods U and V in $\mathcal{M}$ overlap. Their respective maps to $\mathbb{R}^n$, ϕ and φ, give two different coordinate systems to the overlap region. The relation between these coordinates characterizes the differentiability class of the manifold.

$$T_\alpha(A) := Trh_\alpha[A] \equiv Tr\mathcal{P}\exp\int_\alpha Ads \tag{3.51}$$

As defined, these are functions on the space of connections. However, being gauge invariant, they project down naturally to $\mathcal{A}/\mathcal{G}$. The momentum observables, T_S are associated with piecewise analytic strips S, i.e., ribbons which are foliated by a 1-parameter family of loops. For technical reasons, it is convenient to begin with piecewise analytic embeddings $S : (1,1)S17$ and use

$$T_S(A) := \int_S dS^{ab}\eta_{abc}T^c_{\alpha_\tau}(\sigma,\tau) \tag{3.52}$$

where

$$T^c_{\alpha_\tau}(A) := Tr\big(h_{\alpha_\tau}(\sigma,\tau)[A]\tilde{E}^c(\sigma,\tau)\big) \tag{3.53}$$

σ,τ are coordinates on S (with τ labeling the loops within S and σ running along each loop α_τ), η_{abc} denotes the Levi-Civita tensor density on Σ, and, as before h_{α_τ} denotes the holonomy along the loop α_τ. Again, the functions T_S are gauge invariant and hence well-defined on the phase space (cotangent bundle) over $\mathcal{A}/\mathcal{G}$. They are called "momentum variables" because they are linear in $\tilde{E}^a_i$.

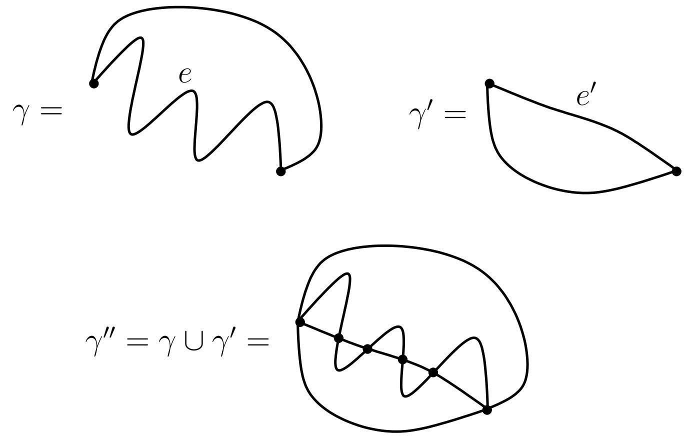

Figure 3.20: The union of the two graphs γ and γ' has a finite number of edges. If the edge e were allowed to "oscillate" arbitarily rapidly the union of the two graphs would have an inifinte number of edges.

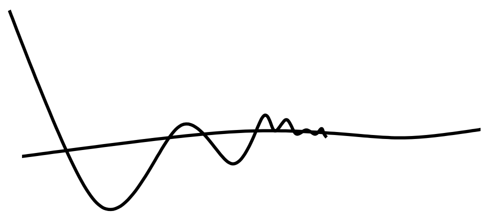

Figure 3.21: smoothIntersecF. Piecwise analytic curves which interesect at least a countable number of times *must* coincide everwise, i.e. be the same edge. However, smooth edges can intersect one another at an infinite number of points without coinciding everywhere and so the union has an infinite number of independent edges.

3.5 Implimentation of Quantization

3.5.1 GNS Constrution for the Holonomy-Flux Representation

we concentrate on the commutative sub-algebra of the W_γ's. These functions serve to separate the points of $\mathcal{A}/\mathcal{G}$ i.e. if two potentials are inequivalent then there exists at least one loop for which the corresponding W_γ's are different. This sub-algebra is the holonomy algebra and is denoted $\mathcal{HA}$. If γ is a trivial loop, H_γ is $\mathbf{1}$ and is the identity element of the sub-algebra. We define a norm:

$$\|H_\gamma\| := \sup_{[A]\in\mathcal{A}/\mathcal{G}} |H_\gamma[A]| \tag{3.54}$$

Completion with respect to this norm

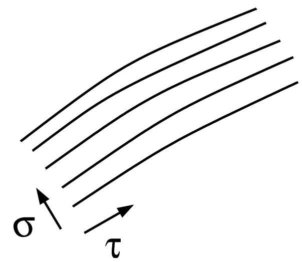

Figure 3.22: CongHolVarbF.

Now one invokes the Gelfand-Naimark theorem which asserts that every C^*-algebra with identity is isomorphic to the C^*-algebra of all continous, bounded, complex functions on a compact, Hausdorff space.

We now have a completed holonmy algebra. We still have construct $\mathcal{H}_{aux}$.

The C^*-algebra allows construction of its representations on Hilbert spaces. For every cyclic representation of $\overline{\mathcal{HA}}$ there is a Borel measuer μ on $\mathcal{A}/\mathcal{G}$ using which we get a Hilbert space:

$$\mathcal{H}_{aux} := L^2(\overline{\mathcal{A}/\mathcal{G}}, \mu). \tag{3.55}$$

Thus $\mathcal{H}_{aux}$ consits of square integrable functions on the "quantum configuration space" $\overline{\mathcal{A}/\mathcal{G}}$. The H_γ's act multiplicatively (just as position operators does in ordinary quantum mechanics) and are bounded operators on $\mathcal{H}_{aux}$.

What about the momentum conjugates?

This characterization of $\mathcal{A}/\mathcal{G}$ as a limit of finite dimensional spaces allows the introduction of integral calculus on $\mathcal{A}/\mathcal{G}$ using integration theory on finite dimensional spaces.

3.6 A Measure for Integration:

A key ingredient for discussing quantum physics is to have at hand an inner product to compute expectation values.

a loop transform,

$$\Psi[s] = \int \mathcal{D}A \Psi[A] W_A[s], \tag{3.56}$$

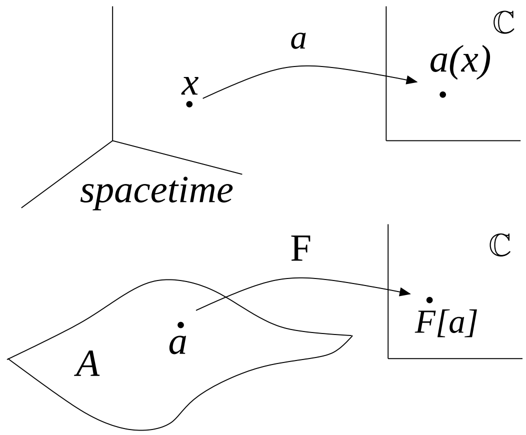

Figure 3.23: funcspace. (a) a is a function on spacetime. It maps points in spacetime to real or complex numbers. (b) a is apoint in the function space $\mathcal{A}$. The functional $F[a]$ maps points in the function space $\mathcal{A}$ to real or complex numbers. That is, the functional $F[a]$ turns functions into numbers.

3.6.1 Primer on Functional Integration

Consider, for definiteness, a scalar field theory. The key step then is that of giving meaning to the Euclidean functional integrals by defining a rigorous version $d\mu$ of the heuristic measure

$$\text{“}[\exp(S(\phi))]\prod_x d\phi(x)\text{”} \tag{3.57}$$

on the space of histories of the scalar field, where $S(\phi)$ denotes the action governing the dynamics of the model.

an approximation scheme to solve the formal equations of the theory and then use the approximation method to define what is meant by one's equations in the first place.

$$\int \mathcal{D}A\delta[A-\xi] = 1. \tag{3.58}$$

We think of $\mathcal{D}A$ as the infinite product,

$$\mathcal{D}A = \prod_{\vec{x}} dA(\vec{x}). \tag{3.59}$$

$$\delta[a-\xi] = \prod_{\vec{x}} \delta(A(\vec{x}) - \xi(\vec{x})), \tag{3.60}$$

may find it helpful to think of $A(x)$ as infinite dimensional column vectors where $\mathbf{x}$ plays the role of an index. A functional integral is an infinite-dimensional limit of ordinary finite-dimensional integrals.

Consider

$$\int \mathcal{D}[A] e^{-\int dx A^2(x)} \tag{3.61}$$

$$\begin{aligned} \int \mathcal{D}[A] &\rightarrow \int \prod dA(x) e^{-\sum_x A^2(x)} \\ &= \int \prod_x dA(x) \prod_x e^{-A^2(x)} \\ &= \prod_x \int dA(x) e^{-A^2(x)} \\ &= \prod_x \sqrt{\pi}, \end{aligned} \tag{3.62}$$

$$\int \mathcal{D}[A] e^{-\int\int dx\, dy A(x) M(x,y) A(y)} = \frac{(\sqrt{\pi})^\infty}{\sqrt{\det M}} \tag{3.63}$$

Motivation for Constructing Measures From of Finite Subspace Measuress?

constructing measures.

whenever a self-consistent sequence $\{\mu_n\}$ is given, does there exist a $\sigma-$additive[1] measure μ which satisfies the condition

$$\mu_n(E) = \mu(\pi_n^{-1}(E)), \quad \text{for all } E \subset X^n. \tag{3.64}$$

$$d\mu := d^d x e^{-\alpha x^2} \tag{3.65}$$

$$\chi(k) = \int \exp(ik \cdot x)(d^d x e^{-\alpha x^2}) = \int \exp(ik \cdot x) d\mu \tag{3.66}$$

Contains all the information on $d\mu$.

[1] $\sigma-$additive means $\mu(\sum_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$

3.6.2 Cylidrical Intergration Theory

$$F_e(\phi) = \int_{\mathbb{R}^{d+1}} \phi(x)e(x)\ d^{d+1}x \tag{3.67}$$

depends on ϕ only through their "n-components" $F_{e_1}(\phi), \ldots, F_{e_n}(\phi)$

$$f(\phi) = \tilde{f}\ (F_{e_1}(\phi), \ldots, F_{e_n}(\phi)). \tag{3.68}$$

where $\tilde{f}$ is a well behaved function on $\mathbb{R}^d$.

A cylindrical measure is a measure that allows to integrate cylindrical functions. Any measure would allow us to integrate cylindrical functions, but the tricky part is that there has to be consistency of these measures as there will be nesting and overlapping between these finite dimensional subspaces.

$$\int_C d\mu(\phi)f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n)d\mu_{\langle e_1,\phi\rangle,\ldots,\langle e_n,\phi\rangle} \tag{3.69}$$

$$f(\phi) = \tilde{f}_1\ (F_e(\phi)) := \exp[i\lambda \int_{\mathbb{R}^{d+1}} e(x)\phi(x)\ d^{d+1}x] \tag{3.70}$$

a function of $F_e(\phi)$ and $F_{\hat{e}}$ that just so happens to not depend on $F_{\hat{e}}$:

$$f(\phi) = \tilde{f}_2\ (F_e(\phi), F_{\hat{e}}(\phi)) := \exp[i\lambda \int_{\mathbb{R}^{d+1}} e(x)\phi(x)\ d^{d+1}x] \tag{3.71}$$

$$\int_{\mathbb{R}} e^{i\lambda\eta} d\mu_e(\eta) = \int_{\mathbb{R}^2} e^{i\lambda\eta} d\mu_{e,\hat{e}}(\eta, \hat{\eta}) \tag{3.72}$$

And therefore one has to have that,

$$d\mu_e(\eta) = \int_{\mathbb{R}} d\mu_{e,\hat{e}}(\eta, \hat{\eta}). \tag{3.73}$$

Any set of measures one finite dimensional spaces satisfying these conditions for any cylindrical function F, defines a cylindrical measure via,

$$\int_C d\mu(\phi)f(\phi) = \int_{R^n} F(\eta_1, \ldots, \eta_n)d\mu_{e_1,\ldots,e_n}(\eta_1, \ldots, \eta_n) \tag{3.74}$$

And conversely, a cylindrical measure defines consistent sets of measures in finite dimensional settings.

