Chapter 7

Semi-Classical Limit

- Introduction
- Relating Loop Representation to Fock-Space Descrition in the Low Energy Limit.
- Coherent States.
- Minkowskian Spacetime and Scattering Amplitudes.
- Noiseless Subsystems.
- Infinite tensor product

7.1 Introduction

how the background independed, non-peturbative quantum and low energy physics described by perturbation field theory in Minkowskian spacetime. In fully non-perturbative approaches to quantum gravity because there is no background spacetime to begin with. Both conceptually .

the relations between dynamical objects give the apperance of a background spacetime in the appropriate limits. We recover a Minkowski spacetime

In his model, he was able to show he cosmological constant paradox appears only if spacetime is regarded as fundemental rather than emergent, [266].

This cannot be done using Fock space states, as the inner product on Fock space depends on a background metric (most of the time Minkowski metric), whose presence breaks diffeomorphism invariance.

to do with the passage from the classical field to the quantum field

Algebraic Quantization

1. The set $\mathcal{S}$ should be a vector space large enough so that every function on Γ can be obtained

2. The set $\mathcal{S}$ should be small enough so that it is closed under Poisson brackets

$$[\hat{A}, \hat{B}] = i\hbar \widehat{\{A, B\}} \tag{7.1}$$

We must now find a vector space V and a representation of the elements of $\mathcal{A}$ as operators on V. Real observables must be represented by Hermition operators. One then completes V to get the Hilbert space $\mathcal{H}$ of the theory.

We give the Schrödinger picture where the Hilbert space is $\mathcal{H} = L_2(\mathbf{R}^3, d^3x)$ and the operators are represented by:

$$(\hat{\mathbf{1}}\Psi)(q) = \Psi(q), \quad (\hat{q}^i\Psi)(q) = q^i\Psi(q), \quad (\hat{p}_i\Psi)(q) = -i\hbar\frac{\partial}{\partial q^i}\Psi(q) \tag{7.2}$$

This is of course just the conventional Schrödinger representation of the CCR.

Ernfest's Theorem

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}_x\rangle}{m}, \tag{7.3}$$

$$\frac{d\langle\hat{p}_x\rangle}{dt} = -\left\langle\frac{\partial\hat{V}}{\partial x}\right\rangle. \tag{7.4}$$

When we Take the Semiclassical Limit Do we Always Get Back the Classical Theory we Quantized??

Perhaps surprisingly the answer is *no*, at least when we have an infinite number of degrees of freedom. When we quantize a classical theory, however, when there is an infinite number of degrees of freedom things get more complicated and not so clear-cut.

no analog of the Stone- von Neuman uniqueness theorem for quantum mechanics on a vector space. Hence there are infinitely many inequivalent representations of the Poisson bracket algebra. The Fock space representation is singled out by the additional requirement of Poincare invariance. This choice is equivalent to the requirement that the Fock vacuum be a zero eigenstate of the Fock space annihilation operators.

The operator algebra is associated with the system. However, the system can be in any number of different phases - with each phase there corresponds a different, mutually exclusive Hilbert space. For example: a ferreomagnetic system can be in a magnetic phase or a non-magnetic phase - each is described by a different Hilbert space, but the operator algebra is the same in either. Now, different phases involve different physics and so different low energy behaviour. Some representations may reduce to the classical theory in its low energy regime, however, others wont!

Analogies with between quantum field theory and statistical field theory.

$$\text{“}\sum_x e^{itS[\Phi(x)]} \quad t \to -i\tau \quad \sum e^{-\tau S[\Phi]}\text{”} \tag{7.5}$$

Aspects of one have counterparts in the other. In statistical field theory different phases - different states - different physics . In quantum field theory different **sectors** - different states - different physics!

The total Hilbert space is non-countable. The action of operators of the algebra, when apllied to a particular state, is countable. Hence any irreducible representation of the operator alegbra is a subset of the total Hilbert space - in fact there are an uncountable number of irreducible representations.

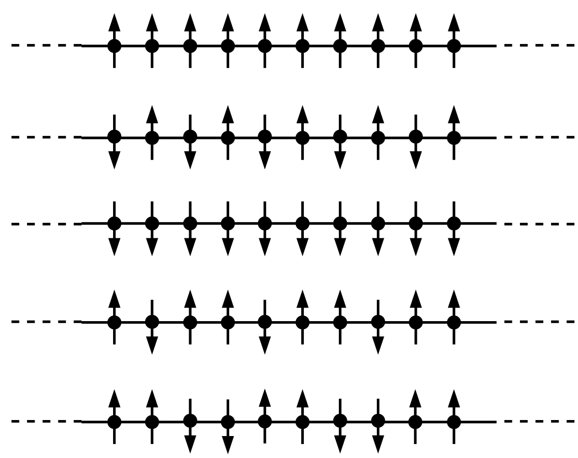


Figure 7.1: NonSepExF. No amount of algebraic action can one of these states into another. The different states describe states ‘infinitly’ different from each other. In statistical mechanics, different representations of the operator algebra describe different phases.

7.1.1 Properties of Semi-Classical States

the quantum representation

a property one need is that the semi-classical states be labelled by points of a classical phase space, i.e. $|\omega>, \omega \in \Sigma$. second property needed is "peaking". given any quantum observable $\hat{A}$, its expectation value $a(\omega) \equiv <\omega|\hat{A}|\omega>$. Quantum fluctuations should be small in a suitable sense. specifying $a_i(\omega) = C_i$ will enable one to obtain a unique point $\omega(C_i)$ in the phase space.

The requirement that the theory admit such semi-classical states is non-trivial as an arbitrarly constructed Hilbert space may or may not admit $|\omega>$.

7.1.2 The problem with establishing the semiclassical limit of LQG has to do with the quantum dynamics:

For graph changing operators such as the Hamiltonian constraints it turns out to be extremely diffcult to define coherent (or semiclassical) states. That is, states labelled by points in the classical phase space with respect to which the operator assumes an expection value which reproduces the value of the corresponding classical function at that point in phase space and with respect to which the (relative) fluctuations are small. The reason for why this happens is that the existing coherent states for LQG [] are defined over a finite collection of finite graphs and these suppress very e ectively the fluctuations of those degrees of freedom that are labelled by the given graph. However, the Hamiltonian constraints add degrees of freedom to the state on which they act and the fluctuations of those are therefore no longer suppressed. Indeed, the semiclassical behaviour of the Hamiltonian constraints with respect to these coherent states is rather bad.

bf cite: "T. Thiemann. Quantum Spin Dynamics (QSD): VII. Symplectic structures and continuum lattice formulations of gauge field theories. Class. Quant. Grav. 18 (2001) 3293-3338. [hep-th/0005232] T. Thiemann. Gauge Field Theory Coherent States (GCS): I. General properties. Class. Quant. Grav. 18 (2001), 2025-2064. [hep-th/0005233]"

Hence we see that the problem of investigating the classical limit of LQG and to verify the quantum algebra of constraints are very much interlinked:

1. Spatial diffeomorphism invariance enforces a weakly discontinuous representation of spatial diffeomorphisms.

2. Anomaly freeness in the presence of only finite diffeomorphisms enforces graph changing Hamiltonian constraints.

3. Graph changing Hamiltonians seem to prohibit appropriate semiclassical states.

7.2 Some of the Schemes

(i) scattering amplitudes in a background independent field theory:

Rovelli *et al* are devising the a formulation for calculate scattering amplitudes in a background independent field theory of gravity plus matter theory. They already have a tentative expression representing the quantum state corresponding to Minkowskian spacetime in terms of a spin foam model. If this scheme works out they will explicitly demonstrate that LQG has the correct semi-classical limit.

(ii) Percolation model from microcausal spin foam models:

It would appear that, as is no notion of background time in GR there can be no background causal structure at the fundamental level. However, one might take the view that the notion of causality is actually more primitive than that of time, more fundamental, being already present in the notion of ordering between events. Smolin and Markopoulou have introduced microcausality into spin foam models - it has resulted into a *directed percolation model*, (however involving amplitude rather than probabilities as have been considered in condensed matter applications). If the "percolation" leads to critical behaviour, there are long established results guarantee that the leading order in the effective action is the Einstein-Hilbert action of classical GR.

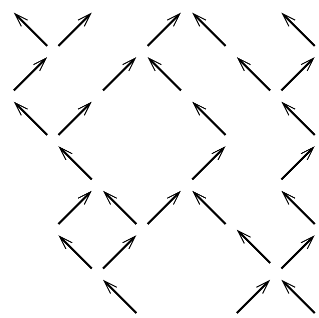


Figure 7.2: "perculation" leads to critical behaviour.

[257]

(iii) Lorentzian three-dimensional dynamical triangulations:

The model of Lorentzian three-dimensional dynamical triangulations provides a non-perturbative definition of three-dimensional quantum gravity. The theory has two phases:

In this approach, (diff-equivalent classes of) smooth metric configurations are approximated by spacetime rtiangulations where 1-simplexes have the same fixed proper length ℓ. The smaller the length scale the better the approximation; therefore, the proper length represents the regulator in the theory to be removed in a certain limit.

(iv) Causal Dynamical Triangulations:

Causal Dynamical Triangulations in four dimensions provide a background-independent definition of the sum over geometries in nonperturbative quantum gravity, with a positive cosmological constant. We present evidence that a macroscopic four-dimensional world emerges from this theory dynamically. [265]

(v) non-graph-changing Master constraint:

A spatially diff invariant operator can not be defined on $\mathcal{H}_{Diff}$ if it is graph changing. can give an ad hoc *non-graph-changing* Master constraint. Can check its semiclassical properties with tools already developed due to recent progress [205]

(vi) Lattice Quantum Gravity and Supercomputers:

non-graph-changing Master constraint arbitrary but fixed graph and study how the theory changes under coarsening of the graph with background independent renormalization schemes, [123], [124].

(vii) quantum geometry in quantum information terms:

Loop Quantum Gravity defines the quantum states of space geometry as spin networks and describes their evolution in time. We reformulate spin networks in terms of harmonic oscillators and show how the holographic degrees of freedom of the theory are described as matrix models. This allow us to make a link with non-commutative geometry and to look at the issue of the semi-classical limit of LQG from a new angle. This work is thought as part of a bigger project of describing quantum geometry in quantum information terms. [255], [256].

(vii) Evolution in Quantum Causal Histories:

[370] "This does mean that the structure of a QCH encompasses a reasonable notion of a quantum field theory, and hence is capable of describing matter degrees of freedom. It also indicates how quantum fields on curved spacetime might be obtained as a limit of some quantum gravity model based on QCH's".

(viii) Noiseless subsystems: Noiseless subsystems are useful for describing the long-term behaviour of the system because they are conserved.

If we divide the quantum gravitational field into subsystems, those properties that are **conserved** under interactions between subsystems will characterize the low energy-limit of spacetime geometry. If we have ... behaving as modes propagating through spacetime then we have spacetime.

(vii) Reduced Phase Space Quantization of LQG: In [223] they perform a canonical, reduced phase space quatization of General Relativity by Loop Quantum Gravity methods.

The kinematic Hilbert space becomes a physical Hilbert space.

We no longer have to deal with the constraints and so no anomalies cannot arise which cast doubt on the semiclassical limit. The physical Hilbert space doesn't need to be derived by complicated methods such as group averaging techniques.

7.3 Semi-Classical Limit

Weaves states we then average over the position of the loops same yield a spacially invariant state. The idea is that in the mean field limit the state approximates the described by the metric used in the construction of the weave state in the firstplace. Such states are by construction, eigenstates of the "momentum operator" $\tilde{E}^a_i(x)$.

In ordinary quantum mechanics

$$\psi(x) = [2\pi(\Delta x)^2]^{-1/4} \exp\left[-\frac{(x- < x >)^2}{4(\Delta x)^2} + \frac{i < p > x}{\hbar} \right] \tag{7.6}$$

7.4 Emergence of Spacetime from Dynamics of Background Independent Quantum Theories

a weak-coupling phase with quantum fluctuations around a "semiclassical" background geometry which is generated dynamically despite the fact that the formulation is explicitly background-independent, and a strong-coupling phase where "classical" space disintegrates into a foam of baby universes. [264]

A Simple Model

In a course-grained low energy approximation

$$\mathcal{H} = J_\perp \sum_{i=1}^{N} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+). \tag{7.7}$$

Is an abstraction of the relationships that exist between physical objects that make up the world. We specify how the spins are in relation with one another without having to talk about these spins sitting in physical space. In this sense it is a background independent theory. The relations are a connectivity between these spins. However, this connectivity does not necessarily have to be represented by points in a pre-existing physical space.

$$\{f_i^\dagger, f_j\} = \delta_{ij}. \tag{7.8}$$

$$\mathcal{H} = \sum_{i=1}^{N} \epsilon(k) f_k^\dagger f_k \tag{7.9}$$

where the energy is given by

$$\epsilon(k) = 4\pi J_\perp \cos\frac{2\pi}{N}k. \tag{7.10}$$

linear dispersion relation

$$\Delta\epsilon = 4\pi J_\perp \frac{2\pi}{N}\Delta k \equiv v_F \Delta k. \tag{7.11}$$

7.5 Semi-Classical Limit of Completely Constrained Systems

Generalized Uncertainty Relation

$$\Delta q\ \Delta p = \hbar/2. \tag{7.12}$$

The generalized uncertainty relation for two arbitrary self adjoint operators $\hat{a}$ and $\hat{b}$:

$$\Delta a \Delta b \geq \frac{\omega([\hat{a},\hat{b}])}{2i} \tag{7.13}$$

where by definition

$$(\Delta a)^2 :=\ \omega((\hat{a}-\omega(\hat{a}))^2), \quad (\Delta b)^2 :=\ \omega((\hat{b}-\omega(\hat{b}))^2). \tag{7.14}$$

Proof: Consider the zero mean operators $A := \hat{a} - \omega(\hat{a})$, $B := \hat{b} - \omega(\hat{b})$ then

$$(\Delta a)^2 = \omega(A^2), \qquad (\Delta b)^2 = \omega(B^2). \tag{7.15}$$

$(\Delta a)^2(\Delta b)^2 = \omega(A^2)\omega(B^2)$. From the Schwartz inequality we have:

$$\omega(A^2)\omega(B^2)\ \geq |\omega(AB)|^2 =\ \omega(AB)\omega(BA). \tag{7.16}$$

We have

$$\begin{aligned} \omega(AB)\omega(AB)^* &= Re(\omega(AB))^2 + Im(\omega(AB))^2 \\ &\geq Im(\omega(AB))^2 \\ &= ((\omega(AB) +\ \overline{\omega(AB)})/2i)^2. \end{aligned} \tag{7.17}$$

and also

$$
\begin{aligned}
\omega(AB) &= \omega\big((\hat{a}-\omega(\hat{a})(\hat{b}-\omega(\hat{b})\big) \\
&= \omega(\hat{a}\hat{b})-\omega(\hat{a})\omega(\hat{b}).
\end{aligned}
\tag{7.18}
$$

