Chapter 4

Dynamics: The Hamiltonian constraint and Spin foams

· **Real Formalism and the Hamiltonian constraint.**

· **Quantization of the Hamiltonian constraint.**

· **Spin foams - the quantum geometry of spacetime.**

· **Unsettled concerns.**

4.1 Introduction

The task before us is:

Construct a sequence of regularized operators, $\mathcal{H}^\epsilon(x)$ to represent the Hamiltonian constraints, in H_{kin}. Prove that the limit as $\epsilon \rightarrow 0$ takes diffeomorphism invariant states to diffeomorphism invariant states, and thus defines a finite operator in H_{diff}. Prove that the limit has a kernel in H_{Diff} that is infinite dimensional. This kernel $H_{Physical} \subset H_{Diff}$ is the physical Hilbert space.

$$\hat{S}_L \equiv \hat{W}\hat{S}_E\hat{W} \quad (4.1)$$

$$\mathcal{H}^{Mink}_{phys} \equiv \hat{W}\mathcal{H}^{Eucl}_{phys} \quad (4.2)$$

With the introduction of a generalized Wick transform to map the constraint equations of Riemannian general relativity to those of the Lorentzian theory. This opens up the possibility within "connection-dynamics" where one can work, throughout, only with real variables. The resulting quantum theory would then be free of complicated reality conditions.

At the quantum mechanical level, we need to impose extra conditions to ensure that at the end of the day we are dealing with real general relativity. One possibility that was suggested was to request that observables (perennials) of the theory be real. At a quantum mechanical level this means they should be self-adjoint operators with respect to the inner product one selects. In particular, this is used as a selection criterion for the inner product. We will also see that it is harder to handle theories with non-compact groups as complex $SO(3)$

We discussed in the first lecture how one could write the Hamiltonian constraint and suprisingly find some solutions. The results were unregulated and formal. Part of the problem stemmed from the fact that we were looking at the Hamiltonian constraint in its double-densitized form,

$$\tilde{\tilde{H}} = \epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j F^k_{ab} \tag{4.3}$$

Suppose one wants to promote this to an operator in the loop representation. We have at hand a maifold and loops. How could one build a density of weight two? The answer is: one can't, without the aid of external structures. Most regularizations of the "early years" did exactly that.

Couldn't we consider the single-densitized Hamiltonian? Then one could represent it as a Dirac delta, which is defined intrinsically.

The reason that stopped people from trying this is the complicated, non-polynomial form of the single-densitized constraint in terms of Ashtekar's new variables,

$$H = \frac{\epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j}{\sqrt{\epsilon^{ijk} \tilde{E}^a_i \tilde{E}^b_j \tilde{E}^c_k}} F^k_{ab} \tag{4.4}$$

4.2 Dynamics: The Hamiltonian constraint

In geometrodynamics la Wheeler and DeWitt, the basic canonically conjugate variables were the 3-metric and extrinsic curvature . The idea was to quantize these, making them into operators acting on wavefunctions on the space of 3-metrics, and then to quantize the Hamiltonian and diffeomorphism constraints and seek wavefunctions annihilated bythese quantized constraints. However, this program soon became regarded as dauntingly difficult for various reasons, one being the non-polynomial nature of the Hamiltonian constraint:

$$\mathcal{H} = \sqrt{\det q}(K_{ab}K^{ab} - (K_a^a)^2 - {}^3R), \tag{4.5}$$

where ${}^3R$ is the scalar curvature of the 3-metric. It is often difficult to quantize non-polynomial expressions in the canonically conjugate variables and their derivatives. The factor of $(\det q)^{1/2}$ is not even an entire function of the 3-metric!

In the 1980's Ashtekar found a new formulation of general relativity in which the canonically conjugate variables are a densitized complex triad field $\tilde{E}_i^a$ and a chiral spin connection A_a^i. When all the constraints are satisfied, these are related to the original geometrodynamical variables by

$$(\det q)q^{ab} = \delta^{ij}\tilde{E}_i^a\tilde{E}_j^b, \quad A_a^i = \Gamma_a^i - iK_a^i, \tag{4.6}$$

where Γ_a^i is built from the Levi-Civita connection of the 3-metric and K_a^i is built from the extrinsic curvature. In terms of these new variables the Hamiltonian constraint appears polynomial in form, reviving hopes for canonical quantum gravity.

Actually, in this formulation one works with the densitized Hamiltonian constraint, given by

$$\tilde{\mathcal{H}} = \epsilon^{ijk}F_{abi}E_j^aE_k^b = \mathrm{tr}(F_{ab}[E^a, E^b]) \tag{4.7}$$

where F_{abi} is the curvature of A_a^i, and the trace and commutator are interpreted by thinking of as indices. Clearly $\tilde{\mathcal{H}}$ is a polynomial in A_a^i, $E_{\underset{\sim}{i}}^a$, and their derivatives. However, it is related to the original Hamiltonian constraint by $\tilde{\tilde{\mathcal{H}}} = (\det q)^{1/2}\tilde{\mathcal{H}}$ so in a sense the original problem has been displaced rather than addressed. It took a while, but it was eventually seen that many of the problems with quantizing $\tilde{\tilde{\mathcal{H}}}$ can be traced to this fact (or technically speaking, the fact that it has density weight 2).

A more immediately evident problem was that because E_i^a is complex-valued, the corresponding 3-metric is alsocomplex-valued unless one imposes extra 'reality conditions'. The reality conditions are easy to deal with in the Riemannian theory, where the signature of spacetime is taken to be . There one can handle them by working with a real densitized triad field and an connection given not by the above formula but by

In the physically important Lorentzian theory, however, no such easy remedy is available.

Despite these problems, the enthusiasm generated by the new variables led to a burst of work on canonical quantum gravity.Many new ideas were developed, most prominently the loop representation. In the Riemannian theory, this allows one to rigorously construct a Hilbert space of wavefunctions on the space of connections on space. The idea is to work with graphs embedded in space, and for each such graph to define a Hilbert space of wavefunctions depending only on the holonomies of the connection along the edges of the graph. Concretely, if the graph has edges, the holonomies along its are summarized by a point in , and the Hilbert space we get is , defined using Haar measure on. If the graph is contained in a larger graph then is contained in and one has . We can thus form the union of all these Hilbert spaces and complete it to obtain the desired Hilbert space.

One can show that has a basis of 'spin networks', given by graphs with edges labeled by representations of –i.e., spins – and vertices labeled by vectors in the tensor product of the representations labeling the incident edges. One can also rigorously quantize geometrically interesting observables such as the total volume of space, obtaining operators on . The matrix elements of these operators can be explicitly computed in the spin network basis.

Thiemann's approach applies this machinery developed for the Riemannian theory to Lorentzian gravity by exploiting theinterplay between the Riemannian and Lorentzian theories. He takes as his canonically conjugate variables an connection and a real densitized triad field , and takes as his Hilbert space as defined above. This automatically deals with the reality conditions, as in the Riemannian case. Then he writes the Lorentzian Hamiltonian constraint in terms of these variables, and quantizes it to obtain a densely defined operator on – modulo some subtleties we discuss below. Interestingly, it is crucial to his approach that he quantizes the Hamiltonian constraint rather than the densitized Hamiltonian constraint . This avoids the regularization problems that plagued attempts to quantize .

He writes the Lorentzian Hamiltonian constraint in terms of and in a clever way, as follows. First he notes that

$$\mathcal{H} = -\mathcal{H}_R + \frac{2}{\sqrt{\det q}} \mathrm{Tr}([K_a, K_b][E^a, E^b]) \qquad (4.8)$$

where the commutators and trace are taken in $su(2)$, and is the Riemannian Hamiltonian constraint, given by

Then he notes that

where

is the total volume of space (which is assumed compact). This observation lets him get rid of the terrifying factors of . Similarly, he notes that

where

Thus he obtains

where

Finally, he eliminates from the formula for using the formula

If we use the standard trick of replacing Poisson brackets by commutators, these formulas reduce the problem of quantizing to the problem of quantizing , , and . As noted, the volume has already been successfully quantized, and the resulting 'volume operator' is known quite explicitly. This leaves the connection and curvature.

Now, a fundamental fact about the loop representation – at least as currently formulated – is that the connection and curvature do not correspond to well-defined operators on , even if one smears them with test functions in the usual way. Instead, one has operators corresponding to the holonomy along paths in space. The holonomy along an open path can be used to define a kind of substitute for , and the holonomy around an open loop to define a substitute for . One cannot, however, take the limit as the path or loop shrinks to zero length. Thus the best one can do when quantizing a polynomial in and is to choose some paths or loops and use the substitutes built from holonomies. This eliminates problems associated with multiplying operator-valued distributions, but it introduces another kind of ambiguity: dependence on the arbitrary choice of path or loop.

So, ironically, while the factors of in the Hamiltonian constraint are essential in Thiemann's approach, the polynomial expressions in and introduce problematic ambiguities! Accepting but carefully minimizing this ambiguity, Thiemann obtains for any lapse function a large family of different versions of the smeared Hamiltonian constraint operator. The ambiguity is such that two different versions acting on a spin network give spin networks differing only by a diffeomorphism of space. Mathematically speaking we may describe this as follows. Let be the space of finite linear combinations of spin networks, and let be the space of finite linear combinations of spin networks modulo diffeomorphisms. Then Thiemann obtains, for any choice of lapse function, a Hamiltonian constraint operator

independent of the arbitrary choices he needed in his construction.

Since these operators do not map a space to itself we cannot ask whether they satisfy the naively expected commutation relations, the 'Dirac algebra'. However, this should come as no surprise, since the Dirac algebra also involves other operator-valued distributions that are ill-defined in the loop representation, such as . Thiemann does check as far as possible that the consequences one would expect from the Dirac algebra really do hold. Thus if one is troubled by how arbitrary choices of paths and loops prevent one from achieving a representation of the Dirac algebra, one is really troubled by the assumption, built into the loop representation, that , , and are not well-defined operator-valued distributions. Ultimately, the validity of this assumption can only be known through its implications for physics. The great virtue of Thiemann's work is that it brings us closer to figuring out these implications.

Remarkably, Thiemann (1996) discovered the identity,

$$\frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} = 2\epsilon^{abc}\{A^k_c, V\} \tag{4.9}$$

Which alows us to write the single densitized Hamiltonian as,

$$\tilde{H} = -2Tr(F_{ab}\{A_c, V\})\epsilon^{abc} \tag{4.10}$$

This is not only remarkably simple, but Thiemann found something else as well...

Recall that when Ashtekar introduced the new variables, there was a free parameter in the canonical transformation (the Immirzi β parameter) and if one did not choose it equal to the imaginary unit the Hamiltonian constraint looked like,

$$\begin{aligned} \tilde{\tilde{S}} &\equiv -\zeta\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j F_{abk} + \frac{2(\beta^2\zeta - 1)}{\beta^2}\tilde{E}^a_{[i}\tilde{E}^b_{j]}(A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) \\ &= \mathcal{H}_E(N) - 2(1+\gamma^2)\mathcal{T}(N) = 0 \end{aligned} \tag{4.11}$$

Thiemann found that the ugly looking second piece (divided by det(q)) can be written as,

$$\mathcal{T}(N) = 4\epsilon^{abc}Tr(\{A_a, K\}\{A_b, K\}\{A_c, V\}) \tag{4.12}$$

where,

$$K = -\{\frac{V}{\kappa}, \int_\Sigma d^3x H^E(x)\} \tag{4.13}$$

That is, not only is the "Euclidean" piece of the Hamiltonian constraint (single densitized) easy. The full Hamiltonian constraint without using complex variables is reasonably simple too!