The situation is strikingly similar to ordinary quantum mechanics, where the Hilbert space of physical states is obtained by suitable completions of square integrable functions on the configuration space. In field theory the situation is more involved. Not every physical state is a function on just the configuration space, but distributions on the time=constant hypersurface are also generically involved.

$$\chi(e) = \text{“}\frac{1}{Z}\int \exp\left(\int_{\mathbb{R}^{d+1}} \Big(ie(x)\phi(x) - \mathcal{L}[\phi(x), \partial\phi(x)] \Big)\, d^{d+1}x\right)\text{”} \tag{3.75}$$

3.6.3 Integrating Gauge Field Theories

Simple example

consider the integral

$$Z = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\; e^{(x-y)^2} \tag{3.76}$$

$$\begin{aligned} x &\to x + a \\ y &\to y + a \end{aligned} \tag{3.77}$$

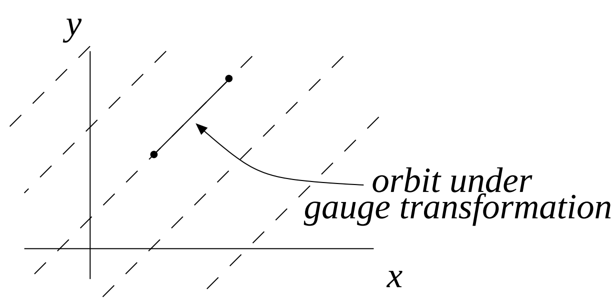

Figure 3.24: The motion of a “configuration,” (x, y), in the configuration space under the “gauge” transformation defined in (D.6). The path is called a gauge orbit. In the simple example here, the gauge orbits are lines of constant $x - y$.

the gauge orbits are lines of constant $x - y$. The “action”, $(x - y)^2$ is a gauge invariant.

$$\overset{-1}{\det}\left(\frac{\partial f}{\partial x}\right)\Bigg|_{f=a} = \int dx\delta(f(x) - a) \tag{3.78}$$

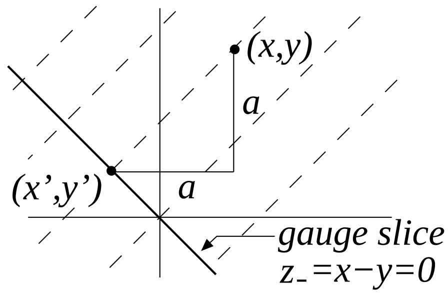

Figure 3.25: The gauge choice $x+y=0$ defines a "gauge slice" through the configuration space. (x', y') is a configuration on the slice, that is, it satifies the gauge condition. (x, y) is a gauge equivalent configuration, since both (x, y) and (x', y') reside on the same gauge orbit. a is the gauge transformation that takes us from the slice (x, y).

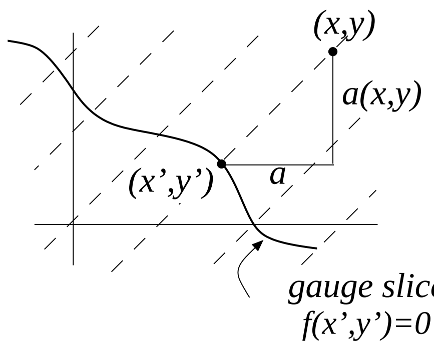

Figure 3.26: Illustration of a general choice of gauge, $f(x, y) = 0$. The desired change of coordinates is from (x, y) to (s, y), s being a variable that runs along the slice,and a deing the gauge transformation that runs from the slice (x, y).

The degeneracy is caused by the fact that we integrate over a redundent set of integration variables which results in an infinite volume factor. This situation occurs because of the way we formulate the theory as based on the principle of a local gauge invariance. The complete physical content is contained by one contribuation out of each equivalance class. One selects to one such member by imposing a condition called a *gauge fixing condition.*

Infinite Dimensional Non-Linear spaces

It is not easy to develop functional measures in infinite dimensional non-linear spaces like the space of connections modulo gauge transformations. Measures have been introduced for connections in the cases like 1+1 Yang-Mills with finite dimensional spaces.

We wish to have a rigorous defintion for the "loop transformation"

$$\Psi[\gamma] = \int \mathcal{D}[A]\Psi[A]W_\gamma[A] \tag{3.79}$$

which is how the loop representation was originally introduced by Rovelli and Smolin in 1988.

As part of the development of the techniques for dealing with quantum gravity, mathematically rigorous measure were introduced in these kinds of spaces, in some cases for the first time ever.

The Rovelli

" The space of physical states must have the structure of a Hilbert space, namely a scalar product, in order to be able to compute expectation values. This Hilbert structure is determined by the requirement that real physical observables correspond to self-adjoint operators. In order to define a Hilbert space of physical states, it is convenient to define first a Hilbert space of unconstrained states. This is because we have a much better knowledge of the unconstrained observables than the physical ones. If we choose a scalar product on the unconstrained state space which is gauge invaiant then there exist standard techniques to "bring it down" to the space of the physical states. Thus, we need a gauge and diffeomorphism invariant scalar product, with respect to which real observable are self-adjoint operators. "

The space of connections is an infite-dimension space is a well defined limit -a projective limit - of a family of finite dimensional spaces. This ***projective*** structure reduces the task of dealing with an infinite dimensional space to a problem of finite dimensional spaces with certain consistency conditions. The idea is to work with graphs embedded in space, and for each graph to define a Hilbert space of wave functions depending only on the holonomies of the connection along the edges of the graph. This work is due to Ashtekar, Lewandowski, Marolf, Mourao, Thiemann. Due to its heavy mathematical nature, we will only give a brief sketch here to highlight the main concepts.

One wishes to compute:

$$(\psi_1, \psi_2) = \int_{\mathcal{A}/\mathcal{G}} d\mu([A])\psi_1([A])\psi_2([A]) \tag{3.80}$$

3.7 How does one apply cylindrical measure theory to non-Abelian connections?

The physical degrees of freedom, the gauge orbits, is the quotient of the space of connections $\mathcal{A}$ by the gauge transformations $\mathcal{G}$, which is neatly written

$$\mathcal{A}/\mathcal{G}. \tag{3.81}$$

The idea is to work with graphs embedded in space, and for each graph to define a Hilbert space of wave functions depending only on the holonomies of the connection along the edges of the graph.If the graph γ has n edges, the holonomies along its edges are summarized by a point in $\mathcal{A}_\gamma \cong SU(2)$, and the Hilbert space we get is $L^2(\mathcal{A}_\gamma)$, is defined using a measure on $SU(2)$.

Any cylindrical function, f', based on a graph Σ' can be written as a cylindrical function, f, based on a larger graph containing Σ. We simply choose the function f to agreee with f' on the links Σ shares with Σ' and to be independent of the links in Σ but not in Σ'.

If the graph γ is contained in a larger graph γ' then $\mathcal{A}_\gamma$ is contained in $\mathcal{A}_{\gamma'}$ and one has $L^2(\mathcal{A}_\gamma) \subseteq L^2(\mathcal{A}_{\gamma'})$. We can thus form the union of all these Hilbert spaces and complete it to obtain the desired Hilbert space.

Pick a graph defined above. For each of the n links γ_i of Σ_n consider the holonomy $U_i(A) \equiv U[A, \gamma_i]$ of the connection A along γ_i. Every (smooth) connection assigns a $SU(2)$ matrix to each link γ_i of Σ_n via the holonomy $g_i \equiv U_i(A) = m\mathcal{P} \exp \int_{\gamma_i} A$. Thus an element of $[SU(2)]^n$ is assigned to the graph Σ_n. The next step is to consider complex-valued functions $f_n(g_1, \ldots, g_n)$ on $[SU(2)]^n$,

$$f_n : [SU(2)]^n \to \mathbb{C} \tag{3.82}$$

These functions are finite with respect to the Haar measure of $[SU(2)]^n$.

Given a graph and a function f_n, we define

$$\Phi_{\Sigma_n, f_n}(A) = f_n(U_1, \ldots, U_n). \tag{3.83}$$

These are fake infinite functions as they depend on the connection only via the graphs finite number of holonomies. They are called cylindrical functions. They form a dense subset of states in $\mathcal{L}$, the space of continuous smooth functions on $\mathcal{A}$. This justifies the exclusive use of this special class of functions for the construction of the Hilbert space.

duality between connection and holonomies of graphs is non-liner :

$$h_\gamma(A_1 + A_2) \neq h_\gamma(A_1) + h_\gamma(A_1) \tag{3.84}$$

as apposed to Eq.(3.67).

We introduce the notion of "hoops" (holonomic loops), that is, loops that yield the same holonomy for any connection. Such quantities form a group (Gambini 1980's) under composition at a given base point x_0.

Let us consider a set of independent hoops $(\beta_1, \ldots, \beta_n)$ (hint: use spin networks). If we consider the holonomy along each of these loops for a given connection, I get a map from the space of connections modulo gauge transformations to n copies of the gauge group modulo the adjoint action.

$$\pi_{\beta_1,\ldots,\beta_n}([A]) \quad \mathcal{A}/\mathcal{G} \to G^n/Ad \tag{3.85}$$

$$\pi_{\beta_1,\ldots,\beta_n}([A]) \quad = \quad [H(\beta_1, A), \ldots, H(\beta_n, A)], \tag{3.86}$$

We can now define a cylindrical function very much as we did before, in this case considering a function on G^n/Ad,

$$\int_{\mathcal{A}/\mathcal{G}} f([A]) d\mu([A]) = \int_{G^n/Ad} F([g_1, \ldots, g_n]) d\mu_{\beta_1,\ldots,\beta_n}([g_1, \ldots, g_n]) \tag{3.87}$$

A particularly simple choice of measure is to consider the Haar measure on each G/Ad. This choice turns out to be consistent (hard to prove with loops, easier with spin nets).

A scalar product is defined on the space of these functions as follows. Given two cylindrical functions defined by the same graph Γ, we define

$$(\Psi_{\Gamma,f}|\Psi_{\Gamma,g}) := \int_{G^n} dU_1 \overline{f(U_1, \ldots, U_L} \; g(U_1, \ldots, U_L) \tag{3.88}$$

Since the measure was defined without reference to any background structure it is naturally diffeomorphism invariant!

The construction looks intimidating but the end result is amazingly simple, especially if one casts it in terms of spin nets. It simply states that

$$\langle s_1 s_2 \rangle = \int_{\mathcal{A}/\mathcal{G}} D[A] W_{s_1}[A] W_{s_2}[A] = i_{s_1;s_2} \tag{3.89}$$

Which means that the inner product of two spin networks states vanishes if the two spin network states are different. More precisely, if no representative of the diffeo-equivalence class of spin networks s_1 is present in the class s_2.

