Chapter 6

The Master Constraint Programme and an Improved Understanding of the Dynamics

MUCH IS TAKEN DIRECTLY FROM PAPERS, TO BE REPLACED

6.1 Introduction

We left chapter 4 with several problems to do with the Hamitonian constraint:

(i) The dual Hamiltonian constraint operator does not preserve the Hilbert space $\mathcal{H}_{Diff}$. As a result the inner product structure of $\mathcal{H}_{Diff}$ cannot be employed in the construction of the physical inner product.

(ii) Classically the collection of Hamiltonians do not form a Lie algebra. We cannot use the group averaging strategy to solve the corresponding quantum constraint equation.

(iii) Their proposal takes a fixed graph in space and uses it to construct coherent states that approximate any given metric/extrinsic curvature pair in the classical theory. However, the Hamiltonian constraint is a graph changing operator - the new graph it generates has degrees of freedom upon which the coherent state does not depend and so their quantum fluctuations are not suppressed. In the Hamiltonian constraint programme the recovery of the low energy physics appears cumbersome.

6.1.1 The Master Constraint

In [77] the Master Constraint Progaremme was launched which proposes to replace $\mathcal{D}$ by a much simpler Master Constraint Algebra $\mathcal{M}$. Basically, the infinite number of

Hamiltonian constraints are replaced by a single constraint, namely the weighted integral of their squares such that the associated Master Constraint $\mathbf{M}$ is spatially diffeomorphism invariant. One can show that $\mathcal{D}$, $\mathcal{M}$ are classically equivalent.

The physical Hilbert is then readily available using standard spectral analysis techniques [77],[208] provided one manages to implement $\mathbf{M}$ as a self-adjoint operator $\hat{\mathbf{M}}$ on either $\mathcal{H}_{aux}$ or $\mathcal{H}_{Diff}$ (and provided that the Hilbert space is a direct sum of separable subspaces invariant for $\hat{\mathbf{M}}$).

To take the sum of squares of constraints rather than the consraints themselves has successfully been tested for various toy models including those with an infinite number of degrees of freedom and with structure functions [209], [210], [211], [212].

2. after having shown that the solution theory is consistent reduces to QTF on a background spacetime in the semiclassical limit of low geometry fluctuations.

This is precisely the purpose of the Master Constraint Programme to complete this task.

• **Possibility of having:**

(i) Control over Physical space of solutions,

(ii) Control over Quantum Dirac Observables of LQG,

(iii) An Answer to Whether LQG has the Correct Semi-classical limit.

In his paper he proposes a solution to this set of problems based on the so-called Master Constraint which combines the smeared Hamiltonian for all smearing functions into a single constraint.

If certain mathematical conditions hold, which still have to be proved (which now have been see [214]),

then not only the problems with the commutator algebra could disappear, also chances are good that one can control the solution space and the (quantum) Dirac observables of LQG. Even a decision on whether the theory has the correct classical limit and connection with the path integral (or spin foam) formulation could be within reach.

Summary:

Hence we see that the problem of investigating the classical limit of LQG and to verify the quantum algebra of constraints are very much interlinked:

1. Spatial diffeomorphism invariance enforces a weakly discontinuous representation of spatial diffeomorphisms.

2. Anomaly freeness in the presence of only finite diffeomorphisms enforces graph changing Hamiltonian constraints.

3. Graph changing Hamiltonians seem to prohibit appropriate semiclassical states.

· Problems with the commuter algebra disappear
· Could have control of the solution space
· Could have control of the (quantum) Dirac observables LQG.
· Even a decision on whether the theory has the correct classical limit.
· and the connection with the path (or spin foam) formulation could be within reach.

Can be traced back to simple facts about the constraint algebra:

1. *The (smeared) Hamiltonian constraint is not a spacially diffeomorphism invariant function.*
2. *The algebra of (smeared) Hamiltonian constraints does not close, it is proportional to a spatial diffeomorphism constraint.*
3. *The coefficient of proportionality is not a constant, it is a non-trivial function on the phase space.*

$$\{\vec{C}(\vec{N}), \vec{C}(\vec{N}')\} = \kappa \vec{C}(\mathcal{L}_{\vec{N}} \vec{N}') \tag{6.1}$$

$$\{\vec{C}(\vec{N}), C(N')\} = \kappa C(\mathcal{L}_{\vec{N}} \vec{N}') \tag{6.2}$$

$$\{C(N), C(M)\} = \int d^3x (N\partial_a M - M\partial_a N) g^{ab} C_b \tag{6.3}$$

where $C(N) = \int_\sigma d^3 N(x) C(x)$ is the smeared Hamiltonian constraint, C_b is the spacial diffeomorphism constraint, $\vec{C}(\vec{N}) = \int_\sigma d^3 N^a(x) C_a(x)$ is the smeared spatial diffeomorphism constraint, q^{ab} is the inverse spatial metric tensor, N, N, N^a, N'^a are smearing functions on the spacial three-manifold σ and κ is the gravitational constant.

The righthand side of the commutator does not obviously resemble the right hand side of the Poisson bracket of two Hamiltonian constraints.

All these problems would disappear if it would be possible to reformulate the Hamiltonian constraint in such a way that it is equivalent to the original formulation but such that it becomes spatially diffeomorphim invariant function with an honest Lie algebra. There is a natural candidate, namely

$$\mathbf{M} = \int_\sigma d^3x \frac{[C(x)]^2}{\sqrt{det(q(x))}} \tag{6.4}$$

This has been called the **Master Constraint** corresponding to the infinite number of constraints $C(x)$, $x \in \sigma$ because due to the positivity of the integrand, the Master Equation $\mathbf{M} = \mathbf{0}$ is equivalent with $C(x) = 0 \; \forall x \in \sigma$ since $C(x)$ is real valued. The factor has been incorporated in order to make the integrand a scalar of density of weight one. This guarantied

1) that **M** is spatially diffeomorphism invariant quantity and
2) that **M** has a chance to survive quantization.

Invariance under active spacial diffeomorphisms follows from section 1.6.1.

Secondly, only scalar densisties of weight 1 can be promoted to spacially diffeomorphism invaraiant operators.

Why was it that anyone didn't think of such a quantity before? There was an a priori problem with which prevented Thiemann to from considering it seriously much earlier: consider the Poisson bracket

$$\begin{aligned}\{O,\mathbf{M}\} &= \{O,\int_\Sigma d^3x \frac{C(x)}{\sqrt{q(x)}}C(x)\} \\ &= \{O,\int_\Sigma d^3x \frac{C(x)}{\sqrt{q(x)}}\}\; C(x) + \int_\Sigma d^3x \frac{C(x)}{\sqrt{q(x)}}\; \{O,C(x)\}\end{aligned} \tag{6.5}$$

(where we used the product rule of the Poisson bracket). On the constraint surface $\mathbf{M}=0$ ($C(x)=0$) we obviously have $\{O,\mathbf{M}\}=0$ for *any* differentiable function O on the phase space. This is a problem because (weak) Dirac observables for first class constraints such as $C(x)=0$ are selected precisely by the condition $\{O,C(x)\}=0$ for all $x\in\sigma$ on the constraint surface. Thus the Master Constraint seems to fail to detect Dirac observables with respect to the original set of Hamiltonian constraints $C(x)=0$, $x\in\sigma$.

if it satisfies the **Master Equation**

$$\{O,\{O,\mathbf{M}\}\}_{\mathbf{M}=0}=0 \tag{6.6}$$

The price we have to pay is this Master Equation is that it is a non-linear condition on O. Now from the theory of differential equations one knows that non-linear differential equations (such as the Hamlitonian-Jacobi equation) are often easier to solve if one transforms them into a system of linear partial differential equations and one might think that in order to find solutions to the Master-Constraint one has to go back to the original infinite system of conditions. However, also this is not the case: As we will show, one can explicitly solve the Master Condition for a subset of *strong* Dirac observables by using **Ergodic Theory Methods**.

i) The Hamiltonian constraints $C(x)$ do not preserve $\mathcal{H}_{Diff}$.

$$\hat{C}^a(x)|\Phi>=0 \tag{6.7}$$

$$[\hat{C}(x),\hat{C}^a(x')]\neq 0 \tag{6.8}$$

$$\hat{C}^a(x')\hat{C}(x)|\Phi>\neq \hat{C}(x)\hat{C}^a(x')|\Phi>=0 \tag{6.9}$$

$$\hat{C}(x)|\Phi >\neq 0. \tag{6.10}$$

$C[N]$ is not diffeomorphism invariant, and therefore $C[N]|s>$ is not in H_{Diff}, because a diffeomorphism modifies N. More precisely it is not invariant under diffeomorphisms that move the positions of the nodes; the state $C[N]|s>$ has a factor $N(x_i)$ where x_i is the position of a node of the state $|s\rangle$. Under a diffeomorphism in which the position x_i is sent to the position x'_i the field $N(x_i)$ is replaced by $N(x'_i)$.