4.3 Regularization of the Hamiltonian Constraint

$$[\hat{C}(N)'\Psi](f) := \Psi(\hat{C}^\dagger(N)f) \tag{4.14}$$

does not preserve $(\Phi^*_{Kin})_{Diff}$

Simple Regularization

$$\mathcal{C}^{Eucd}(N) = -\frac{2N(v_\Box)}{k^2\gamma^{3/2}}\sum_i Tr\Big((\overline{A}(\beta_i) - \overline{A}(\beta_i^{-1}))\overline{A}(s_i)^{-1}\{\overline{A}(s_i), V\}\Big) \tag{4.15}$$

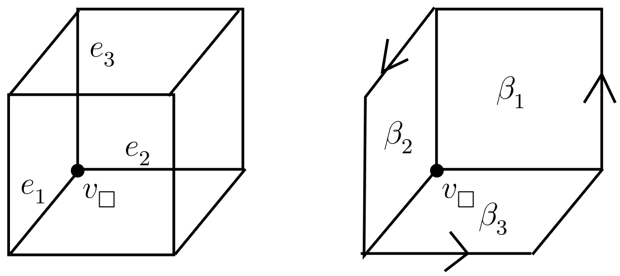

Figure 4.1: regfigHam1

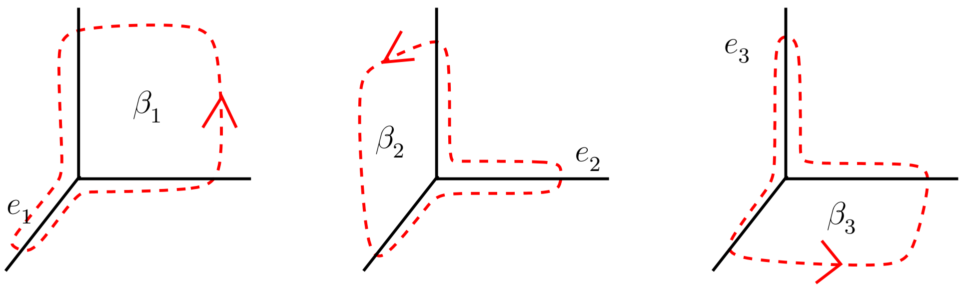

Figure 4.2: regfigHam2

State-Dependant Triangularization

Thiemann proceeds to quantize this expression. He starts by discretizing it on a lattice, obtained by triangulating the spatial manifold in terms of tetrahedra,

$$\tilde{H} = -2Tr(F_{ab}\{A_c, V\})\epsilon^{abc} \tag{4.16}$$

$$H^E_\Delta[N] = -\frac{2}{3}N_v\epsilon^{ijk}tr(h_{\alpha_{ij}}h_{s_k}(\Delta)\{h^{-1}_{s_k}(\Delta), V\}) \tag{4.17}$$

When $\Delta \to 0$, $h_{e_{ij}} = 1 + F_{ij}s^is^j\tilde{E}^2$; $\quad h_{s_k} = 1 + A_ks^k\tilde{E}$.

So the expression tends (pointwise) to the above one (times Δ^3) when the triangulation is shrunk. Moreover it is manifestly gauge invariant since loops are all closed in the end. Therefore,

$$H^E_T[N] = \sum_{\Delta T} H^E_\Delta[N] \tag{4.18}$$

Is a good approximation to the “smeared” classical Hamiltonian.

Since both the holonomy and the volume operator are well defined quantities in the space of cylindrial functions of spin networks, it is immediate to promote H to a quantum operator.

via a state-dependent triangulation T on Σ??

$$\hat{H}^E_\Delta[N] = -2\frac{N(v(\Delta))}{3il_p^2}\epsilon^{ijk}tr(h_{\alpha_{ij}}h_{s_k}(\Delta)[h^{-1}_{s_k}(\Delta),\hat{V}]) := N_v\hat{H}^E_\Delta \tag{4.19}$$

clearer?:

$$\hat{H}_\tau[N] = \frac{1}{\kappa}\sum_{\Delta\in\tau} N(p_\Delta)\epsilon^{IJK}Tr((A(\alpha_{IJ}(\Delta))A_{s_K}(\Delta)[A_{s_K}(\Delta)^{-1},\hat{V}(R_{p(\Delta)}]) \tag{4.20}$$

Here $T(\gamma, v)$ is the number of ordered triples of edges incident at v (taken with outgoing orientation), see fig. (**??**). This is

$$T(\gamma, v) = \frac{n_v(n_v-1)(n_v-2)}{6}$$

where n_v is the valence of vertex v. Proof. First assume that the three vectors e_1, e_2, e_3 and the 'empty edges' don't come in any particular order - there are $n_v!$ ways of arranging these. Now, we are only interseted in only one particular ordering of the vectors e_1, e_2, e_3 so we must divide by 3!. Similarly, we must divide by $(n_v - 3)!$ because the 'empty edges' have only one ordering themselves.

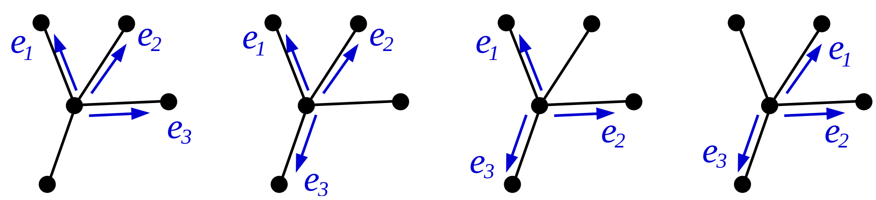

Figure 4.3: NumTripF. .

It is at this point where we must regularise (??). We consider a triangulation τ of Σ by tetrahedra $\triangle$. For each $\triangle$, let us single out a corner$p(\triangle)$ and denote the edges of $\triangle$ outgoing from $p(\triangle)$ by $s_I(\triangle)$, $I = 1, 2, 3$.

$$\int_\Sigma = [\int_{\Sigma - W^\epsilon_\gamma}] + \sum \frac{1}{T(\gamma, v)} \sum_{e_1\cap e_2\cap e_3 = v} \{[\int_{W^\epsilon_\gamma - W^\epsilon_\gamma(e_1,e_2,e_3)}] + [\int_{W^\epsilon_\gamma(e_1,e_2,e_3)}]\} \tag{4.21}$$

tetSaturate1ChD

4.4 Details of Thiemann's Hamiltonian Constraint

The first task here is to rewrite Thiemann's Hamiltonian (Eq.) in terms of the Poisson braket the connection and the volume and with the function K (see Eq.). This task will be broken down into managible parts.

$$\tilde{H} := \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}}F_{abk} + \frac{2}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}}\tilde{E}^a_{[i}\tilde{E}^b_{j]}(A^i_a - \Gamma^i_a)(A^j_b - \Gamma^j_b) \tag{4.22}$$

$$\tilde{H} = -2Tr(F_{ab}\{A^i_c(x), V\})\epsilon^{abc} + 4\epsilon^{abc}Tr(\{A^i_a(x), K\}\{A^j_b(x), K\}\{A^k_c(x), V\}) \tag{4.23}$$

First we need the important identity,

Detailed calculation (I1.1)———————————————————————

$$2\epsilon^{abc}\{A^k_c, V\} = \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.24}$$

where

$$V[\sigma]_\epsilon = \int_\sigma \sqrt{g} = \int \sqrt{\epsilon_{abc}\epsilon_{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k} \tag{4.25}$$

Proof:

$$\{A^b_j(x), V\} = \frac{\delta A^b_j(y)}{\delta A^a_i(x)}\frac{\delta V}{\delta\tilde{E}^a_i(x)} + 0 \tag{4.26}$$

the second part of the Poisson bracket is zero because $\delta A^b_j/\delta\tilde{E}^b_j = 0$ as they are independent variables.

$$\frac{\delta V}{\delta\tilde{E}^b_j} = \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.27}$$

$$\{A^k_c, V\} = \frac{\epsilon_{abc}\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.28}$$

We now use

$$\epsilon_{abe}\epsilon^{cde} = \delta^c_a\delta^d_b - \delta^c_b\delta^d_a \tag{4.29}$$

to get,

$$2\epsilon^{abc}\{A^k_c, V\} = \frac{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j}{\sqrt{\epsilon^{ijk}\tilde{E}^a_i\tilde{E}^b_j\tilde{E}^c_k}} \tag{4.30}$$

$$\begin{aligned} e^j_a(x) &= -\frac{2}{\kappa\beta}\{A^j_a, V(R_x)\} \\ K^j_a(x) &= -\frac{1}{\kappa\beta}\{A^j_a, \{C_E(1), V\}\} \\ V(R_x) &= \int_\sigma \chi_{R_x}(y)d^3y\sqrt{det(q(x))}(y) \\ C_E(1) &= C_E(N)_{|N=1} \end{aligned} \tag{4.31}$$

Thiemann proceeds to quantize this expression. He starts by discretizing it on a lattice, obtained by triangulating the spatial manifold in terms of tetrahedra,

$$\tilde{H} = -2Tr(F_{ab}\{A_c, V\})\epsilon^{abc} \tag{4.32}$$

$$H^E[N] = \frac{2}{\kappa}\int_\Sigma d^3xN(x)\epsilon^{abc}Tr(F_{ab}\{A_c, V\}) \tag{4.33}$$

$$H^E_\Delta[N] = -\frac{2}{3}N_v\epsilon^{ijk}tr(h_{\alpha_{ij}}h_{s_k}(\Delta)\{h^{-1}_{s_k}(\Delta), V\}) \tag{4.34}$$

When $\Delta \to 0$, $h_{e_{ij}} = 1 + F_{ij}s^is^j\tilde{E}^2$; $\quad h_{s_k} = 1 + A_ks^k\tilde{E}$.

So the expression tends (pointwise) to the above one (times Δ^3) when the triangulation is shrunk. Moreover it is manifestly gauge invariant since loops are all closed in the end. Therefore,

$$H^E_T[N] = \sum_{\Delta\in\mathcal{T}(\epsilon)} H^E_\Delta[N] \tag{4.35}$$

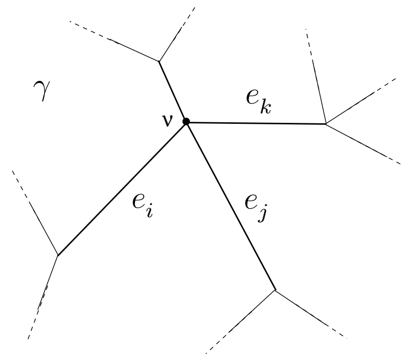

Figure 4.4: graphofnet. Given a graph γ we will construct one triangularization of the triple of edges meeting at the given vertex ν.

An elamentary tetrahedron tetrahedra shrink to their base points the expression (Eq. 4.34) converges (pointwise) to the expression (Eq. 4.33)

Is a good approximation to the “smeared” classical Hamiltonian.

In order to prepare the classsical expression for quantizaton, we triagulate space-like hypersurface Σ in terms of an elementary tetradra $\triangle$. The triangulation has the following properties: we select a finite set of distict points of Σ, denoted as $\{v\}$. At each of these points we choose three independent directions $(\hat{u}_1), (\hat{u}_2), (\hat{u}_3)$ and construct the eight tetrhedra with vertex v and edges $(\pm u_1), (\pm u_2), (\pm u_3)$, with $u_i = \epsilon \hat{u}_i$. The eight terahedra saturating v define a closed region of scale length ϵ.

The rest of the region $\Sigma -$is triangulated by arbitrary tetrahedra Σ'. The motivation for this particular discretizaton is the when we promote the classical expression to a quantum operator, we will adapt the triangularizaton to the spin network of the state in question by choosing the points v to coincide with the vertices of the spin network.

We define the regulated operators on different $\mathcal{H}'_\alpha$ separately, then remove the regulator ϵ so that the limit operator is defined on $\mathcal{H}_{kin}$ cylindrically consistenly.

then we average with respect to the triples of edges meeting at the given vertex.

For each vertex v of γ and each three distinct edges (e_I, e_J, e_K) incident to v, the number $C_{n(v)}$ of such distinct triples of edges is easily seen to be

$$C_{n(v)} = \frac{n(v)}{3!}(n(v) - 1)(n(v) - 2) \tag{4.36}$$

where $n(v)$ is the valence of the vertex.

• The graph γ in $T(\epsilon)$ for all , so that every vertex v of γ coincides with a vertex $v(\triangle)$ in $T(\epsilon)$.