That is, not only have we made sense precisely of the infinite dimensional integral present in the inner product, but the result is remarkably simple at the time of doing calculations.

the inner product was obtained on this set of states by requiring that the classical reality conditions be implemented as adjointness conditions on the corresponding quantum operators.

The Hamiltonian constraint takes diff invariant wave functions and maps them onto non-diff invariant wave functions. Master constraint is diff invariant and so can make full use of...

3.8 Weyl Rather than Heisenberg

The only way to construct diffeomorphism invariant theories is to start with exponentiated objects, like holonomies: $h_e(A) = \mathcal{P}\exp(\oint A)$. The quantum theory is discontinous. This means that for a system with $\hat{p}$ and $\hat{q}$ as fundermental coordinates, one of them becomes ill-defined.

Say if the Hamiltonian is of the form $H = p^2 + V(q)$, we can not define it on the kinematic Hilbert space $\mathcal{H}$.

We approximate the non-existing operator by a different (finite) operator that does exist.

− − − − − − − − − −

$$[P, Q] = i\hbar \tag{3.90}$$

The algebraic relations between Q and P expressed in are replaced by

$$U(a)V(b) = e^{i2\pi ab/\hbar}V(b)U(a). \tag{3.91}$$

and the product is

$$U(a)U(b) = U(a+b), \qquad V(a)V(b) = V(a+b) \tag{3.92}$$

This is the Weyl form of the CCR for one degree of freedom. We can then ask *formally* what the algebra for "generated",

$$\begin{aligned} U(a)QU^{-1}(a) &= \exp(iaP)Q\exp(-iaP) \\ &= Q + ia[P,Q] + \frac{(ia)^2}{2!}[P,[P,Q]] + \dots \\ &= Q + a\hbar \quad (\text{as } [P,Q] = i\hbar \text{ a scalar}). \end{aligned} \tag{3.93}$$

$$(U(a)\Psi)(x) = \Psi(xa) \text{ and } (V(b)\Psi)(x) = e^{-i2\pi bx/\hbar}\Psi(x), \tag{3.94}$$

The formula involving bounded operators will typically imply the one for unbounded operators but not vice versa.

$$[e^{i\mu\hat{x}}, e^{i\nu\hat{p}}] = \tag{3.95}$$

The Weyl algebra $\mathcal{W}$ is generated by taking finite linear combinations of the generators $U(a)$ and $W(b)$. Quantization means finding a unitary representation of the Weyl algebra $\mathcal{W}$ on a Hilbert space.

(exponetiated) diffeomorphisms only. It is given by

$$\begin{aligned} U(\varphi)U(\varphi')U(\varphi)^{-1} &= U(\varphi \circ \varphi' \circ \varphi^{-1}) \\ U(\varphi)\hat{H}(N)U(\varphi)^{-1} &= \hat{H}(N \circ \varphi^{-1}) \end{aligned} \tag{3.96}$$

3.9 Projective Limits

Summarize the basic ideas which used to construct the measure on $\overline{\mathcal{A}/\mathcal{G}}$ from the measure on cylindrical functions.

The collection of onto projections,

$$p_{SS'} \; \chi_{S'} \rightarrow \chi_S \tag{3.97}$$

The continuum theory will be recovered in the limit as one considers lattices of increasing number of loops of arbitrary complexicity.

$$\begin{aligned} \overline{\chi} \;\; := \;\; \{ \text{elements}, x_S, \text{ of the cartesian porduct, } \times_{S\in L} \chi_S, \\ \text{such that } : S' \geq S \Rightarrow p_{SS'} x_{S'} = x_S \}. \end{aligned} \tag{3.98}$$

This limit gives us the gives us the quantum configuration space $\overline{\mathcal{A}/\mathcal{G}}$.

the minimal graph the function is defined on.

$$\bigcup_{S\in L} C^0(\chi_S) \tag{3.99}$$

$$\text{Cyl}(\overline{\chi}) := (\bigcup_{S\in L} C^0(\chi_S)) \; / \; \sim . \tag{3.100}$$

quantum configuration space. [2]

3.9.1 Measure Theory on Infinite Dimensional Manifolds

$\sigma-$algebra

Let X be a set. Then a collection of subsets $\mathcal{U}$ of X is called a $\sigma-$algebra provided that

1) $X \in \mathcal{U}$

2) $U \in \mathcal{U}$ implies $X - U \in \mathcal{U}$ and

3) $\mathcal{U}$ is closed under countable unions, that is, $U_n \in \mathcal{U},\ n = 1, 2, \ldots$ then also is $\cup_{n=1}^{\infty} U_n \in \mathcal{U}$.

Let M be a metric space with metric d. Suppose $U \subseteq M$ is a subset of M. U is called open or an open set, if, given $x \in U$, there is $\delta > 0$ so that

$$B_\delta \subseteq U \subseteq M$$

where $B_\delta(x) = \{y : y \in M, d(x, y) < \delta\}$.

A topology on X is a family of open sets, containing $\emptyset$ and X, which is closed under the uncountible union and finite intersection.

Any metric space is a topological space, with open sets defined above. Our three conditions are satisfied.

In general, a givenn set admits many different topologies, that is different families of open sets which satify the definition.

[2]Typical measures used in quantum field theories are. The classical configuration spaces are contained in sets of measure zero. As a result, the action S inside the functional integral is ill-defined

In the Hilbert space language, this is the origin of field theoretic infinites, e.g., the reson why we can not naively multiply field operators.

The presnce of an infinite number of degres of freedom causes only one major modificatoin: the classical configuration space $\mathcal{C}$ of smooth fields is enlarged to the quantum configuration space $\mathcal{S}'$ of (tempered) distributions. Quantum theoretical difficulties associated with defining products of operators can be directly traced back to this enlargement [34].

This enlaregment from $\mathcal{A}$ to $\overline{\mathcal{A}}$ which occurs in the passage to the quantum theory is ver similar to the enlargement from $\mathcal{C}$ to $\mathcal{S}'$ in the case of scalar fields. This enlargement plays a key role in the quantum theory (especially in the discussion of surface states of the quantum horizon).

3.9.2 Tychonov' theorem

In rough terms, one topology is weaker than another if it has fewer open sets, and stronger than another if it has more open sets. topology on X which makes all the f_i's are continuous mappings. This is clearly a topology on X which makes all the f_i's continuous, and it is weaker than any topology which has this property.

The compactness of each Hausdorff space will imply that every X_α is compact and Haudorff. Now on a direct product space (independent of the cardinality of the index set) in which each factor space is compact and Haudorff one can naturally define a topology, the Tychonov topology, such that X_∞ is itself compact. If $\overline{X}$ is closed in X_∞ then $\overline{X}$ will be compact and Hausdorff as well in the subspace toplogy. However, for compact Hausdorff spaces powerful measure theoretic theorems hold which enable one to equip to relavent infinite dimensional spaces associated to background independent gauge theories with the structure of the so-called $\sigma-$algebra and to develop measure theory thereon.

Borel sets

Let X be a topological space. The smallest $\sigma-$algebra on X that contains all open set of X is called the Borel $\sigma-$algebra of X.

" $\mathcal{A}$ lies topologically dense, but measure theoretically thin in $\overline{\mathcal{A}}$ this section we will see that $\mathcal{A}$ (similar results apply to $\mathcal{A}/\mathcal{G}$ with respect to $\overline{\mathcal{A}/\mathcal{G}} = \overline{\mathcal{A}}/\overline{\mathcal{G}}$) with respect to the uniform measure μ_0. More precisely, there is a dense embedding (injective inclusion) $\mathcal{A} \to \overline{\mathcal{A}}$ but $\mathcal{A}$ is embedded into a measurable subset of $\mathcal{A}$ of measure zero. The latter result demonstrates that the measure is concentrated on non-smooth (distributional) connections so that $\overline{\mathcal{A}}$ is indeed much larger than $\mathcal{A}$. "

Riesz representation theorem

Let X be a locally compact Hausdorff space and let $\Lambda : C_0(X) \to \mathbb{C}$ be a positive linear functional on the space of continuous, complex-valued functions of compact support in X. Then there exists a $\sigma-$algebra $\mathcal{U}$ on X which contains the Borela $\sigma-$algebra and a unique positive meausre μ on $\mathcal{U}$ such that Λ is represented by μ, that is,

$$\Lambda(f) = \int_X d\mu(x) f(x) \tag{3.101}$$

for all $f \in C_0(X)$.

3.9.3 Functional Calculus

3.10 Generalized Eigenstates as Solutions to Constraint Equations

This may seem it may sound outlandish but... outlandish: looking or sounding very strange or foreign, bizarre.

Consider a finite Hilberts pace upon which we want to impose a constraint.

$$\hat{C}|\psi_n >= 0 \tag{3.102}$$

The subset $(\psi(), \psi(), \ldots, \psi())$ of the unconstrained Hilbert space that have zero eigenvalue with respect to the constraint. For calculating expectation values between these we simply use the inner-product of the unconstrianed Hilbert space, restricted to states.

However, we run into trouble even in simple quantum mechanical systems constraint equation. Say we have quauntum mechanics of a particle a two dimensional plane. The inner-product is

$$||\psi(x,y)|| := \int dx \int dy \overline{\psi(x,y)} \psi(x,y) \tag{3.103}$$

Consider the constraint

$$\hat{P}_y \psi(x,y) = -i\hbar \frac{\partial}{\partial y} \psi(x,y) = 0 \tag{3.104}$$

we easily see that the solutions are

$$\psi(x,y) = \phi(x) \tag{3.105}$$

These solutions have are non-normalizable in the natural inner-product:

$$||\psi(x,y)|| = \int dx \overline{\phi(x)} \phi(x) \times \int dy = \int dx \overline{\phi(x)} \phi(x) \times \infty = \infty. \tag{3.106}$$

Therefore there are no solutions to the constraint eq(3.104) that lie in this Hilbert space (the Hilbert space, by definition, being the space of square integrable wavefunctions). The constraint is imposing the symmetry that nothing changes in the $y-$direction. The translation group in the $y-$direction and this group volume diverges: $\int_{-\infty}^{\infty} dy = \infty$. This circumstance often occurs with constrained systems for the same simply mathematical reason: by satisfying the constraints, the physical wavefunctions must be constant on some degrees of freedom on which unconstrained wavefunctions can depend. If the "volume" obtained by integrating over these degrees of freedom diverges, the wavefunction will be non-normalizable.

The physical states that solve the "Wheeler-DeWitt" equation are infinite-norm states in the natural Hilbert space structure of the unconstained states space.

In the continuous part of the spectrum of an observable does not charaterize the state of the system because the corresponding state is non-normalizable, and hence does not belong to the Hilbert space. However, there is always at least one state in which the distribution of values around the mean value is as sharp as we want.

We shall use the Dirac notation for vectors and vector duals in $\mathcal{H}$: a vector will be written as a 'ket' $|\psi>$, and a linear functional (something that maps vectors to a complex number) on $\mathcal{H}$ is written as a 'bra' $<\psi|$.