Swapping $\hat{a}$ and $\hat{b}$ around gives $\omega(BA) = \omega\big((\hat{b}-\omega(\hat{b})(\hat{a}-\omega(\hat{a})\big) = \omega(\hat{b}\hat{a})-\omega(\hat{a})\omega(\hat{b})$. Putting all this together:

$$
\begin{aligned}
(\Delta a)^2(\Delta b)^2 \geq &= ((\omega(AB)-\omega(BA))/2i)^2 \\
&= ((\omega(\hat{a}\hat{b})-\omega(\hat{b}\hat{a}))/2i)^2 \\
&= \left(\frac{\omega([\hat{a},\hat{b}])}{2i}\right)^2 .
\end{aligned}
\tag{7.19}
$$

Coherent states for the subalgebra of observables $\mathcal{S}$

$$
\Delta a \ \Delta b = \hbar\omega([\hat{a},\hat{b}])/2i. \tag{7.20}
$$

there exists a certain non-self adjoint operator

$$
\hat{z} = \hat{A} + i\lambda\hat{B} \tag{7.21}
$$

with $\hat{A},\hat{B}\in\mathcal{S}$ and a certain 'squeezing' paramter λ.

$$
\omega(z^*z)\geq 0, \qquad \hat{z} = \hat{A} + i\lambda\hat{B}, \tag{7.22}
$$

for all $\lambda > 0$, that is

$$
\begin{aligned}
\omega\big((\hat{A}+i\lambda\hat{B})^*(\hat{A}+i\lambda\hat{B})\big) &= \omega(\hat{A}^2)+\omega(\lambda^2\hat{B}^2)+\omega(i\lambda\hat{A}\hat{B}-i\lambda\hat{B}\hat{A}) \\
&= (\Delta a)^2+\lambda^2(\Delta b)^2+\lambda\omega(i[\hat{a},\hat{b}]) \\
&\geq 0
\end{aligned}
\tag{7.23}
$$

or

$$
(\Delta a)^2-\lambda\hbar\omega(i[\hat{a},\hat{b}])+\lambda^2(\Delta b)^2\geq 0. \tag{7.24}
$$

rewritten

$$\left(\lambda-\frac{\hbar\omega(i[\hat{a},\hat{b}])}{2(\Delta b)^2}\right)^2+\frac{(\Delta a)^2}{(\Delta b)^2}-\frac{\hbar^2\omega(i[\hat{a},\hat{b}])^2}{4(\Delta b)^4}\geq 0. \tag{7.25}$$

implying

$$(\Delta a)^2(\Delta b)^2\geq\hbar^2(\omega([\hat{a},\hat{b}])/2i)^2 \tag{7.26}$$

so for this ineqlality to hold for all λ implies the Heisenberg uncertainty relation (7.20), also that equality holds only if for some positive λ

$$\left(\frac{\hat{a}+i\lambda\hat{b}}{\sqrt{2\lambda\hbar}}\right)|\psi_0>=0. \tag{7.27}$$

For $\{q,p\}=i\hbar$

$$[q,p]/2i=\hbar/2. \tag{7.28}$$

the state

$$(\hat{q}+i\lambda\hat{p})|\psi_0>=0. \tag{7.29}$$

for $\lambda=1$ $|\psi>_0$ is the ground state (for other values the state is what's called a squeezed state).

This implies that the minimal uncertainty relation is saturated for the pair of elements $(\hat{a},\hat{b})$, i.e.,

$$\Psi_m([\hat{a}-\Psi_m(\hat{a})]^2)=\Psi_m([\hat{b}-\Psi_m(\hat{b})]^2)=\frac{1}{2}|\Psi_m([\hat{a},\hat{b}])|. \tag{7.30}$$

□

An important issue is whether the constraint operators have the correct semi-classical limit. This has to be done by using the kinematic semiclassical states in $\mathcal{H}_{Kin}$.

The physical Hilbert space must contain enough semiclassical states to guaratee that the quantum theory one obtains returns the correct classical theory when $\hbar\to 0$.

The semiclassical states in a Hilbert space should have the following properties

Given a class of observables $\mathcal{S}$ which comprises a subalgerba in the space $\mathcal{L}(\mathcal{H})$ of linear operators on the Hilbert space, a family of (pure) states $\{\omega_m\}_{m\in\mathcal{M}}$ are said to be semiclassical with respect to $\mathcal{S}$ if and only if

(1) The observables in $\mathcal{S}$ should have the correct semiclassical limit on semiclassical states and the fluctuations should be small, i.e.,

$$\begin{aligned} \lim_{\hbar\to 0}\left|\frac{\omega_m(\hat{a}) - a(m)}{a(m)}\right| &= 0, \\ \lim_{\hbar\to 0}\left|\frac{\omega_m(\hat{a}^2) - \omega_m(\hat{a})^2}{\omega_m(\hat{a})^2}\right| &= 0, \end{aligned}$$

for all $\hat{a}\in\mathcal{S}$.

• Coherent states for QGR, based on the general complexifier method, with built-in semiclassical properties.

• Polymer-like states for Maxwell theory and linearized gravity constructed by Varadarajan.

By making explicit use of the Minkowski background metric, he was able to construct an image of the usual Fock states on a distributional extension of the type of background independent Hilbert space $\mathcal{H}_0$ on which quantum general relativity currently is based. Varadarajan's states are complexifier coherent states.

7.6 Coherent States

7.6.1 Reminder of Coherent States

It is well known that coherent states provide a useful bridge between a classical theory and the corresponding quantum theory. Consider quantum mechanics of a particle on the real line, without specifying the potential. The basic observables are configuration and momentum, $\hat{X}$, $\hat{P}$, with

$$[\hat{X},\hat{P}] = i\hbar I$$

From these, we can build an annihilation operator

$$\hat{a} = \sqrt{\frac{\omega}{2\hbar}}\hat{X} + i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}$$

whose classical counterpart we denote by z:

$$z = \sqrt{\frac{\omega}{2\hbar}} X_0 + i\frac{1}{\sqrt{2\hbar\omega}} P_0.$$

Here (X_0, P_0) is a point in the classical phase space.

Let $(\hat{a}^\dagger)$ be the creation operator. If we set $N := a^\dagger a$ (the number operator), then

$$[N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \quad [a^\dagger, a] = -1. \tag{7.31}$$

Let H be a Fock space generated by a and $a^\dagger$. The actions of a and $a^\dagger$ on H are given by

$$\begin{aligned} a|n> &= \sqrt{n}|n-1> \\ a^\dagger|n> &= \sqrt{n+1}|n+1> \\ N|n> &= n|n> . \end{aligned} \tag{7.32}$$

$$|n> = \frac{(a^\dagger)^n}{\sqrt{n!}}|0> . \tag{7.33}$$

where the state $|0>$ is defined by $\hat{a}|0>= 0$. These states satisfy the orthogonality and completeness conditions

$$< m|n> = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n>< n| = I. \tag{7.34}$$

the following three conditions are equivalent

$$(i) \quad a|z> = z|z> \text{ and } < z|z> = 1 \tag{7.35}$$

$$(ii) \quad |z> = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n> = e^{-|z|^2/2} e^{za^\dagger}|0> \tag{7.36}$$

$$(iii) \quad |z> = e^{za^\dagger - \bar{z}a}|0> . \tag{7.37}$$

The equivalence of (ii) and (iii) follows from the *Baker-Campbell-Hausdorff formula*

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \tag{7.38}$$

which holds whenever $[A,[A,B]]=[B,[A,B]]=0$.

We prove that (ii) implies (i). We first consider

$$\hat{a}(z)=e^{-z\hat{a}^\dagger}\hat{a}e^{z\hat{a}^\dagger} \tag{7.39}$$

which obeys

$$\begin{aligned}\frac{\partial}{\partial z}\hat{a}(z) &= e^{-z\hat{a}^\dagger}(\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a})e^{z\hat{a}^\dagger}\\ &= e^{-z\hat{a}^\dagger}e^{z\hat{a}^\dagger}=1.\end{aligned} \tag{7.40}$$

Hence

$$\hat{a}(z)=\hat{a}(0)+z=\hat{a}+z. \tag{7.41}$$

Now from (ii) we have

$$|z>=e^{-|z|^2/2}e^{z\hat{a}^\dagger}|0>$$

then

$$\begin{aligned}\hat{a}|z> &= e^{z\hat{a}^\dagger}e^{-z\hat{a}^\dagger}\hat{a}e^{z\hat{a}^\dagger}|0>\\ &= e^{z\hat{a}^\dagger}(\hat{a}+z)|0>\\ &= z|z>.\end{aligned} \tag{7.42}$$

We give a prooof that (i) implies (ii). As the states $|n>$ form a complete set we can write

$$|z>=\sum_{n=0}^{\infty}c_n|n>. \tag{7.43}$$

From $\hat{a}|z>=z|z>$ we have

$$\sum_{n=0}^{\infty}c_n\hat{a}|n>=z\sum_{n=0}^{\infty}c_n|n>$$

or

$$\sum_{n=0}^{\infty} c_n \sqrt{n} |n-1>= z \sum_{n=0}^{\infty} c_n |n>$$

which can then be rewritten

$$\sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n>= z \sum_{n=0}^{\infty} c_n |n> .$$

So that

$$c_{n+1} = \frac{c_n}{\sqrt{n+1}} z.$$

If we set $c_0 = \mathcal{N}$ then

$$\begin{aligned}
c_1 &= c_0 z = \mathcal{N} z \\
c_2 &= \frac{1}{\sqrt{2}} c_1 z = \mathcal{N} \frac{1}{\sqrt{2}} z^2 \\
c_3 &= \frac{1}{\sqrt{3}} c_2 z = \mathcal{N} \frac{1}{\sqrt{3!}} z^3
\end{aligned}$$

etc. Then (7.43) becomes

$$|z>= \mathcal{N} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n> .$$

The condition $< z|z >= 1$ implies

$$\begin{aligned}
1 &= < z|z > \\
&= |\mathcal{N}|^2 \sum_{n,n'=0}^{\infty} \frac{z^{n+n'}}{\sqrt{n!}\sqrt{n'!}} < n|n' > \\
&= |\mathcal{N}|^2 \sum_{n,n'=0}^{\infty} \frac{z^{n+n'}}{\sqrt{n!}\sqrt{n'!}} \delta_{nn'} \\
&= |\mathcal{N}|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} = |\mathcal{N}|^2 e^{|z|^2}
\end{aligned}$$

so we have

$$|z>=^{-|z|^2/2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n> .$$

See also worked exercises on how (i) implies (ii).

Overcompleteness

It can be shown that (see worked exercises)

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z>e^{-z\overline{z}}<z^*| &= \sum_n\frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0><0|\frac{\hat{a}^n}{\sqrt{n!}}\\ &= \sum_n|n><n|. \end{aligned} \tag{7.44}$$

Therefore the coherent states form a complete set.

Let us compute the inner-product of two coherent states:

$$\begin{aligned}<z_1|z_2> &= e^{-\frac{|z_1|^2+|z_2|^2}{2}}\sum_{n,n'=0}^{\infty}\frac{\overline{z}_1^n z_2^{n'}}{\sqrt{n!n'!}}<n|n'>\\ &= e^{-\frac{|z_1|^2+|z_2|^2}{2}}\sum_{n=0}^{\infty}\frac{(\overline{z}_1 z_2)^n}{n!}\\ &= e^{-\frac{|z_1|^2+|z_2|^2}{2}+\overline{z}_1 z_2} \end{aligned} \tag{7.45}$$

and so

$$\begin{aligned}|<z_1|z_2>|^2 &= e^{-\left(\frac{|z_1|^2+|z_2|^2}{2}+\overline{z}_1 z_2+z_1\overline{z}_2\right)}\\ &= e^{-(\overline{z}_1+\overline{z}_2)(z_1+z_2)}\\ &= e^{-|z_1+z_2|^2}. \end{aligned} \tag{7.46}$$

So coherent states are not orthogonal. Making the coherent states an overcomplete basis.

Configuration space and momentum space representation:

In configuration space and momentum space representation:

$$\psi_z(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, \quad (7.47)$$

$$\psi_z(p) = \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \exp - \left\{ \frac{\hbar}{2\omega} \frac{(p - P_0)^2}{\hbar^2} + \frac{i}{\hbar} p X_0 \right\}. \quad (7.48)$$

substituting $t = \hbar/\omega$ this is rewritten

$$\psi_z(x) = \frac{1}{(\pi t)^{\frac{1}{4}}} \exp - \left\{ \frac{1}{2t}(x - X_0)^2 - \frac{i}{\hbar} x P_0 \right\}, \quad (7.49)$$

$$\psi_z(p) = \left(\frac{t}{\pi}\right)^{\frac{1}{4}} \exp - \left\{ \frac{t}{2}(p - P_0)^2 + \frac{i}{\hbar} p X_0 \right\}. \quad (7.50)$$

From these formulae we can read off the most important properties of coherent states: In both configuration and momentum representation the wavefunctions are Gaussian distributions, centered at

$$< \hat{X} >_{\Psi_z} = X_0, \quad \text{and} \quad < \hat{P} >_{\Psi_z} = P_0, \quad (7.51)$$

respectively. Furthermore we can see that the width of the distribution in the configuration representtion is inversely proportional to that on the momentum representation.

$$(\Delta \hat{q}_i)^2 \equiv < \Psi_\alpha | \hat{q}_i^2 | \Psi_\alpha > - [< \Psi_\alpha | \hat{q}_i | \Psi_\alpha >]^2 = \frac{1}{2} \ell_i^2,$$

$$(\Delta \hat{p}_i)^2 \equiv < \Psi_\alpha | \hat{p}_i^2 | \Psi_\alpha > - [< \Psi_\alpha | \hat{p}_i | \Psi_\alpha >]^2 = \frac{1}{2} \hbar^2 / \ell_i^2 \quad (7.52)$$

$$< \Psi_\beta | : F(\hat{a}_i^\dagger, \hat{a}_i) : | \Psi_\alpha > = F(\overline{\beta}_i, \alpha_j) < \Psi_\beta | \Psi_\alpha > \quad (7.53)$$

We come to an important property. However, we must first derive the result

$$e^{t\Delta/2} \delta_y(x) = \frac{1}{\sqrt{2\pi t}} e^{\frac{1}{2t}(x-y)^2}. \quad (7.54)$$

To do this we use the Dirac delta-function representation

$$\delta_y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)}.$$

Then note

$$\begin{aligned}
e^{t\Delta/2}\delta_y(x) &= e^{t\Delta/2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip\cdot y}\exp\left\{\frac{t}{2}\frac{d^2}{dx^2}\right\} e^{ip\cdot x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ip\cdot y}\left[1-\frac{t}{2}p^2+\frac{1}{2!}\left(\frac{t}{2}\right)^2 p^4-\dots\right] e^{ip\cdot x} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ip\cdot(x-y)}e^{-tp^2/2} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\frac{t}{2}\left\{p^2-\frac{2ip}{t}(x-y)\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp -\frac{t}{2}\left\{\left(p-\frac{i}{t}(x-y)\right)^2+\frac{1}{t^2}(x-y)^2\right\} \\
&= e^{-\frac{1}{2t}(x-y)^2}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-tp^2/2} \\
&= \frac{1}{\sqrt{2\pi t}}e^{-\frac{1}{2t}(x-y)^2}.
\end{aligned} \tag{7.55}$$

Now the coherent state can be obtained as an *analytic continuation* of the heat kernel:

$$\begin{aligned}
\left[\frac{1}{\sqrt{2\pi t}}e^{\frac{1}{2t}(x-y)^2}\right]_{y\to z}(x) &\propto \exp\left\{-\frac{1}{2t}\left(x-X_0-\frac{it}{\hbar}P_0\right)^2\right\} \\
&= \exp\left\{-\frac{1}{2t}\left[(x-X_0)^2-2\frac{it}{\hbar}(x-X_0)P_0-\frac{t^2}{\hbar^2}P_0^2\right]\right\} \\
&\propto \exp\left\{-\left[\frac{1}{2t}(x-X_0)^2-\frac{i}{\hbar}xP_0\right]\right\}
\end{aligned}$$

that is,

$$\psi_z(x) \sim \left[e^{-t\Delta/2}\delta_y(x)\right]_{y\to z}(x). \tag{7.56}$$

These eigenstates have many interesting properties.