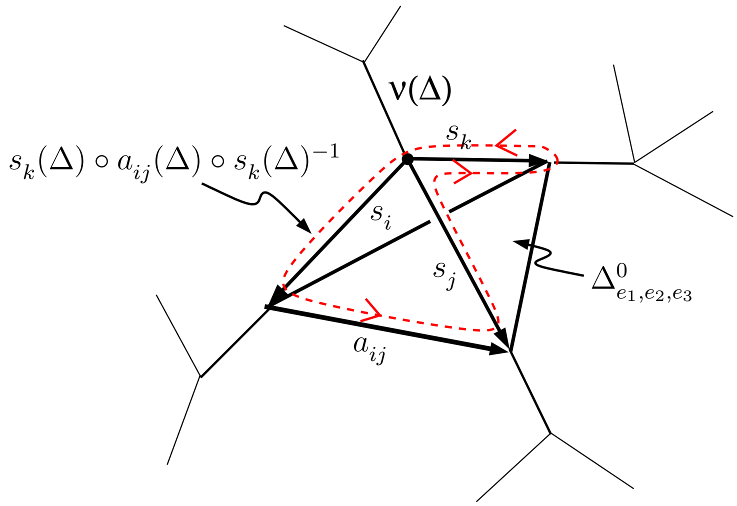

Figure 4.5: tetrahedron. An elementary tetrahedron and the choice of edges. Where $s_i(\Delta) \subset e_i$ for all $i = 1, 2, 3$. $a_{ij}(\Delta) := s_i(\Delta) \circ a_{ij}(\Delta) \circ s_j(\Delta)^{-1}$ is a "loop with a nose". It is clear that the regulated formular for $S(N)$, based on such paths, is invariant under internal gauge transformations.

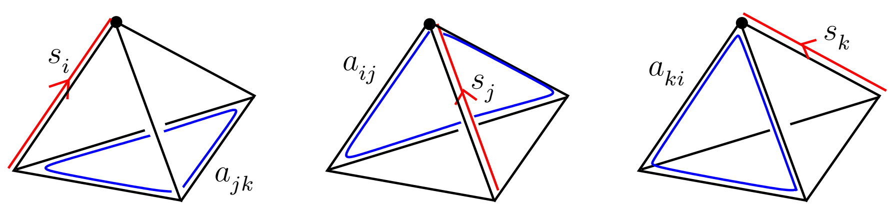

Figure 4.6: ThiemHam2. An elamentary tetrahedron and the edges.

• The triangulation must be fine enough so that the neighbourhoods $U(\nu) := \cup_{e_1,e_2,e_3} U_{e_1,e_2,e_3}(\nu)$ are disjoint for different vertices ν and ν' of γ. Thus for any open neigbourhood U_γ of the graph γ, there exists a triangulation $T(\epsilon)$ such that $\cup_{\nu \in V(\gamma)} U(\nu) \subseteq U_\gamma$.

• The distance between a vertex $\nu(\Delta)$ and the corresponding arcs α_{ij} is described by the parameter ϵ. For any two different ϵ and ϵ', the arcs $\alpha_{ij}(\Delta^\epsilon)$ and $\alpha_{ij}(\Delta^{\epsilon'})$ with respect to one vertex $\nu(\Delta)$ are semi-analytically diffeomorpic with each other .

• With the triangulations $T(\epsilon)$, the integral over Σ is replaced by the Riemanian sum:

$$\int_{U(v)} = \frac{1}{C^3_{n(v)}} \sum_{e_1,e_2,e_3} [\int_{U_{e_1,e_2,e_3}(v)} + \int_{U(v)-U_{e_1,e_2,e_3}(v)}] \tag{4.37}$$

where $n(v)$ is the valence of the vertex $v = s(e_1) = s(e_2) = s(e_3)$. Observe that

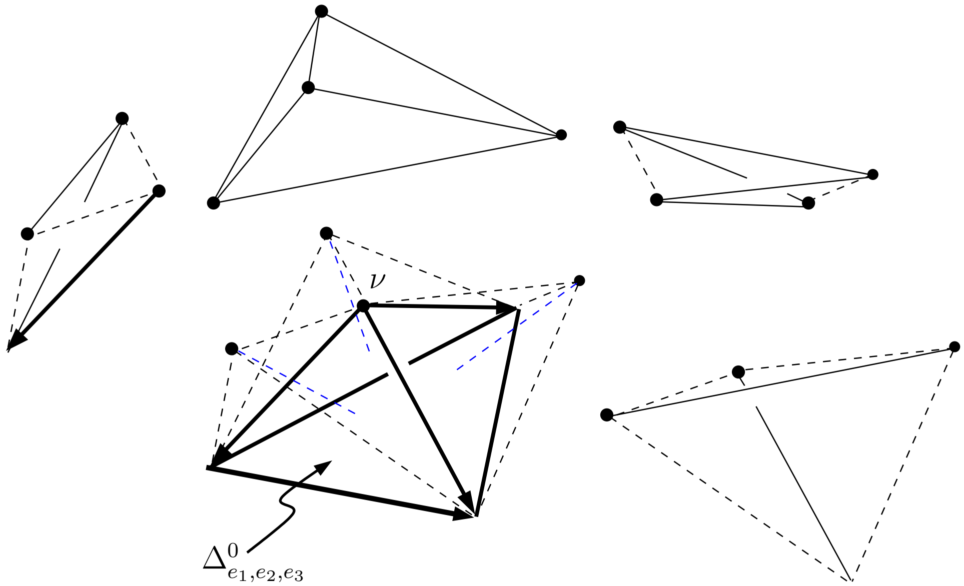

Figure 4.7: tetSaturate1. Visualizing the 7 additional tetrahedra $\Delta^{\mathfrak{p}}_{e_1,e_2,e_3}$, $\mathfrak{p}= 1,\ldots,7$, which together with $\Delta^0_{e_1,e_2,e_3}$, form a neighbourhood, $U_{e_1,e_2,e_3} := \cup_{\mathfrak{p}=0}^{7}\Delta^{\mathfrak{p}}_{e_1,e_2,e_3}$, of ν.

$$\int_{U_{e_1,e_2,e_3}(v)} = 8\int_{\Delta^0_{e_1,e_2,e_3}(v)} \tag{4.38}$$

in the limit $\epsilon \to 0$, that is when the tetrahedra shrink to there base points. As we will see because of this the dependency on the additional tetrahedra will drop out.

- Triangulation in the regions:

$$\begin{aligned} &U(v) - U_{e_1,e_2,e_3}(v)\\ &U_\gamma - \cup_{v\in V(\gamma)}U(v)\\ &\Sigma - U_\gamma \end{aligned} \tag{4.39}$$

are arbitrary. These regions do not contribute to the construction of the operator, since the commutator term $[A(s_i), V_{R_{v(\Delta)}}]\psi_\alpha$ vanishes for all tetrahedron Δ in the regions (4.39).

$$H(N) = \frac{2}{G}\lim\delta \to 0\int d^3y \sum_{\in\{,\Delta\}} \tag{4.40}$$

$$\mathcal{V}\epsilon^{abc} = \frac{\alpha}{6}\epsilon^{ijk}u_i^a u_j^b u_k^c, \tag{4.41}$$

where$\alpha = 8$ if $square =$, and $\alpha = 1$ if $square = \Delta$. In this last case, the u_i' represent the edges of the tetrahedra Δ' adjacent to $v_{\Delta'}$. We also have adapted the regularizaton to the Dirac delta funcion to the tetrahedral decomposition by defining,

$$f_d(y) = \frac{\Omega_d(y)}{\mathcal{V}_d} \tag{4.42}$$

where $\Omega_d(y)$ is one if $y \in D$ and zero otherwise.

$$\hat{H}^E_{T(\gamma)}[N]f := \hat{H}^E_\gamma[N]f = \sum_{v \in V(\gamma)} N_v \frac{8}{C_{n(v)}} \sum_{v(\Delta)=v} \hat{H}^E_\Delta f =: \sum_{v \in V(\gamma)} N_v \hat{H}^E_v f \tag{4.43}$$

there is no state in $\mathcal{H}_{Kin}$ that could be interpreted as "$\lim_{\epsilon \to 0} \hat{H}[N, \epsilon]\Psi$"

We assume we are acting on a state given by a loop transform,

$$\Psi[s] = \int \mathcal{D}A\Psi[A]W_A[s], \tag{4.44}$$

$$H(N) = \lim D \to 0 \int d^3y \sum \frac{\alpha}{3G} \epsilon_{ijk} u^a_i u^b_j N(y) Tr[F_{ab}() h(u_k)\{h^{-1}(u_k), V\}] f_D(y), \tag{4.45}$$

and we realize the Hamiltonian operator over spin network wavefunctions by promoting the classiacal expression Eq.(4.45) as an operator acting on the Wilson net appearing in the loop transform.

$$\mathcal{H}^\Delta_T[N] := \frac{2}{3GC(m)} N(v(\Delta)) \epsilon^{ijk} \mathrm{tr}\left[h_{\alpha_{ij}} h_{s_k} \{h^{-1}_{s_k(\Delta)}, V\}\right], \tag{4.46}$$

4.4.1 Quantization of the Regulated Constraint

In order to quantize this expression one now replaces all apearing quantities by operators and Poisson brackets by commutator divided by $i\hbar$.

$$\begin{aligned} &A(e) \mapsto \hat{A}(e), \quad V_R \mapsto \hat{V}_R, \quad \{\ ,\ \} \mapsto \frac{[\ ,\ ]}{i\hbar} \\ &K \mapsto \hat{K}^\epsilon = \frac{\gamma^{-2}}{i\hbar}[S^\epsilon_1(1), \hat{V}_\Sigma]. \end{aligned} \tag{4.47}$$

to arrive at an unambiguous result one had to make the triagulation *state dependent*. That is, the regulated operator is defined on a certain (so called spin-network??) basis elements T_s of the Hilbert space in terms of an adapted triangularization τ_s and extented by linearity. This is justified because the Riemann sum that enters the definition of $C_\tau(N)$ converges to $C(N)$ no matter how we refine the triangulation.

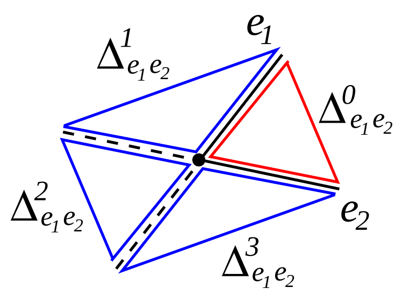

Figure 4.8: TriangHam. so that $U_{e_1,e_2}; = \cup^3_{\mathcal{P}=0} \triangle^{\mathcal{P}}_{e_1,e_2}$ is a neighborhood of v.

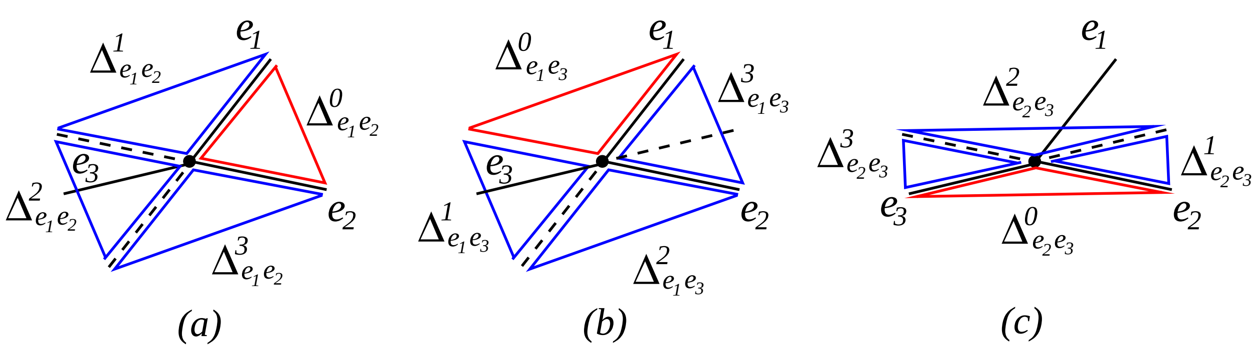

Figure 4.9: TriangHam2. trivalent node we average over the three triangles based on using only one such quadrupel of triangles.