Solving the Hamiltonian constraint using a projection operator. The idea if the Hilbert space were finite: say we the equation

$$\hat{H}|v>=0 \tag{3.107}$$

$$\hat{H}|v_\lambda>=\lambda|v_\lambda> \tag{3.108}$$

$$\hat{H}(\hat{P}|v>) = \hat{H}(\exp \hat{H}|v>) = \tag{3.109}$$

Since these physical states do not belong to H_{kin}, the scalar product between them is well-defined - they are too distributional to be normalizable states in $L^2(\overline{\mathcal{A}/\mathcal{G}}, d\mu_0)$.

this can be expressed as

$$<\Psi_{phys}|\Phi_{phys}>=<P\Psi|P\Phi>=<P\Psi|\Phi>=\int \tag{3.110}$$

position eigenstates

We call ϕ a distribution or dual state

We think about think of this states in the momentum representation in which we find it is a delta function in p.

$$\hat{p}|\psi>=p|\psi> \tag{3.111}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{ipx} \tilde{\psi}(p), \text{ where } \mathcal{L}_2(R). \tag{3.112}$$

$$\begin{aligned} \left(-i\frac{d}{dx}\right)\psi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp \left(-i\frac{d}{dx}\right) e^{ipx}\tilde{\psi}(p) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp\, p\, e^{ipx}\tilde{\psi}(p) \end{aligned} \tag{3.113}$$

for the function $\tilde{\psi}(p)$we, for example, use $\psi(p) = \frac{1}{2\pi d} e^{-(p-p_0)^2 d}$ $d >> 1$ approximate the delta fucntion $\delta(p - p_0)$ centered at p_0,

$$\begin{aligned} \left(-i\frac{d}{dx}\right)\psi(x) &\approx p_0 \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{ipx}\tilde{\psi}(p) \\ &= \approx p_0 \psi(x). \end{aligned} \tag{3.114}$$

a smeared momentum eigenstate

$$< \omega| \iff e^{ipx}[.] := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dp e^{ipx}(.) \tag{3.115}$$

We think about this state as a dual state. For this construction to be well defined we must use a smaller subset $\mathcal{D} \subset \mathcal{H}$ for which

$$e^{ipx}[\psi] < \infty \text{ for every } \psi \in \mathcal{D}. \tag{3.116}$$

The space of functionals is called the dual of $\mathcal{D}$ and is denoted $\mathcal{D}^*$. The dual space contains all the functions in $\mathcal{H}$ i.e. $\mathcal{H} \subset \mathcal{D}^*$. Altogother we have that

$$\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^* \tag{3.117}$$

This construction is called a Gel'fand triple or a rigged Hilbert space. It turns out that there is always an analogous construction for any self-adjoint operator. In fact, it can be shown that there is a sence in which all self-adjoint operators have a complete set of generalized eigenststes, (this is refered to as the **Nuclear spectral theorem**).

This isn't a state in the Hilbert space itself but has a finite inner product with some large class of states in the Hilbert space. An example as the Schwartz space - which are functions that fall off rapidly and are smooth (we can differentiate them as many times as we want) and correspondingly have smooth Fourier transforms and the inner product of our position independent state with one of these nice rapidly falling off smooth states should be well defined.

3.10.1 Induced Hilbert Space and the New Inner Product

$$-i\frac{d}{dx}\psi(x,y) = 0 \tag{3.118}$$

are functions $\psi(x,y)$ constants in the y and are non-normalizable in $\mathcal{K}$. However, the decomposition

$$\mathcal{K} = \int_{\mathbb{R}} dy H_y. \tag{3.119}$$

where $H(y) = L^2[\mathbb{R}, dx]$.

$$(\psi, \phi)_{\mathcal{K}} = \int_{\mathbb{R}} d^2x\overline{\psi(x,y)}\phi(x,y) = \int_{\mathbb{R}} (\psi_y, \phi_y)_{H_y}, \tag{3.120}$$

where $\psi_y(x) = \psi(x,y)$ and

$$(\psi_y, \phi_y)_{H_y} = \int_{\mathbb{R}} dx\overline{\psi_y(x)}\phi_y(x). \tag{3.121}$$

The space of solutions of (L.5) is $H(0)$ and has the natural Hilbert structure $H(0) = L^2[\mathbb{R}, dx]$. Introduce some notation,

3.11 Spacial Diffeomorphism Inner-Product Structure

3.11.1 Solving the Diffeomorphism Constraint

Unlike the strategy in solving Gaussian constraint, one cannot define an operator for quantum diffeomorphism constraint as the infinitesimal generator of finite diffeomorphism transformations (unitary operators since the measure is diffeomorphism invariant) represented on $\mathcal{H}_{kin}$. The representation of finite diffeomorphisms is a family of unitary operators $\hat{U}_\varphi$ acting on cylindrical functions ψ_α by

$$\hat{U}_\varphi \psi_\alpha := \psi_{\varphi \circ \alpha}, \tag{3.122}$$

for any spatial diffeomorphism φ on Σ.

There is a subset of Diff(Σ) which leaves the curve γ invariant, and only reparamertizes it. These elements of this subset are (finite) diffeomorphisms generated by the vector fields on Σ that are tangent to γ.

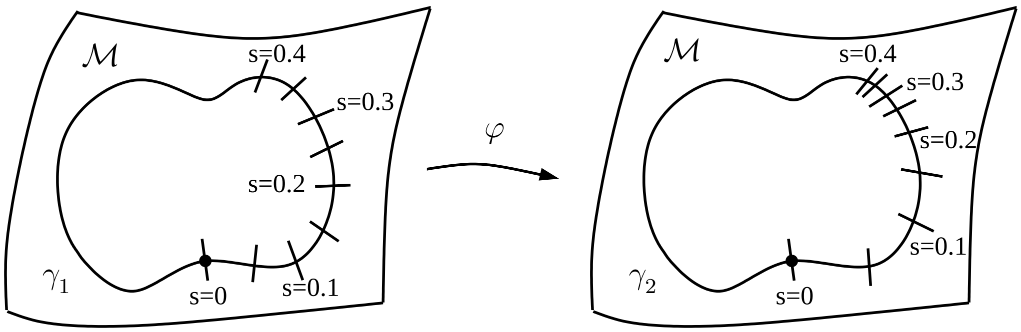

Figure 3.27: GraSymmFig0. The diffeomorphism φ maps a loop to itself resulting in a reparametrization of the loop γ_1.

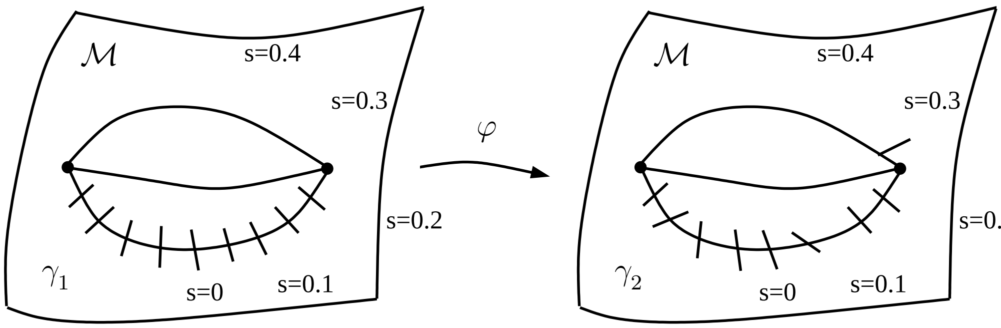

Figure 3.28: GraSymmFig0a. The diffeomorphism φ maps a loop to itself resulting in a reparametrization of the loop γ_1.

Averaging over the group of graph symmetries

Also let $GS(\gamma)$ be the group of graph symmetries of γ, that is, the group $Iso(\gamma)/TA(\gamma)$, where $Iso(\gamma)$ is the group of diffeomorphisms mapping γ to itself, and $TA(\gamma)$ is the subgroup fixing each edge of γ . We may define an element

If we to average over these two graphs

$$\frac{1}{2}(\Psi_{\gamma_1} + \Psi_{\gamma_2}) \tag{3.123}$$

will be a state that remains unchanged by the diffeomorphism. Let us denote the effect induced on the state Ψ_α by the diffeomorphism φ as $\varphi \star \Psi_\alpha$, this is known as the pull-back of Ψ_α under φ. In our example

$$\mathbf{1} \star \Psi_{\gamma_1} = \Psi_{\gamma_1} \quad \text{and} \quad \varphi \star \Psi_{\gamma_1} = \Psi_{\gamma_2}$$

where $\mathbf{1}$ is a diffeomorphism which maps each edge to themselves.

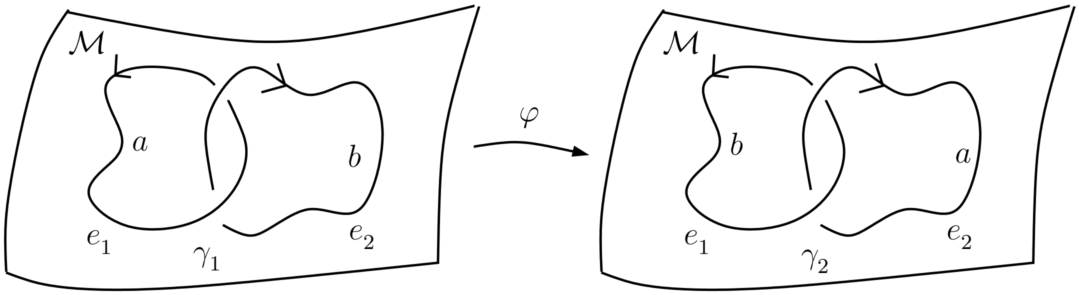

Figure 3.29: GraSymmFig. The diffeomorphism φ maps the edge e_1 to e_2 and at the same time the edge e_2 to the edge e_1. This has the effect of swaping around the labels on the two edges.

This obviously has the structure of a group, this is called the *group of graph symmetries*. We can collect all the information on group operations in a group table.

	I	a	b
I	I	b	b
a	a	b	b
b	b	b	b

For each row (and in each column) of a group table every group element appears exactly once. The statement that b occors in row a and column x is the equation

$$ax = b,$$

this is solved by $x = a^{-1}b$. Say there were another element $y \in G$ such that $ay = b$, then $ax = ay$. Multiplying both sides of this from the left by a^{-1} gives $x = y$. So there is a unique solution for x, i.e. given by an element b, there is precisely one column x in which it occurs, i.e. each element occurs exactly once in each row. Now, this implies that if we multiply the sum of all group elements by any one group element the result will be the same summation, but now in a different order. This observation allows us to factorize out the group of graph symmetries.

First, given any $\Psi_\alpha \in \mathcal{H}'_\alpha$, we average it using *only the group of graph symmetries* and obtain a projection map $\hat{P}_{Diff,\alpha}$ from $\mathcal{H}'_\alpha$ to its subspace which is invariant under $\hat{GS}_\alpha$:

$$\hat{P}_{Diff,\alpha}\Psi_\alpha := \frac{1}{N_\alpha}\sum_{\varphi\in GS_\alpha}(\varphi \star \Psi_\alpha) \tag{3.124}$$

where N_α is the number of the elements of GS_α (the volume of the obit of GS_α).