Overcompleteness

There is a resolution of unity

$$1_{\mathcal{H}} = \int_{\mathcal{M}} d\nu(m)\psi_m \ < \psi, \cdot > \tag{7.57}$$

for some measure ν on $\mathcal{M}$.

Saturation of the Heisenberg uncertainty relation

For self-adjoint operators $\hat{x} := (\hat{g}+\hat{g}^\dagger)/2$, $\hat{y} := (\hat{g}-\hat{g}^\dagger)/2i$ uncertaintly relation is saturated

$$< (\hat{x}- < \hat{x} >_m)^2 >_m = < (\hat{y}- < \hat{y} >_m)^2 >_m = \frac{1}{2}| < [\hat{x}, \hat{y}] >_m | \tag{7.58}$$

Annihilation operator property

Peakedness in phase space

Ehrenfest theorems

7.6.2 Quantum Gravity Coherent states

One introduces coherent states for the harmonic oscillator as eigenvalues of the annihilation operator in terms of superposition of energy eigenstates. Here one had a preferred Hamiltonian (unlike quantum gravity) and the problem is straightforward because of the linearity of the system.

The construction of coherent states for full nonlinear, non-Abelean Quantum General Relativity with all the desired properties like overcompleteness, saturation of the Heisenberg uncertainty relation, peakedness in phase space (thus both connection and electric flux are well approximated), construction of annihilation and creation operators and corresponding Ehrenfest theorems.

They are non-normalizable.

Cut-off Coherent states

The coherent state ψ_m with respect to a finite graph as a graph dependent coherent state in $\mathcal{H}_{Kin}$. These are then normalizable, graph dependent states produced by the complexifier method as well.

For a graph γ containg edges $e_1, \ldots, e_N$ the coherent state is constructed by taking the tensor product for all edges,

$$\psi^t_{g_1,\ldots,g_N}(h_1, \ldots, h_N) = \prod_{n=1}^{N} \psi^t_{g_n}(h_n). \tag{7.59}$$

However, as cut-off coherent states are defined for each graph separately, for graph changing operators, the new graph generated has degrees of freedom upon which the coherent state does not depend and so whose fluctuations are not supressed. This makes it difficult to investigate the semmiclassical behaviour of graph changing operators like the Hamiltonian constraint.

Coherent states for the subalgebra of observables $\mathcal{S}$

Given a Hilbert space $\mathcal{H}$ for a dynamical system with constrints and a subalgebra of observables $\mathcal{S}$ in the space $\mathcal{L}(\mathcal{H})$ of linear operators on $\mathcal{H}$, the semiclassical states with respect to $\mathcal{S}$ are defined in ....??

7.6.3 The Complexifier Approach

Given a phase space $\mathcal{M} = T^*\mathcal{Q}$ for some dynamical system with configuration coordinates q and momentum coordinates p, a complexifier $C_q(p) = C(q, p)$, is a positive smooth function on $\mathcal{M}$, such that

(1) $C/\hbar$ is dimensionless;

(2) $\lim_{\|p\|\to\infty} \frac{|C(m)|}{\|p\|} = \infty$ for some suitable norm on the space of momentum;

(3) Certain complex coordinates $(z(m), \overline{z}(m))$ of $\mathcal{M}$ (given $z(m)$ and $\overline{z}(m)$ we can invert them to find real coordinates m for $\mathcal{M}$) can be constructed from C.

As a simple example, in the case of one-dimensional harmonic ascillator with Hamiltonian $H = \frac{1}{2}\left(\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2\right)$, with complexifier $C = p^2/(2m\omega)$, we illustrate the construction and obtain the usual harmonic ascillator coherent state up to phase factor.

Given a well-defined complexifier C on phase space Γ, we construct coherent states associated with C:

(i) Complex polarization

$$z(m) := \sum_{n=0}^{\infty} \frac{i^n}{n!}\{q, C\}_{(n)}(m). \tag{7.60}$$

For the simple harmonic oscialltor this is simply

$$z(m) = q + \frac{i}{m\omega}p \tag{7.61}$$

which together with the complex conjugate $\overline{z}(m) = q - ip/m\omega$ form complex coordinates $(z(m), \overline{z}(m))$ for the phase space $\Gamma = \mathbb{R}^2$.

(ii) Defining the annihilation operator

Given a suitable Hilbert space $\mathcal{H} = L^2(\mathcal{Q}, d\mu)$, we assume C can be defined as a positive self-adjoint operator $\hat{C}$ on $\mathcal{H}$. Then a corresponding operator $\hat{z}$ can be defined by replacing the Poisson brackets in (7.60) by commutators,

$$\hat{z} := \sum_{n=0}^{\infty} \frac{i^n}{n!}\frac{1}{(i\hbar)^n}[\hat{q}, \hat{C}]_{(n)}(m), \tag{7.62}$$

This can also be written as $e^{-\hat{C}/\hbar}\hat{q}e^{\hat{C}/\hbar}$ as can be proven by employing the quantity

$$e^{-\alpha\hat{C}/\hbar}\hat{q}e^{\alpha\hat{C}/\hbar} = \sum_{n=0}^{\infty} \hat{X}_{(n)}\frac{\alpha^n}{n!}. \tag{7.63}$$

It is obvious that $\hat{X}_{(0)} = \hat{q}$, as can be seen from setting $\alpha = 0$. In order to determine the other terms in the expansion, we will first prove by induction that

$$\frac{d^k}{d\alpha^k}(e^{-\alpha\hat{C}/\hbar}\hat{q}e^{\alpha\hat{C}/\hbar}) = \frac{1}{\hbar^k}e^{-\alpha\hat{C}/\hbar}[\hat{q}, \hat{C}]_{(k)}e^{\alpha\hat{C}/\hbar}. \tag{7.64}$$

holds for all $k \geq 1$. First assume (7.64) for fixed k and then differentiate both side with respect to α,

$$\begin{aligned}
\frac{d^{k+1}}{d\alpha^{k+1}}(e^{-\alpha\hat{C}/\hbar}\hat{q}e^{\alpha\hat{C}/\hbar}) &= \frac{d}{d\alpha}\left(\frac{1}{\hbar^k}e^{-\alpha\hat{C}/\hbar}[\hat{q},\hat{C}]_{(k)}e^{\alpha\hat{C}/\hbar}\right) \\
&= -\frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}\hat{C}[\hat{q},\hat{C}]_{(k)}e^{\alpha\hat{C}/\hbar}+\frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}\hat{C}[\hat{q},\hat{C}]_{(k)}\hat{C}e^{\alpha\hat{C}/\hbar} \\
&= \frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}([\hat{q},\hat{C}]_{(k)}\hat{C}-\hat{C}[\hat{q},\hat{C}]_{(k)})e^{\alpha\hat{C}/\hbar} \\
&= \frac{1}{\hbar^{k+1}}e^{-\alpha\hat{C}/\hbar}[\hat{q},\hat{C}]_{(k+1)}e^{\alpha\hat{C}/\hbar}. \qquad (7.65)
\end{aligned}$$

This shows that if the relation holds for k then it also holds for $k+1$. Obviously we have

$$\frac{d}{d\alpha}(e^{-\alpha\hat{C}/\hbar}\hat{q}e^{\alpha\hat{C}/\hbar}) = \frac{1}{\hbar}e^{-\alpha\hat{C}/\hbar}[\hat{q},\hat{C}]e^{\alpha\hat{C}/\hbar}$$

so the relation holds for $k=1$, which completes the proof that (7.64) holds for all $k \geq 1$. Now we differentiate both sides of (7.63) k times with respect to α and set $\alpha = 0$, and arrive at

$$\hat{X}_{(k)} = \frac{1}{\hbar^k}[\hat{q},\hat{C}]_{(k)},$$

Substituting this into (7.63) and setting $\alpha = 1$ gives

$$\hat{z} = \sum_{n=0}^{\infty}\frac{i^n}{n!}\frac{1}{(i\hbar)^n}[\hat{q},\hat{C}]_{(n)}(m) = e^{-\hat{C}/\hbar}\hat{q}e^{\hat{C}/\hbar}. \qquad (7.66)$$

For the simple harmonic oscialltor this is just

$$\hat{z} = \hat{q} + \frac{1}{m\omega\hbar}\hat{p}. \qquad (7.67)$$

which is the annihilation operator. $\hat{z}$ in (7.62) will be called the annihilation operator.

(iii) Constructing coherent states

Now, let $\delta_{q'}(q)$ be the delta function (distribution) on C with respect to the measure $d\mu$, i.e. $\int \delta_{q'}(q)f(q)d\mu = f(q')$.

One may analytically extend the variable q' in $e^{-\hat{C}/\hbar}\delta_{q'}(q)$ to complex values $z(m)$

$$\psi'_m(q) := [e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)}, \qquad (7.68)$$

such that one has

$$\begin{aligned}
\hat{z}\;\psi'_m(q) &:= e^{-\hat{C}/\hbar}\hat{q}e^{\hat{C}/\hbar}\;[e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \\
&= [e^{-\hat{C}/\hbar}\hat{q}\delta_{q'}(q)]_{q'\to z(m)} \\
&= [q'e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \\
&= z(m)\;[e^{-\hat{C}/\hbar}\delta_{q'}(q)]_{q'\to z(m)} \\
&= z(m)\psi'_m(q).
\end{aligned} \tag{7.69}$$

We define our coherent states $\psi_m(q)$ as the normalization of $\psi'_m(q)$.

We illustrate how to form an eigenstate of the annihilation operator of the simple harmonic oscillator. Let $\delta_{q'}(q)$ be the delta function (distribution) on $\Gamma = \mathbb{R}^2$. Consider

$$\psi'_m(q) := [e^{-\hat{p}^2/2m\omega\hbar}\delta_{q'}(q)]_{q'\to\hat{q}+i\hat{p}} \tag{7.70}$$

$$\begin{aligned}
\psi'_m(q) &:= [e^{-\hat{p}^2/2m\omega\hbar}\sum_n e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= [e^{-\hat{H}(\hat{q})/\hbar\omega}e^{\hat{q}^2}\sum_n e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= [e^{q^2}\sum_n e^{-E_n/\hbar\omega}e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= [e^{q^2}\sum_n e^{-(n+\frac{1}{2})}e_n(q')e^*_n(q)]_{q'\to\hat{q}+i\hat{p}} \\
&= e^{q^2}\sum_n e^{-(n+\frac{1}{2})}e_n(q+i\hat{p})e^*_n(q)
\end{aligned} \tag{7.71}$$

Constructing kinematic coherent states for LQG

The complexifier appraoch can be used to construct kinematic coherent states in loop quantum gravity. Given a suitable complexifier C, for each path $e \subset \Sigma$ one can define

$$A^C(e) := \sum_{n=0}^{\infty}\frac{i^n}{n!}\{A(e), C\}_{(n)}(m). \tag{7.72}$$

where $A(e) \in SU(2)$ is assigned to e.

the $\delta-$distribution on the quantum configuration space $\overline{\mathcal{A}}$ can be formally expressed as

$$\delta_{A'}(A) = \sum_s T_s(A')\overline{T_s(A)}$$

Thus by () we obtain the coherent states

$$\psi'_{A^C}(A) = \sum_s e^{-\tau\lambda_s} T_s(A^C)\overline{T_s(A)}. \tag{7.73}$$

Since this is an uncountable sum of an infinite number of terms, the norm of $\psi'_{A^C}(A)$ will be divergent. So $\psi'_{A^C}(A)$ does not belong to $\mathcal{H}_{Kin}$ but rather to the dual of a dense subset of $\mathcal{H}_{Kin}$.

A candidate complexifier C

A candidate complexifier C is considered in [??] such that the corresponding operator acts on on cylindrical functions f_γ by

$$(\frac{\hat{C}}{\hbar})f_\gamma = \frac{1}{2}(\sum_{e\in E(\gamma)} l(e)\hat{J}_e^2)f_\gamma, \tag{7.74}$$

where $\hat{J}_e^2$ is the Casimir operator

Complexification of a Lie Group

The complexification of a Lie group $\mathbf{g}$ is denoted $\mathbf{g}_{\mathbb{C}}$. Recall the defining properties of a Lie algebra: antisymmetric $[X,Y] = -[Y,X]$, bilinearity $[\alpha X, Y] = \alpha[X,Y] = [X, \alpha Y]$ for any real number α, and satisfies the Jacobi identity $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] \equiv 0$. $X_1 + iX_2$ where $X_1, X_2 \in \mathbf{g}$

$$[X_1 + iX_2, Y_1 + iY_2] = ([X_1, Y_1] - [X_2, Y_2]) + i([X_1, Y_2] + [X_2, Y_1]). \tag{7.75}$$

It is clear that (B.44) is *real* bilnear and skew-symmetric. If we prove that it is *complex* linear in the first factor, it will be complex linear in the second because of the skew-symmetry. As we already know it is real linear in the first factor, it suffices to show that it is *imaginary* linear. This is not difficult to prove, all we need to do is verify that

$$[i(X_1 + iX_2), Y_1 + iY_2] = i[X_1 + iX_2, Y_1 + iY_2] \tag{7.76}$$

is true. This is easily done by expanding each side and seeing they are indeed equal.

It remains to check the Jacobi identity. First consider Y and Z to be in $\mathbf{g}$ but take X to be in $\mathbf{g}_{\mathbb{C}}$. Now, $X = X_1 + iX_2$ is linear in the Jacobi identity and the Jacobi identity holds separately for X_1 and X_2,

$$[[X_1, Y], Z] + [[Z, X_1], Y] + [[Y, Z], X_1] + i([[X_2, Y], Z] + [[Z, X_2], Y] + [[Y, Z], X_2]) \equiv 0, \tag{7.77}$$

and so the Jacobi identity holds for $X \in \mathbf{g}_{\mathbb{C}}$ and $Y, Z \in \mathbf{g}$. Similarly for Y and Z. Therefore we have shown that the elements of the complexification $\mathbf{g}_{\mathbb{C}}$ satisfy the Jacobi identity.