Thus the inner product structure of H_{Diff} cannot be employed, via the same powerful techniques used to construct inner product structure of H_{Diff} from the kinematic inner product structure ... , in the construction of the physical inner product.

ii) *Physical States and the Physical Inner Product*

The constraint operators are defined on a common dense domain, S, consisting of the space of infinitely differentiable wavefunctions, ψ, for which $\hat{x}^j\hat{p}^k\psi$ is normalizable for all positive integers j and k. This is small so it has a very large dual space, S', called the space of *tempered distributions*, to which the operators can be transposed.

$$\mathbf{\Phi}^*{}_{Aux} \tag{6.11}$$

We cannot define the Hamiltonian constraint on

$$(\mathbf{\Phi}^*{}_{Kin})_{Diff} \tag{6.12}$$

iii) *Strong Dirac Observables*

$$\boxed{[\hat{O}] := \lim_{T\to\infty}\int_{-T}^{T} dt\hat{U}(t)\hat{O}_{Diff}\hat{U}(t)^{-1}} \tag{6.13}$$

Hence, anomaly freeness has been transformed into the issue of the size of the $\mathcal{H}_{Phys}$

Instead of solving all the (possibly infinitely many) equations $C_i(m) = 0$, one can also define the so-called master constraint []

$$M := \sum_i C_i K_{ij} C_i. \tag{6.14}$$

Here K_{ij} is a symmetric, positive definite matrix in the case of i being a discrete set. Otherwise, Kij has to be a positive definite operator kernel and the summation over i turns into an integration. It is straightforward to see that

$$M(m) = 0 \qquad \Leftrightarrow \qquad C_i(m) = 0 \text{ for all } i. \tag{6.15}$$

Furthermore, for any phase space function f weakly commuting with the constraints:

$$\{\{M, f\}, f\} \approx 0 \qquad \Leftrightarrow \qquad \{C_i, f\} \text{ for all } i. \tag{6.16}$$

So M enables us to derive the complete set of observables on the physical pase space. This means that the physical phase space itself can be constructed from the knowledge of M, so one does not lose any information if one goes over to M from the C_i. The final classical systems defined by both are in fact equivalent. This is in fact independent of the actual choice of K_{ij}, so there are a priori many possible master constraints. One can choose the one that is most useful, makes the sum (6.14) converge and is the most convenient to compute. This freedom is quite useful in the quantized theories [].

6.2 The Master Constraint Programme versus the Hamiltonian Constraint Programme for GR.

Dirac programme:

· given (first class) constraints C_i

· implement C_i on $\mathcal{H}_{kin}$

· look for solutions $\hat{C}_i\phi = 0$ (RAQ: in $\Phi^* \rightarrow \mathcal{H}_{kin} \rightarrow \Phi^*$

· construct inner product (rigging map) $\rightarrow \mathcal{H}_{phys}$

Master Constraint programme:

· given (first class) constraints $M = \sum K_{ij} C_i C_j$ (finite case)
$M(m) =< C(m), KC(m) >_{\mathcal{H}}$ (field theory)

· quantize M on $\mathcal{H}_{kin}$ (pos., s.a.)

· determine direct integral decomposition wrt $\hat{M}$
$\mathcal{H}_{kin} = \int_{R_+}^{\oplus} d\mu(\lambda) \mathcal{H}_{kin}^{\oplus}(\lambda)$

· $\mathcal{H}_{phys} = \mathcal{H}_{kin}^{\oplus}(\lambda_{min})$ with induced inner product

in order to quantize this expression one no replaced all appearing quantities by operators and the Poisson bracket by a commutator divided by $i\hbar$.In addition, in order to arrive at an unambiguous result one had to make the triangularization *state dependent.* That is, the regulated operator is defined on a certain spin network basis elements T_s of the

Hilbert space in terms of an adapted triangularization τ_s and extended by linearity. This is justified because the Riemann sum that enters the definition of $C_\tau(N)$ converges to $C(N)$ no matter how we refine the triangularization.

Quantum Dirac Algebra

We may compute the commmutator $[\hat{C}, \hat{C}(N')]$ on Ψ_{Kin} corresponding to the Poisson braacket $\{\hat{C}, \hat{C}(N')\}$ which is proportional

Classically the constraint algebra $\{C_I, C_J\} = f_{IJ}{}^K C_K$ for structure functions $f_{IJ}{}^K$ that are constants for all cases other than the Hamiltonian Poison bracket.

$$[\hat{H}_I, \hat{H}_J] = i\hbar \hat{H}_K \hat{f}_{IJ}{}^K = i\hbar\{[\hat{H}_K, \hat{f}_{IJ}{}^K] + \hat{f}_{IJ}{}^K \hat{H}_K\} \tag{6.17}$$

and it follows that any $l \in \mathcal{D}^*_{phys}$ also solves the equation $([\hat{H}_K, f_{IJ}{}^K])l = 0$ for all I, J. If that commutator is not itself a constraint again, then it follows that l solves more that only the equations $\hat{C}_I l = 0$. The quantum solutions are subject to more conditions than the classical solutions and thus the quantum theory has less physical degrees of freedom than the classical theory.

The Master Constraint approach improves on these issues:

1) Quantizaton of the Regulated Constraint

$$\hat{C}^a(x)|\Phi >= 0 \tag{6.18}$$

$$[\hat{M}(x), \hat{C}^a(x')] = 0 \tag{6.19}$$

$$\hat{C}^a(x')\hat{M}(x)|\Phi >= \hat{M}(x)\hat{C}^a(x')|\Phi >= 0 \tag{6.20}$$

therefore

$$\hat{C}^a(x')\hat{M}(x)|\Phi >= 0. \tag{6.21}$$

So we either have that:

$$\hat{M}(x)|\Phi >= 0, \quad \text{or} \quad \hat{M}(x)|\Phi > \in \mathcal{H}_{Diff}. \tag{6.22}$$

We say the master constraint preserves the space of Diff solutions.

We can quantize $\mathbf{M}$ directly on $(\mathbf{\Phi}^*{}_{Kin})_{Diff}$,

2) Removal of the Regulator

By the same methods of the last talk one can remove the triangulation dependence. Diffeomorphism invariance ensures that the limit does not depend on the representative index set $\mathcal{I}$.

3) Quantum Dirac algebra.

If states have to solve additional constraints then the quantum theory would not have as many physical degrees of freedom as the classical theory. To ensure that this is so with the Hamiltonian constraint, we have to restrict the way it acts upon spinnetwork states: The way an operator acts on a state depends on the triangularisation prescription one is adhering to, the requirement for there to be no anomalies places a restriction on this triangularisatoin prescription.

Hamiltonian constraint was to have an anomaly free constraint algebra among the smeared Hamiltonian constraints $\hat{C}(N)$. This motivation is void with respect to $\hat{\mathbf{M}}$ since there is only one $\hat{\mathbf{M}}$ so there cannot be any anomaly (at most in the sense that H_{Phys} is too small, that is, has an unsucient number of semiclassical states). So we have more freedom in the way the Master Constraint acts on spin network states.

However with the Master Constraint one does no. So we have more freedom in the way the Master Constraint acts on spin network states.