For the simply example above this is

$$\hat{P}_{Diff,\alpha}\Psi_\alpha = \frac{1}{2}(1 \star \Psi_{\gamma_1} + \varphi \star \Psi_{\gamma_1}) = \frac{1}{2}(\Psi_{\gamma_1} + \Psi_{\gamma_2}) \tag{3.125}$$

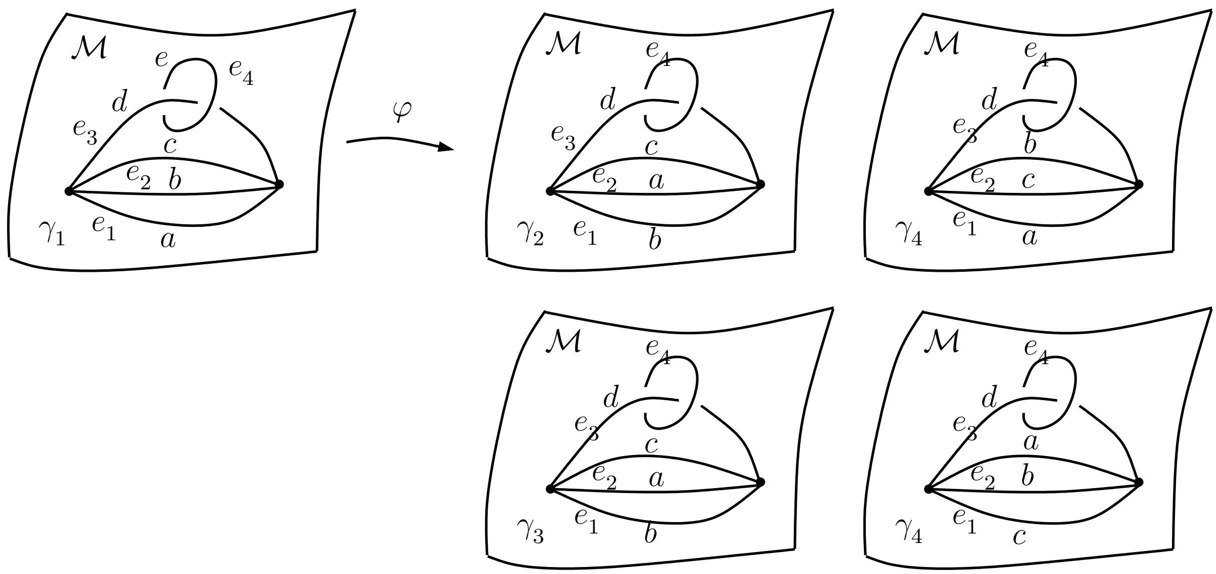

Figure 3.30: GraSymmFig2. The diffeomorphism φ maps swaping around the labels on the two edges.

Averaging over the remaining diffeormorphisms

test function space - $\Phi^\times$ is the topological dual of Φ - it corresponds to the complete space of continuous (bounded) linear functionals. $\Phi^\times$ denotes the space of distributions. Because the elements of Φ are so well-behaved $\Phi^\times$ is very large and contains solutions to the operators through their adjoint action.

$$(\mathbf{\Phi}^*_{Kin})_{Diff} \tag{3.126}$$

$$\mathcal{H}_{Diff} \tag{3.127}$$

Construct a space of diffeomorphism invariant states $H_{Diff} \subset H^*_{kin}$, which are invariant under the action of $\hat{U}(\phi)$. These are the diffeomorphism invariant states and they live inside the dual of the kinematical Hilbert space.

A continous index set for diff-invariant states

Finally we mention that $\mathcal{H}_{Diff}$ just like $\mathcal{H}$ is still not separable because the set of singular knot classes $[\gamma]$ has uncountably infinite cardinality [??].

This is easy to understand. By the definition of tangent space T_p loops define directions in T_p and diffiomorphisms act linearly on it. Thus the equivalence loops under diffeomorphisms implies the corresponding equivalence of T_p under liner transformtions induced by the diffeomorphism. So from the fact that the group of semianalytic diffeomorphism reduces to $GL(3,\mathbb{R})$ at each vertex. This transformation depends on nine parameters (per point).

Hence, for vertices of valence higher than nine one cannot arbitrarily change, in a coordinate chart, all the angles between the tangents of the adjacent edges. Any two vertices with the same valence which is above nine are not related through a semianalytic diffeomorphism and so belong to inequivalence classes. This leads to the emergence of equaivalence classes labelled by continuous parameters. It turns out that valence five is already sufficient, that is, there are diffeomorphism invariant "angles", called moduli θ in all vertices of valence five or higher.

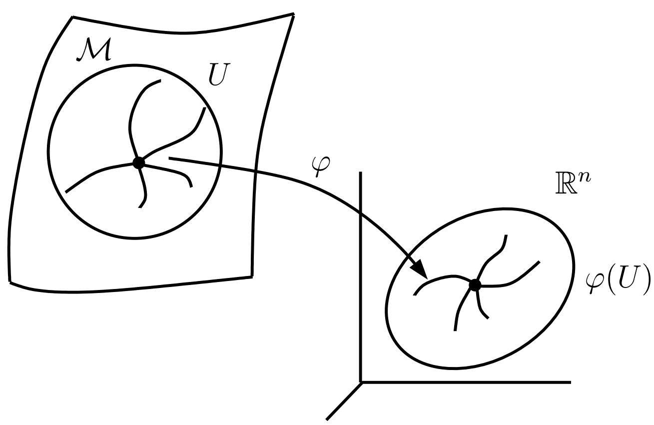

Figure 3.31: tangnodeF. The map .

There are several proposals for an enlargement of the group of diffeomorphisms [??], however, these groups do not interact well with certain crucial operators in the the theory such as the volume operator which depend on at least $C^{(1)}(\Sigma)$ structures while those extensions basically replace diffeomorphisms by homeomorphisms or even mor general bijective maps on Σ. We will see, however, that the non separability of $\mathcal{H}_{Diff}$ is immaterial when we pass to the physical Hilbert space $\mathcal{H}_{Phys}$.

[102]:

"We investigate the action of diffeomorphisms in the context of Hamiltonian Gravity. By considering how the diffeomorphism-invariant Hilbert space of Loop Quantum Gravity should be constructed, we formulate a physical principle by demanding, that the gauge-invariant Hilbert space is a completion of gauge- (i.e. diffeomorphism-)orbits of the classical (configuration) variables, explaining which extensions of the group of di eo-morphisms must be implemented in the quantum theory. It turns out, that these are at least a subgroup of the stratified analytic diffeomorphisms. Factoring these stratified dif-feomorphisms out, we obtain that the orbits of graphs under this group are just labelled by their knot classes, which in turn form a countable set. Thus, using a physical argu-ment, we construct a separable Hilbert-space for di eomorphism invariant Loop Quantum Gravity, that has a spin-knot basis, which is labelled by a countable set consisting of the combination of knot-classes and spin quantum numbers. It is important to notice, that this set of di eomorphism leaves the set of piecewise analytic edges invariant, which ensures, that one can construct flux-operators and the associated Weyl-operators."

3.11.2 Non Weak Continuity of Diffeomorphisms in LQG

We consider an aspect of the action of diffeomorphisms on the quantum theorey. The space of quantum configurations $\mathcal{A}$, i.e. the space of (distributional) connections on Σ carries a natural action of the diffeomorphism group Diff Σ. An element $\phi \in$ Diff Σ simply acts by $A \to \phi^* A$ on a (distributional) connection A. With this, one can simply define the action of Diff Σ on $\mathcal{H}_{kin}$ by

$$\alpha_\phi f(A) := f(\phi^* A)$$

where $\phi^* A$ is the pullback of the connection A under the diffeomorphism ϕ. By this definition, under a diffeomorphism ϕ the kinematic Hilbert space transforms as

$$\alpha_\phi \mathcal{H}_\gamma \to \mathcal{H}_{\phi(\gamma)}. \tag{3.128}$$

where $\phi(\gamma)$ is the image of γ under ϕ. Note the action (3.128) is not weakly continuous in ϕ, since two graphs can be arbitrary "close" to each other, but still be not intersecting, which means that their corresponding Hilbert spaces are mutually orthogonal subspaces of $\mathcal{H}_{kin}$. This is natural in the LQG picture, since the notion of "being close to each other" has no meaning as the metric can be transformed into another one which imposes a differernt geometry by an active diffeomorphism.

We say that the representation is not weakly continuous with respect to the holonomies, by which we mean that matrix elements of cylindrical functions are not continuous under continuous deformations of edges. While the Weyl representation is not weakly continuous with respect to the holonomies it is with respect to the flux.

3.11.3 Ergodicity of Spacial Diffeomorphisms

3.12 Uniqueness Theorem for the Holonomy-Flux Representation

Under reasonable physical assumptions, there is a unique representation of the Holonomy-Flux algebra. This means that, once the Holonomy-Flux algebra has been chosen, we can be confident to use that unique, kinematic representation as a basis for the constraint quantization programme. Uniqness will lead to make definiative predictions that may be confronted with experiments.

In section ?? we introduced the Ashtekar-Isham-Lewendowski state ω_0 - existence of a representation is established.

One will be interested in those representation which are distinguished by physical selection criteria. One such criterion is a unitary representation of the diffeomorphism group (rather than an infintesimal version). It turns out that such a representaion is actually unique. We need to introduce some defintions and technical results.

In general any representation of a $*-$algebra is a direct sum of cyclic representations (the proof is analogous to the proof of the existence of a orthonormal basis for a Hilbert space). But every cyclic represenation comes from a state ω (positive linear functional) on the algebra $\mathcal{O}$ via the GNS construction.

If the state is invariant the associated representation is automatically unitary. Hence, it sufficse to look for invariant states and it has now been proven [][??] that the only such state is the Ashtekar-Isham-Lewendowski state ω_0.

The uniqueness theorem not only requires that the expenentiated fluxes are represenated weakly continuously but actually smoothly. So as compared to the Stone-von Neumann theorem of ordinary quantum mechanics, one considers representations in which continuity is slightly relaxed in one direction and slightly tightened in the other.

Theorem 3.12.1 *There is a unique, semi-weakly smooth, diffeomorphism-invariant state on (equivalently, cyclic representation of)* $\mathfrak{A}$*. Moreover, the corresponding cyclic GNS representation is irreducible.*

3.12.1 Assumptions of the Uniqueness Theorem

Quantum Geometry: Representation on $L_2(\overline{\mathcal{A}}/\overline{\mathcal{G}}, \mu_0)$ is unique if

1. diffeomorphism invariant;

2. semianalytic.

Diffeomorphism invariance

A natural idea is to first look at irreducible or at least cyclic representations as the simple building blocks, out of which more complicated representations could eventually be built.

If $(\mathcal{H}, \pi, \Omega)$ is a cyclic representation of a $*-$algebra, and $\pi(G)$ is the representation of a group symmetry then

$$< \Omega|\pi(g)^{-1}\pi(A)\pi(g)\Omega > = < \Omega|\pi(A)\Omega >$$

As $\pi(g)$ is unitary, this condition is equivalent to

$$\omega(gA) = \omega(A)$$

for all $g \in G$ and $A \in \mathcal{O}$.