The cut-off state of the corresponding coherent state,

$$\psi_{A^C,\gamma}(A) = \psi'_{A^C,\gamma}(A)/\|\psi'_{A^C,\gamma}(A)\|, \tag{7.78}$$

where

$$\psi'_{A^C,\gamma}(A) := \sum_{s,\gamma(s)=\gamma} e^{-\frac{1}{2}\sum_{e\in E(\gamma(s))} l(e) j_e(j_e+1)} T_s(A^C)\overline{T_s(A)}. \tag{7.79}$$

There are various choices for the complexier coherent states on a compact Lie group G are given by the different possible complexifier $\hat{C}$. We have seen one possible complexiefier: the negative Laplacian on G.

$$\psi_g^t(h) = \sum_\pi e^{l_\pi t/2} d_\pi tr\pi(gh^{-1}) \tag{7.80}$$

where we are summing over all irreducible finite-diemnsional representations π of G.

7.6.4 $U(1)$ Coherent states

Since all irreducible representations of $U(1)$ are know to be e^{-in} with $n \in \mathbb{Z}$, $d_\pi = 1$ and $l_\pi = n^2$ (7.80) becomes (for $n \to -n$).

$$\psi_z^t(\phi) = \sum_{n\in\mathbb{Z}} e^{-n^2 t/2} e^{-in(z-\phi)} \tag{7.81}$$

where $g = e^{iz}$ and $h = e^{i\phi}$. By the Poisson resummation theorem (see worked exercises), for a function

$$\psi(y+nt) = \sum_{n=-\infty}^{\infty} f(y+nt)$$

we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f(y+nt) &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{iny} \int_{-\infty}^{\infty} f(\tau) e^{-in\tau} \\
&= \frac{2\pi}{t} \sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{t}\right) \cdot \exp(2\pi i y n/t), \qquad (7.82)
\end{aligned}$$

where $\tilde{f}(k) = \int_{\mathbb{R}} (dx/2\pi) e^{-ikx} f(x)$. Now

$$\begin{aligned}
\exp(n^2 t/2 + yn) &= \exp(\frac{1}{t}(nt+y)^2/2) \exp(-y^2/2t) \\
&= \exp(-y^2/2t) f(nt+y) \qquad (7.83)
\end{aligned}$$

where $f(x) = \exp(x^2/2t)$. In our case $y = -i(z-\phi)$.

$$\begin{aligned}
\sum_{-\infty}^{\infty} \exp(ny) \exp(n^2 t/2) &= \exp(-y^2/2t) \sum_{-\infty}^{\infty} \exp((y+nt)^2/2t) \\
&= \exp(-y^2/2t) \, \frac{2\pi}{t} \sum_{n=-\infty}^{\infty} \tilde{f}(\frac{2\pi n}{t}) \cdot \exp(2\pi i y n/t) \qquad (7.84)
\end{aligned}$$

As

$$\tilde{f}(z) = \frac{1}{2\pi} \int f(x) e^{izx} dx = \frac{1}{2\pi} \int e^{-x^2/2t} e^{izx} dx = \sqrt{\frac{t}{2\pi}} e^{-z^2 t/2}$$

$$\tilde{f}(\frac{2\pi n}{t}) = \sqrt{\frac{t}{2\pi}} e^{-(2\pi n)^2 t/2} \qquad (7.85)$$

The Poisson summation formula (7.84) converts a term in the exponention of t to an exponation where $1/t$ appears instead of t. This allows us to investigate the classical limit, $t \to 0$.

$$
\begin{aligned}
\psi_z^t(\phi) &= e^{-(z-\phi)^2/2t}\ \frac{2\pi}{t} \sum_{n=-\infty}^{\infty} \sqrt{\frac{t}{2\pi}} e^{-(2\pi n)^2/2t} \ \cdot\ e^{-4\pi n(z-\phi)/2t} \\
&= \sqrt{\frac{2\pi}{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{(z-\phi-2\pi n)^2}{2t}}. \qquad (7.86)
\end{aligned}
$$

inner product

$$
\begin{aligned}
<\psi_g^t|\psi_{g'}^t> &= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} e^{-m^2t/2} e^{-n^2t/2}\ e^{im\overline{z}-inz'} \int \frac{d\phi}{2\pi} e^{i(-m+n)\phi} \\
&= \sum_{n=-\infty}^{\infty} e^{-n^2t} e^{in(\overline{z}-z')} \\
&= \sqrt{\frac{\pi}{t}} \sum_{n=-\infty}^{\infty} e^{-(\overline{z}-z'-2\pi n)^2/t}. \qquad (7.87)
\end{aligned}
$$

where again we have used the Poisson summation formula.

7.6.5 $SU(2)$ Coherent states

Coherent states are cylindrical states, obtained as a product of cylindrical over edges of the graph. In contrast to weave states, the functions of the edges are not eigenstates of the geometry, but are chosen to have good classical behaviour for both configuration and momentum degrees of freedom. Cylindrical functions on the edges are chosen as coherent states on $SU(2)$.

the coherent states of the simple harmonic oscillator coherent states can be obtained as analytic continuation of the heat kernel on $\mathbb{R}^n$:

$$
\psi_z^t(x) = e^{-t\Delta} \delta_{x'}(x)\big|_{x'\to z} \quad x \in, z \in \mathbf{C}, \qquad (7.88)
$$

the Laplacian Δ playing the role of a *complexifier*.

It was shown by Hall [242] that coherent states on a connected compact Lie group G can analogously be defined as an analytic continuation of the heat kernel

$$
\psi_g^t(x) = e^{-t\Delta_G} \delta_{h'}^{(G)}(h)\Big|_{h'\to u}, \qquad (7.89)
$$

to an element u of the complexification $G^{\mathbb{C}}$ of G.

the complexification of $SU(2)$ is $SL(2,\mathbb{C})$.

$$u = \exp[i\tau_j^j/2] \tag{7.90}$$

ψ_u^t is exponentially (Gaussin) peaked with respect to multiplication operator $\hat{h}$ on the group at the point h. The width of the peak is approximately given by $\sqrt{t}$,

ψ_u^t is Gaussian peaked with respect to the invariant vector fields at a point p/t in the associated momentum representation. The width of the peak is approximately given by $1/\sqrt{t}$.

Feature of $\mathcal{H}_{Diff}$ is separable as compared to $\mathcal{H}_\omega$: (can be made) separable! $\Longrightarrow$ Can construct *normalized* coherent states in $\mathcal{H}_{Diff}$ with support on all knot classes.
$\Longrightarrow$ Chance to take the classical limit on knot class changing operators just like in Varadarajans's polymer states do for the graph changing operators in his polymer version of Maxwell's theory.

We have to better understand mathematics of $\mathcal{H}_{Diff}$ (**generalized knot theory**)

Coherent states of a compact manifold

$$\begin{aligned} h_e^{\mathbf{C}} &:= g_e = \sum_{n=0}^{\infty} \frac{i^n}{n!}\{h_e, C\}_{(n)} \\ &= \sum_{n=0}^{\infty} \frac{i^n}{n!}(-{}_j^e\frac{\tau_j}{2})^n\ h_e \\ &= e^{-i\tau_j p_j^e/2} h_e \end{aligned} \tag{7.91}$$

polar decomposition.

$$\psi_g^t(h) = \sum_{j\in\frac{1}{2}\mathbb{N}} e^{-j(j+1)t/2}(2+1)tr_j(g^{-1}h) \tag{7.92}$$

with $g \in SU(2)^{\mathbb{C}} = SL(2,\mathbb{C})$. It has been shown in [243], [244], [**?**], that these states are sharply peaked around their labels $g \in SL(2,\mathbb{C})$ i.e. the overlap

$$\frac{|<\psi_{g_1}^t|\psi_{g_2}^t>|^2}{\|\psi_{g_1}^t\|^2\|\psi_{g_2}^t\|^2} \tag{7.93}$$

equals 1 for $g_1 = g_2$, and tends to zero faster that any power of t as $t \to 0$.

7.6.6 Gauge-invariant Coherent states for LQG

from [253], [254].

This is a first step to construct *physical* coherent states.

project corresponding complexifier coherent states to the gauge-invariant Hilbert space.

form (7.59) it is straightforward to see that it is not invariant under Gause-gauge transformations. One can write the projector onto the gauge-invariant Hilbert space

$$Pf(h_{e_1}, \ldots, h_{e_N}) := \int_{G^V} d\mu_H(U_1, \ldots, U_V)\, f(U_{b(e_1)} h_{e_1} U^{-1}_{f(e_1)}, \ldots, U_{b(e_N)} h_{e_N} U^{-1}_{f(e_N)}) \tag{7.94}$$

7.7 Relating Loop Representation to Fock-Space Description in the Low Energy Limit.

We now turn to the second approach. As we have seen, loop quantum gravity is based on quantum geometry, where the fundamental exctitations are one-dimensional polymer-like. On the other hand, low energy physics based on quantum field theories which are constructed in flat Minkowski spacetime.

The configuration variables are holonomies along curves in the spacial slices of the spacestime, the basic momentum variables are integrals of the triad filed over surfaces in the spacial slices of spacetime. This is in contrast to ordinary background dependent quantum field theories, where both the basic configuration and momentum variables are three-dimension fields in the spacial slices of the spacetime.

We must understand how to recover background-dependent ordinary QFT in order to make contact with low energy physics. How can the techniques of $\overline{\mathcal{A}}, \mu_0$ can be used to describe the Fock states of Maxwell theory and linearized gravity on a Minkowski background spacetime?

7.7.1 Polymer representation of Maxwell's Field

The space $\overline{\mathcal{A}}$ carries a diffeomorphism and gauge invariant measure μ_0 induced by the Haar measure on $U(1)$, which gives rise to the Hilbert space $\mathcal{H}_0 := L^2(\overline{\mathcal{A}}, \mu_0)$, of polymer states.

the set of cylindrical functions

$$\mathcal{N}_{\alpha,\vec{n}}(A) = [h(e_1)]^{n_1}[h(e_2)]^{n_2}\cdots[h(e_N)]^{n_N} \tag{7.95}$$

Fock space:

• 3-dimensional excitations,

• Basic operators are the connection $\hat{A}_a(x)$ and ‘electric field” $\hat{E}$,

• Holonomies do have well defined operator in the Fock space.

Quantum geometry:

• One dimensional excitations,

• Basic operators are holonomies h_A and the “electric field” E^a,

• The connection operator is not well defined on this representation.

$\hat{A}(e)$ fails to be well defined. Introduce a test function using the Euclidean background metric on $\mathbb{R}^2$,

$$f_r(x,y) = \frac{1}{(2\pi)^{3/2}}\frac{e^{|x-y|^2/2r^2}}{r^3} \tag{7.96}$$

approximates the Dirac delta function for small r. The Gaussian smeared form factor for an edge e is defined

$$X^a_{\gamma(r)}(\vec{x}) = \int_e ds f_r(\vec{e}(s)-\vec{x})\dot{e}^a. \tag{7.97}$$

Then, the smeared holonomy is defined as

$$H_{\gamma(r)}(A) := \exp i\int_{\mathbb{R}^3} X^a_{\gamma(r)}(x)A_a(x)d^3x. \tag{7.98}$$

where $A_a(\vec{x})$ is the $U(1)$ connection one-form of the Maxwell field on Σ. Similarly one can define Gaussian smeared electric fields by

$$E_{\gamma(r)}(g) := \int_{\mathbb{R}^3} g_a(\vec{x})d^3x\int_{\mathbb{R}^3} f_r(\vec{y}-\vec{x})E^a(\vec{y})d^3y. \tag{7.99}$$

Poisson bracket algebra:

We are given that

$$\{A_a(x), E^b(y)\} = \frac{1}{q_0}\delta^b_a \delta(x, y) \tag{7.100}$$

$$\begin{aligned} \{H_{\gamma(r)}, H_{\alpha(r)}\} &= \{E^a(x), E^b(y)\} = 0, \\ \{H_{\gamma(r)}, E^a(x)\} &= \frac{1}{q_0} X^a_{\gamma(r)}(x) H_{\gamma(r)}. \end{aligned} \tag{7.101}$$

we define the classical Gaussian smeared electric field $E^a_r(x)$ by

$$E^a_r(x) := \int d^3y f_r(y - x) E^a(y). \tag{7.102}$$

The Poisson bracket algebra generated by the (unsmeared) holonomies and the Gaussian smeared electric field is

$$\begin{aligned} \{H_\gamma, H_\alpha\} &= \{E^a_r(x), E^b_r(y)\} = 0, \\ \{H_{\gamma(r)}, E^a_r(x)\} &= \frac{1}{q_0} X^a_{\gamma(r)}(x) H_\gamma. \end{aligned} \tag{7.103}$$

the **r-Fock representation**

In the r-Fock representation the holonomies are *well defied operators* and so we can

The relation betwen the Fock and r-Fock representations. There is an isomorphism between them,

$$I_r : (A_{(r)}(e), E(g)) \mapsto (A(e), E_{(r)}(g)). \tag{7.104}$$

For a measurement at a given length scale there always exists a sufficiently small r such that the prediction from r-Fock theory is experimently indistinguishable from that of usual Fock theory.

The image of the Fcok vacuum can be shown to be the following element of Cyl^*,

$$(V| = \sum_{\alpha,\vec{n}} \exp\big(-\frac{\hbar}{2}\sum_{IJ} G_{IJ} n_I n_J\big)(\mathcal{N}_{\alpha,\vec{n}}|, \tag{7.105}$$

where $(\mathcal{N}_{\alpha,\vec{n}}| \in Cyl^*$ maps the flux network function $|\mathcal{N}_{\alpha,\vec{n}} >$ to one and every other flux network functions to zero.

$$G_{IJ} = \int_{e_I} dt\ \dot{e}^a_I(t) \int_{e_J} dt'\ \dot{e}^b_J(t') \int d^3x \delta_{ab}(\vec{x})[f_r(\vec{x} - \vec{e}_I(t))|\Delta|^{-1/2} f(\vec{x}, \vec{e}_J(t'))], \quad (7.106)$$

where δ_{ab} is th eflat Euclidean metric and Δ its Laplacian. Therefore, one can single out the Fock vacuum state directly in the polymer representation.

the action of the Fock vacuum $(V|$ on $(\mathcal{N}_{\alpha,\vec{n}}|$ reads

$$(V|\mathcal{N}_{\alpha,\vec{n}} > = \int_{\overline{A}_\alpha} d\mu^0_\alpha \overline{V}_\alpha \mathcal{N}_{\alpha,\vec{n}}, \quad (7.107)$$

where the state V_α is in the Hilbert space $\mathcal{H}_\alpha$ for the graph α and given by

$$V_\alpha(A) = \sum_{\vec{n}} \exp[-\frac{\hbar}{2} \sum_{IJ} G_{IJ} n_I N_J] \mathcal{N}_{\alpha,\vec{n}}(A). \quad (7.108)$$

Thus for any cylindrical functions φ_α associated with α,

$$(V|\varphi_\alpha > = < V_\alpha|\varphi_\alpha >, \quad (7.109)$$

where the inner product in the right hand side is taken in $\mathcal{H}_\alpha$. $V_\alpha(A)$ are refered to as "shadows" of $(V|$ on the graphs α.

7.8 Minkowskian Spacetime and Scattering Amplitudes

In the traditional in the traditional perturbative framework, the gravitational attration between two point masses aries from an exchange of virtual gravitons, described by the Feynman propagator.