There is no constraint algebra anymore, the issue of mathematical consistency (anomoly freenes) is trivalized. However, the issue of physical consistency as not answered yet. The operator ordering choices will have influence on the sizze of the physical Hilbert space and thus on the number of semiclassical states, see below.

4) Classical Limit

The issue could improve on the level of $\mathcal{H}_{Diff}$ for two resons:

i) First of all, $\mathcal{H}_{Diff}$ in contrst to $\mathcal{H}_{Kin}$ does carry an inner product.

ii) The Hilbert space is separable hence coherent states are not distributional but honest elements of $\mathcal{H}_{Diff}$.

Finally, there is a less ambitious programme where $\hat{\mathbf{M}}$ exists as a diffeomorphism invariant operator on $\mathcal{H}_{Kin}$ and where one can indeed try where one could try to answer the queston about the corectness of classical limit using existing semiclassical tools.

Construct semiclassical, spacially diffeomorphism invariant states, maybe by applying the map η to the states consructed in [], and compute expectation values and fluctuations of the Master constraint operator. Show that these quantities coincide with the expected clasical values up to $\hbar$ corrections. This is the second most important step because the existence of suitable semiclassical states at the spatially diffeomorphism invariant level is not a priori granted. Once this step is established, we would have shown that the classical limit of $\hat{\mathbf{M}}$ is the correct one and therfore the quantization really qualifies as a quantum field theory of GR.

Solution of all the Quantum Constraints

The Hilbert space is not a priori separable because there are continous moduli associated with intersecting knot classes with vertices of valence higher than four. It turns out that there is a simple way to remove those moduli by performing additional averaging in the rigging map η mentioned above. This should not affect the classical limit

Thus, if $\mathcal{H}_{Diff}$ is separable, then we can construct the direct integral representation of $\mathcal{H}_{Diff}$ associated with the self-adjoint operator $\hat{\mathbf{M}}$, that is,

$$\mathcal{H}_{Diff} = \int_R^{\oplus} d\mu(\lambda) \mathcal{H}^{\oplus}_{Diff}(\lambda) \tag{6.23}$$

and since $\hat{\mathbf{M}}$ acts on the Hilbert space $\mathcal{H}^{\oplus}_{Diff}(\lambda)$ by multiplication with λ it follows that

$$\mathcal{H}_{Phys} = \mathcal{H}^{\oplus}_{Diff}(0) \tag{6.24}$$

is the physical Hilbert space and a crucial open quesion to be answered is whether it is large enough (has sufficient number of smiclassical solutions).

The physical inner product is given by

$$< s, s' >:= \lim_{T \to \infty} < s, \int_{-T}^{T} dt e^{it\,\hat{\mathbf{M}}} s' > \tag{6.25}$$

Simple example: "Master constraint" direct decompostion of the kinnematic Hilbert space.

We wish to find the solution space for the constraint

$$\hat{C}\psi(p_1, p_2) := \hat{p}_2 \psi(p_1, p_2) = 0 \tag{6.26}$$

and an induced inner product. Let us first write,

$$\hat{C}\psi(p_1, p_2) = \lambda \psi(p_1, p_2) \tag{6.27}$$

the solutions to the constraint equation (M.-19) are the eigenstates with zero eigenvector, $\lambda = 0$.

$$\int dp_1 dp_2 \overline{\psi(p_1,p_2)}\phi(p_1,p_2) = \int d\lambda \left[\int dp_1 dp_2 \delta(\lambda - p_2)\overline{\psi(p_1,p_2)}\phi(p_1,p_2) \right]$$
$$= \int d\lambda \left[\int dp_1 \overline{\psi(p_1,\lambda)}\phi(p_1,\lambda) \right] \tag{6.28}$$

$$(\psi,\phi)_K = \int d\mu(\lambda)(\psi_\lambda,\phi_\lambda)_{\mathcal{H}}, \quad d\mu(\lambda) = d\lambda \tag{6.29}$$

$$(\psi_\lambda,\phi_\lambda)_{\mathcal{H}} = \int dp\overline{\psi_\lambda(p)}\phi_\lambda(p) \text{ where } \phi_\lambda(p) := \phi(p,\lambda) \tag{6.30}$$

and so the kinematic Hilbert space is the direct summation:

$$\mathcal{H}_{Kin} = \int d\mu(\lambda)\mathcal{H}^{\oplus}_{Kin}(\lambda) \tag{6.31}$$

where $\mathcal{H}^{\oplus}_{Kin}(\lambda)$ is the subset of the kinematic Hilbert space on which $\hat{C}$ operates by multiplication, i.e. for every $\psi(p) \in \mathcal{H}^{\oplus}_{Kin}(\lambda)$, $\hat{C}\psi(p) = \lambda\psi(p)$.

Now the "Master constraint"

We wish to find the solution space for the constraint

$$\hat{\mathbf{M}}\psi(p_1,p_2) = \hat{p}_2^2\psi(p_1,p_2) = 0 \tag{6.32}$$

and an induced inner product. Let us first write,

$$\hat{\mathbf{M}}\psi(p_1,p_2) = \lambda\psi(p_1,p_2). \tag{6.33}$$

the solutions are the eigenstates with zero eigenvector, $\lambda = 0$.

$$\int dp_1 dp_2 \overline{\psi(p_1,p_2)}\phi(p_1,p_2) = \int d\lambda \left[dp_1 dp_2 \delta(\lambda - p_2^2)\overline{\psi(p_1,p_2)}\phi(p_1,p_2) \right] \tag{6.34}$$

with a change of variables

$$u = p_2^2, \;\; dp_2 = \frac{du}{2\sqrt{u}} \tag{6.35}$$

eq(M.-19) evaluates as

$$
\begin{aligned}
\int dp_1 dp_2 \overline{\psi(p_1,p_2)}\phi(p_1,p_2) &= \int d\lambda \left[\int dp_1 \int \frac{du}{2\sqrt{u}}\delta(\lambda-u)\overline{\psi(p_1,u)}\phi(p_1,u)\right] \\
&= \int d\lambda \frac{1}{2\sqrt{\lambda}}\left[\int dp_1 \overline{\psi(p_1,\sqrt{\lambda})}\phi(p_1,\sqrt{\lambda})\right] \\
&= \int d\mu(\lambda)\left[\int dp_1 \overline{\psi(p_1,\sqrt{\lambda})}\phi(p_1,\sqrt{\lambda})\right] \qquad (6.36)
\end{aligned}
$$

$$
(\psi,\phi)_K = \int d\mu(\lambda)(\psi_\lambda,\phi_\lambda)_{\mathcal{H}}, \quad \text{with } d\mu(\lambda) = d\lambda \frac{1}{2\sqrt{\lambda}} \qquad (6.37)
$$

$$
(\psi_\lambda,\phi_\lambda)_{\mathcal{H}} = \int dp \overline{\psi_\lambda(p)}\phi_\lambda(p) \text{ where } \phi_\lambda(p) := \phi(p,\sqrt{\lambda}) \qquad (6.38)
$$

and

$$
\mathcal{H}_{Kin} = \int d\mu(\lambda)\mathcal{H}^{\oplus}_{Kin}(\lambda). \qquad (6.39)
$$

$$
\mathcal{H}_{phys} = \mathcal{H}^{\oplus}_{Kin}(0) \qquad (6.40)
$$

with induced inner product $L_2[R, dp]$. Where $\mathcal{H}^{\oplus}_{Kin}(\lambda)$ is the subset of the kinematic Hilbert space on which $\hat{\mathbf{M}}$ operates by multiplication, i.e. for every $\psi(p) \in \mathcal{H}^{\oplus}_{Kin}(\lambda)$, $\hat{\mathbf{M}}\psi(p) = \lambda\psi(p)$.