Now we consider the converse of the above statement. A morphism is a map from an algebra to itself that, that is $g : \mathcal{O} \to \mathcal{O}$. A morphism that has an inverse g^{-1} is called an automorphism. We say a state ω is invariant under a group of automorphisms G if

$$\omega(gA) = \omega(A)$$

for all $g \in G$ and $A \in \mathcal{O}$. If the state is invariant the associated representation is automatically unitary.

Let G be a group of automorphisms of the $C*-$algebra $\mathcal{O}$ and ω a corresponding $G-$invariant state on $\mathcal{O}$. Then there is a cyclic representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$

$$\pi_\omega(gA) = U_\omega(g)\pi_\omega(A)U_\omega(g)^{-1}, \quad U_\omega(g)\Omega_\omega = \Omega_\omega, \tag{3.129}$$

for all $g \in G$ and $A \in \mathcal{O}$.

A simple formulation of these properties can be given by asking for a state (i.e. a positive, normalized, linear functional) on $\mathcal{U}$ that it is invariant under the classical symmetry automorphisms of $\mathcal{U}$. Given a state ω on $\mathcal{U}$ one can define a representation via the GNS construction. This representation will be cyclic by construction, $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$,. If the state is invariant under some automorphism of $\mathcal{U}$, its action is automatically unitarily implemented in the representation.

Semianalyticity

Now we move to condition 3. A necessary condition for the Poisson bracket between the variables to be finite is that every edge intersects every face in at most a finite number of isolated intersection points plus a finite number of connected segments (i.e. the edges themselves). It is vital for the uniqueness proof that the intersection between paths with surfaces contain finitely many isolated points. A simple condition that ensures this property uses a real analytic structure on Σ, analytic paths and analytic surfaces. We could consider the class , analytic diffeomorphisms. However, analytic functions are uniquely determined by the value of it and all its derivatives at one point and, thus, in particular, is uniquely determined by its value in an arbitrarily small neighbourhood of a point. This implies that analytic diffeomorphisms, in a sense, have no local degrees of freedom. To alleviate this restriction while retaining the finite intersection property of analytic structures, one considers the larger group of diffeomorphisms - semianalytic diffeomorphisms.

Briefly, 'semianalytic' means 'piecewise analytic'. For example, a semianalytic sub-manifold would be analytic except for on some lower dimensional sub-manifolds, which in turn have to be piecewise analytic. We have already met the idea of semianalycity, see fig (K.9) (a). To convey the general idea, fig (K.9) (b) depicts a semi-analytic surface in $\mathbb{R}^3$.

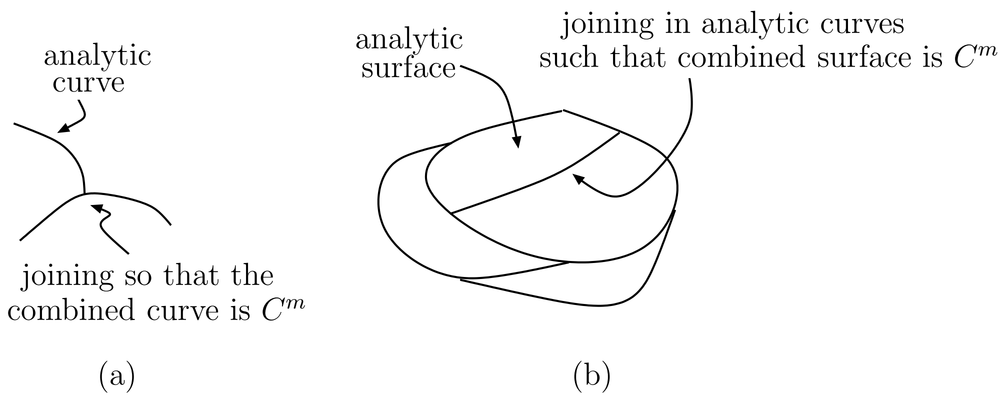

Figure 3.32: (a) A semianalytic curve in $\mathbb{R}^3$. (b) A semianalytic surface in $\mathbb{R}^3$.

analytic paths are determined everywhere once they are kniwn on an open set, thus making them non-local. If we make it semianalytic then these data only determine the path up to the next point where analyticity is reduced to C^n, $n > 0$. This is important because we need to make sure that certain local constructions do not have an impact on rdegions far away from the region of interest.

Semianalytic diffeomorphisms do have local degrees of freedom, in the sense that for every point $x \in \mathcal{M}$ and its neighbourhood $\mathcal{U}'$, there is a semianalytic diffeomorphism of $\mathbb{R}^n$ which moves x, but restricted to $\mathbb{R}^n/\mathcal{U}'$ is the identity map. With the semianalytic case, one is not requiring that the diffeomorphisms we consider be analytic everywhere but, roughly speaking, analytic only up to submanifolds of lower dimension. Some care has to be taken in the precise definition of this notion (see section I.19), mainly to insure that they form a group and that application of these diffeomorphisms produce surfaces and edges that still have finitely many isolated intersection points. The analyticity of the entire analytic patches will ensure the finite intersection property with piecewise analytic edges discussed before. The important point is that this larger symmetry group now contains local diffeomorphisms, and this is instrumental for proving the uniqueness result.

3.12.2 Outline of Proof

We have already seen that a diffeomorphism invariant state exists, the Ashtekar-Lewandowski state ω_0. Given a $^*-$algebra and a symmetry group, assuming the existence of a diffeomorphism invariant state is a strong condition and it turns out that ω_0 is the only one - this is the result of the uniqueness theorem.

One defines a $*-$algebra $\mathfrak{A}$ called the Sahlmann holonomy-flux $*-$algebra whose elements correspond to cylindrical functions and fluxes, and then studies $*-$representations of the algebra on a Hilbert space.

Consider the space of cylindrical functions on $\mathcal{A}$, that is, functions of the form

$$\Phi[A] \equiv \phi(h_{e_1}[A], h_{e_2}[A], \ldots, h_{e_n}[A])$$

where ϕ is a complex-valued continuous function. From the Poisson brackets of A and E one can compute the Poisson brackets for the Φ, $E_{S,f}$.

We then consider the Hamiltonian vector fields $X_{S,f}$ of the fluxes $E_f(S)$ defined by

$$X_{S,f} \cdot A(p) = \{E_f(S), A(p)\}.$$

Then the association of the classical functional E with elements of $\mathfrak{A}$ is given by

$$E_{S,f} \quad \mapsto \quad \hat{X}_{S,f}. \tag{3.130}$$

We let $\mathfrak{A}$ be generated by

$$\Phi \quad \mapsto \quad \hat{\Psi}\Phi := \Psi\Phi, \tag{3.131}$$

$$\Phi \quad \mapsto \quad \hat{X}_{S,f}\Phi, \tag{3.132}$$

where Φ, Ψ are C^∞ cylindrical functions. We introduce the $*$ operation on $\mathfrak{A}$ by specifying its action on the generating set:

$$\hat{\Psi}^* := \overline{\hat{\Psi}}, \quad \hat{X}^*_{S,f} := \hat{X}_{S,f}.$$

Every element of the algebra $\mathfrak{A}$ is a finite linear combination of elements of the form

$$\hat{\Psi},\ \hat{\Psi}_1\hat{X}_{S_{11},f_{11}},\ \hat{\Psi}_2\hat{X}_{S_{21},f_{21}}\hat{X}_{S_{22},f_{22}},\ \ldots,\ \hat{\Psi}_k\hat{X}_{S_{k1},f_{k1}}\ldots\hat{X}_{S_{kk},f_{kk}},\ \ldots\ , \tag{3.133}$$

$$\begin{aligned} a &= \hat{X}_{S,f}\hat{\Psi}\hat{X}_{S',f'} \\ &= -i\widehat{\{\hat{X}_{S,f},\hat{\Psi}\}}\hat{X}_{S',f'} + \hat{\Psi}\hat{X}_{S,f}\hat{X}_{S',f'} \\ &= -i\widehat{X_{S,f}(\Psi)}\hat{X}_{S',f'} + \hat{\Psi}\hat{X}_{S,f}\hat{X}_{S',f'}. \end{aligned} \tag{3.134}$$

Note that since $\mathfrak{A}$ is generated by the elements of Cyl and X, a represention π of $\mathfrak{A}$ is completely determined once the representors π (Cyl) and $\pi(X)$ are known.

Let us use the GNS notation

$$[a] = \pi_\omega(a)\Omega_\omega,$$

then

$$[\hat{\Psi}],\ [\hat{\Psi}_1\hat{X}_{S_{11},f_{11}}],\ [\hat{\Psi}_2\hat{X}_{S_{21},f_{21}}\hat{X}_{S_{22},f_{22}}],\ \ldots,\ [\hat{\Psi}_k\hat{X}_{S_{k1},f_{k1}}\ldots\hat{X}_{S_{kk},f_{kk}}],\ \ldots\ , \tag{3.135}$$

where $[a] := \{a + b : b \in \mathfrak{A} \text{ such that } \omega(b^*b) = 0\}$ is the equivalence class of $a \in \mathfrak{A}$ with respect to the Gel'fand ideal of null vectors, discussed above. For the $*$ operation to be well defined we must have that b^* in the equivalence class of a (we go into the details of this in appendix P). This algebra is a unital one with a unit given by a constant cylindrical function of the value equal to 1.

The product provides a norm $\|a\|_\omega = \sqrt{< [a], [a] >}$ in $[\mathfrak{A}]$, and the completion

$$\mathcal{H}_\omega := \overline{[\mathfrak{A}]} \tag{3.136}$$

together with the inner product $< \cdot\,, \cdot >$ is a Hilbert space.

To every element a of $\mathfrak{A}$ we asign a linear but in general unbounded operator $\pi_\omega(a)$ acting in $[\mathfrak{A}]$

$$\pi_\omega(ab)\Omega_\omega = \pi_\omega(a)\pi_\omega(b)\Omega_\omega, \quad \text{for every} \quad b \in \mathfrak{A}$$

or

$$\pi_\omega(a)[b] := [ab], \quad \text{for every} \quad b \in \mathfrak{A} \tag{3.137}$$

The action of π_ω preserves the suspace $[\mathfrak{A}]$, hence a dense set of vectors in $\mathcal{H}_\omega$ is given by the linear span of all the vectors of the form

$$\begin{aligned}[][\hat{\Psi}],\ \pi_\omega(\hat{\Psi}_1)[\hat{X}_{S_{11},f_{11}}],\ \pi_\omega(\hat{\Psi}_2\hat{X}_{S_{21},f_{21}})[\hat{X}_{S_{22},f_{22}}],\dots \\ \dots,\pi_\omega(\hat{\Psi}_k\hat{X}_{S_{k1},f_{k1}}\dots)[\hat{X}_{S_{kk},f_{kk}}]\end{aligned} \tag{3.138}$$

The space of cylindrical functions used to construct $\mathfrak{A}$ leads naturally to the space of generalized connections $\overline{\mathcal{A}}$.

In the Ashtekar-Lewandowski representation the fluxes vanish. It turns out that the only representation with this property is indeed the Ashtekar-Lewandowski representation. By vanishing of fluxes we mean that for any face S and any smearing vector field f in any GNS-representation coming from the invariant state

$$[X_{S,f}] = 0. \tag{3.139}$$

Once this rather technical result is established the rest of the proof of uniqueness is fairly straightforward.

This is proved by first making a certain decomposition and proving that each individual term satisfies this condition.