4.4.2 Removal of the Regulator

The *weak operator topology* on $\mathcal{B}(H)$ is the weak topology generated by all functions of the form $T \to (Tx, y)$; that is, it is the weakest topology with respect to which all these functions are continuous. It is easy to see from the inequality $|(Tx, y) - (T_0x, y)| \leq \|T - T_0\| \, \|x\| \, \|y\|$ that this topology is weaker than the usual norm topology, so that its closed sets are also closed in the usual sense.

(from intro topology and modern analysis)

The action of the Hamiltonian consraint operator on ψ_γ adds an arc with a 1/2-representation with respect to each vertex $\nu(\Delta)$ of γ. - the action of $\hat{S}^\epsilon(N)$ on cylindrical functions is graph changing. Hence the operator does not converge with respect to the weak operator topology in $\mathcal{H}_{kin}$ when $\epsilon \to 0$, since different $\mathcal{H}'_{\alpha(\gamma)}$ with different graphs γ are mutually orthogonal. Thus on has to define a weaker operator topology to make the operator limit meaningful.

To take the infinite refinement limit (or continuum) $\tau \to \sigma$ of the resulting regulated operator $\hat{C}^{\dagger}_{\tau}(N)$ is non-trival because the holonomy operators are not even weakly continuous represented on the Hilbert space, hence the limit cannot exist in the weak oerator topology

The weak operator topology on $\mathcal{L}(X, Y)$ is the weakest topology such that the maps

$$E_{x,\ell} : \mathcal{L}(X, Y) \to \mathbf{C} \tag{4.48}$$

given by $E_{x,\ell}(T) = \ell(Tx)$ are all continuous for all $x \in X$, $\ell \in Y^*$. A basis at the origin is given by sets of the form

$$\{S|S \in \mathcal{L}(X, Y),\ |\ell_i(Tx_j)| < \epsilon,\ \ i = 1, \ldots, n,\ \ j = 1, \ldots, m\} \tag{4.49}$$

where $\{x_i\}_{i=1}^n$ and $\{x_j\}_{j=1}^m$ are finite families of elements of X and Y^* respectively.

It turns out that it exists in the, what one would cal the *weak Diff* topology* [32], [33]:

Let $\mathbf{\Phi}_{Kin}$ be a dense invariant domain for the closable operator $\hat{C}^{\dagger}_{\tau}(N)$ on the Hilbert space $\mathcal{H}_{Kin}$ and let $(\mathbf{\Phi}^*_{Kin})_{Diff}$ be the set of all spacially diffeomorphism invariant distributions over Φ_{Kin} (equipped with the topology of pointwise convergence - so algebraic dual of Φ_{Kin}?). Then $\lim_{\tau\sigma} \hat{C}_{\tau}(N) = \hat{C}_{\tau}(N)$ if and only if for each $\epsilon > 0$, T_s, $\ell \in (\Phi^*_{Kin})_{Diff}$ there exists $\tau_s(\epsilon)$ *independent of* ℓ such that

$$|\ell([\hat{C}_{\tau_s}(N) - \hat{C}(N)]T_s)| < \epsilon \ \text{ for all } \ \tau_s(\epsilon) \subset \tau_s \tag{4.50}$$

That the limit is uniform in ℓ is crucial because it excludes the existence of the limit on spaces larger than $(\Phi^*_{Kin})_{Diff}$ [] which would be unphysical because the space of physical solutions to all the constraints must obviously be a subspace of $(\Phi^*_{Kin})_{Diff}$. Notice that the limit is required refinements of adapted triangulations only.

Action of the regularised Hamiltonian

Given a spin network S, the operator (4.20) in two ways:

The graph is modified by the two operators $h_{\alpha_{ij}}$ and h_{s_k}. The first superposes a path of lenth ϵ to links of Γ. The second superposes a triangle two sides length ϵ alond links of Γ, and a third side that is not on Γ.

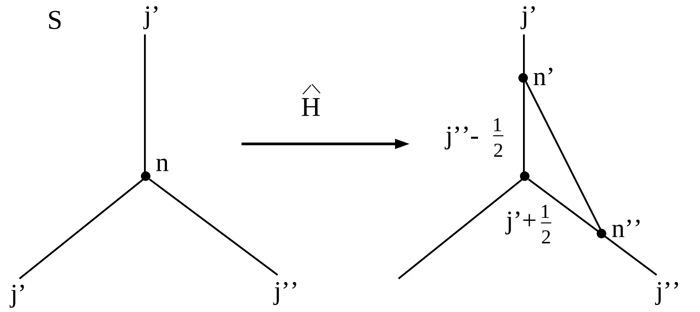

Figure 4.10: The action of the Hamiltonian constraint on a trivalent node.

The action of the Hamiltonian constraint on spin network states is the creation of a new edge of the underlying graph and rerouting of the quanta of angular momentum.

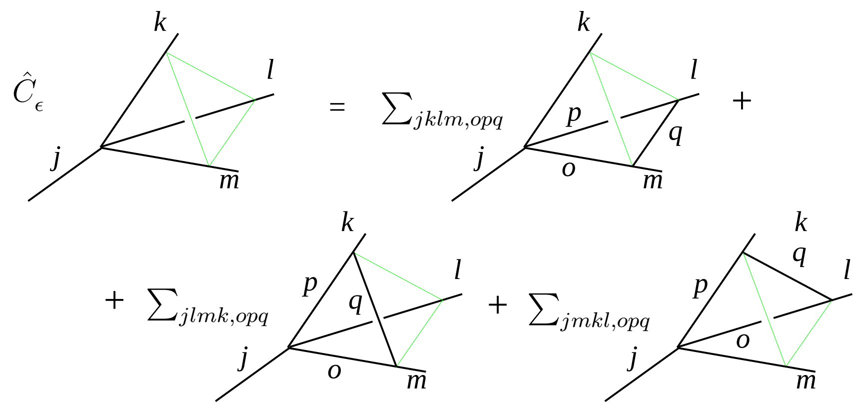

Figure 4.11: actionham2. The action of the Hamiltonian constraint on a node of valence 4.

$$(\phi|\ \hat{C}(N)\ |\psi >= \lim_{\epsilon \to 0} (\phi|\ \hat{C}_\epsilon(N)\ |\psi > \tag{4.51}$$

The operator is finite. It is well defined if we are acting on diffeomorphism invariant states. Otherwise we would have to specify where to add the "crossings" line of the loop α.

4.5 Concerns About the Hamiltonian Constraint

Quantum anomalies

There are complications. The classical condition for first class constraints,

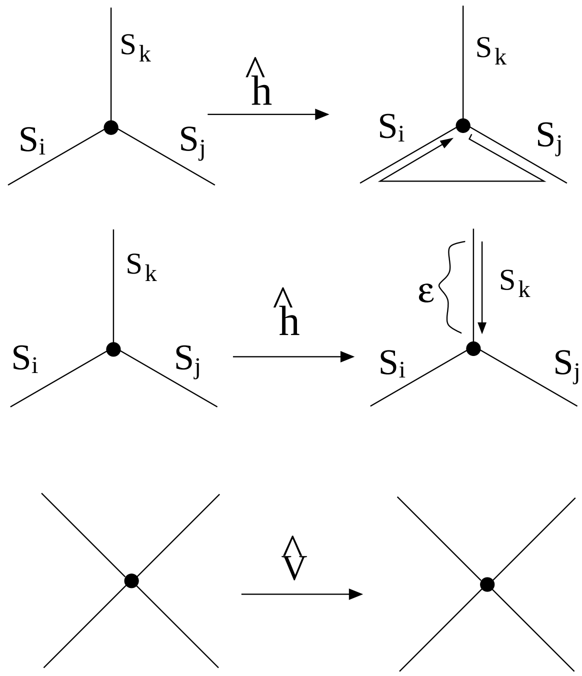

Figure 4.12: actionofham. The action of the Hamiltonian constraint on a trivalent node.

$$\{C_a, C_b\} = C^c_{ab} C_c, \tag{4.52}$$

could be spoiled by quantum corrections of order $\hbar^2$,

$$[\hat{C}_a, \hat{C}_b] = C^c_{ab} \hat{C}_c + \hbar^2 \hat{D}_{ab}. \tag{4.53}$$

In this case () would imply

$$\hat{D}_{ab} |\psi >= 0. \tag{4.54}$$

This could restrict the physical subspace.

Also, the classical relation

$$\{\mathcal{H}, C_a\} = V_a{}^b C_b \tag{4.55}$$

$$[\bar{\mathcal{H}}, \hat{C}_a] = V_a{}^b \hat{C}_b + \hbar^2 \hat{C}_a. \tag{4.56}$$

To apply the Dirac quantization method, one has to asume $\hat{D}_{ab} = \hat{C}_c = 0$

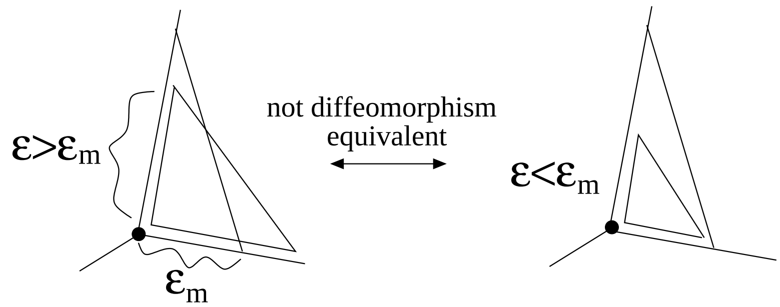

Figure 4.13: ϵ_{max}.

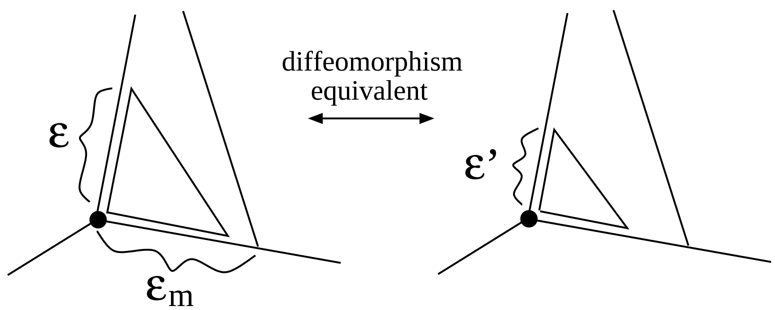

Figure 4.14: rescale. ϵ_{max}.

Due to this property, the operator is also anomaly-free. That is, the commutator of two Hamiltonians, vanishes, which agrees with the classical Poisson algebra on diffeomorphism invariant functions.

$$[\hat{H}(N), \hat{H}(M)] = 0 \quad \{\hat{H}(N), \hat{H}(M)\} = \int d^3x(N\partial_a M - M\partial_a N)g^{ab}C_b \tag{4.57}$$

If states have to solve additional constraints then the quantum theory would not have as many physical degrees of freedom as the classical theory. To ensure that this is so with the Hamiltonian constraint, we have to restrict the way it acts upon spinnetwork states: The way an operator acts on a state depends on the triangularisation prescription one is adhering to, the requirement for there to be no anomalies places a restriction on this triangularisatoin prescription.

Notice the crucial role that diffeomorphism invariance played in the construction. For ϵ and ϵ', sufficently small, $\hat{H}_\epsilon|\Psi>$ and $\hat{H}_{\epsilon'}|\Psi>$ are equivalent under diffeomorphisms. If the functions were not diffeomorphism invariant, the added line would have to be shrunk to the vertex and possible divergences could appear.

The same construction can be applied to the Hamiltonian of general relativity coupled to matter: scalar fields, Yang-Mills fields, fermions. In all cases the theory is finite, anomaly free and well defined. Gravity indeed appears to be acting as a fundemental regulator" of theories of matter.

A similar construction, applies in 2+1 dimensions, yeilds a theory of quantum gravity that correctly contains the physical states found by Witten through his quantization.