Quantum Dirac Observables

They are much cleaner than the ones involved in the Hamiltonian Constraint Programme.

$$
[O]_T := \frac{1}{2T}\int_{-T}^{T} dt e^{t\mathcal{L}_{\chi_{\mathbf{M}}}} O \qquad (6.41)
$$

$$
\{[O]_T, \mathbf{M}\} = \frac{1}{2T}\int_{-T}^{T} dt \{e^{t\mathcal{L}_{\chi_M}} O, \mathbf{M}\} \qquad (6.42)
$$

$$
\frac{df}{dt} := \mathcal{L}_{\chi_{\mathbf{M}}} \equiv \{f, \mathbf{M}\}. \qquad (6.43)
$$

$$
\{[O]_T, \mathbf{M}\} = \frac{1}{2T}\int_{-T}^{T} dt \frac{d}{dt} e^{t\mathcal{L}_{\chi_M}} O = \frac{e^{T\mathcal{L}_{\chi_{\mathbf{M}}}} - e^{-T\mathcal{L}_{\chi_{\mathbf{M}}}}}{2T} O \qquad (6.44)
$$

Since O is bounded (in sup-norm) on by assumption so is $e^{\pm T\mathcal{L}_{\chi_{\mathbf{M}}}}O$, hence

$$\lim T \to \infty\{[O]_T, \mathbf{M}\} = 0 \tag{6.45}$$

Thus provided that we can interchange the limit $\lim T \to \infty$ with the Poisson bracket, we get $\{[O], \mathbf{M}\}$

Summary of Master Constraint

$$\mathbf{M} := \int_\sigma d^3x \frac{[C(x)]^2}{\sqrt{\det(q)(x)}} \tag{6.46}$$

Properties:

i) Positive: $\mathbf{M} = 0$ if and only if $C(x) = 0$ for all $x \in \sigma$.

ii) Weak Dirac observables: $\{O, \{O, \mathbf{M}\}\}_{\mathbf{M}=0} = 0$ if and only if $\{O, C(x)\}_{\mathbf{M}=0} = 0$ for all $x \in \sigma$.

iii) Spatially diffeomorphism invariant: $\{\mathbf{M}, C_a(x)\}$ for all $x \in \sigma$.

iv) Trivial commutator: $\{\mathbf{M}, \mathbf{M}\} = 0$.

4) Since $\mathbf{M}$ spatial diffeomorphic we can define $\mathbf{M}$ directly on $\mathcal{H}_{Diff}$! - we can solve one constraint onto the other.

6.3 Quantization of the Master Constraint Operator for GR.

Clasical preliminaries

$$\mathbf{M} := \int_\sigma d^3x \frac{C(x)^2}{\sqrt{\det(q)}}(x) = \int_\sigma d^3x (\frac{C}{\det(q)^{1/4}})(x) \int_\sigma d^3y \delta(x,y)(\frac{C}{\det(q)^{1/4}})(y) \tag{6.47}$$

A slight variant of

$$\begin{aligned}
\frac{C(x)}{\det(q)^{1/4}} &= H^{(1)} + H^{(2)} \quad \text{where} \\
H^{(1)}_\epsilon &= -2Tr(F_{ab}\{A_c, V^{1/2}\})\epsilon^{abc} && (6.48) \\
H^{(2)}_\epsilon &= \frac{\gamma^2+1}{\gamma^2}\epsilon^{abc}Tr(\{A_a, K\}\{A_b, K\}\{A_c, V^{1/2}\}) && (6.49)
\end{aligned}$$

A regularized expression for the **Master constraint**:

$$\mathbf{M} = \lim \epsilon \to 0 \int_\sigma d^3x Tr H_\epsilon(x) \int_\sigma d^3y \chi_\epsilon(x,y) Tr H_\epsilon(y) \tag{6.50}$$

the integrands will be *precisely those* of (M.-19) but with $\{A_c^k, V^{1/2}\}$ instead of $\{A_c^k, V\}$ and with $H_\epsilon = H_\epsilon^{(1)} + H_\epsilon^{(2)}$ where

Here $\chi_\epsilon(x,y)$ is any one parameter family of functions such that $\lim \epsilon \to 0\ \chi_\epsilon(x,y)/\epsilon^3 = \delta(x,y)$ and $\chi_\epsilon(x,x) = 1$.

$$V_\epsilon(x) := \int_\sigma d^3y \chi_\epsilon(x,y) \sqrt{det(q)}(y) \tag{6.51}$$

We recognize that the integrands of the two integrals in are *precisely* those of Eq.(), the only difference being that the last factor in the wedge product is given by $\{A_c, V_\epsilon^{1/2}\}$ rather than $\{A_c, V\}$ which comes from the additional factor of $(det(q))^{-1/4}$ in the point-split expression. Thus we proceed exactly as in the last talk and introduce a partition $\mathcal{P}$ of σ into cells ∇, splitting both integrals into sums $\int_\sigma = \sum_{\nabla \in \mathcal{P}}$

$$H(N) = \lim_{\epsilon \to 0} \sum_{\triangle \in \mathcal{T}(\epsilon)} N(v(\Delta))\ H(\Delta) \tag{6.52}$$

where $H(\Delta) = H(\chi_\Delta) = \int_\sigma d^3x\ \chi_\Delta(x)\ H(x)$

$$\mathbf{M} = \lim_{\epsilon \to 0} \sum_{\triangle \in \mathcal{T}(\epsilon)} \frac{H(\Delta)^2}{V(\Delta)} \tag{6.53}$$

$$C(\Delta) := \frac{H(\Delta)}{\sqrt{V(\Delta)}} = \int_\Delta d^3x\ \epsilon^{abc} Tr(F_{ab} \frac{\{A_c, V(\Delta)\}}{\sqrt{V(\Delta)}}) = 2 \int_\Delta d^3x\ \epsilon^{abc} Tr(F_{ab} \{A_c, \sqrt{V(\Delta)}\}) \tag{6.54}$$

where we have used $\{\ .\ , V(\Delta)\}/\sqrt{V(\Delta)} = 2\{\ .\ , \sqrt{V(\Delta)}\}$.

$$\mathbf{M} = \lim_{\epsilon \to 0} \sum_{\triangle \in \mathcal{T}(\epsilon)} C(\Delta)^2 \tag{6.55}$$

6.3.1 Spatially Diffeomorphism Invariant Operators on $\mathcal{H}_{Kin}$

Non graph changing operators only change the labels of a graph but not change the graph it self. Diffeomorphism invariant operators which are graph-changing cannot exist on $\mathcal{H}_{Kin}$. This is due to the "infinite volume of the diffeomorphism group". We sketch the proof here missing out some of the technical details. Recall that for each φ in the group of diffeomorphisms of Σ there is a unitary operator $\hat{U}(\varphi)$ acts on the kinematical Hilbert space and as it is a unitary operator it is in the collection of bounded operators on $\mathcal{H}_{Kin}$. We call

$$Q(f,f') := \left\langle f, \hat{O} f' \right\rangle_{Kin}$$

the quadratic form of $\hat{O}$. Let $Q_{s,s'} := Q(T_s, T_{s'})$

$$Q_{s,s'} = 0 \ \text{ whenever } \ \gamma(s) \neq \gamma(s') \tag{6.56}$$

A spacially diffeomorphism invariant quadratic form is defined by

$$Q(\hat{U}(\varphi)f, \hat{U}(\varphi)f') = Q(f,f')$$

for all f, f' in the domain of Q and for all φ in the difeomorphism group. Note that if an operator $\hat{O}$ is graph changing then there will exist a spin network state T_s such that $\left\langle T_s, \hat{O} T_{s'} \right\rangle_{Kin} \neq 0$ for $T_s \neq T_{s'}$. Consider any $\gamma(s) \neq \gamma(s')$ where $\gamma(s)$ is the graph underlying spin network s.

Recall that at the kinematic level two spin network states are orthogonal if their underlying graphs are topologically distinct or positioned at different places in σ. It should be obvious that one can find a countable infinite number of diffeomorphisms φ_n, $n = 0, 1, 2, \dots$ such that $T_{s'}$ is invariant but $T_{s_n} = \hat{U}(\varphi_n) T_s$ are mutually orthogonal spin network states, that is, we can find φ_n, $n = 0, 1, 2, \dots$ such that

$$U(\varphi_n) T_{s'} = T_{s'} \quad \text{and} \quad \left\langle \hat{U}(\varphi_m) T_s, \hat{U}(\varphi_n) T_s \right\rangle_{Kin} = 0 \quad \text{for } m \neq n.$$

Suppose now that Q is the quadratic form of a spacially diffeomorphism invariant operator $\hat{O}$ on $\mathcal{H}_{Kin}$, that is, $\hat{U}(\varphi) \hat{O} \hat{U}(\varphi)^{-1} = \hat{O}$ for all φ in the diffeomorphism group.