The proof of this result relies crucially on the local character of semianalycity. By local character is meant there is a partition of unity subordinate to a local finite covering $\{\mathcal{U}_I\}$, i.e., there exists a family of differentiable functions $\phi_I(x)$ such that

(i) $0 \leq \phi_I(x) \leq 1$

(ii) $\phi_I(x) = 0$ if $x \neq U_I$

(iii) $\sum_I^N \phi_1(x) = 1$ for any point $x \in \mathcal{M}$.

Even though a partition of unity does not exist in the analytic case, it does for the semianalytic one, just as in the smooth case.

Analytic diffeomorphisms - an analyic function is already determined by its values in an arbiary small neighbourhood of any point. The local aspect of semianalytic diffeomorphisms is required in the above result.

Because of this result, only the terms of the form $[\hat{\Psi}]$ in (3.138) are non-zero. It follows that all the information on the state ω is determined by its restriction to smooth cylindrical functions Cyl^∞. As Cyl is a unital Abelian C^*-algebra we can now use some powerful results of representation theory. Due to the theorem of Gelfand, since Cyl is Abelian, it is isomophic to the algebra of continuous functions on the spectrum $\overline{\mathcal{A}}$, on a compact Hausdorff space, of Cyl. From this and the representation theorem of Riesz and Markow, there exists a measure μ on $\overline{\mathcal{A}}$ such that

$$\omega(\hat{\Psi}) = \int_{\overline{\mathcal{A}}} \Psi d\mu.$$

Note that (3.139) implies in what follows

$$\begin{aligned}
\int_{\overline{\mathcal{A}}} \overline{\Psi} X_{S,f}(\Psi') d\mu &= \big\langle [\overline{\Psi}], [\widehat{X_{S,f}(\Psi')}] \big\rangle_{\mathcal{H}_\omega} \\
&= i\big\langle [\hat{\Psi}], [X_{S,f}\Psi'] - [\Psi' X_{S,f}] \big\rangle_{\mathcal{H}_\omega} \\
&= i\big\langle [\hat{\Psi}], [X_{S,f}\Psi'] \big\rangle_{\mathcal{H}_\omega} \\
&= i\big\langle [X_{S,f}\hat{\Psi}], [\Psi'] \big\rangle_{\mathcal{H}_\omega} \\
&= -\int_{\overline{\mathcal{A}}} \overline{X_{S,f}(\Psi)} \Psi' d\mu.
\end{aligned}$$

Setting $\Psi = I$ (i.e. the constant function on $\overline{\mathcal{A}}$ on the value 1) it follows that for any face S and any smearing vector field f and for any function $\Psi' \in \mathrm{Cyl}$

$$\int_{\overline{\mathcal{A}}} X_{S,f}(\Psi') d\mu = 0.$$

As was shown in [88] the only measure satifying the above condition coincides with the measure defined on $\overline{\mathcal{A}}$ by the Ashtekar-Lewandowski state ω_0.

3.12.3 Irreducibility

If the theory has non-trivial closed invariant subspaces signals the possibility that the representation space $\mathcal{H}_0$ is too large because interesting physics can already be captured by one of its invariant subspaces [114]. It has been proven that this is not the case for π_0, further strenghing physical assumption undrlying LQG.

Irreduciblity: In this situation, it is worthwhile to note that there is a strong analogy between the AIL representation of $\mathcal{U}$ and the Schrödinger representation of the Heisenberg algebra in quantum mechanics. In both representations, the representation spaces are roughly speaking L^2 spaces over the configuration space, the configuration variables act by multiplication and the momenta by derivations. The Heisenberg algebra is again an algebra of unbounded operators, which makes the definition of irreducibility diffcult. Moreover it is dubious that its Schrödinger representation can be irreducible in any sense, since for example the sub-spaces generated by functions which vanish on fixed open sets are invariant under action with multiplication operators and differentiation (if defined). To see this consider the space of C^∞ functions non-vanishing in the interval $I = [-a, a]$ of the real line, which is non-empty by example of

$$f(x) = \begin{cases} \exp(-1/(x-a)^2) + \exp(-1/(x+a)^2) & |x| \leq a \\ 0 & |x| > a \end{cases} .$$

Any function in C_I^∞ acted on by $\hat{x}, \hat{p}$ an arbitary number of times results in another function belonging to C_I^∞. Obviously, this space of functions is orthogonal to any function belonging to C_J^∞ where $J \subset \mathbb{R}$ with $I \cap J = \emptyset$. Hence there are many invariant subspaces under the operators $\hat{x}, \hat{p}$.

However,

$$(V(b)f)(x) = f(x+b)$$

changes the support and hence there is a chance that the Weyl algebra is represented in an irreducible fashion.

However, the Schrödinger representation of the Heisenberg algebra can be obtained from the Schrödinger representation of the corresponding Weyl algebra, and it is this representation that is irreducible. That is, if $\rho(a, b)$ $(a, b \in \mathbb{R})$ is the Schrödinger representation of the Heisenberg algebra, then the only nonzero closed invariant subspace of $L^2(\mathbb{R}^n)$ is $L^2(\mathbb{R}^n)$ itself.

3.12.4 Outline of Proof

Introduce a flux vector field satisfying

$$Y_\gamma(t_\gamma) f_\gamma = \sum_{e \in E(\gamma)} t_e^j R_J^e f_\gamma$$

$$\begin{aligned} M_{\psi,\psi'}(t_\gamma, I_\gamma) &:= \; < \psi, T_{\gamma,I_\gamma} W_\gamma(t_\gamma) \psi' >_{\mathcal{H}_0} \\ &= \int_{\overline{A}} d\mu_0(A) \; T_{\gamma,I_\gamma}(A) W_\gamma(t_\gamma)(A) \end{aligned} \qquad (3.140)$$

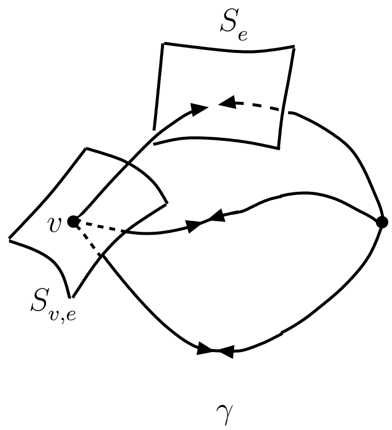

Figure 3.33: Action of $Y_\gamma(t_\gamma)$ on T_s

The inner product of the type

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma := \int_{D_\gamma} d\mu(t_\gamma) \sum_{I_\gamma} \overline{M_{\psi_1,\psi_1'}(t_\gamma, I_\gamma)} M_{\psi_2,\psi_2'}(t_\gamma, I_\gamma)$$

is a crucial ingredient in an elementary irreducibility proof of the Schrödinger representation of ordinary quantum mechanics.

i) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ we have

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \le \|\psi_1\| \, \|\psi_1'\| \, \|\psi_2\| \, \|\psi_2'\| \tag{3.141}$$

ii) For any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_{0,\gamma}$ we have

$$(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma = <\psi_2, \psi_1>_{\mathcal{H}_0} <\psi_1', \psi_2'>_{\mathcal{H}_0} \tag{3.142}$$

where $\mathcal{H}_{0,\gamma}$ denotes the closure of the cylindrical functions over γ.

In the evaluation of $M_{\psi,\psi'}(t_\gamma, I_\gamma)$ this the cylindrical delta function

$$\delta_\gamma(A, A') = \prod_{e \in E(\gamma)} \delta_{\mu_H}(h_e[A], h_e[A'])$$

arises from the Plancherel formula

$$\delta_{\mu_H}(g, g') = \sum_{\pi,m,n} \overline{T_{\pi,m,n}(g)} T_{\pi,m,n}(g')$$

(recall the formula $\delta(x,y)=\sum_n e_n^*(x)e_n(y)$ for a complete basis $e_n(x)$).

We subdivide the degrees of freedom of $A\in\overline{\mathcal{A}}$ into the set $\overline{\mathcal{A}}_\gamma$ and the complement. We need to be more presice about this definition which we do in appendix P but here we will be slightly heuristic. For any function $f\in\mathcal{H}_0$

$$f(A)=F(A_{|\overline{\gamma}},A_{|\gamma}).$$

We have

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma})\;\delta_\gamma(A,A')F(A_{|\overline{\gamma}},A'_{|\overline{\gamma}},A_{|\gamma},A'_{|\gamma})$$
$$=\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma})\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})F(A_{|\overline{\gamma}},A'_{|\overline{\gamma}},A_{|\gamma},A_{|\gamma})$$

First simply estimate we can make is

$$\begin{aligned}|(M_{\psi_1,\psi_1'},M_{\psi_2,\psi_2'})_\gamma| \;\leq\; & \int_{D_\gamma} d\mu(t_\gamma)\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \\ \times\; & \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\;\left|\Psi_1(A_{|\overline{\gamma}},A_\gamma)\right|\;\left|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}},A_\gamma)\right| \\ \times\; & \int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A'_{|\overline{\gamma}})\;\left|\Psi_2'(A'_{|\overline{\gamma}},A_\gamma)\right|\;\left|[W_\gamma(t_\gamma)\Psi_2](A'_{|\overline{\gamma}},A_\gamma)\right|\end{aligned} \tag{3.143}$$

Then by the Cauchy-Schwartz inequality

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\;\left|\Psi_1(A_{|\overline{\gamma}},A_\gamma)\right|\;\left|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}},A_\gamma)\right|$$
$$\leq\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\left|\Psi_1(A_{|\overline{\gamma}},A_\gamma)\right|^2\Big)^{1/2}\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}})\left|[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}},A_\gamma)\right|^2\Big)^{1/2}$$

We expand Ψ_1' in terms of spin-network functions

$$\Psi_1'(A_{|\overline{\gamma}},A_\gamma)=\sum_{n=1}^{\infty}T_{s_n}(A_{|\overline{\gamma}},A_\gamma).$$

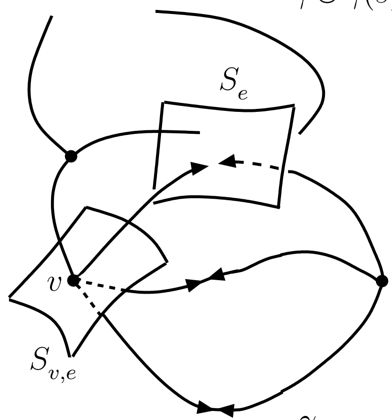

Figure 3.34: Action of $Y_\gamma(t_\gamma)$ on T_s. T_s is composed of degrees of freedom $A_{|\gamma}$ of A related to the edges $E(\gamma)$ and degrees of freedom $A_{|\overline{\gamma}}$, the complement of $A_{|\gamma}$, related to the edges $E(\gamma\cup\gamma(s)-\gamma)$.

This is where our choice of flux vector field $Y_\gamma(t_\gamma)$ simplifies the rest of the calculation.