This technique can be applied, in particular, to the standard model coupled to gravity [110], [?]. In particular, it works for *Lorentzian* gravity while all other earlier proposals could at best work in the Euclidean context only (see [25] and references therein). The algebra of important operators of the resulting field theories was shown to be consistent [?]. Notice that as far as these operators are concerned, is stronger than the believed but unproven finiteness of scattering amplitudes order by order in perturbation theory of the five critical string theories, in a sense we claim the perturbation series converges. (Scatering amplitudes for background independent theories are conceptually very difficult and it is only recently that substantial progress has been made. How to define, interpret and calculate background independent scattering amplitudes are currently being developed [?].) The absence of the divergences that usually plague interacting field theories in a Minkowskian background spacetime can be understood intuitively from the diffeomorphism invariance of the theory "short and long distances are gauge equivalent".

What we therefore have a well defined theory of quantum gravity. Is this "THE" theory?

Von-Neumann's uniques theorem tells us that there is only one representation of the canonical commutation relations In the context in the field theories, on the other hand, the operator algebra admit infinely many inequivalent representations, that is, there may be For example, a ferromagnetic material is described by different spaces, $\mathcal{H}_E$, according to whether it is magnetized or not. The algebra is the same. These diferent representations have different "vacuum" states and their low energy behaviour is different. One does not know a priori which of them will provide the physical situation under consideration. There is concern that they are working in an unphysical sector.

Although this could only be settled in detail when the semiclasssical limit is worked out, there are certain worries.

4.5.1 A "Failure to Propagate"?

The first one is the sort of action the Hamiltonian has. It only acts at vertices and it acts by "dressing up" the vertex with lines. It does not interconnect vertices nor change the valences of the lines (outside the "dressing"). This imediately suggests superselection rules and quantities that are anomalously conserved, that is, relations, amongst the quantum mechanical degrees of freedom, that have no classical counterpart. For instance, one can consider surfaces that enclose a vertex (diffeomorphically invariantly defined). The area of such surfaces would commute with the Hamiltonian.

This hints at the theory "failing to propagate" (we will clarify what to "propagate" in the next section).

However, this is inconclusive because the constraint acts everywhere.

Infact, it is actually technically incorrect that the actions of the Hamiltonian constraints $\hat{H}_v$, $\hat{H}_{v'}$ at different vertices v, v' do not influence each other: In fact, these two operators do not commute, for instance if v, v' are next neighbour, because for any choice function $\gamma \mapsto \epsilon_\gamma$ what is required is that the loop attachments at v, v' do not intersect which requires that the action at v' after the action at v attaches the loop at v' closer to v' than it would before the action at v and vice versa.

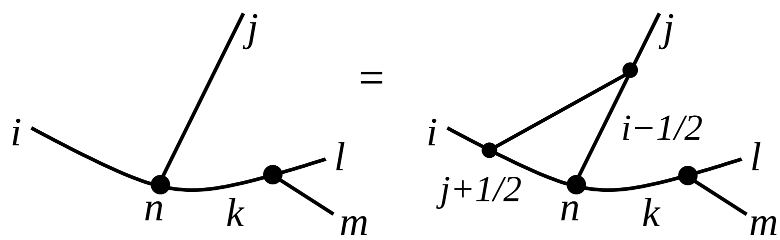

Figure 4.15: HamNonComfig0. .

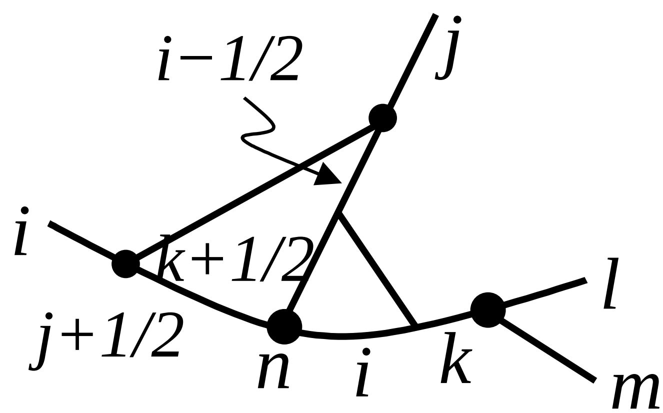

Figure 4.16: HamNonComfig0a. .

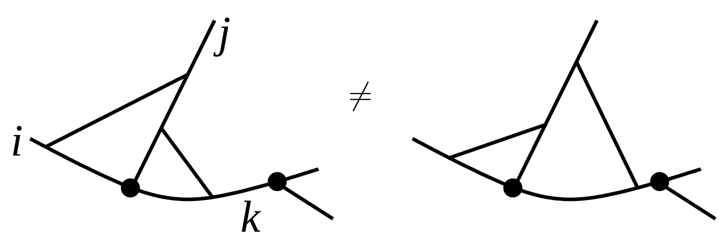

Figure 4.17: HamNonComfig. .

4.5.2 Anomaly-freeness

• First, we consider the subalgebra of the quantum diffeomorphism constraint. Recall

$$\hat{U}_{\varphi}\psi_{\gamma} := \psi_{\varphi\circ\gamma},$$

for any spatial diffeomorphism φ on Σ. Therefore, this subalgebra is free of anomaly by construction:

$$\hat{U}_{\varphi}\hat{U}_{\varphi'}\hat{U}_{\varphi}^{-1}\hat{U}_{\varphi'}^{-1} = \hat{U}_{\varphi\circ\varphi'\circ\varphi^{-1}\circ\varphi'^{-1}} \tag{4.58}$$

which coincides with the exponentiated version of the Poisson bracket between two diffeomorphism constraints generating the transformations $\varphi, \varphi' \in Diff(\Sigma)$.

• The quantum constraint algebra between the dual Hamiltonian constraint $S'(N)$ and the finite diffeomorphism transforamtion $\hat{U}_{\varphi}$ on diffeomorphism-invariant states coincides with the classical Poisson algebra between $\mathcal{V}(\vec{N})$ and $S(M)$.

Given a cylindrical fucntion ϕ_{γ} associated with a graph γ and triangulations $T(\epsilon)$ adapted to the graph α??, triangulations $T(\varphi\circ\epsilon) \equiv \varphi\circ T(\epsilon)$ are compatible with the graph $\varphi\circ\gamma$.

$$\begin{aligned}
\Big(-([\hat{S}(N),\hat{U}_{\varphi}])'\Psi_{Diff}\Big)[\phi_{\gamma}] &= ([\hat{S}'(N),\hat{U}'_{\varphi}]\Psi_{Diff})[\phi_{\gamma}] \\
&= \Psi_{Diff}[\hat{S}^{\epsilon}(N)\phi_{\gamma} - \hat{S}^{\epsilon}(N)\phi_{\varphi\circ\gamma}] \\
&= \sum_{\nu\in V(\gamma)} \big(N(\nu)\Psi_{Diff}[\hat{S}^{\epsilon}_{\nu}\phi_{\gamma}] - N(\varphi\circ\nu)\Psi_{Diff}[\hat{S}^{\varphi\circ\epsilon}_{\varphi\circ\nu}\phi_{\varphi\circ\gamma}]\big) \\
&= \sum_{\nu\in V(\gamma)} [N(\nu) - N(\varphi\circ\nu)]\Psi_{Diff}[\hat{S}^{\epsilon}_{\nu}\phi_{\gamma}] \\
&= \big(\hat{S}'(N-\varphi^{*}N)\Psi_{Diff}\big)[\phi_{\gamma}]
\end{aligned} \tag{4.59}$$

Thus there is no anomoly.

• The commutator between two Hamiltonian constraint operators

$$\{H(M), H(N)\} = \int_{\Sigma} d^3x (M_{,a}N - MN_{,a})q^{ab}V_b \tag{4.60}$$

where V_b is the vector constraint.

$$
\begin{aligned}
[\hat{H}(N),\hat{H}(M)]\phi_\gamma &= \sum_{v,v'\in V(\gamma),v\neq v'} [M(v)N(v')-N(v)M(v')]\hat{H}^{\epsilon'}_{v'}\hat{H}^{\epsilon}_{v}\;\phi_\gamma \\
&= \frac{1}{2}\sum_{v,v'\in V(\gamma),v\neq v'} [M(v)N(v')-N(v)M(v')][\hat{H}^{\epsilon'}_{v'}\hat{H}^{\epsilon}_{v}-\hat{H}^{\epsilon}_{v}\hat{H}^{\epsilon'}_{v'}]\;\phi_\gamma \\
&= \frac{1}{2}\sum_{v,v'\in V(\gamma),v\neq v'} [M(v)N(v')-N(v)M(v')][(U_\varphi-U_\varphi)\hat{H}^{\epsilon}_{v}]\;\phi_\gamma
\end{aligned}
\tag{4.61}
$$

It is mathematically consistent with the classical expression of two Hamiltonian constraint operators commute on diffeomorphism states.

Obviously, we have in the uniform Rovelli-Smolin topology

$$
([\hat{S}(N),\hat{S}(M)])'\Psi_{Diff}=0 \tag{4.62}
$$

for all $\Psi_{Diff}\in Cly^*_{Diff}$.

4.5.3 Recovering Poisson bracket between two Hamiltonian constraints

While the commutator of two Hamiltonian constraints then is anomaly free in the sense explained, in addition one would like to check that the classical limit of the commutator between quantum Hamiltonian constraints is precisely the corresponding Poisson bracket between the classical constraints. Failure to do so could raises concerns that the theory does not have the correct semiclassical limit. At present this can't be checked.

Summary?

The Hamiltonian constraint remains the major unsolved problem in LQG. These unsettled questions that concern the algebra of commutators among smeared Hamiltonian constraints which must be faced in order to make progress. We will see in chapter on the Master constrain program aimed at doing just that.

4.5.4 Solutions and Physical Inner Product

Solutions to all constraints can be constructed algorithmically []. The Hamiltonian constraint acts on spin network nodes by adding *exceptional* edges. These exceptional edges are added in the vicinity of nodes and are anihilated by subsequent applications of the Hamiltonian constraint. It follows that solutions of the Hamiltonian constraint can be built by appropriate superpositions of states corresponding to graphs with 'spiderweb' nodes as illustrated in fig. 4.18.

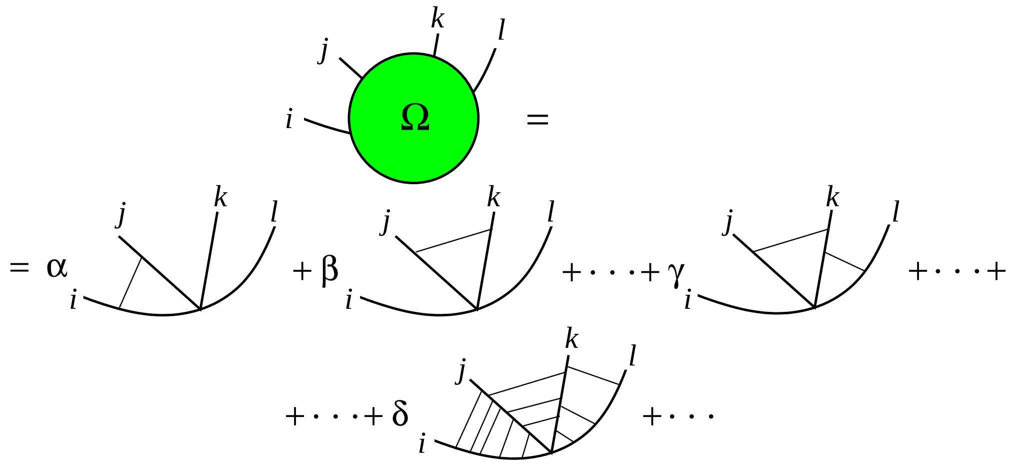

Figure 4.18: spinderfig. (spiderweb) around each vertex.

where we have left out the spin labels for clarity and the coefficients $\alpha, \beta, \ldots, \delta$ depend on the defition of the Hamiltonian constraint and the spin labelling of the corresponding spin networks.