Let us expand $\hat{O} T_s'$ as an uncountable formal summation

$$\hat{O} T_{s'} = \sum_t a_t T_t$$

The coefficents a_t are obtained from

$$
\begin{aligned}
Q_{s,s'} = \left\langle T_s, \hat{O} T_{s'} \right\rangle_{Kin} &= \sum_t a_t \left\langle T_s, T_t \right\rangle_{Kin} \\
&= \sum_t a_t \delta_{st} = a_s
\end{aligned}
$$

Now we consider the norm square of $\hat{O} T_{s'}$

$$
\begin{aligned}
\|\hat{O} T_{s'}\|^2 &= \| \sum_t Q_{t,s'} T_t \|^2 \\
&= \sum_{t,t'} Q^*_{t,s'} Q_{t',s'} \left\langle T_t, T_{t'} \right\rangle_{Kin} \\
&= \sum_s |Q_{s,s'}|^2. \qquad (6.57)
\end{aligned}
$$

As summation over all graph labels s includes summation over the countable collection $\{s_n\}$, $n = 0, 1, 2, \ldots$, we have

$$
\sum_s |Q_{s,s'}|^2 \geq \sum_{n=0}^{\infty} |Q_{s_n,s'}|^2
$$

Now we use the spacial diffeomorphism invariance on the RHS

$$
\begin{aligned}
\sum_{n=0}^{\infty} |Q_{s_n,s'}|^2 &= \sum_{n=0}^{\infty} | \left\langle \hat{U}(\varphi_n) T_s, T_{s'} \right\rangle |^2 \\
&= \sum_{n=0}^{\infty} | \left\langle T_s, \hat{U}(\varphi_n)^{-1} \hat{O} \hat{U}(\varphi_n)\, \hat{U}(\varphi_n)^{-1} T'_s \right\rangle |^2 \\
&= \sum_{n=0}^{\infty} | \left\langle T_s, \hat{O} T'_s \right\rangle |^2 = \sum_{n=0}^{\infty} |Q_{s,s'}|^2 \qquad (6.58)
\end{aligned}
$$

where we have used that $\hat{U}(\varphi_n)$ has an inverse which is also a diffeomorphism. Hence

$$
\|\hat{O} T'_s\|^2 \geq |Q_{s,s'}|^2 \left[\sum_{n=0}^{\infty} 1 \right] \qquad (6.59)
$$

diverges unless $Q_{s,s'} = 0$. We conclude that diffeomorphism invariant operators which are graph changing cannot exist on the kinematic Hilbert space $\mathcal{H}_{Kin}$.

Diffeomorphism invariant operators that are graph changing must be defined on $\mathcal{H}_{Diff}$ and not on $\mathcal{H}_{Kin}$. Roughly speaking, this works because all the terms in the infinite summation (6.59) are equivalent under diffeomorphisms, hence we only need one of them, whence the infinite sum becomes finite.

For an operator to be well defined on $\mathcal{H}_{Kin}$ it must be non graph changing, but of course, this isn't a sufficient condition.

6.3.2 Diffeomorphism Invariant Hilbert Space

6.3.3 Regularizaton and Quantization of the Master Constraint

Basic building blocks of (??) are the integrals

$$C_{\epsilon\mathcal{P}()} = Tr\left(\left[F_{ab} + \frac{\gamma^2+1}{\gamma^2}\{A_a,\{C_E(1),V\}\}\{A_b,\{C_E(1),V\}\})\right]\{A_c,V_\epsilon^{1/2}\})\right)\epsilon^{abc} \quad (6.60)$$

It frequently happens in quantum mechanics that an operator is not closed but has an extension which is closed.

6.3.4 Non Graph Changing (Extended) Master Constraint

In order to have control on the semiclassical limit one must currently use a non graph changing operator and an operator which can be defined on $\mathcal{H}_{Kin}$. The advantage of having a non-graph changing Master Constraint Operator is that one can quantize it directly as a positive operator on $\mathcal{H}_{Kin}$ and check its semi-classical properties by testing it with the semi-classical tools developed in [], [], [].

In order to define such an operator we need the notion of a minimal loop: Given a vertex v of a graph γ and two edges e, e' outgoing from v, a loop $\alpha(\gamma, v, e, e')$ within γ based at v, outgoing along e and incoming along e_2 is said to be minimal if there is no other loop within γ with the same properties and fewer edges traversed (see fig. 6.3.4). Let $L(\gamma, e_1, e_2, e_3)$ be the set of minimal loops with the data indicated. Notice that this set is always non empty but may consist of more than one element. If $L(\gamma, e_1, e_2, e_3)$ has more than one element then the corresponding Master constraint operator averages over the finite number of elements of $L(\gamma, e_1, e_2, e_3)$. We now define

Here $T(\gamma, v)$ is the number of ordered triples of edges incident at v (taken with outgoing orientation)

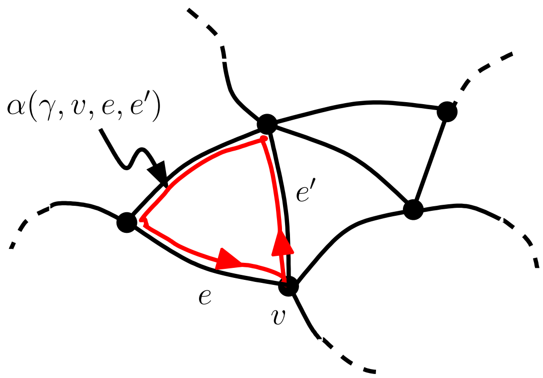

Figure 6.1: minimalloop1. minimal loop.

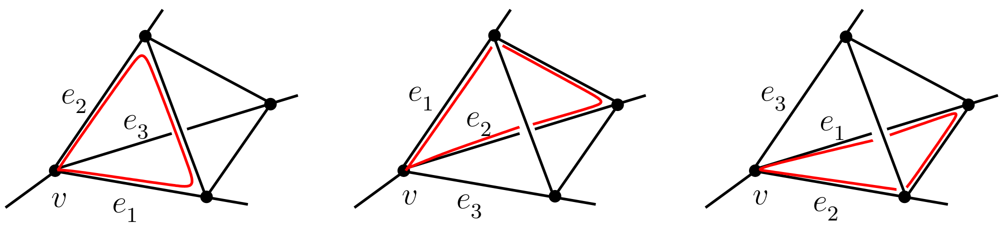

Figure 6.2: minimaloop2. minimal loop. $|L(\gamma, e_1, e_2, e_3)| = 3$

$$T(\gamma, v) = \frac{n(n-1)(n-2)}{6}$$

The expression below is a good approximation to the classical Master constraint

$$\begin{aligned}
\hat{M}_\gamma &:= \sum_{v \in V(\gamma)} \hat{\tilde{C}}_v^\dagger \hat{\tilde{C}}_v \qquad (6.61)\\
\hat{\tilde{C}}_v &:= \frac{1}{|T(\gamma, v)|} \sum_{e_1, e_2, e_3 \in T(\gamma, v)} \frac{\epsilon_v(e_1, e_2, e_3)}{|L(\gamma, e_1, e_2, e_3)|} \times \\
&\qquad \times \sum_{\alpha \in L(\gamma, e_1, e_2, e_3)} Tr\big([A(\alpha) - A(\alpha)^{-1}] A(e_3) [A(e_3)^{-1}, \sqrt{\hat{V}_v}]\big)
\end{aligned}$$

whose tangents are linearly independent and $\epsilon_v(e_1, e_2, e_3) = \mathrm{sgn}(\det(\dot{e}_1(0), \dot{e}_2(0), \dot{e}_3(0))$. The volume operator is given explicitly by

$$\hat{V}_v = \sqrt{|\frac{i}{48} \sum_{e_1, e_2, e_3 \in T(\gamma, v)} \epsilon_v(e_1, e_2, e_3) \epsilon_{jkl} X^j_{e_1} X^k_{e_2} X^l_{e_3}|} \qquad (6.62)$$