Our conctrete vector field $Y_\gamma(t_\gamma)$ involves a finite collection of surfaces to which edges $e\in E(\gamma)$ are already adapted to. From fig (3.34) it is easy to see that the action of $Y_\gamma(t_\gamma)$ on T_s is given by

$$Y_\gamma(t_\gamma)T_s=[\sum_{e'\in E(\gamma\cup\gamma(s))-E(\gamma)} t_j^{e'}(t_\gamma)R^j_{e'}+\sum_{e\in E(\gamma)} t^e_j R^j_e]F.$$

where $T_j^{e'}(t_\gamma)$ is a certain linear combination of the t^e_j depending on e' and the concrete sutfaces $S_e, S_{v,e}$ used in the construction of $Y_\gamma(t_\gamma)$.

The right action R^j_e on T_s is easily computed

$$\begin{aligned} R^j_e T_s &= [\tau^j\pi(h_e)]_{mn}\frac{\partial T_s}{\partial[\pi(h_e)]_{mn}} \\ &= \sqrt{d_{\pi_{e_1}}}[\pi_{e_1}(h_{e_1})]_{m_{e_1}n_{e_1}}\dots[\tau_j]_{m_e l_e}\sqrt{d_{\pi_e}}[\pi_e(h_e)]_{l_e n_e}\dots\sqrt{d_{\pi_{e_N}}}[\pi_{e_N}(h_{e_N})]_{m_{e_N}n_{e_N}} \end{aligned}$$

$$\begin{aligned} W_\gamma(t_\gamma)\Psi'_1(A_{|\overline{\gamma}},A_\gamma) &= \sum_{n=1}^\infty W_\gamma(t_\gamma)T_{s_n}(A_{|\overline{\gamma}},A_\gamma) \\ &= \sum_{n=1}^\infty F_{s_n}(\{e^{t_j^{e'}(t_\gamma)\tau_j}h_{e'}\}_{e'\in E(\gamma)\cup E(\gamma(s))-E(\gamma)},\{e^{t^e_j\tau_j}h_e\}_{e\in E(\gamma)}) \end{aligned}$$

The integral

$$\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \big| [W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}}, A_\gamma)\big|^2$$

can be written as a countable linear combination of integrals over spin-network functions T_s and then the prescription is to integrate over the degrees of freedom $A(e'), e' \in E(\gamma(s) \cup \gamma) - E(\gamma)$ for each individual integral with the corresponding product Haar measure. By the left invariance of the Haar measure

$$\begin{aligned}
& \int_G d\mu_H(h_{e'}) \overline{T_{s_m}(\{e^{t_j^{e'}(t_\gamma)\tau_j} h_{e'}\}, \{e^{t_j^e \tau_j} h_e\})} T_{s_n}(\{e^{t_j^{e'}(t_\gamma)\tau_j} h_{e'}\}, \{e^{t_j^e \tau_j} h_e\}) \\
= & \int_G d\mu_H(h_{e'}) \overline{T_{s_m}(\{h_{e'}\}, \{e^{t_j^e \tau_j} h_e\})} T_{s_n}\{(h_{e'}\}, \{e^{t_j^e \tau_j} h_e\}).
\end{aligned}$$

This allows us to make the replacement

$$[W_\gamma(t_\gamma)\Psi_1'](A_{|\overline{\gamma}}, A_\gamma) \to \Psi_1'(A_{|\overline{\gamma}}, \{e^{t_j^e \tau_j} h_e\}_{e \in E(\gamma)})$$

in the integral.

Introduce the notation

$$\Big(\int_{\overline{\mathcal{A}}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) \big| \Psi_1(A_{|\overline{\gamma}}, A_\gamma)\big|^2 \Big)^{1/2} = \|\Psi_1(A_\gamma)\|_{\overline{\gamma}}.$$

In (3.143) we introduce new integration variables $h_e' := g(t_e) h_e$ where $g(t_e) = \exp(t_j^e \tau_j)$. Since by defintion

$$d\mu(t_\gamma) = \prod_{e \in E(\gamma)} d\mu(t_e) = \prod_{e \in E(\gamma)} d\mu_H(g(t_e))$$

$$\begin{aligned}
|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})_\gamma| \;\; \leq & \;\; \int_{D_\gamma} d\mu(t_\gamma) \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \; \|\Psi_1(A_\gamma)\|_{\overline{\gamma}} \; \|\Psi_1'(\{g_e A(e)\}_{e \in E(\gamma)})\|_{\overline{\gamma}} \\
& \times \|\Psi_2(A_\gamma)\|_{\overline{\gamma}} \; \|\Psi_2'(\{g_e A(e)\}_{e \in E(\gamma)})\|_{\overline{\gamma}} \\
= & \;\; [\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \; \|\Psi_1(A_\gamma)\|_{\overline{\gamma}} \; \|\Psi_2(A_\gamma)\|_{\overline{\gamma}}] \\
& \times [\int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A'_{|\gamma}) \;\; \|\Psi_1'(A_\gamma')\|_{\overline{\gamma}} \;\; \|\Psi_2'(A_\gamma')\|_{\overline{\gamma}}] \\
\leq & \;\; \| \, \|\Psi_1\|_{\overline{\gamma}} \, \|_\gamma \;\; \| \, \|\Psi_1'\|_{\overline{\gamma}} \, \|_\gamma \;\; \| \, \|\Psi_2\|_{\overline{\gamma}} \, \|_\gamma \;\; \| \, \|\Psi_2'\|_{\overline{\gamma}} \, \|_\gamma
\end{aligned}$$

where we have used the Cauchy-Schwartz inequality again. Noting

$$
\begin{aligned}
\| \, \|\Psi_1\|_{\overline{\gamma}} \, \|_\gamma &= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \mid \Psi_1(A_{|\gamma}) \mid^2 \\
&= \int_{\overline{\mathcal{A}}_\gamma} d\mu_{0\gamma}(A_{|\gamma}) \int_{\overline{A}_{\overline{\gamma}}} d\mu_{0\overline{\gamma}}(A_{|\overline{\gamma}}) |\Psi_1(A_{|\overline{\gamma}}, A_{|\gamma})|^2 = \int_{\overline{\mathcal{A}}} d\mu_0(A) |\psi_1(A)|^2 \\
&= \|\psi_1\|^2_{\mathcal{H}_0}
\end{aligned}
\tag{3.144}
$$

completes the proof of the inequality i). The proof of ii) is simply as the integrals over $\overline{A}_{|\overline{\gamma}}$ are trivial.

With these results established the rest of the proof is straightforward. Suppose the representation π_0 is not irreducible, that is, not every vector is cyclic. Thus we can find non-zero vectors $\psi, \psi' \in \mathcal{H}_0$ such that

$$
< \psi, a\psi' > = 0 \tag{3.145}
$$

for all $a \in \mathfrak{M}$.

Since the cylindrical functions are dense in $\mathcal{H}_0$, for any $\epsilon > 0$ we can find a graph γ and functions f, f' cylindrical over γ such that

$$
\|\psi - f\| < \epsilon, \quad \|\psi' - f'\| < \epsilon. \tag{3.146}
$$

$$
\begin{aligned}
(M_{\psi-f,\psi'}, M_{\psi,\psi'}) &= (< \psi, \psi >_{\mathcal{H}_0} - < \psi, f >_{\mathcal{H}_0}) < \psi', \psi' >_{\mathcal{H}_0} \\
(M_{\psi,\psi'-f'}, M_{\psi,\psi'}) &= < \psi, f >_{\mathcal{H}_0} (< \psi', \psi' >_{\mathcal{H}_0} - < f', \psi' >_{\mathcal{H}_0}) \\
(M_{f,f'}, M_{\psi-f,\psi'}) &= (< \psi, f >_{\mathcal{H}_0} - < f, f >_{\mathcal{H}_0}) < f', \psi' >_{\mathcal{H}_0} \\
(M_{f,f'}, M_{f,\psi'-f'}) &= < f, f >_{\mathcal{H}_0} (< f', \psi' >_{\mathcal{H}_0} - < f', f' >_{\mathcal{H}_0}) \\
(M_{f,f'}, M_{f,f'}) &= < f, f >_{\mathcal{H}_0} < f', f' >_{\mathcal{H}_0}
\end{aligned}
\tag{3.147}
$$

$$
\begin{aligned}
0 &= (M_{\psi,\psi'}, M_{\psi,\psi'})_\gamma \qquad (3.148) \\
&= (M_{\psi-f,\psi'}, M_{\psi,\psi'})_\gamma + (M_{f,\psi'-f'}, M_{\psi,\psi'})_\gamma + (M_{f,f'}, M_{\psi-f,\psi'})_\gamma + (M_{f,f'}, M_{\psi,\psi'-f'})_\gamma \\
&\quad + \|f\|^2 \, \|f'\|^2
\end{aligned}
$$

Obviously

$$\|f\|^2\,\|f'\|^2 \le |(M_{\psi-f,\psi'}, M_{\psi,\psi'})| + \cdots + |(M_{f,f'}, M_{f,\psi'-f'})|$$

Recall for any $\psi_1, \psi_1', \psi_2, \psi_2' \in \mathcal{H}_0$ we have

$$|(M_{\psi_1,\psi_1'}, M_{\psi_2,\psi_2'})| \le \|\psi_1\|\,\|\psi_1'\|\,\|\psi_2\|\,\|\psi_2'\|.$$

hence

$$\begin{aligned}\|f\|^2\,\|f'\|^2 \;\le\; & \|\psi - f\|\,\|\psi'\|\,\|\psi\|\,\|\psi'\| + \|f\|\,\|\psi' - f'\|\,\|\psi\|\,\|\psi'\| \\ & + \|f\|\,\|f'\|\|\psi - f\|\,\|\psi'\| + \|f\|\,\|f'\|\,\|f\|\,\|\psi' - f'\|\end{aligned}$$

therefore

$$\begin{aligned}(\|\psi\| - \epsilon)^2(\|\psi'\| - \epsilon)^2 \;\le\; & \epsilon\big\{\|\psi'\|^2\,\|\psi\| + (\|\psi\| + \epsilon)\|\psi\|\,\|\psi'\| \\ & + (\|\psi\| + \epsilon)\,(\|\psi'\| + \epsilon)\,\|\psi'\| + (\|\psi\| + \epsilon)^2\,(\|\psi'\| + \epsilon)\big\}\end{aligned}$$

Since this inequality holds for all ϵ we can take $\epsilon \to 0$ and find

$$\|\psi'\|^2\,\|\psi\|^2 = 0 \tag{3.149}$$

that is, either $\psi = 0$ or $\psi' = 0$ in contradiction to our assumption. Hence π_0 is irreducible.

3.12.5 Alternative Uniqueness Theorem of Fleischhack

Regular: We wish to require that the Weyl algebra is represented weakly continuously. States whose GNS representation have this property are said to be regular.

Stone-von Neuman theorem says that if a representation is regular and irreducible then the representation is unique.

Quantum geometry:

1. diffeomorphism invariant;

2. regular;

3. irreducible;

4. semianalytic - stratified diffeomorphisms.

3.13 Summary:

· Significant problem in majour technical issues that plagued the formalism (and related ones) from the beginning.

· Spin networks provide an elegant and powerful calculational tool.

· Discreteness of areas and volumes.

· Well-defined functional intergation via the cyclindrical measure theory.

· Well-defined and well understood Kinematic Hilbert space.

· Solving the spacial diffeomorphism constraint by the group averaging technique.

· Uniqueness Theorem for the Ashtekar-Isham-Lewendowski representation of the holonomy-flux algebra.