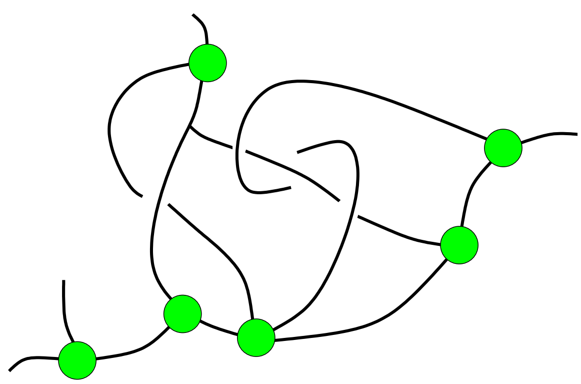

Figure 4.19: solutionfig. Solutions to all the constraints of canonical quantum gravity as dressed (Diff-invariant) spin networks.

Finding solutions to all the constraints of LQG reduces to solving algebraic relations between coefficients. They are the first rigorous solutions ever constructed in canonical quantum gravity, have non zero volume and are labelled by fractal knot classes because the iterated action of the Hamiltonian constraint creates a self-similar structure (spiderweb) around each vertex.

These are the full LQG analogs of the LQC solutions of the difference equation that results from the single Hamiltonian constraint of LQC which will be discussed in chapter 5. However, as in LQC these solutions are not systematically derived from a rigging map which is why a physical inner product is currently missing for these solutions.

4.5.5 Semi-Classical Limit

The second is that the existing semiclassical tools are only appropriate for non-graph-changing operators such as the volume operator. Namely, as we will see in chapter 7, in order to be normalisable, coherent states are (superpositions of) states defined on specific graphs. The Hamiltonian constraint operator, however, is graph changing. This means that it creates new modes on which the coherent state does not depend and whose fluctuations are therefore not suppressed. Therefore the existing semiclassical tools are insufficient for graph changing operators such as the Hamiltonian constraint. The development of improved tools is extremely diffcult and currently out of reach.

4.6 Spin Foams

Spin foam models [152] are an attempt at a path integral definition of LQG. They were heuristically defined in the seminal work [151][92] which attempted at the construction of the physical inner product via the formal exponentiation of the Hamiltonian constraints of [85].

After the construction of Hamiltonian constraint operator, formal, Euclidean functional integral was constructed in [M. Reisenberger, C. Rovelli, Phys.**D56** (1997) 3490-3508] and gave rise to the so-called spinfoam models (a spin foam is a history of a graph with faces as the history of edges).

4.6.1 Obtaing Physical Solutions with the "Projection" Operator

One is naturally lead to consider a functional evolution formalism to describe the dynamics of the quantum theory.??

$$W(x,t;x',t') \sim \int_{x(t')=x'}^{x(t)=x} \mathcal{D}[x(t)] e^{iS[x]}, \tag{4.63}$$

The amplitude to go from one 3 geometry to another is given by summing over a phase factor (as usual calculated from the action of classical general relativity) over all 4-spacetimes interpolating the two 3-spaces.

This amplitude does not depend upon any times.

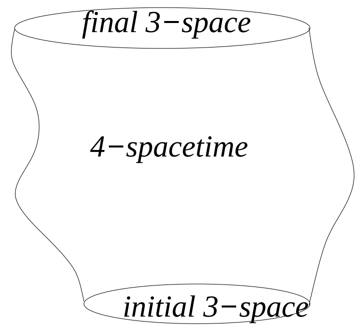

Figure 4.20: histories. “Sum over histories” formalism in quantum gravity.

It can be viewed as a mathematical well defined and possibly divergence free version of Hawking’s “sum over geometries” formulation of quantum gravity.

$$W(s,s') =< s|P|s' > \tag{4.64}$$

$W(s,s')$ giving the probability amplitude of measuring the quantized three geometry described by the spin network s is the three geometry described by the spin nework s' has been measured.

$$P\int [\mathcal{D}N] e^{i\hat{\mathcal{H}}[N]} = \int [\mathcal{D}N] e^{iN\hat{\mathcal{H}}}. \tag{4.65}$$

In the spin network basis, the matrix elements of P are

$$< s|P|s' >=< s|\int [\mathcal{D}N] e^{iN\hat{\mathcal{H}}}|s' > \tag{4.66}$$

It can be shown that a diffeomorphism invariant notion of integration exists for this functional integral.

$$< s|P|s' >\sim< s|s' > + \int [\mathcal{D}N] \left(N < s|\hat{\mathcal{H}}|s' > + NN < s|\hat{\mathcal{H}}\hat{\mathcal{H}}|s' > + \ldots \right) \tag{4.67}$$

$$W(s,s') =< s|P|s' > \tag{4.68}$$

we can now . Inserting resolutions of identity $I = \sum_s |s >< s|$, we obtain an expansion of the form

$$
\begin{aligned}
W(s,s') &= \lim_{N\to\infty} \sum_{s_1,\ldots,s_N} < s|e^{-\int d^3x\mathcal{H}(x)dt}|s_N >< s_N|e^{-\int d^3x\mathcal{H}(x)dt}|s_{N-1} > \\
&\quad \cdots < s_1|e^{-\int d^3x\mathcal{H}(x)dt}|s' >
\end{aligned} \tag{4.69}
$$

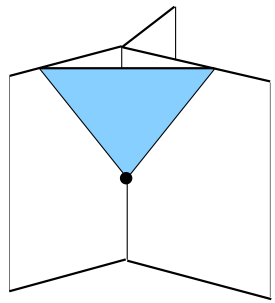

Figure 4.21: A vertex of a spinfoam.

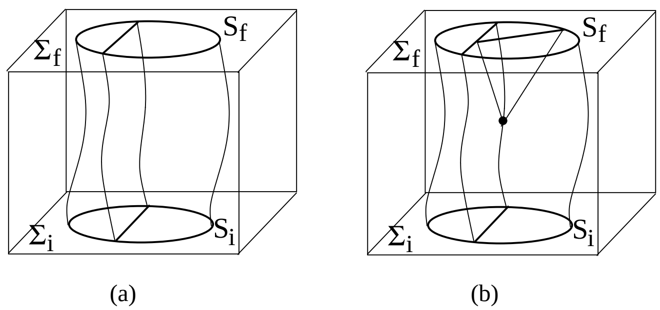

Figure 4.22: (a) Surface corresponding to a term of order zero. (b) Surface corresponding to first order term.

This is "unfreezing the frozen formalism" - in the classical theory this would be like reintroducing the unphysical time coordinate. So we can think of evolution of spin networks.

4.6.2 Difficulties with this Spin Foams Model

Both canonical and spin foam programme try to construct $\mathcal{H}_{Phys}$. Heuristic "projector" onto physical states,

$$
\begin{aligned}
& P\cdot|s>:=\delta[C']|s>:=[\prod_{x\in\sigma}\delta(C'(x))]|s>:=\int_{\mathcal{N}}[dN]e^{iC'(N)}|s> \\
& < P\cdot, P\cdot l' >_{Phys} := < l, P\cdot l' >_{Diff}
\end{aligned} \tag{4.70}
$$

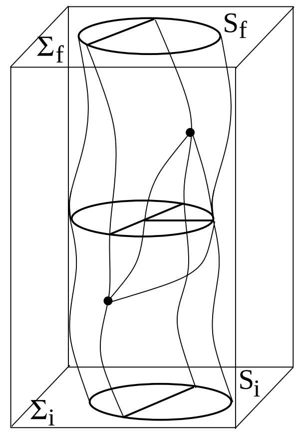

Figure 4.23: A term of second order.

Open issues:

1. $C'(N)$ does not preserve $\mathcal{H}_{Diff}$, hence $< l, P \cdot l' >_{Diff}$ is ill-defined.

2. Possibly final expression well-defined after intergrating over N, however, convergence issues.

The reason that this approach was formal is that the Hamiltonian constraints do not form a Lie algebra and they are not even self-adjoint. Thus, there are mathematical (exponentiation of non normal operators) and physical (non Lie group structure of the constraints prohibiting the possibility that functional integration over N of $\exp(i(N))$ leads to a (generalised) projector) issues with this proposal.

It tries to define a generalized projector of the form

$$\prod_{x\in\sigma} \delta(\hat{C}(x)) \tag{4.71}$$

at least formally where $\hat{C}(x)$ is the Hamiltonian constraint of [2, 3, 4??]. However, this is quite diffcult to turn into a technically clean procedure for several reasons:

1. First of all the $\hat{C}(x)$ are not self-adjoint whence the exponential is defined at most on analytic vectors of $\mathcal{H}_{Kin}$.

$C'(N)$ not self-adjoint, hence $\exp(iC'(N))$ cannot be defined via spectral theorem. Only the formal power expansion of the exponetial can be defined.

Defintion analytic vectors: Let A be an (unbounded) linear operator on a Banach space B. An *analytic vector* for A is an element $u \in B$ such that $A^n u$ is defined for all n and

$$\sum_{n=0}^{\infty} \frac{\|A^n u\|}{n!} t^n < \infty \tag{4.72}$$

for some $t > 0$, that is, the power series expansion of $e^{tA}u$ is defined and has positive radius of absolute convergence.

2. Secondly there is an infinite number of constraints and thus the generalized projector must involve a path integral over a suitable Lagrange multiplier N and one is never sure which measure to choose for such an integral without introducing anomalies.

Possibly final expression well-defined after intergrating over N, however, convergence issues.

3. Thirdly and most seriously, the $\hat{C}(x)$ are not mutually commuting and since products of projections define a new projection if and only if the individual projections commute, the formal object $\prod_{x \in \sigma} \delta(\hat{C}(x))$ is not even a (generalized) projection.

Proof: Projection operators are meant to be Hermitian and equal to their square (this second condition needs to be modified for generalised projection operators). Note that since P_1 and P_2 are projection operators they are Hermitian, $P_1 = P_1^\dagger$ and $P_2 = P_2^\dagger$. Therefore

$$(P_1 P_2)^\dagger = P_2^\dagger P_1^\dagger = P_2 P_1$$

So that a necessary condition for the product of two projection operators to be a projection operator is that

$$P_1 P_2 = P_2 P_1.$$

So we have $[P_1, P_2] = 0$. Now consider the square

$$(P_1 P_2)^2 = (P_1 P_2)(P_1 P_2)$$

If P_1 and P_2 commute, we have

$$(P_1 P_2)^2 = (P_1 P_2)(P_1 P_2) = P_1^2 P_2^2 = P_1 P_2.$$

So we see that the necessary and sufficient condition for the product of two projection operators to be itself a projection operator is that the two operators commute.

□

If one defines it somehow on diffeomorphism invariant states (which might be possible because, while the individual $\hat{C}(x)$ are not diffeomorphism invariant, the product might be up to an (infinite) factor) then that problem could disappear because the commutator of two Hamiltonian constraints annihilates diffeomorphism invariant states [2, 3, 4??], however, this would be very hard to prove rigorously.

Fourthly there is a somewhat subtle problem: $\hat{C}(x)$, while defined on $\mathcal{H}_{Kin}$ are not explicitly known (they are known up to a diffeomorphism; they exist by the axiom of choice).

It is probably due to these diffculties and the non-manifest spacetime covariance of the amplititudes computed in [20??] for the Euclidean Hamiltonian constraint that the spin foam approach has chosen an alternative route that, however, has no clear connection with Hamiltonian formalism so far. Such an approach is the Barret-Crane model which we come to in the next section.

Another approach has been proposed which could not only remove the above four problems but also has the potential to combine the canonical and spin foam programme rigorously. This is the Master constraint programme which we will come to in chapter 6.

How about using $C(N) \to \frac{1}{2}(C(N) + C^{\dagger}(N))$.

5. Power expansion $\approx$ state sum model (sum over representation labels of SNW), motivaes PI with respect to Palantinin action (=spin foam models). Connection to canonical theory?

4.7 Canonical Reduced Phase Space Quantization of LQG

4.7.1 Introduction

The attractive feature of this reduced phase space approach is that we no longer need to deal with the constraints: No anomalies can arise, no master constraint needs to be constructed (this is an alternative way of formulating the dynamics, to be covered in the next chapter), no physical Hilbert space needs to be derived by complicated group averaging techniques. We map a conceptually complicated gauge system to the conceptually safe realm of an ordinary dynamical Hamiltonian system. The kinematical results of LQG such as discreteness of spectra of geometric operators now become physical predictions.