Loop quantum gravity has no background metric. Therefore one cannot even begin to calculate begin to calculate the the propagator along these traditional lines.

the problem of deriving scattering amplitudes is still an open one. formulism available yet for deriving particle's scattering amplitudes from boundary amplitudes.

The aim of a semiclassical analysis would be to show that, for suitable choices of initial and final states, the transition amplitudes of LQG reduce to the transition amplitudes of an effective field theory on a background geometry. If this effective field theory contained gravity, this would be the prove that LQG is indeed a quantum theory of gravity.

7.8.1 Quantum states Representing Minkowskian Spacetime

dynamically generated quantum geometry that acts as a background geometry around which small quantum fluctuations take place.

corresponding object in the loop representation.

$$\Psi_{Mink}(A) = \sum_{\gamma} \Psi_{Mink}[\gamma] \, \overline{W}_{\gamma}[A] \tag{7.110}$$

where $\Psi_{Mink}[\gamma]$ are invariants for the flat-space vacuum.

7.8.2 Scattering Amplitudes

In the standard formulism on associates A Hilbert space of states with each time-slice of a global foliation of space-time. An evolution takes place between two such time-slices and is represented by a unitary operator. Associated with states in the two such time-slices is a transition amplitude, whose modulus square determines the probability of finding the final state given that the initial one was prepared.

$$\mathcal{A}_{12} =< \psi_{int}|\hat{U}(t_1,t_2)|\psi_{fin} >, \quad \text{where } < \psi_{int}| \in \mathcal{H}^* \text{ and } |\psi_{fin} >\in \mathcal{H}. \tag{7.111}$$

$$\mathcal{A}_{12} = \int d^3x \int d^3y \; \psi^*_{int}(x,t_{int}) W(x,y) \psi_{fin}(y,t_{fin}) \tag{7.112}$$

Just as GR doesn't determine the distance between spacetime points, it doesn't determine this probability; the only way to preserve general covariance is if $W(x,y)$ is constant.

Transition amplitudes are associated with regions of spacetime and states are associated with their boundaries.

The boundary value of the gravitational field determine the geometry of the boundary surface Σ.

Scattering probabilities are determined internally. Scattering amplitude and the spacetime geometry both encoded in the state. Reflects the fact that there are no *external* reference bodies.

Introduce an S Cauchy data (q_{ab}, K^{ab}) induced by a flat Euclidean 4-metric. By 'evolving' this data via Einstein's equation one would recover an Euclidean 4-metric g_{ab} in the interior of S. However, this 4-metric, of course, will be unique up to active diffeomorphisms (which are identity at the boundary S). Here we have used the reference systems defining the region of the space, to do a partial gauge fixing. The propagator is then obtained by fixing points x, x' on the boundary S and summing over all configurations in the interior bulk which agree with the boundary data (q_{ab}, K^{ab}). In the quantum then one fixes a LQG boundary state $\Psi_{q,k}(s)$ peaked at some flat space initial data (q_{ab}, K^{ab}), where s is a (diffeomorphism equavalence class of) spin network(s) on the 3-sphere S.

Wave function on time-slice

A wave function at constant "time" has no physical meaning in a generally covariant setting.

$$\mathcal{H}^* \otimes \mathcal{H} \tag{7.113}$$

in which no reference is made to infinitely extended spacial surfaces. it suggests a way to derive particles' scattering amplitudes from a spinfoam model.

$$W[\varphi, \Sigma] = W[\varphi] \tag{7.114}$$

$$\Psi_M[s] =< s|0_M >= \lim_{T\to\infty} \int \mathcal{D}\Phi f_{s\#s_T}[\Phi] e^{-S[\Phi]}. \tag{7.115}$$

$$W[s] = \int \mathcal{D}\Phi f_s[\Phi] e^{-S[\Phi]}, \tag{7.116}$$

The spin foam polynomial is defined as

$$f_s[\Phi] = n \int dg_{n_1} \ldots dg_{n_4} D_\alpha \beta_{n_1} \ldots C^{i_n}_{\beta\ldots\beta} l \delta^{l_1 l_2} \tag{7.117}$$

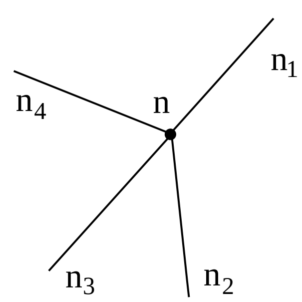


Figure 7.3: node.

7.9 Semiclassicallity from Spin Foams

large spin limit

The vertex is dominated in the large spin limit by semiclassical states satisfying the closure condition for each tetrahedron and the relation adjacent tetrahedra in the same 4-simplex.

7.10 The Infinite Tensor Product Extension

Quantum field theory on curved spacetimes is best understood if the spacetime is flat Minkowski space on the manifold $\mathcal{M} = \mathbb{R}^4$. Thus, when one wants to compute the low-energy limit of canonical Quantum General Relativity to show that one gets the standard model (pluss corrections) on a background metric one should do this first for the Minkowski background metric. In order that we can define a semiclassical limit for all the initial value data slice σ we must necessarily work with at least countably infinite embeded graphs.

However, the Hilbert spaces used in LQG have as dense subspace the space of cylindrical functions labelled by graphs with a finite number of edges.

When the number of edges of graphs are infinite, it turns out, that a much larger Hibert space is required. Actually the construction ofthe appropriate structure was already developed by von Neumann more that 60 years ago, know as the Infinite Tensor Product (ITP).

7.10.1 Von Neumann's Infinite Tensor Product (ITP)

We first consider the tensor product of a finite number of Hilbrt spaces. Say $f_k, g_k \in \mathcal{H}_k$ with inner product $(\cdot,\cdot)_k$ on $\mathcal{H}_k$. If an element of $\otimes_k \mathcal{H}_k$ is f, the inner product of the tensor product is defined as

$$(f, g) = \prod_{k=1}^{n} (f_k, g_k)_k$$

and the norm

$$\|f\| = \sqrt{\prod_{k=1}^{n} (f_k, f_k)_k} = \sqrt{\prod_{k=1}^{n} \|f_k\|_k^2} = \prod_{k=1}^{n} \|f_k\|_k.$$

We are lead to the consideration of the mathematics of products of arbitrary complex numbers.

Now when one forms the infinite tensor product of a collection of Hilbert spaces, a physical requirement is that this product must not depened on the order of the individual Hilbert spaces (whether the collection is countably or uncountably infinite).

Hence we are interested in the convergence properties of a countable or uncountable product of complex numbers which are independent of the ordering of the product. This was developed in the paper by von Neumann (availble at http://www.numdam.org/item?id=1939 0).

As we will see, convergence of products is related to convergence of corresponding summations. Now, it is a remarkable fact that whether an infinite series converges or not can depend on the ordering of the terms of that series. From which it follows that whether or not the coresponding product of complex numbers converges depends on the ordering of the terms of that product.

Absolutely and conditionally convergent series have completely different behaviours under rearrangement

Theorem 7.10.1 *(***Riemann's Rearrangement Theorem***) Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. For each real number s, there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ that converges and has sum s.*

Proof:

The nonnegative series $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty}(-q_n)$ diverge. In fact, if both were to converge, it would follow that $\sum_{n=1}^{\infty} |a_n|$ converges, that is, $\sum_{n=1}^{\infty} a_n$ would be absolutely convergent. On the other hand, if one of these series converged and the other diverged, it would follow that the partial sums of $\sum_{n=1}^{\infty} a_n$ diverge to either $+\infty$ or to $-\infty$. The convergence of $\sum_{n=1}^{\infty} a_n$ itself implies that both $\{p_n\}$ and $\{q_n\}$ have limit zero.

Now we construct the rearrangement. Choose terms $p_1, p_2, \dots$ up to the first index k_1 such that

$$p_1 + p_2 + \cdots + p_{k_1} > s.$$

This will occur , because $\sum_{n=1}^{\infty}$. Note

$$|p_1 + p_2 + \cdots + p_{k_1} - s| < p_{k_1}.$$

Next, we choose $q_1, q_2, \dots$ up to the first index l_1

such that

$$(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) < s.$$

Note

$$|(p_1 + p_2 + \cdots + p_{k_1}) + (q_1 + q_2 + \cdots + q_{l_1}) - s| < \max\{p_{k_1}, |q_{l_1}|\}.$$

We then add just enough new p's to make the left hand side greater than s, followed by just enough q's to make it less than s, and continue. At each phase of the $2n-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_n}, |q_{l_n}|\}$, and at each phase of the $2n+1-$th step, the difference between s and the partial sum of the new series has absolute value smaller than $\max\{p_{k_{n+1}}, |q_{l_n}|\}$. As these have the limit 0, the rearranged series has sum s.

□

Corollary 7.10.2 *Suppose that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series. It has a rearrangement that diverges to $+\infty$.*

Proof:

□

Theorem 7.10.3 *Suppose that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and that $\sum_{n=1}^{\infty} b_n$ is a rearrangement. Then $\sum_{n=1}^{\infty} b_n$ converges and has the same sum as $\sum_{n=1}^{\infty} a_n$.*

Proof: Let $\{s_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} a_k$, i.e,

$$s_n = \sum_{k=1}^{n} a_k,$$

and let s be the limit. Let $\{t_n\}$ be the sequence of partial sums of $\sum_{k=1}^{\infty} b_k$. Given $\epsilon > 0$, choose M so large that

$$\sum_{k=M+1}^{\infty} |a_k| < \epsilon/2. \tag{7.118}$$

It follows from this that $|s - s_M| < \epsilon/2$,

$$|s - s_M| = |\sum_{k=M+1}^{\infty} a_k| \leq \sum_{k=M+1}^{\infty} |a_k| < \epsilon/2.$$

Choose N so large that every one of the first M terms of $\{a_k\}$ occurs among the first N terms of $\{b_k\}$. So that for any $n \geq N$

$$|t_n - s_M| = |t_n - \sum_{k=1}^{M} a_k| \leq \sum_{k=M+1}^{\infty} |a_k|.$$

Therefore

$$|t_n - s| \leq |t_n - s_M| + |s_M - s| < \epsilon/2 + \epsilon/2 = \epsilon.$$

□

The inifinite product

$$\prod_e z_e$$

of complex numbers $z_e = |z_e| e^{i\varphi_e}$ is defined by

$$\prod_e z_e := \left[\prod_e |z_e|\right] \, e^{i \sum_e \varphi_e}$$

$\varphi_e \in [-\pi, \pi)$ provided that both

i) $\sum_e \big||z_e| - 1\big|$

ii) $\sum_e |\varphi_e|$

converge, in which case we also say that $\prod_e z_e$ is convergent. One would naively expect the product to converge if $\sum_e \varphi_e$ converged - this is a consequence of the defintion of convergence which is indepndent of ordering. too vehement oscillation of the φ_e. We say that $\prod_e z_e$ is quasi-convergent if $\prod_e |z_e|$ converges, and asign to $\prod_e z_e$ the value zero.

Two vectors $\otimes_f$, $\otimes_{f'}$ are said to be strongly equivalent if and only if

$$\sum_e |(f_e, f'_e)_{\mathcal{H}_e} - 1|$$

converges. We denote by $[f]$ the strong equivalence class of f. It follows that

$$(\otimes_f, \otimes_{f'}) = 0$$

if either $[f] \neq [f']$ or $[f] = [f']$ and $(f_e, f'_e)_{\mathcal{H}_e}$ for at least one e.

If we set

$$(z \cdot f)_e := z_e f_e$$

then the product formula

$$\otimes_{z \cdot f} = (\prod_e z_e) \otimes_f$$

fails to hold if $\prod_e z_e$ is (quasi-convergent but) not convergent. We say that f, f' are weakly equivalent provided that there exists z such that

$$[z \cdot f] = [f'].$$

This is equivalent to the convergence of

$$\sum_e \left| |(f_e, f'_e)_{\mathcal{H}_e}| - 1 \right|$$

We denote by (f) the weak equivalemce class of f. Obviously, strong equivalence implies weak equivalence. One can show that the closure of the linear span of all vectors in the same strong equivalence class $[f]$, denoted by $\mathcal{H}^{\otimes}_{[f]}$ is separable, consisting of the completion of the finite linear span of the vectors of the form $\otimes_{f'}$ where $f'_e = f_e$ for all but finitely many e. The ITP Hilbert space $\mathcal{H}^{\otimes}$ is the direct sum of the $\mathcal{H}^{\otimes}_{[f]}$.

Let also $\mathcal{H}^{\otimes}_{(f)}$ be the closure of the finite linear span of the $\otimes_{f'}$ with $(f') = (f)$. Then the strong equivalence subspaces of $\mathcal{H}^{\otimes}_{(f)}$ are unitarily equivalent, the corresponding unitary operators being of the form $U_z \otimes_f := \otimes_{z \cdot f}$ with $\prod_e z_e$ quasi-convergent.

A state f in the infinite tensor product Hilbert space which is a direct product of normalized states, one for each edge of the graph, generates so-called strong and weak equivalence classes of so-called C_0-sequences. It turns out that the corresponding C_0-vector plays the role of a cyclic vector (vacuum state) for a Fock-like tiny closed subspace of the complete ITP Hilbert space.

Those Fock-like spaces that correspond to the same weak class but different strong classes are unitarily equivalent while those that correspond to different strong and weak classes are unitarily inequivalent. This way the ITP gives rise to an uncountably infinite number of mutually unitarily inequivalent representations of the canonical commutation relations.

Infinite spin chain.

Now suppose our system becomes infinitely large. The dimension of the system will be infinite, of course, but it will be a larger infinity than those to which we are used - specifically it will be $2^{\aleph_0}$, the cardinality of the continuum, which is strictly larger than the cardinality $\aleph_0$ of the integers.

It follows that systems with infinitely many components have a Hilbert space which is non-separable (i.e., has uncountable dimension). To see the consequences of this, consider the operator algebra of our set of two-state systems. It consists of the set of linear combinations of spin operators, and hence has countably many linearly independent elements. The action of this algebra on any given state will generate only countably many linearly independent states, hence the action of the operator algebra on the total, non-separable space is highly reducible.

Our label set will be the integers $I = \mathbb{Z}$ and for each $n \in Z$ we have the Hilbert space $\mathcal{H}_n = \mathbb{C}^2$ with standard inner product

$$< f_n, f_n' >_n = \overline{f}_n^+ f^{+\prime}_n + \overline{f}_n^- f_n^{-\prime}.$$

In each Hilbert space we have the standard orthogonal basis of vectors $e_n^\pm$ and spin operators $\sigma_n = \sigma_3$ (Pauli matrix) so that $\sigma_n e_n^\pm = \pm e_n^\pm$ corresponds to spin up/down. We also have ladder operators $\sigma_n^\pm = \frac{1}{2}[\sigma_1 \pm \sigma_2]$ so that $\sigma_n^\pm e_n^\pm = 0$ and $\sigma_n^\pm e_n^\mp = e_n^\pm$. Consider the positive semi-definite, self-adjoint Hamiltonian

$$\mathcal{H} := \frac{1}{2}\sum_n [1 + \sigma_n] = \sum (\sigma_n^-)^\dagger \sigma_n^-$$

on the ITP Hilbert space

$$\mathcal{H}^\otimes = \otimes_n \mathcal{H}_n$$

which is non-separabe even though each $\mathcal{H}_n$ has finite dimension two.