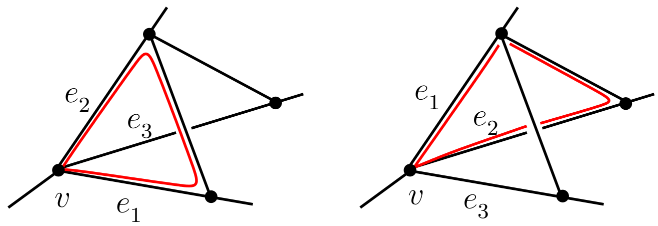

Figure 6.3: minimaloop3. minimal loop. $|L(\gamma, e_1, e_2, e_3)| = 2$

It is easy to see that the definition (6.61) is spatially diffeomorphism invariant. Moreover, the results of [220], [221], [222] imply that expectation values with respect to the coherent states constructed in [][86] defined on graphs which are suffieciently fine, the zeroth order in $\hbar$ of $\hat{M}_\gamma$ coincides with the classical expression. In other words, the correctness of the classical limit of $\hat{M}$ has been established recently.

The results of [220], [221], [222] also imply that the commutator between the $\sum_v N_v \hat{C}_v$ reproduces the third relation (6.3) in the sense of expectation values with respect to coherent states where $\hat{C}_v$ is the same as $\hat{\hat{C}}_v$ in (6.61) just that $\sqrt{\hat{V}_v}$ is replaced by $\hat{V}_v$. This removes a further criticism mentioned in section 4.5.3, namely we have off-shell closure of the Hamiltonian constraints to zeroth order in $\hbar$. Possible higher order corrections (anomalies) are no obstacle for the Master constraint programme as already said.

The disadvantage of a non-graph-changing operator is that it uses a prescription like the above minimal loop prescription as an ad hoc quantization step. While it is motivated by the more fundamental quantization procedure of the previous section and is actually not too drastic a modification thereof for sufficiently fine graphs, the procedure of the previous section should be considered as more fundamental. Maybe one could call the operator as formulated in this section an effective operator since it presumably reproduces all semiclassical properties.

6.3.5 Brief Note on the Volume Operator

In order to show this one has to calculate the matrix elements of (6.62) which is non trivial because the spectrum of that operator is not accessible exactly (The matrix elements of the square root are known in closed form [205]). However, one can perform an error controlled $\hbar$ expansion within coherent state matrix elements and compute the matrix elements of every term in that expansion analytically [220], [221], [222].

The idea is extremely simple: In applications we are interested in expressions of the form Q^r where Q is a positive operator, $0 < r \leq 1/4$ rational number and its relation to the volume operator is

$$V = Q^{1/4.}$$

The matrix elements of Q in coherent states can be computed in closed form. Now use the Taylor expansion of the function $f(x) = (1+x)^r$ up to some order N including the remainder with $x = Q/ < Q > 1$ where $< Q >$ is the expectation value of Q with respect to the coherent state of interest.

$$\begin{aligned} Q^{1/r} &= 1 + rx + \frac{r(r-1)}{2!}x^2 + \cdots + R_{N+1}(x) \\ &= 1 + r\left(\frac{Q}{< Q >} - 1\right) + \frac{r(r-1)}{2!}\left(\frac{Q}{< Q >} - 1\right)^2 + \cdots + R_{N+1}\left(\frac{Q}{< Q >} - 1\right) \end{aligned}$$

The operators x^n in that expansion can be explicitly evaluated in the coherent state basis

$$\begin{aligned} < \psi| \left(\frac{Q}{< Q >} - 1\right)^n |\psi > &= \sum_{i_1, i_2, \ldots i_N} < \psi|x|\psi_{i_1} >< \psi_{i_1}|x|\psi_{i_2} >< \psi_{i_2}|x|\psi > \\ &= \sum_{i_1, i_2, \ldots i_n} x_{ii_1} x_{i_1 i_2} x_{ii_1} \ldots x_{i_n i} \end{aligned}$$

while the remainder $R_{N+1}(x)$ can be estimated from above and provides a higher $\hbar$ correction than any of the x^n, $0 \leq n \leq N$.

6.4 Spectral Decomposition

One of the basic results of linear algebra is the theorem of the existence of a complete system of eigenvectors for any self-adjoint linear operator A in an n-dimensional Eucldean space $\mathbb{R}^n$. This theorem states that if A is a self-adjoint operator in an n-dimensional Eucldean space $\mathbb{R}^n$, then an orthonormal basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$ can be found, each vector of which is an eigenvector of the operator A:

$$Ae_k = \lambda_k e_k,$$

where λ_k is a real number. Expanding any vector f of the spaace $\mathbb{R}^n$ by means of the vectors $e_1, \ldots, e_n$:

$$f = a_1 e_1 + \cdots + a_n e_n,$$

where $a_k = (f, e_k)$, we can write the operator in the form:

$$Af = \sum_{k=1}^{n} \lambda_k (f, e_k) e_k. \tag{6.63}$$

The situtation beccomes complicated upon passing from the finite to the infinite dimensional case.

For a self-adjoint and unitary operators the eigenvalues are ordered in a natural way. For a self-adjoint operator this is so because the eigenvalues are real numbers and for a unitary operators the eigenvalues are represented by points on the unit circle of the complex plane.

Let us first consider self-adjoint operators. We assume that $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, and we use the P_i's to define new projections:

$$\begin{aligned}
E_{\lambda_0} &= 0 \\
E_{\lambda_1} &= P_1 \\
E_{\lambda_2} &= P_1 + P_2; \\
&\vdots \\
E_{\lambda_m} &= P_1 + P_2 + \cdots + P_m.
\end{aligned} \tag{6.64}$$

$$\begin{aligned}
A &= \lambda_1 P_1 + \lambda_2 P_2 + \cdots \lambda_m P_m \\
&= \lambda_1 (E_{\lambda_1} - E_{\lambda_0}) + \lambda_2 (E_{\lambda_2} - E_{\lambda_1}) + \cdots \lambda_m (E_{\lambda_m} - E_{\lambda_{m-1}}) \\
&= \sum_{i=1}^{m} \lambda_i (E_{\lambda_i} - E_{\lambda_{i-1}}).
\end{aligned} \tag{6.65}$$

$$A = \sum_{i=1}^{m} \lambda_i \Delta E_{\lambda_i} \tag{6.66}$$

Theorem For each self-adjoint operator there is a unique spectral family of projection operators P_x such that

$$(\phi, A\varphi) = \int_{\infty}^{\infty} x d(\phi, P_x \varphi) \tag{6.67}$$

for all vectors φ and ϕ.

The spectral decomposition of A

$$A = \int_{-\infty}^{\infty} dE_x \tag{6.68}$$

$$\begin{aligned}
(\phi, \psi) &= (\phi, P_1\psi) + (\phi, P_2\psi) + \cdots (\phi, P_m\psi) \\
&= (\phi, (E_{\lambda_1} - E_{\lambda_0})\psi) + (\phi, (E_{\lambda_2} - E_{\lambda_1})\psi) + \cdots (\phi, (E_{\lambda_m} - E_{\lambda_{m-1}})\psi) \\
&= \sum_{i=1}^{m} (\phi, \Delta E_{\lambda_i}\psi).
\end{aligned} \tag{6.69}$$

$$\begin{aligned}
(\phi, A\psi) &= (\phi, \lambda_1 P_1\psi) + (\phi, \lambda_2 P_2\psi) + \cdots (\phi, \lambda_m P_m\psi) \\
&= (\phi, (E_{\lambda_1} - E_{\lambda_0})\psi) + (\phi, (E_{\lambda_2} - E_{\lambda_1})\psi) + \cdots (\phi, (E_{\lambda_m} - E_{\lambda_{m-1}})\psi) \\
&= \sum_{i=1}^{m} \lambda_i (\phi, \Delta E_{\lambda_i}\psi).
\end{aligned} \tag{6.70}$$

for any vectors φ and ϕ we have

$$(\phi, \psi) = \int_{-\infty}^{\infty} d(\phi, E_x\psi) \tag{6.71}$$

and

$$(\phi, A\psi) = \int_{-\infty}^{\infty} x d(\phi, E_x\psi). \tag{6.72}$$

Unitary operators can be treated in a similarly. For a unitary operator U let its eigenvalues be $u_i = e^{i\theta_i}$ labelled in the order

$$0 < \theta_1 < \theta_2 < \cdots < \theta_{m-1} < \theta_m \leq 2\pi \tag{6.73}$$