There is, however, a difference between the constraint quantization programme and the reduced phase space quantization programme which leads to physically different predictions: As we shall see in the reduced phase space programme the clock variables are replaced by real numbers and their conjugate momenta by the functions. Thus the clock variables are not quantized. On the other hand, in the Dirac quantization programme also the clock variables are quantized as well as the conjugate momenta which are not replaced, via the constraints, in terms of q^a, p_a, T_j . Hence, the representations of the Dirac observables that come from constraint quantization know about the quantum fluctuations of the clock variables while those of the reduced phase space quantization do not.

4.7.2 Reduced Phase Space Quantization of Some Toy Systems

For systems with one constraint

$$F^{\tau}_{f,T} := \sum_{k=0} \frac{(\tau - T)^k}{k!}(X)^k \cdot f \tag{4.73}$$

where $X \cdot f$ is defined by

$$X \cdot f = \{A^{-1}C, f\}, \qquad A := \{C, T\}.$$

$$\alpha^{\tau}_{C}(f(x)) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!}\{C, f(x)\}_k. \tag{4.74}$$

Example 1 A point in phase space $x = (q^1, p_1, q^2, p_2)$. We have constraint equation

$$\mathcal{C} = p_2 + \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2 q_1^2$$

There are canonical coordinates (q, p) and (T, P_T) on the unreduced phase space,

$$\{q, p\} = 1, \qquad \{T, P_T\} = 1$$

Here $q^1 = q$, $p_1 = p$, $q^2 = t$, $p_2 = p_t$, and the constraint equation is then

$$\mathcal{C} = p_t + \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$$

We define with $F_T(\cdot) := F^{\tau=0}_{\cdot,T}$ the weak Dirac observables at multi-fingered time $\tau = 0$ (or any other fixed allowed value of τ)

$$Q := F_T(q) = q\cos\omega t - \frac{p}{m\omega}\sin\omega t \tag{4.75}$$

$$P := F_T(p) = q\cos\omega t - \frac{p}{m\omega}\sin\omega t \tag{4.76}$$

F_T is a homomorphism with respect to pointwise operations:

$$F_T(f+f') = F_T(f) + F_T(f'), \qquad F_T(ff') = F_T(f)F_T(f') \tag{4.77}$$

where f, f' are arbirary phase space functions, we have

$$F_T(P_T) \approx E(F_T(q), F_T(p), F_T(T)) \approx E(Q, P, \tau) \tag{4.78}$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q, P, namely

$$\frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2 =$$

Due to the homomorphism property (4.77)

$$\{P, Q\} \approx F^0_{\{p,q\}^*,T} = F^0_{1,T} = 1, \qquad \{Q, Q\} \approx \{P, P\} \approx 0. \tag{4.79}$$

In other words, even though the functions P, Q are complicated expressions in terms of q, p, T they nevertheless have canonical brackets at least on the constraint surface.

Now the reduced phase space quantization consists in quantizing the subalgebra of $\mathcal{D}$, spanned by our preferred Dirac observables Q, P evaluated on the constraint surface. As we have just seen, the algebra $\mathcal{D}$ itself is given by the Poisson algebra of the functions of the Q, P evaluated on the constraint surface. Hence all the weak equalities now become exact. We are therefore looking for a representation

$$\pi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$$

of that subalgebra of $\mathcal{D}$ as self-adjoint, linear operators on a Hilbert space such that

$$[\pi(P), \pi(Q)] = i\hbar.$$

The constraint has disappeared, it has been solved and reduced. Instead of a constraint on the gauge variant phase space coordinatised by gauge variant canonical pairs one has a gauge invariant phase space coordinatised by (4.75) and (4.76). At this point it looks as if we have completely trivialized the reduced phase space quantization problem of our constrained Hamiltonian system because there is no Hamiltonian constraint to be considered and so it seems that we can just choose the standard kinematical representations for quantizing the phase space coordinatized by the q, p (representations unitarily equivalent to the free particle Schrodinger representation) and simply use it for Q, P because the respective Poisson algebras are (weakly) isomorphic. However, this is not the case. In addition to satisfying the canonical commutation relations we want that the parameter group of automorphisms α_τ on $\mathcal{D}$ be represented unitarily on $\mathcal{H}$. In other words, we want that the parameter group of automorphisms be represented unitarily on $\mathcal{H}$, i.e., that there exist a one parameter group of unitary operators $U(\tau)$ on $\mathcal{H}$ such that

$$\pi(\alpha^\tau(Q)) = U(\tau)\pi(Q)U(\tau)^{-1} \qquad \text{and} \qquad \pi(\alpha^\tau(P)) = U(\tau)\pi(P)U(\tau)^{-1}.$$

Notice that due to the relation (which is exact on the constraint surface)

$$\alpha_\tau(Q) = F_{\alpha'_\tau(q),T} = \sum_{k=0} \frac{\tau^k}{k!} F_{X^k \cdot q,T} \tag{4.80}$$

and where on the right hand side we may replace any occurence of P_T, T by functions of Q, P according to the above rules. Hence the automorphism α^τ preserves the algebra of functions of the Q, P, although it is a very complicated map in general and in quantum theory will suffer from ordering ambiguities. On the other hand, for short time periods (4.80) gives rise to a quickly converging perturbative expansion.

One way to implement the time evolution unitarily is by quantizing the Hamiltonian that generates the Hamiltonian flow. The constraint is of the form

$$C = P_T + E(q, p, T)$$

We now set the

$$H(Q, P) := F^0_{E,T} \approx E(F^0_{q,T}, F^0_{p,T}, F^0_{T,T}) \approx E(Q, P, 0)$$

$$\begin{aligned}
\{H, F^0_{f,T}\} &\approx F^0_{\{E,T\}^*,T} = F^0_{\{E,T\},T} = F^0_{\{C,T\},T} \\
&= \sum_k \frac{(\tau - T)^k}{k!} X^k \cdot \tilde{X} \cdot f \qquad \text{as } \{T, E\} = \{T, f\} = 0 \\
&\approx \tilde{X} \cdot F^0_{f,T} - \sum_k (\tilde{X} \cdot \frac{(\tau - T)^k}{k!} X^k \cdot f)
\end{aligned} \tag{4.81}$$

The problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the function H.

$$H(Q,P) \approx E(Q,P,0) = \frac{P^2}{2m} + \frac{1}{2}m\omega^2 Q^2$$

$$[\pi(P),\pi(Q)] = i\hbar, \qquad U(\tau) = \exp i\left(\frac{\pi(P)^2}{2m} + \frac{1}{2}m\omega^2\pi(Q)^2\right)\tau \tag{4.82}$$

$$\pi(Q) = Q, \qquad \pi(P) = -i\hbar\frac{\partial}{\partial Q}$$

As $U(\tau)$ is unitarily implemented on $\mathcal{H}$ for an orthonormal basis $\psi_n \in \mathcal{H}$

We no longer have a gauge symmetry but rather a symmetry group of the physical Hamiltonian H_{ph}. We require a covariant representation of the group symmetry.

$$U(\tau)\psi(Q,t) = \psi(Q,t-\tau).$$

The function $\psi(t)$ can be expanded uniquely into a Fourier integral

$$\psi(Q,t) = \int_{-\infty}^{\infty} \tilde{\psi}(Q,E)e^{iEt}dE$$

The functions $e_E(Q,t) = \psi(Q)e^{iEt}$ thus form a basis. Moreover,

$$(U(\tau)e_E)(t) = e^{iE(t-\tau)} = e^{-iE\tau}e_E(t),$$

Example 2

The double harmonic oscillator was first studied by Rovelli [277] as a toy model to help understand the "problem of time".

The Hamiltonian for the double harmonic oscillator

$$C = \lambda\left(\frac{1}{2}(p_1^2 + \omega^2 q_1^2) + \frac{1}{2}(p_2^2 + \omega^2 q_2^2) - E\right) \tag{4.83}$$

One of the oscillators can be thought of as a "clock" for the other oscillator.

Here we choose $q^1 = q$, $p_1 = p$, $q^2 = t$, $p_2 = p_t$. These are canonical coordinates

$$(q,p) \qquad \text{and} \qquad (T, P_T) \tag{4.84}$$

on the unreduced phase space,

$$\{q,p\} = 1, \qquad \{T, P_T\} = 1$$

and the constraint equation is then

$$\mathcal{C} = \lambda \left(\frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}(p_t^2 + \omega^2 T^2) - E \right).$$

As before we define the weak Dirac observables at multi-fingered time $\tau = 0$ (or any other fixed allowed value of τ)

$$Q := F_T(q) = \tag{4.85}$$

$$P := F_T(p) = \tag{4.86}$$

Since at least locally we can solve the constraint C for the momentum P_T, that is

$$p_t = -\sqrt{\left(\frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}\omega^2 T^2) - E \right)}$$

and F_T is a homomorphism property (4.77)

$$\begin{aligned} F_T(P_T) &\approx E(F_T(q), F_T(p), F_T(T)) \\ &\approx E(Q, P, \tau) \\ &= \sqrt{\left(\frac{1}{2}(P^2 + \omega^2 Q^2) + \frac{1}{2}\omega^2 \tau^2) - E \right)} \end{aligned} \tag{4.87}$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q, P.

Again by the homomorphism property

$$\{P, Q\} \approx F^0_{\{p,q\}^*,T} = F^0_{1,T} = 1, \qquad \{Q,Q\} \approx \{P,P\} \approx 0,$$

we have produced gauge invariant canonical pairs from gauge variant canonical pairs.

This can be done as follows: The original constraint C can be solved for the momenta P_T conjugate to T and we get an equivalent constraint

$$\tilde{C} \approx p_t + \sqrt{\left(\frac{1}{2}(p^2 + \omega^2 q^2) + \frac{1}{2}\omega^2 T^2) - E\right)}$$

We now set the

$$H(Q,P) := F^0_{E,T} \approx E(F^0_{q,T}, F^0_{p,T}, F^0_{T,T}) \approx E(Q,P,0) \tag{4.88}$$

$$\{H, F^0_{f,T}\} \approx (\frac{\partial}{\partial \tau})_{\tau=0} \; \alpha^\tau(F_T(f))$$

This means that the strongly Abelian group of Poisson bracket automorphisms α^τ is generated by the "Hamiltonians" H_j . Thus, if we interpret the T_j as clocks then we have a multi-fingered time evolution with Hamiltonians H_j.

The problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the function H.

The constraint has disappeared, they have been solved and reduced. Instead of a constraint on the gauge variant phase space coordinatised by (4.84) which generates gauge transformations, there is a physical Hamiltonian (4.88) which generates physical time evolution on the gauge invariant phase space coordinatised by (4.85) and (4.86).

4.7.3 The General Scheme

except in asymptoticy flat spacetimes where all metrics under consideration approach a fixed metric because here all active diffeomorphisms universally reduce to active Poincare transformations. we have at infinity we have the symmetry group the Poincare group the Poincare charges at spacial infinity

The Hamiltonian constraints are not spacially diffeomorphism independent and this has meant solving the Hamiltonian constraint has been slow. One cannot first solve the spatial diffeomorphism constraints and then solve the Hamiltonian constraint because the latter do not preserve the space of solutions to the spacially diffeomorphism constraint.

f and T_j functions on phase space. Weak Dirac observable there are n T_j's

$$F^{\tau}_{f,T} := \sum_{k_1,\ldots,k_n=0} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \ldots \frac{(\tau_n - T_n)^{k_n}}{k_n!} (X_1)^{k_1} \ldots (X_n)^{k_n} \cdot f. \tag{4.89}$$

where $X_r \cdot f$ is defined as

$$X_j \cdot f := \{(A^{-1})_{jk} C_k, f\}, \quad A_{jk} := \{C_j, T_k\}. \tag{4.90}$$

Poisson algebra

$$\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\} = F^{\tau}_{\{f,f'\}^*,T} \tag{4.91}$$

defining the automorphism on ?? generated by the Hamiltonian vector field of $\sum_j \tau^j C'_j$

$$\alpha'_{\tau}(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_j \tau^j X_j \right)^n \cdot f \tag{4.92}$$

$$\{\alpha^{\tau}(F^{\tau}_{f,T}), \alpha^{\tau}(F^{\tau}_{f',T})\} \approx \alpha^{\tau}(\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\}) \tag{4.93}$$

In other words, α^{τ} is a weak, abelian, multi-parameter group of automorphisms on the each $F^{\tau_0}_{f,T}$.