We will first consider a C_0-vector $\otimes_f$ with $\|f_n\|_n = 1$ and a second one $\otimes_{f'}$ with $f_n = -f_n$. Are the corresponding C_0-sequences in the same strong (weak) eqivalence class? Since $(f_n, f_n)_n = -1$ we see that

$$\sum_n |(f_n, f_n)_n - 1| = \sum_n 2 = \infty$$

but

$$\sum_n \big||(f_n, f_n)_n| - 1\big| = \sum_n 0 = 0,$$

thus they are in different strong classes within the same weak class. In fact, the unitary operator $\hat{U}$ on $\mathcal{H}^{\otimes}$ defined densely on arbitrary C_0-vectors by $\hat{U}\otimes_g = \otimes_{g'}$ with $g_n = -g_n$ maps the two unit C_0-vectors into each other and thus the strong equivalence class Hilbert spaces built from them will be unitarily equivalent subspaces of the whole ITP. Notice that indeed $(\otimes_f, \otimes_{f'}) = \prod_n (-1)^n = 0$ since the product of numbers $z_n = -1$ is only quasi-convergent.

To see what these sectors are, suppose we start with all components having spin up. Then the action of any element of the algebra can, at most, cause finitely many components to have spin down. So no amount of algebraic action can transform such a state into one in which, say, every second component has spin up. This state, in turn, can be transformed into other states differing from it in finitely many places, but not into a state in which all components are spin down... or every third component is spin down... or where half the states are spin up but the spin-up states are grouped in pairs...

7.11 Emergent Coherent Excitations as Noiseless Subsystems

A few of preliminaries first

Open quantum system

We have a complete quantum experiment, which we want to divide into the system S and environment $\mathcal{E}$. We want to understand what quantum properties of the system may survive stably in spite of continual and uncontrollable interactions with the environment. The joint Hilbert space decomposes into the product of system and environment,

$$\mathcal{H}^{total} = \mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{E}} \tag{7.119}$$

while the Hamiltonian decomposes into the sum

$$H = H^{\mathcal{S}} + H^{\mathcal{E}} + H^{int} \tag{7.120}$$

where $H^{\mathcal{S}}$ acts only on the system, $H^{\mathcal{E}}$ acts only on the environment and all the interactions between them are contained in H^{int}.

We are interested in the state of the system alone, and want to disregard the state of the environment. If total is the state of the whole system + environment, then the state of the system alone is

$$\begin{aligned} \rho_{sys} &= Tr_{env}\left[\rho_{total}\right] \\ &= \sum_{env} < env|\rho_{total}|env > . \end{aligned} \tag{7.121}$$

Noiseless sub-systems

When an environment has a symmetry to it, there exists a subspace of the Hilbert space which is protected against decoherence effect of the environment - this is called a noiseless subsystem.

• Noiseless Subsystems are useful for describing long-term behaviour of the system because they are conserved.

• If we divide the quantum gravitational field into subsystems, those properties that are **conserved** under interactions between subsystems will characterize the low energy-limit of spacetime geometry.

• The commutant $A^{int\prime}$ should include the symmetries that characterize classical spacetime (e.g. Poincare).

Particles in quantum geometry

It is argued that a particle could be some kind emergent excitation, some subsystem, of microscopic quantum dynamics. The quantum spacetime, being dynamical, is constantly changing. In order for the emergent excitation to behave as if it were a particle moving through a fixed, non-dynamical background spacetime it must be a noiseless sub-system. So Markopoulou identified emergent particles as noiseless subsystems of quantum geometry corresponding to a Lorentz invariance symmetry at the appropriate scale.

7.11.1 The Standard Model - a Reminder

Matter

The Standard Model fermion are the leptons and quarks

(i) Leptons:

$$\begin{pmatrix} e \\ \nu_e \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu_\mu \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \nu_\tau \end{pmatrix} \tag{7.122}$$

The electron, muon and tau electron all have $-1e$, their anti particles have charge $+1e$, but the neutrinos are all uncharged.

They form doublets under $SU(2)$

(ii) Quarks:

$$\begin{pmatrix} u \\ d \end{pmatrix}, \quad \begin{pmatrix} c \\ s \end{pmatrix}, \quad \begin{pmatrix} t \\ b \end{pmatrix} \tag{7.123}$$

Left handed fermions in the Standard Model

Fermion left-handed	symbol	spin	electric charge	colour charge	mass
electron	e	1/2	-1	1	0.510999 Mev
electron neutrino	ν_e	1/2	0	1	~ 0 ev $< 2ev$ Mev
positron	e^c	1/2	$+1$	1	0.510999 Mev
electron antineutrino	μ_e^c	1/2	0	1	~ 0 ev < 2 ev
up quark	u	1/2	$+2/3$	3	~ 3 Mev
down quark	d	1/2	$-1/3$	3	~ 6 Mev
anti-up quark	u^c	1/2	$-2/3$	$\overline{3}$	~ 3 Mev
anti-down quark	d^c	1/2	$+1/3$	$\overline{3}$	~ 6 Mev

Forces

(i) Forces mediated by bosons (spin $0, 1, 2, \dots$)

(ii) The electromagnetic force is mediated by the photon (γ)

(iii) The weak force is mediated by the $W^{\pm}$ (posite, negative charge) and Z^0 (uncharged) bosons

(iv) The strong force is mediated by gluons (g)

Discrete symmetries

$$
\begin{aligned}
\mathcal{P} &: \mathbf{x} \rightarrow -\mathbf{x} \\
\mathcal{C} &: e \rightarrow -e \\
T &: t \rightarrow -t
\end{aligned} \tag{7.124}
$$

Parity transfomation:

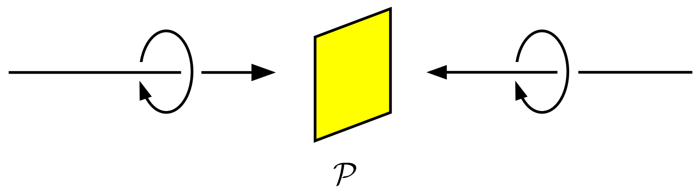


Figure 7.4: parityPartF. The operation $\mathcal{P}$ reverses the momentum of the particle whithout flipping its spin.

Charge conjugation $\mathcal{C}$: is the particle-antiparticle symmetry. Charge conjugation is defined to take a fermion with a given spin orientaion into an antifermion with the same spin orientaion.

• $\mathcal{P}$ converts a left-handed electron into a right-handed electron,

and

• $\mathcal{C}$ converts a left-handed electron into a left-handed positron.

the combination of these two operations interchanges left-handed particles with right-handed antiparticles.

zero-mass fermions

A wave function of negative energy and momentum $-\mathbf{p}$ corresponds to an anti-fermion with positive energy and momentum $\mathbf{p}$. Zero mass fermions and anti-zero mass ferminons with positive chirality both also have helicity (they are "right-handed"). Similarly, fermions and anti-fermions with negative chirality both barry negative helicity (they are "left-handed").

Although strong and electromagnetic forces make no distinction between right-handed or left-handed particles (particle invariance), particles subject to weak forces do make this distinction. (A right-handed particle is a particle spinning in the direction the right-hand fingers curl when the particle is traveling in the direction pointed-to by the right thumb). Thus, left-handed neutrinos are matter, whereas right-handed neutrinos are antimatter.

Gauge symmetry group of fermions

$SU(3) \times SU(2) \times U(1)$

Electroweak

iso-spin doublets

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_e \\ e \end{pmatrix} = \tag{7.125}$$

Strong

What is conserved in the Standard Model

(i) Energy, momentum, but not mass

(ii) Angular momentum

(iii) Charge and colour

(iv) Lepton flavour number (and therefore lepton number)

(v) Quark number (but not quark flavour number)

(vi) Charge conjugation ($\mathcal{C}$), parity ($\mathcal{P}$), and time reversal (T) are conserved by the strong and electromagnetic interactions, but not by the weak interaction ($\mathcal{CP}T$ is always conserved).

7.11.2 Conventional Preon Models and their Problems

Unlike hadrons, the light quarks and leptons are much lighter than the inverse of the largest experimentally allowed binding scale for their proposed subcomponents. This made it challenging to bind such hypothetical subcomponents by means of ordinary gauge interactions.

The binding mechanism proposed here operates at Planck scales, below the scales at which effective field theory would be a good description. The states are bound here, not by fields, but by quantum topology, because the configurations that we interpret as quarks and leptons are conserved under the dynamics of the quantum geometry.

7.12 The Standard Model from Loop Qunatum Gravity?

7.12.1 Introduction

These theories already contain elementary particles which can be identified with particles of the standard model so that the symmetries of the standard model are symmetries of the dynamics of the quantum geometry.

When they are continually in interaction with the quantum fluctuations of the microscopic theory - they are protected by symmetries in the dynamics.

7.12.2 The Standard Model from Loop Qunatum Gravity?

Preons:

"Quarks, leptons and heavy vector bosons are suggested to be composed of a stable spin-1/2 preons, existing in three flavours, combined according to simple rules."

Now onto the emergence of (part) of the Standard Model:

Sundance O. Bilson-Thompson (who was working on preons, not quantum gravity) noticed that a simple braiding of ribons that captured precisely the structure of the preon models of particle physics.

From this insight of Bilson-Thompsons', Markopoulou that the different ways of braid and knot the edges of graphs in a quantum spacetime must be different kinds of elementary particles!!! - So loop quantum gravity is not just about quantum spacetime - it already has elemenmtary particle physics in it.

From abstract of "Quantum Gravity and the Standard Model" (hep-th/0603022):

"We show that a class of background independent models of quantum spacetime have local excitations that can be mapped to the first generation fermions of the standard model of particle physics. These states propagate coherently as they can be shown to be noiseless subsystems of the microscopic quantum dynamics[2]. These are identified in terms of certain patterns of braiding of graphs, thus giving a quantum gravitational foundation for the topological preon model proposed in [1]."

7.12.3 Substitution Moves

Ribbon graph theories

Quantum $SU(2)$ group.

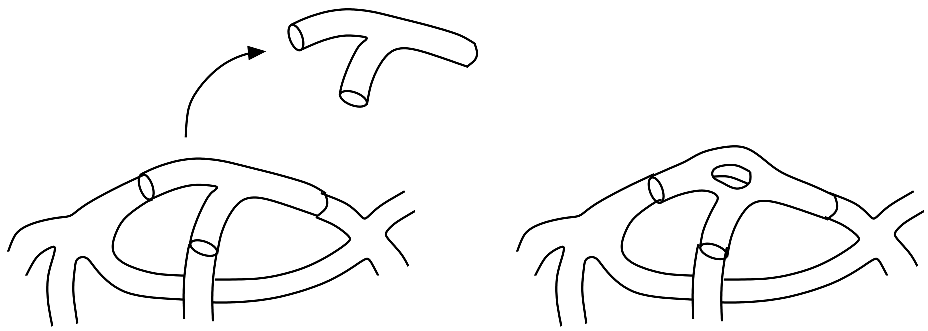


Figure 7.5: submoveF. An example of a substitution move.

The result is a sequence of states, $|0>, |1>, \ldots, |N>$, esch following from the previous one by a substitution move.

7.12.4 Discrete Symetries of Braids

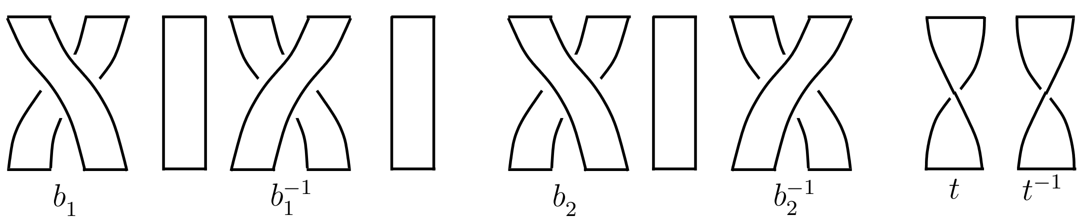


Figure 7.6: basucMovesF. The basic braiding moves on three strands, and the basic twisting moves.

Parity. A parity transformation is defined as a reflection of a braid, while not effecting the handedness of the twists on the strands.

Charge conjugation.

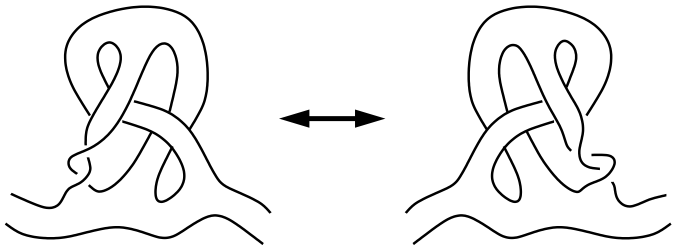


Figure 7.7: paritytranF. The effect of a parity transformation on a braid.

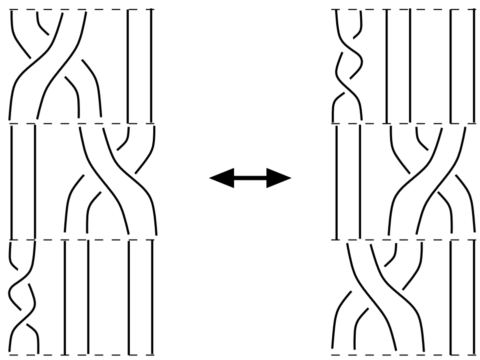


Figure 7.8: chargeDefF. The definition of a charge conjugation transformation on a braid.

7.12.5 Identification of the First Generation Fermions of the Standard model

Assumptions

As we will see, in order to arrive at the standard model fermions we have to make two assumptions:

• The lightest states are the simplest non-trivial braids made of ribbons with no twists (no charge) or one full twist (positive or negative charge).

• Quantum numbers are assigned only to braids with no positive and negative charge mixing. Such a rule is necessary in preon models as discussed [275].

These should be justified in a fundamental theory. At the level that an effective dynamics in spacetime emerges a notion of mass of emergent excitations will arise. This requires that there be in the low energy limit an emergent translation invariance in space and time. This will imply the conservation of energy and momentum for small excitations around the ground state. When the effective Hamiltonian H is evaluated on the states described here at zero momentum it will give us a mass matrix.

(1) Given two braids as just described, which have the same number of strands and

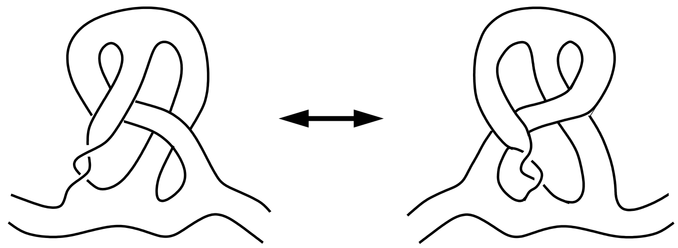


Figure 7.9: chargetranF. The effect of a charge conjugation transformation on a braid.

twistings, but differ by the number of crossings, the mass will increase with the number of crossings.