For each real number x let

$$E_x = \sum_{\theta_i \leq x} P_i \tag{6.74}$$

This is the progection operator onto the space spanned by all eigenvectors for eigenvalues $e^{i\theta_i}$ with $\theta_i \leq x$. If $x \leq 0$, then $E_x = 0$. If $x \geq 2\pi$, then $E_x = 1$. Evidently E_x increses by incriments P_i the same as for the Hermitian operator with eigenvalues θ_i. For

$$U = \sum_{i=1}^{m} u_i P_i = \sum_{i=1}^{m} e^{i\theta_i} P_i \tag{6.75}$$

we can write

$$U = \int_0^{2\pi} e^{ix} dE_x. \tag{6.76}$$

For any vectors ψ and ϕ we have

$$(\phi, U\psi) = \int_0^{2\pi} e^{ix} d(\phi, E_x\psi). \tag{6.77}$$

6.5 Rigged Hilbert Space and Direct Integral Decomposition

The concept of a direct integral of a Hilbert space is a generalization of the concept of the orthogonal direct sum of a countable family of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots$.

6.6 Symmetric Operators and their Extensions

In the construction of operators a in physics one often starts from its matrix elements $Q_a(\psi, \psi')$, which should equal $< \psi, a\psi' >$. However, this is not enough to define an operator in inifite dimensions because given an ortho-normal basis (b_n) we must have

$$\|a\psi\| = \sum_n |Q_a(b_n, \psi)|^2 < \infty$$

in order that $\psi \in D(a)$. Hence it may happen that the quadratic form $Q_a(\psi, \psi')$ exists for ψ, ψ' in a dense subset of $\mathcal{H}$ but on the other hand it could be that $D(a) = \{0\}$.

For symmetric operators, there are always closed extensions. A smallest closed extension always exists (the double adjoint), but it is possible that none of these closed extensions is self-adjoint. On the other hand, semibounded forms need not have any closed extensions (defintions for quadratic forms to be given below), but when such extensions exsit and are semibounded, they are the quadratic forms associated with self-adjoint operators.

6.7 Weak Dirac Observables *a la* Dittrich

f and T_j functions on phase space. Weak Dirac observable there are n T_j's

$$F^{\tau}_{f,T} := \sum_{k_1,\ldots,k_n=0} \frac{(\tau_1 - T_1)^{k_1}}{k_1!} \cdots \frac{(\tau_n - T_n)^{k_n}}{k_n!} (X_1)^{k_1} \ldots (X_n)^{k_n} \cdot f. \qquad (6.78)$$

where $X_r \cdot f$ is defined as

$$X_j \cdot f := \{(A^{-1})_{jk} C_k, f\}, \quad A_{jk} := \{C_j, T_k\}. \qquad (6.79)$$

Poisson algebra

$$\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\} = F^{\tau}_{\{f,f'\}^*,T} \qquad (6.80)$$

defining the automorphism on ?? generated by the Hamiltonian vector field of $\sum_j \tau^j C'_j$

$$\alpha'_{\tau}(f) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_j \tau^j X_j \right)^n \cdot f \qquad (6.81)$$

$$\{\alpha^{\tau}(F^{\tau}_{f,T}), \alpha^{\tau}(F^{\tau}_{f',T})\} \approx \alpha^{\tau}(\{F^{\tau}_{f,T}, F^{\tau}_{f',T}\}) \qquad (6.82)$$

In other words, α^{τ} is a weak, abelian, multi-parameter group of automorphisms on the each $F^{\tau_0}_{f,T}$.

6.7.1 Equivalent Hamiltonians that are Spacially Diffeomorphism Invariant

An unfortunate feature of the Hamiltonian constraint was that it could not be implemented on the spatially diffeomorpism invariant Hilbert space because it would map spatially diffeomorpism invariant states onto non-diffeomorpism invariant states - i.e. it did not close on the space.

In [300] it was shown that if one is given a constraint algebra of the form

$$\{C_J, C_K\} = f_{JK}{}^{L} C_L, \quad \{C_J, C_k\} = f_{Jk}{}^{l} C_l, \quad \{C_j, C_k\} = f_{jk}{}^{L} C_L \qquad (6.83)$$

$$A_{lj} := \{C_l, T_j\}. \qquad (6.84)$$

$$\tilde{C}_j := \{M, T_j\} \approx \sum_{k,l} Q_{kl} C_k A_{lj} \qquad (6.85)$$

the constraint algebra can be simplified to

$$\{C_J, C_K\} = f_{JK}{}^L C_L, \quad \{C_J, \tilde{C}_k\} = 0, \quad \{\tilde{C}_j, \tilde{C}_k\} = \tilde{f}_{jk}{}^L C_L + \tilde{f}_{jk}{}^l C_l. \qquad (6.86)$$

Using the Master constraint we can generate *we may generate new Hamiltonian constraints for GR which are spatially diffeomorphism invariant.* Employing these new Hamiltonian constraints it is possible to first solve the spatially diffeomorpism constraints and then implement the new Hamiltonian on the spatially diffeomorpism invariant Hilbert space where they close among themselves. In []

(Lie algebras an commutators???)

The powerful methods of [**?**]an then be used to induce the physical Hilbert space and its inner product.

6.8 Spin Foams From the Master Constraint

New proposal:

Instead of an infinite number of constraints there is only one, then there is an ordiary integral. One splits the T-parameter into discrete steps and writes

$$e^{it\,\hat{\mathbf{M}}} = \lim_{N\to\infty} [e^{it\,\hat{\mathbf{M}}/N}]^N = \lim_{N\to\infty} [1 + it\hat{\mathbf{M}}/N]^N. \qquad (6.87)$$

The action of $1 + it\hat{\mathbf{M}}/N$ on a spin network can be written as a linear combination of of new spin networks whose graphs whose labels have been modified by the creation of new nodes.

6.8.1 Removal of Difficulties of the Sum over Histories

Formally $\hat{\mathbf{M}}$ solved by bf rigging map with physical scalar product.

$$\eta \ : \ \Phi_{Diff} \rightarrow \Phi_{phys}; l \rightarrow \int_R \frac{dT}{2\pi} < e^{iT\hat{M}} l, . > \tag{6.88}$$

$$< \eta(l), \eta(l') >_{Phys} := \eta(l')[l] = \int_R \frac{dT}{2\pi} < e^{iT\hat{M}} l', l > \tag{6.89}$$

The integration variable T is nothing more than a variable of integration.

An approximate calculation:

$$\begin{aligned} & < \eta(l)| \ \exp[-it\frac{\hat{\mathbf{M}}}{\mathbf{N}}] \ |\eta'(l) >_{Diff} \\ = & < \eta(l)| \ [1 - it\frac{\hat{\mathbf{M}}}{\mathbf{N}}] \ |\eta'(l) >_{Diff} + \mathcal{O}(\frac{1}{N^2}) \\ = & \ \delta_{[s][s']} - \frac{it}{N} < \eta(l)| \ \hat{\mathbf{M}} \ |\eta'(l) >_{Diff} + \mathcal{O}(\frac{1}{N^2}) \end{aligned} \tag{6.90}$$

$$< \eta(l)| \ \exp[-it\frac{\hat{\mathbf{M}}}{\mathbf{N}}] \ |\eta'(l) >_{Diff} = \ \delta_{[s][s']} - \frac{it}{N} Q_{\hat{\mathbf{M}}}(\eta(l), \eta'(l)) + \ \mathcal{O}(\frac{1}{N^2}) \tag{6.91}$$

Observaions:

1. $\hat{\mathbf{M}}$ is positive

2. Strong existence theorems for PI's (Osterwalder-Schrader Reconstrusction).

3. Definition of Spin Foam model has precise connection to the Hamiltonian formulism and better convergence prop.($\hat{\mathbf{M}}$ = pos). Formally

4.