In [223] they perform a canonical, reduced phase space quatization of General Relativity by Loop Quantum Gravity methods.

It is assumed that it is possible to to choose the functions T_j as canonical coordinates. In other words, we choose a canonical coordinate system consisting of canonical pairs (q^a, p_a) and (T_j, P^j) where the first system of coordinates has vanishing Poisson brackets with the second so that the only non vanishing brackets are

$$\{p_a, q^b\} = \delta^b_a, \qquad \{P^j, T_k\} = \delta^j_k.$$

The virtue of this assumption is that the Dirac bracket reduces to the ordinary Poisson bracket on functions which depend only on q^a, p_a. We will shortly see why this is important.

Usually the Dirac brackets make the Poisson structure so complicated that one cannot find representations thereof. However, as we see, if the system deparametrises, if one uses as clocks T the configuration varaibles conjugate to the momenta P in $C = P + H$ and if one considers functions f which dont depend on T, P then F_T becomes a Poisson bracket isomorphism. This is no loss of generality because P can be eliminated in terms of the other degrees of freedom via the constraints and T is pure gauge.

We define with $F_T(\cdot) := F^0_{\cdot,T}$ the weak Dirac observables at multi-fingered time $\tau = 0$ (or any other fixed allowed value of τ)

$$Q^a := F_T(q^a), \qquad P_a := F_T(p_a) \tag{4.94}$$

Notice that

$$F_{T_j,T} \approx \tau_j,$$

so the Dirac observable corresponding to T_j is just a constant and thus not very interesting (but evolves precisely as a clock). Likewise

$$F^\tau_{Cj,T} \approx 0$$

is not very interesting. Since at least locally we can solve the constraints C_j for the momenta P^j, that is

$$P^j \approx E_j(q^a, p_a, T_k)$$

and F_T is a homomorphism with respect to pointwise operations:

$$F_T(f + f') = F_T(f) + F_T(f'), \qquad F_T(ff') = F_T(f)F_T(f') \tag{4.95}$$

where f, f' are arbirary phase space functions, we have

$$F_T(P_j) \approx E_j(F_T(q^a), F_T(p_a), F_T(T_k)) \approx E_j(Q^a, P_a, \tau_k) \tag{4.96}$$

and thus also does not give rise to a Dirac observable which we could not already construct from Q^a, P_a. The importance of our assumption is now that due to the homomorphism property

$$\{P_a, Q^b\} \approx F^0_{\{p_a,q^b\}^*,T} = F^0_{\delta^b_a,T} = \delta^b_a, \qquad \{Q^a, Q^b\} \approx \{P_a, P_b\} \approx 0. \tag{4.97}$$

In other words, even though the functions P_a, Q^a are very complicated expressions in terms of q^a, p_a, T_j they nevertheless have canonical brackets at least on the constraint surface. If we would have had to use the Dirac bracket then this would not be the case and the algebra among the Q^a, P_a would be too complicated and no hope would exist towards its quantization. However, under our assumption there is now a chance.

Now the reduced phase space quantization consists in quantizing the subalgebra of $\mathcal{D}$, spanned by our preferred Dirac observables Q_a, P_a evaluated on the constraint surface. As we have just seen, the algebra $\mathcal{D}$ itself is given by the Poisson algebra of the functions of the Q_a, P_a evaluated on the constraint surface. Hence all the weak equalities now become exact. We are therefore looking for a representation

$$\pi : \mathcal{D} \to \mathcal{L}(\mathcal{H})$$

of that subalgebra of $\mathcal{D}$ as self-adjoint, linear operators on a Hilbert space such that

$$[\pi(P_a), \pi(Q^b)] = i\hbar\delta_a^b.$$

At this point it looks as if we have completely trivialized the reduced phase space quantization problem of our constrained Hamiltonian system because there is no Hamiltonian to be considered and so it seems that we can just choose any of the standard kinematical representations for quantizing the phase space coordinatized by the q^a, p_a and simply use it for Q^a, P_a because the respective Poisson algebras are (weakly) isomorphic. However, this is not the case. In addition to satisfying the canonical commutation relations we want that the multi parameter group of automorphisms α_τ on $\mathcal{D}$ be represented unitarily on $\mathcal{H}$ (or at least a suitable, preferred one parameter group thereof). In other words, we want that there exists a multi parameter group of unitary operators $U(\tau)$ on $\mathcal{H}$ such that

$$\pi(\alpha_\tau(Q^a)) = U(\tau)\pi(Q^a)U(\tau)^{-1}$$

and similarly for P_a.

Notice that due to the relation (which is exact on the constraint surface)

$$\alpha_\tau(Q^a) = F_{\alpha'_\tau(q^a),T} = \sum_{\{k\}} \prod_j \frac{\tau_j^{k_j}}{k_j!} F_{\prod_j X_j^{k_j} \cdot q^a, T} \tag{4.98}$$

and where on the righthand side we may replace any occurence of P_j, T_j by functions of Q^a, P_a according to the above rules.

Hence the automorphism α^τ preserves the algebra of functions of the Q^a, P_a, although it is a very complicated map in general and the quantum theory will suffer from ordering ambiguities. On the other hand, for short time periods (4.98) gives rise to a quickly converging perturbative expansion. Hence we see that the representation problem of $\mathcal{D}$ will be severly constrained by our additional requirement to implement the multi time evolution unitarily, if at all possible. Whether or not this is feasible will strongly depend on the choice of the T_j.

A possible way to implement the multi-fingered time evolution unitarily is by quantizing the Hamiltonians $\mathcal{H}_j$ that generate the Hamiltonian flows $\tau_j \mapsto \alpha^\tau$ where $\tau_k = \delta_{jk}\tau_j$. This can be done as follows: The original constraints C_j can be solved for the momenta P^j conjugate to T_j and we get equivalent constraints

$$C_j = P^j + E_j(q^a, p_a, T_k).$$

These constraints have a strongly Abelean constraint algebra. Proof: We must have $\{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk} l \tilde{C}_l$ for some new structure functions $\tilde{f}$ by the first class property. The left hand side is independent of the functions P^j, thus must be the right hand side, which may therefore be evaluated at any value of P^j. Set $P^j = -E_j$.

We may write

$$C'_j = K_{jk}\tilde{C}_k$$

for some regular matrix K. Since

$$\{C'_j, T_k\} \approx \delta_{jk} = \{\tilde{C}_j, T_k\}$$

it follows that $K_{jk} \approx \delta_{jk}$. In other words $C'_j = \tilde{C}_j + \mathcal{O}(C^2)$ where the notation $\mathcal{O}(C^2)$ means that the two constraints set differ by terms quadratic in the constraints. It follows that the Hamiltonian vector fields $X_j, \tilde{X}_j$ of $C'_j, \tilde{C}_j$ are weakly commuting. We now set

$$H_j(Q^a, P_a) := F^0_{E_j,T} \approx E_j(F^0_{q^a,T}, F^0_{p_a,T}, F^0_{T_k,T}) \approx E_j(Q^a, P_a, 0)$$

$$\{H_j(\tau), F^0_{f,T}\} = (\frac{\partial}{\partial\tau_j})_{\tau=0}\ \alpha^\tau(F_T(f)) \tag{4.99}$$

where in the derivation one uses that $\{T_j, E_k\} = \{T_j, f\} = 0$, $\{P_j, f\} = 0$, that the $X_j, \tilde{X}_k$ are weakly commuting, that $F_{f,T}$ is a weak observable, and the definition of the flow. We conclude that the Dirac observables H_j generate the multifingered flow on the space of functions of the Q^a, P_a when restricted to the constraint surface. The algebra of the H_j is weakly Abelean because the flow α^τ is a weakly Abelean group of automorphisms.

Thus, the problem of implementing the flow unitarily can be reduced to finding a self adjoint quantization of the functions H_j. Preferred one parameter subgroups will be those for which the corresponding Hamiltonian generator is bounded from below. Notice, however, that in () we have computed the infinitesimal flow at $\tau = 0$ only. For an arbitrary value of τ the infintesimal generator $H_j(Q^a, P_a, \tau)$ defined by

$$\{H_j(\tau), F^{\tau}_{f,T}\} := \frac{\partial}{\partial \tau_j} \alpha^{\tau}(F_T(f)) \tag{4.100}$$

4.7.4 Outline of a Reduced Phase Space Quantization of LQG

Since the resulting algebra of observables is very simple, one can quantise it using the methods of LQG. Basically, the kinematical Hilbert space of non reduced LQG now becomes a physical Hilbert space and the kinematical results of LQG such as discreteness of spectra of geometrical operators now have physical meaning. The constraints have disappeared, however, the dynamics of the observables is driven by a physical Hamiltonian which is related to the Hamiltonian of the standard model (without dust) and which we quantise.

In the quantum theory we are looking for representations of the Poisson *-algebra generated by the which supports a quantised version of the physical Hamiltonian H_{ph}.

One wants a unitary representation of the spacial diffeomorphism group of the spacial coordinate manifold which is a *gauge group*. Since all our observables are gauge invariant, we have no diffeomorphism gauge group any longer, hence that physical selection criterion is absent. However, it is replaced by another: it turns out that the physical Hamiltonian has the difeomorphism group of the dust label space as a *symmetry group*. These difeomophisms change our observables, they map between physically distinguishable dust label space. This allows us to apply the same selection criterion.

In our case we do not have a diffeomorphism gauge group but rather a diffeomorphism symmetry group $Diff(X)$ of the physical Hamiltonian H_{ph}. This is physical input enough to also insist on cyclic $Diff(S)$ covariant representations and correspondingly we can copy the uniqueness result.

Thus we simply choose the background independent and active diffeomorphism covariant Hilbert space representation of LQG used extensively in [??] and we ask whether that representation supports a quantum operator corresponding to H_{ph}.

4.7.5 Physical Hilbert Space

Like in unreduced LQG we consider the holonomy-flux algebra.

In unreduced LQG representation by covariance under Diff(σ).

In reduced LQG the representation is fixed by covariance under DiffΣ.

Uniqueness result applies: $\mathcal{H}_{phys}$ = usual LQG Hilbert space but here is physical.

In particular: usual kinematic coherent states now physical coherent states.

4.8 Consistent Discrete Classical and Quantum General Relativity

One of the alternatives, to be presented in chapter 6, is thhe "master constraint" project of Thiemann and collaborators. Others considered covariant "spin foam" spproach as an alternative, sinse one may bypass the construction of the canonical algebra entirely, at least in some settings. In this section we introduce another approach.

However, when one discretizes the equations, the resulting system of algebraic equations is in general incompatable.

We are explicitly working at the *physical* level.

4.8.1 Semi-Discrete Approach

Significant departure from what everyone else was doing in LQG. They could not make use of of kinematic tools of LQG, like spin-network states, Astekar-Lewendowski diff-invariant measure, ... Thiemann paper *"One way out could be to look at constraint quantization from an entirely new point of view which proves useful also in discrete formulations of classical GR, that is, numerical GR. While being a fascinating possibility, such a procedure would be a rather drastic step in the sense that it would render most results of LQG obtained so far obsolete."*, [77].

discrete time but keep space continuous in the classical action. But would be not be a discord with GR in which space and time on same footing?

Get coupled non-linear PDEs.

Same kinematics but deal with spacial constraint in the usual way.

$$\begin{aligned} S = & \int dt d^3x \Big[Tr \Big(\tilde{P}^a (A_a(x) - V(x) A_{n+1,a}(x) V^{-1}(x) + \partial_a (V(x)) V^{-1} \Big) \\ & - N^a C_a - NC + \mu Tr(V(x) V^\dagger(x) - 1) \Big] \end{aligned} \quad (4.101)$$

4.9 Summary:

· Real variables.

· Construction of a anomoly free, mathematically rigorous finite Hamiltonian constraint.

· But is it a physically viable theory of quantum GR?

4.10 Biblioliographical notes

In this chapter I have relied on the following references:

[].