(2) Non-trivially braided states with both positive and negative twisted strands incident on the same vertex should have a heavy mass M. All other states are light relative to the scale M.

Identification of fermions

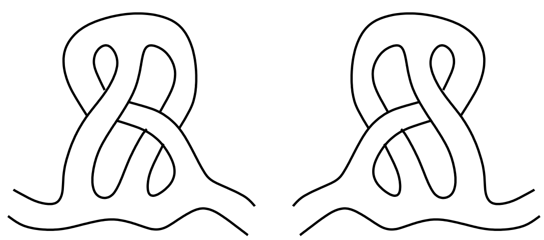


Figure 7.10: elecneutrinoF. The electron neutrino and anti-neutrino - two unchaged states.

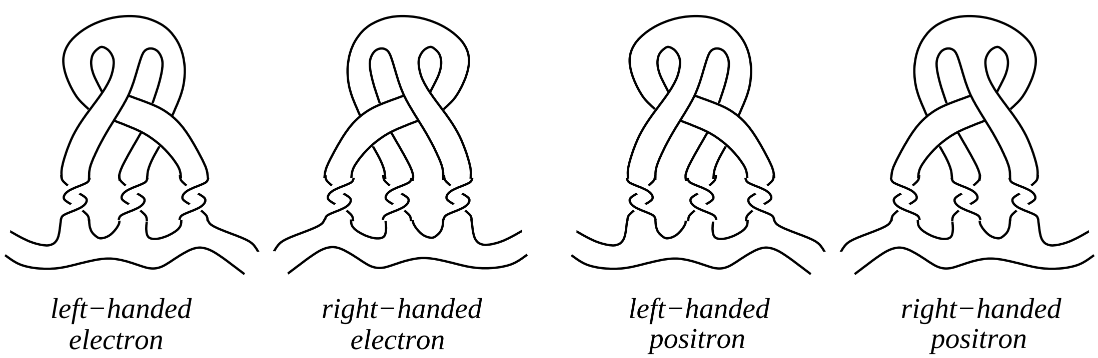
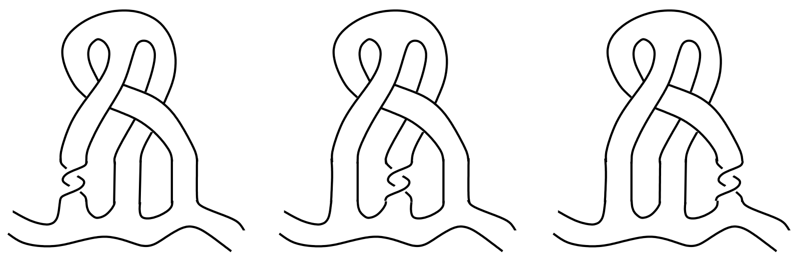


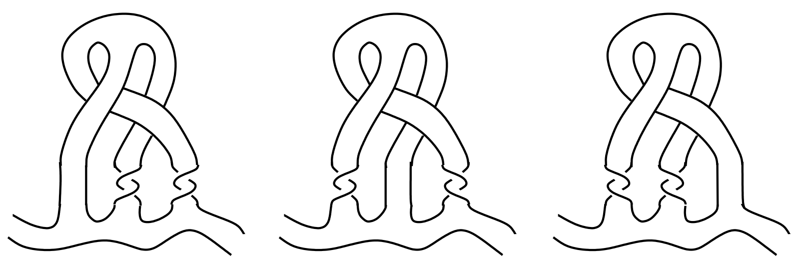
Figure 7.11: elecpositronF. The electron and positron - four maximally charged states.

These are partially charged states, with one and two +'s, and the rest zeroes. Theses are shown in figures 7.12 and 7.13. These are the quarks, with total charges $\pm\frac{1}{3}$ and $\pm\frac{2}{3}$. In each of these there is an "odd strand out" - one that is different from the other two. The non-trivial nature of the braid means that there are three distinct positions in the braid which the "odd strand out" can occupy. Hence each of the partially charged states comes in three versions. We will equate these permutations with colour.



$\overline{d}_L^r$ $\overline{d}_L^g$ $\overline{d}_L^b$

Figure 7.12: dquarkleftF. The left-handed down states - showing tripling of states for fractional charge.



u_L^r u_L^g u_L^b

Figure 7.13: uquarkleftF. The three left-handed up states.

The correspondence between braids and particles suggests that more properties are waiting to be derived from the theory. The most substantial achievement would be to calculate the masses of the elementary particles from first principles.

It is natural to hypothesize then that the second generation standard model fermions come from the next most complicated states, which have three crossings. There are two kinds of three crossing states. There are three stranded braids, such as shown in Figure M.-19??. It is straightforward to see that by adding twists to this state one gets a repeat of the pattern for the first generation. However, this appears to imply no upper bound on the number of allowed generations.

generation = crossings - 1

one new $SU(2) + SU(3)$ singlet appears for all higher generations

Weak vector bosons come from unbraided triplets of lines.

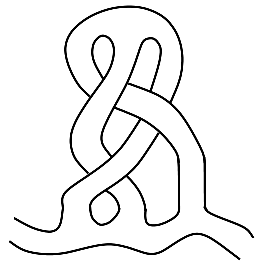


Figure 7.14: muonneutrF. A possible identification of the muon neutrino as a three crossing state.

With one more rule to suppress quark-lepton mixing weak interactoins are a consequence of loacl moves.

From all allowed twists we get a copyh of the 1st generation.

These give additional states which are $SU(2) + SU(3)$ singlets but come in left and right versions. Could these be the right handed neutrinos.

7.13 Semi-Classical Limit in Constitent Discrete Quantum Gravity

7.14 Worked Exercises

Prove that the states

$$|n >= \frac{(a^\dagger)^n}{\sqrt{n!}}|0 >$$

are orthogonal:

$$< m|n >= \delta_{mn} \tag{7.126}$$

Proof:

Using $[a, a^\dagger] = 1$ we obtain

$$
\begin{aligned}
<m|n> &= \frac{1}{\sqrt{m!n!}} <0|(a)^m(a^\dagger)^n|0> \\
&= \frac{1}{\sqrt{m!n!}} <0|(a)^{m-1}(a^\dagger a+1)(a^\dagger)^{n-1}|0> \\
&= \frac{1}{\sqrt{m!n!}} <0|(a)^{m-1}a^\dagger a(a^\dagger)^{n-1}|0> + \frac{1}{\sqrt{mn}} <0|(a)^{m-1}(a^\dagger)^{n-1}|0> \\
&= \frac{1}{\sqrt{m!n!}} <0|(a)^{m-1}a^\dagger(a^\dagger a+1)(a^\dagger)^{n-2}|0> + \frac{1}{\sqrt{mn}} <0|(a)^{m-1}(a^\dagger)^{n-1}|0> \\
&= \frac{1}{\sqrt{m!n!}} <0|(a)^{m-1}(a^\dagger)^2 a(a^\dagger)^{n-2}|0> + 2\frac{1}{\sqrt{mn}} <0|(a)^{m-1}(a^\dagger)^{n-1}|0> \\
&\vdots \\
&= \frac{1}{\sqrt{m!n!}} <0|(a)^{m-1}(a^\dagger)^n a|0> + n\frac{1}{\sqrt{mn}} <0|(a)^{m-1}(a^\dagger)^{n-1}|0> \\
&= n\frac{1}{\sqrt{m!n!}} <0|(a)^{m-1}(a^\dagger)^{n-1}|0>
\end{aligned}
$$

Continuing in this manner we find the end result will bo zero if $m \neq n$. However, if $m = n$ then the end result will be

$$
\begin{aligned}
<m|n> &= n(n-1)\times\cdots\times 2\frac{1}{n!} <0|aa^\dagger|0> \\
&= n(n-1)\times\cdots\times 2\frac{1}{n!} <0|(a^\dagger a+1)|0> \\
&= n!\frac{1}{n!} \\
&= 1.
\end{aligned} \tag{7.127}
$$

$\square$

Using the assumption that

$$f(za^\dagger)|0>=|z>$$

where f can be Taylor expanded, and that

$$a|z>=z|z> \quad \text{and} \quad <z|z>=1$$

prove that

$$|z>= e^{-|z|^2/2} e^{za^\dagger}|0> .$$

Proof:

Note

$$af(za^\dagger)|0>= a|z>= z|z>= [a, f(za^\dagger)]|0> .$$

Write

$$f(za^\dagger) = N\left(1 + c_1 za^\dagger + c_2 z^2 (a^\dagger)^2 + \ldots\right)$$

We will need the result,

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1}$$

which is easily proved by induction: First we prove it explicitly for $n = 2$:

$$\begin{aligned}
[a, (a^\dagger)^2] &= a(a^\dagger)^2 - (a^\dagger)^2 a \\
&= (aa^\dagger - a^\dagger a)a^\dagger + a^\dagger(aa^\dagger - a^\dagger a) \\
&= [a, a^\dagger]a^\dagger + a^\dagger[a, a^\dagger] \\
&= 2a^\dagger
\end{aligned}$$

Now assume $[a, (a^\dagger)^m] = m(a^\dagger)^{m-1}$ and then consider

$$\begin{aligned}
[a, (a^\dagger)^{m+1}] &= a(a^\dagger)^{m+1} - (a^\dagger)^{m+1} a \\
&= [a, a^\dagger](a^\dagger)^m + a^\dagger[a, (a^\dagger)^m] \\
&= (a^\dagger)^m + m(a^\dagger)^m \\
&= (m+1)(a^\dagger)^m.
\end{aligned}$$

Now we can calculate the commutator,

$$\begin{aligned}
[a, f(za^\dagger)] &= N[a, 1 + c_1 za^\dagger + c_2 z^2 (a^\dagger)^2 + c_3 z^3 (a^\dagger)^3 + c_4 z^4 (a^\dagger)^4 + \ldots] \\
&= N\left(c_1 z[a, a^\dagger] + c_2 z^2 [a, a^\dagger)^2] + c_3 [a, (a^\dagger)^3] + \ldots\right) \\
&= Nz\left(c_1 + c_2 za^\dagger + 3c_3 z^2 (a^\dagger)^2 + 4c_4 z^3 (a^\dagger)^3 \ldots\right).
\end{aligned}$$

We now use $[a, f(za^\dagger)]|0>= zf(za^\dagger)|0>$:

$$
\begin{aligned}
&Nz\left(c_1 + 2c_2 za^\dagger + 3c_3 z^2 (a^\dagger)^2 + 4c_4 z^3 (a^\dagger)^3 \ldots\right)|0>= \\
&\qquad\qquad = zN\left(1 + c_1 za^\dagger + c_2 z^2 (a^\dagger)^2 + c_3 z^3 (a^\dagger)^3 + \ldots\right)|0>
\end{aligned}
$$

and read off

$$c_1 = 1, \quad 2c_2 = c_1, \quad 3c_3 = c_2, \quad 4c_4 = c_3, \ldots$$

or

$$c_1 = 1, \quad c_2 = \frac{1}{2!}, \quad c_3 = \frac{1}{3!}, \quad c_4 = \frac{1}{4!}, \ldots$$

So we have $|z>= f(za^\dagger)|0>= Ne^{za^\dagger}|0>$. We now use $<z|z>= 1$ to find N,

$$
\begin{aligned}
1 &= <z|z> \\
&= N^2 <0|e^{\overline{z}a} e^{za^\dagger}|0> \\
&= N^2 <0|\left(1 + a\overline{z} + \frac{1}{2!}a^2\overline{z}^2 + \ldots\right)\left(1 + za^\dagger + \frac{1}{2!}(a^\dagger)^2 + \ldots\right)|0> \\
&= N^2\left(<0| + \overline{z} <1| + \overline{z}^2 \frac{1}{\sqrt{2!}} <2| + \ldots\right)\left(|0> + z|1> + z^2\frac{1}{\sqrt{2!}}|2> + \ldots\right) \\
&= N^2\left(1 + |z|^2 + \frac{|z|^4}{2!} + \ldots\right) \\
&= N^2 e^{|z|^2}
\end{aligned}
$$

where we have used (7.126). We have $N = e^{-|z|^2/2}$.

$\square$

Coherent states in position representation

Use

$$\hat{a}|z>= |z>$$

to find the wavefunction in the position represenation.

Proof:

Let $|X>$ and $|P>$ be the eigenstates of $\hat{X}$ and $\hat{P}$. We write

$$\hat{a}|z>=\left(\sqrt{\frac{\omega}{2\hbar}}\hat{X}+i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}\right)|z>=z|z>$$

where

$$z=\sqrt{\frac{\omega}{2\hbar}}X_0+i\frac{1}{\sqrt{2\hbar\omega}}P_0 \tag{7.128}$$

and act on the right with a state $<X|$ and obtain

$$<X|\left(\sqrt{\frac{\omega}{2\hbar}}\hat{X}+i\frac{1}{\sqrt{2\hbar\omega}}\hat{P}\right)|z>=z<X|z>$$

Since $\hat{X}|X>=X|X>$ and $\hat{P}=-i\hbar\frac{\partial}{\partial X}$, we have $(\hat{X}|X>)^\dagger=<X|\hat{X}^\dagger=<X|(\hat{X}=(X|X>)^\dagger=<X|X$ and $(i\hat{P}|X>)^\dagger=-i<X|\hat{P}^\dagger=-i<X|\hat{P}=(\hbar\frac{\partial}{\partial X}|X>)^\dagger=-\hbar\frac{\partial}{\partial X}<X|$. That is,

$$<X|\hat{X}=<X|X \qquad \text{and} \qquad <X|i\hat{P}=\hbar\frac{\partial}{\partial X}<X|,$$

from which we obtain a differential equation

$$\sqrt{\frac{\omega}{2\hbar}}<X|z>X+\frac{1}{\sqrt{2\hbar\omega}}\hbar\frac{\partial}{\partial X}<X|z>=z<X|z>$$

or

$$\frac{\partial}{\partial X}\psi_z(X)=-\frac{\omega}{\hbar}\left(X-\sqrt{\frac{2\hbar}{\omega}}z\right)\psi_z(X)$$

This has the solution

$$\psi_z(X)=\mathcal{N}\exp-\left\{\frac{\omega}{2\hbar}\left(X-\sqrt{\frac{2\hbar}{\omega}}z\right)^2\right\}$$

or upon subsituting (7.128),

$$
\begin{aligned}
\psi_z(X) &= \mathcal{N} \exp - \left\{ \frac{\omega}{2\hbar} \left(X - X_0 - i \frac{P_0}{\omega} \right)^2 \right\} \\
&= \mathcal{N} \exp - \frac{\omega}{2\hbar} \left\{ (X - X_0)^2 - 2i (X - X_0) \frac{P_0}{\omega} - \frac{P_0^2}{\omega^2} \right\} \\
&= \mathcal{N} \exp - \left\{ \frac{\omega}{2\hbar} (X - X_0)^2 - i (X - X_0) \frac{P_0}{\hbar} - \frac{P_0^2}{2\hbar\omega} \right\} \\
&= \mathcal{N} e^{P_0^2/2\hbar\omega} e^{-i\frac{X_0 P_0}{\hbar}} \exp - \left\{ \frac{\omega}{2\hbar} (X - X_0)^2 - \frac{i}{\hbar} X P_0 \right\}.
\end{aligned}
$$

Normalisation requires

$$
\begin{aligned}
1 &= \int_{-\infty}^{\infty} |\psi_z(X)|^2 dX \\
&= |\mathcal{N}|^2 e^{P_0^2/\hbar\omega} \int_{-\infty}^{\infty} \exp - \left\{ \frac{\omega}{\hbar} (X - X_0)^2 \right\} dX \\
&= |\mathcal{N}|^2 e^{P_0^2/\hbar\omega} \sqrt{\frac{\pi\hbar}{\omega}}.
\end{aligned}
$$

We can choose $\mathcal{N}$ to be

$$
\mathcal{N} = \left(\frac{\omega}{\pi\hbar} \right)^{\frac{1}{4}} e^{-P_0^2/2\hbar\omega} e^{i\frac{X_0 P_0}{\hbar}}
$$

so that

$$
\psi_z(X) = \left(\frac{\omega}{\pi\hbar} \right)^{\frac{1}{4}} \exp - \left\{ \frac{\omega}{2\hbar} (X - X_0)^2 - \frac{i}{\hbar} X P_0 \right\}.
$$

□

Prove coherent states saturate uncerntainty

Prove the states

$$
|z>
$$

which is the eigenvalue of $\hat{a}$ with

$$\hat{a}|z>= z|z>$$

saturate uncerntainty, i.e. we have

$$\Delta X \Delta P = \frac{\hbar}{2}.$$

For z we will have

$$z = \sqrt{\frac{\omega}{2\hbar}} X_0 + i \frac{1}{\sqrt{2\hbar\omega}} P_0$$

Proof:

Note, for example,

$$\begin{aligned} <z|\hat{X}|z> &= \int dX <z|X> X <X|z> \\ &= \int dX \psi_z^*(X) X \psi_z(X) \\ <X>. \end{aligned}$$

Therefore we can avoid integrals, instead just use $\hat{a}$ and $\hat{a}^\dagger$.