5.

6. If can be constructed this way lead to rigorous impleentation of Reisenberger-Rovelli idea, including **sum over all triangulations**.

Hamiltonian and vector constraints

$$M = \int_\Sigma d^3x \frac{C(x)^2 - q^{ab} V_a(x) V^b(x)}{\sqrt{\det q(x)}} \tag{6.92}$$

Obviously, $\hat{\mathbf{M}}_E = 0$ if and only if $C(x) = C_a(x) = 0$ for all $a = 1, 2, 3; \ x \in \sigma$.

6.9 Summary:

· They are much cleaner than the ones involved in the Hamiltonian Constraint Programme.

6.9.1 Problems

(i) $\mathcal{H}_{phys} \not\subset \mathcal{H}_{Kin}$ in general.

(ii) Non-commuting constraints (spectral analysis not possible)

GR

(i) Cannot define $\hat{C}(N)$ directly on $\mathcal{H}_{Diff}$, but concrete implementation uses $\mathcal{H}_{Diff}$.

(iii) Makes semi-classical analysis difficult.

6.9.2 The Master Constraint

Simplifies constraint algebra!

Finite dimension system

$$M = \sum_{i,j} C_i K_{ij} C_j$$

where K_{ij} is strictly positive matrix.

Field theory has infinitely many constraints

$$M = \int d^n x C(x)(K \cdot C)(x)$$

$(K \cdot C)$ is a strictly positive operator.

Gravity

$$M = \int d^3 x \frac{C(x)C(x)}{\sqrt{\det q}}$$

3-dffeomorphism invariant.

$$M = 0 \iff \begin{cases} C_i = 0 & \text{for all } i \\ C = 0 & \text{for all } x \end{cases}$$

We are left with one constraint!

6.9.3 MCP: Physical Hilbert Space

$\mathcal{H}_{Kin}$ (separable), $\hat{M}$ (positive, self-adjoint) then:

Direct integral decomposition ('generalized eigenspace' decomposition)

$$\mathcal{H}_{Kin} \simeq \int_{spec(\hat{M})}^{\oplus} \mathcal{H}_{Kin}^{\oplus}(\lambda)\, d\mu(\lambda) \tag{6.93}$$

$\mathcal{H}_{Kin}^{\oplus}$ carries an induced inner product and $d\mu(\lambda)$ the spectral measure

$$\mathcal{H}_{phys} := \mathcal{H}_{Kin}^{\oplus}(0) \tag{6.94}$$

6.9.4 Recipe for $\mathcal{H}_{phys}$

(i) Find cyclic ON system $\{\Omega_j\}_{j\in\mathcal{J}}$

$$\mathcal{H}_{Kin} = \oplus_j \overline{\text{span}\{\hat{M}^k \Omega_j | k \in IN\}}$$

(ii) Calculate spectral measures.

$$\mu_j(\lambda) =< \Omega_j, \Theta(\lambda - \hat{M})\Omega_j >$$

(iii) Calculate spectral measure $\mu(\lambda)$

$$\mu(\lambda) = \sum_j a_j \mu_i(\lambda), \qquad \sum_j a_j = 1$$

(iv) Find the Radon-Nikodym derivatives

$$\rho_j(\lambda) = \frac{d\mu_j(\lambda)}{d\mu(\lambda)}$$

$\mathcal{H}^{\oplus}_{Kin}(\lambda)$ has an orthonormal basis $\{e_j(\lambda) \mid \rho_j(\lambda) > 0\}$

$\Rightarrow$ determines $\dim\big(\mathcal{H}^{\oplus}_{Kin}(\lambda)\big)$ = multiplicity of λ.

(v) Result does not depend on choice of $\{\Omega_j\}$ (in the following sense).

6.9.5 Uniqueness

$$\mathcal{H}_{Kin} \simeq \int^{\oplus} \mathcal{H}^{\oplus}_{Kin}(\lambda)\, d\mu(\lambda)$$

- measure theoretical formula

(given $\hat{M}$) uniqueness (of $\dim\mathcal{H}^{\oplus}_{Kin}(\lambda)$) μ a.e.

(i) for mixed spectrum points λ_i have **finite** measure

$\Rightarrow$ contributions to $\mathcal{H}^{\oplus}_{Kin}(\lambda_j)$ from ac-sectrum are suppressed

- decompose $\mathcal{H}_{Kin} = \mathcal{H}_{pp} \oplus \mathcal{H}_{ac} \oplus \mathcal{H}_{sing}$

(ii) dependence of $\mathcal{H}_{phys}$ on $\hat{M}$ (that is K)?

6.9.6 Example: Abelian Constraints

$\mathcal{H}_{Kin} = \mathcal{L}^2(\mathbb{R}^2, d^2x)$, $\hat{C}_1 = \hat{p}_1$, $\hat{C}_2 = \hat{p}_2$

$$\hat{M} = \hat{p}_1^2 + \hat{p}_2^2 = -\hbar^2\nabla$$

soc: harmonic functions

'eigenfunctions' $|k_1, k_2 >= \exp(i\vec{k}\cdot\vec{x})$

(i) change to $\mathcal{L}^2(\mathbb{R}^2, d^2p)$

$$\Omega_j = N_j p^{|j|}\exp(ij\varphi)\exp(-\frac{p^2}{2}), j \in$$

p and φ spherical coordinates in $\mathbb{R}^2$

$$\rho_j(\lambda) = \frac{\lambda^{|j|}/|j|!}{2\exp(\frac{\lambda}{2}) - 1} \qquad \rightarrow \lim_{\lambda\to 0}\begin{cases} 0 & |j| \geq 1 \\ 1 & j = 0 \end{cases}$$

$\Rightarrow \mathcal{H}^{\oplus}_{Kin}(0) = \mathcal{H}_{phys} \simeq \mathbb{C}$

$\mathcal{H}^{\oplus}_{Kin}(\lambda > 0) \simeq \mathcal{L}^2(S^1, dp)$

6.9.7 Observables

Classical: How to obtain them?

(i) framework available CBD. this chapter and chapter 1

(ii) gives in general **weak** Dirac observables.

Quantum: for 'strong' Dirac observables, i.e. $[\mathcal{O}, M] = 0$

$$(A|\psi >)_{Kin} \simeq \sum_j a_j(\lambda)(A(\lambda)|\psi >)_{\mathcal{H}(\lambda)} \;\; d\mu(\lambda)$$

can be calculated explicitly

(i) representation for 'weak' Dirac observables?

6.9.8 Examples

(i) Finite dimensional examples

(ii) Maxwell theory, linearized gravity:

squaring of constraints worsened

UV behaviour $\rightarrow$ choose k such that $\hat{M}$ is densely defined

k provides regularization of $\hat{M}$

(iii) Gauss constraints in Einstei - YM

$$M = \int d^3x \, \frac{G_z G^z}{\sqrt{\det(q)}}$$

background independent theories regulate themselves.

result: gauge invariant states

metric-degenerate states

6.9.9 Conclusions and Outlook

(i) method is widely applicable, in articular to open algebras.

(ii) Provides construction of physical inner product and representation of 'strong observables'

(iii) representation of 'weak' observables?

(iv) dependence on $\hat{M}$?

LQG:

(i) $\hat{M}$ can be defined on $\mathcal{H}_{Diff}$

(ii) need separable Hilbert spaces

(iii) semiclassical limit is easier now

(iv) 3-diffeomorphism invariant semiclassical states.

(v) MCP gives construction principle for $\mathcal{H}_{phys}$.

6.10 Biblioliographical notes

In this chapter I have relied on the following refferences:

Fundamental structure of Loop Quantum Gravity [214].

Introduction to Loop Quantum Gravity and Spin foams A. Perez [28].