Recall that

$$\begin{aligned} (\Delta x)^2 &= <(x-<x>)^2> = <x^2> - <x>^2 \\ (\Delta p)^2 &= <(p-<p>)> = <p^2> - <p>^2. \end{aligned}$$

It is easier verified via the definitions

$$(\Delta X)^2 =<\hat{X}^2> - <\hat{X}>^2, \qquad (\Delta P)^2 =<\hat{P}^2> - <\hat{P}>^2$$

As

$$\hat{a} = \sqrt{\frac{\omega}{2\hbar}} \hat{X} + i \frac{1}{\sqrt{2\hbar\omega}} \hat{P}, \qquad \hat{a}^\dagger = \sqrt{\frac{\omega}{2\hbar}} \hat{X} - i \frac{1}{\sqrt{2\hbar\omega}} \hat{P}$$

we obtain

$$\begin{aligned}\hat{X} &= \sqrt{\frac{\hbar}{2\omega}}(\hat{a}+\hat{a}^\dagger)\\ \hat{P} &= -i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}-\hat{a}^\dagger).\end{aligned}$$

Recall that

$$\hat{a}|z>= z|z>, \qquad <z|\hat{a}^\dagger = \overline{z}<z|.$$

First we find $<z|\hat{X}|z>$,

$$\begin{aligned}<z|\hat{X}|z> &= \sqrt{\frac{\hbar}{2\omega}}<z|(\hat{a}+\hat{a}^\dagger)|z>\\ &= \sqrt{\frac{\hbar}{2\omega}}(z<z|z>+\overline{z}<z|z>)\\ &= \sqrt{\frac{\hbar}{2\omega}}(z+\overline{z})\\ &= \sqrt{\frac{\hbar}{2\omega}}2\sqrt{\frac{\omega}{2\hbar}}X_0 = X_0.\end{aligned}$$

Now consider $<z|\hat{P}|z>$,

$$\begin{aligned}<z|\hat{P}|z> &= -i\sqrt{\frac{\hbar\omega}{2}}<z|(\hat{a}-\hat{a}^\dagger)|z>\\ &= -i\sqrt{\frac{\hbar\omega}{2}}(z<z|z>-\overline{z}<z|z>)\\ &= -i\sqrt{\frac{\hbar\omega}{2}}(z-\overline{z})\\ &= -i\sqrt{\frac{\hbar\omega}{2}}2(i\frac{1}{\sqrt{2\hbar\omega}}P_0)\\ &= P_0.\end{aligned}$$

Now let us compute $<\hat{X}^2>$,

$$\begin{aligned}
< z|\hat{X}^2|z > &= \frac{\hbar}{2\omega} < z|(\hat{a}+\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} < z|(\hat{a}^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}+(\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega} < z|(\hat{a}^2+2\hat{a}^\dagger\hat{a}+1+(\hat{a}^\dagger)^2|z > \\
&= \frac{\hbar}{2\omega}\left(z^2+2\overline{z}z+1\overline{z}^2\right) \\
&= \frac{\hbar}{2\omega}\left((z+\overline{z})^2+1\right) \\
&= < z|\hat{X}|z >^2 + \frac{\hbar}{2\omega}
\end{aligned}$$

where we have used $[\hat{a},\hat{a}^\dagger]=1$. We can now write

$$\begin{aligned}
\Delta X &= \sqrt{< z|\hat{X}^2|z > - < z|\hat{X}|z >^2} \\
&= \sqrt{\frac{\hbar}{2\omega}}.
\end{aligned}$$

Now let us compute $< \hat{P}^2 >$,

$$\begin{aligned}
< z|\hat{P}^2|z > &= -\frac{\hbar\omega}{2} < z|(\hat{a}-\hat{a}^\dagger)^2|z > \\
&= -\frac{\hbar\omega}{2} < z|(\hat{a}^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}+(\hat{a})^\dagger)^2|z > \\
&= -\frac{\hbar\omega}{2} < z|(\hat{a}^2-2\hat{a}^\dagger\hat{a}-1+(\hat{a}^\dagger)^2|z > \\
&= -\frac{\hbar\omega}{2}\left(z^2-2\overline{z}z+\overline{z}^2-1\right) \\
&= -\frac{\hbar\omega}{2}\left((z-\overline{z})^2-1\right) \\
&= < z|\hat{P}|z >^2 + \frac{\hbar\omega}{2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Delta P &= \sqrt{< z|\hat{P}^2|z > - < z|\hat{P}|z >^2} \\
&= \sqrt{\frac{\hbar\omega}{2}}.
\end{aligned}$$

For coherent states

$$\Delta X \Delta P = \sqrt{\frac{\hbar}{2\omega}}\sqrt{\frac{\hbar\omega}{2}} = \frac{\hbar}{2}.$$

therefore we have equality in the Heisenberg uncertainty relation.

$\square$

Minimum uncertainty wavefunction

Directly derive the wavefunctions which minimises the Heisenberg uncertainty relations in configuration and momentum space respectively. They are given by

$$\psi_z(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}} \exp - \left\{\frac{\omega}{2\hbar}(x - X_0)^2 - \frac{i}{\hbar}xP_0\right\}, \tag{7.129}$$

$$\psi_z(p) = \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp - \left\{\frac{\hbar}{2\omega}\frac{(p - P_0)^2}{\hbar^2} + \frac{i}{\hbar}pX_0\right\}. \tag{7.130}$$

where

$$< \hat{X} >_{\Psi_z} = X_0, \quad \text{and} \quad < \hat{P} >_{\Psi_z} = P_0. \tag{7.131}$$

Denote the fluctuation of an observable $\hat{O}$ by

$$\Delta_{\Psi_z}(\hat{O}) = \left(< \hat{O}^2 >_\Psi - < \hat{O} >^2_\Psi\right)^{\frac{1}{2}},$$

and find that

$$\Delta_{\Psi_z}(\hat{X})\Delta_{\Psi_z}(\hat{P}) = \frac{\hbar}{2}, \qquad \frac{\Delta_{\Psi_z}(\hat{P})}{\Delta_{\Psi_z}(\hat{X})} = \omega. \tag{7.132}$$

Proof:

$$\begin{aligned}(\Delta x)^2 &= < (x - < x >)^2 > = < x^2 > - < x >^2 \\ (\Delta p)^2 &= < (p - < p >) > = < p^2 > - < p >^2\end{aligned}$$

Define

$$\alpha = x- < x >, \qquad \beta = -i\hbar\left(\frac{d}{dx} - \left\langle\frac{d}{dx}\right\rangle\right) \tag{7.133}$$

then

$$\begin{aligned}(\Delta x)^2(\Delta p)^2 &= \int_{-\infty}^{\infty}\psi^*\alpha^2\psi dx\int_{-\infty}^{\infty}\psi^*\beta^2\psi dx \\ &= \int_{-\infty}^{\infty}(\alpha^\dagger\psi^*)(\alpha\psi)dx\int_{-\infty}^{\infty}(\beta^\dagger\psi^*)(\beta\psi)dx \end{aligned}\tag{7.134}$$

where for β we have performed integration by parts and used that the normalised wave-function vanishes at $x = \pm\infty$.

We have the iequality

$$\int\left|f - g\frac{\int fg^* dx}{\int |g|^2 dx}\right|^2 dx \geq 0.$$

Equality holds only if $f = \gamma g$. Expanding gives

$$\begin{aligned}\int\left|f - g\frac{\int fg^* dx}{\int |g|^2 dx}\right|^2 &= \int\left(f^* - g^*\frac{(\int fg^* dx)^*}{\int |g|^2 dx}\right)\left(f - g\frac{\int fg^* dx}{\int |g|^2 dx}\right)dx \\ &= \int|f|^2 dx + \int|g|^2 dx\frac{\left|\int fg^* dx\right|^2}{\left(\int |g|^2 dx\right)^2} - 2\frac{\left|\int fg^* dx\right|^2}{\int |g|^2 dx} \geq 0\end{aligned}$$

so

$$\int|f|^2 dx\int|g|^2 dx \geq |\int fg^* dx|^2$$

If we know replace f by $\alpha\psi$ and g by $\beta\psi$, (7.134) becomes

$$(\Delta x)^2(\Delta p)^2 \geq |\int(\alpha^\dagger\psi^*)(\beta^\dagger\psi)dx|^2 = |\int\psi^*\alpha\beta\psi|^2. \tag{7.135}$$

The last term can be wrtten

$$\left| \int \psi^* \left[\frac{1}{2}(\alpha\beta - \beta\alpha) + \frac{1}{2}(\alpha\beta + \beta\alpha) \right] \psi dx \right|^2$$
$$= \frac{1}{4} \left| \int \psi^*(\alpha\beta - \beta\alpha)\psi dx \right|^2 + \frac{1}{4} \left| \int \psi^*(\alpha\beta + \beta\alpha)\psi dx \right|^2$$
$$+ \frac{1}{2} Re \left(\int \psi^*(\alpha\beta - \beta\alpha)\psi dx \int \psi^*(\alpha\beta + \beta\alpha)\psi dx \right).$$

The third term vanishes because $\int \psi^*(\alpha\beta - \beta\alpha)\psi dx \int \psi^*(\alpha\beta + \beta\alpha)\psi dx$ is purely imaginary. In deriving the Heisenberg uncertainty principle we note that the second term $\frac{1}{4} \left| \int \psi^*(\alpha\beta + \beta\alpha)\psi dx \right|^2$ is positive and write

$$\begin{aligned}
(\Delta x)^2 (\Delta p)^2 \geq \left| \int \psi^* \alpha\beta\psi dx \right|^2 \quad &\geq \quad \frac{1}{4} \left| \int \psi^*(\alpha\beta - \beta\alpha)\psi dx \right|^2 \\
&= \quad \frac{1}{4} \left| \int \psi^*(i\hbar)\psi dx \right|^2 \\
&= \quad \frac{\hbar^2}{4}.
\end{aligned}$$

The minimum uncertainty product is obtained onlt when the two conditions are fulfilled:

$$\begin{aligned}
\alpha\psi &= \gamma\beta\psi && (7.136) \\
\int \psi^*(\alpha\beta + \beta\alpha)\psi dx &= 0. && (7.137)
\end{aligned}$$

Equations (7.136) and (7.133) give the differential equation

$$(x - <x>)\psi_0 = \gamma(-i\hbar \frac{d\psi_0}{dx} - <p> \psi_0)$$

or

$$\frac{d\psi_0}{dx} = \left[\frac{i}{\gamma\hbar}(x - <x>) + \frac{i<p>}{\hbar} \right] \psi_0$$

which is readily integrated

$$\int \frac{d\psi_0}{\psi_0} = \left[\frac{i}{\gamma\hbar}(x - <x>) + \frac{i<p>}{\hbar} \right] dx$$

so that

$$\ln \psi_0 = \frac{i}{2\gamma\hbar}(x- < x >)^2 + \frac{i < p > x}{\hbar} - \ln \mathcal{N}$$

(where $\mathcal{N}$ is an arbitrary constant) or

$$\psi_0(x) = \mathcal{N} \exp\left[\frac{i}{2\gamma\hbar}(x- < x >)^2 + \frac{i < p > x}{\hbar}\right] \tag{7.138}$$

Equation (7.137), using (7.136), becomes

$$\begin{aligned}
\int \psi_0^*(\alpha\beta + \beta\alpha)\psi dx &= \int \left(\psi_0^* \frac{1}{\gamma}\alpha^2 + (\beta^\dagger \psi_0^*)(\alpha\psi_0)\right) dx \\
&= \int \left(\psi_0^* \frac{1}{\gamma}\alpha^2 + (\frac{1}{\gamma^*}\alpha^\dagger \psi_0^*)(\alpha\psi_0)\right) dx \\
&= \left(\frac{1}{\gamma} + \frac{1}{\gamma^*}\right) \int \psi_0^* \alpha^2 \psi_0 dx = 0
\end{aligned} \tag{7.139}$$

which implies that γ is purely imaginary. Requiring the wavefunction (7.138) to be normalisable means γ is negative imaginary. We now obtain $\mathcal{N}$ from the mormalisation condition

$$\int_{-\infty}^{\infty} |\psi_0(x)|^2 dx = 1,$$

that is

$$\begin{aligned}
1 &= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[\left(\frac{i}{2\gamma\hbar} - \frac{i}{2\gamma^*\hbar}\right)(x- < x >)^2\right] dx \\
&= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[-\frac{1}{|\gamma|\hbar}(x- < x >)^2\right] dx \\
&= \mathcal{N}^2 \int_{-\infty}^{\infty} \exp\left[-\frac{1}{|\gamma|\hbar}x^2\right] dx \\
&= \mathcal{N}^2 \sqrt{|\gamma|\hbar\pi}
\end{aligned}$$

so

$$\mathcal{N} = (|\gamma|\hbar\pi)^{-\frac{1}{4}}. \tag{7.140}$$

The value of γ can be found from

$$\int_{-\infty}^{\infty}(x-<x>)^2|\psi_0(x)|^2dx = (\Delta x)^2$$

that is

$$\begin{aligned}
(\Delta x)^2 &= \frac{1}{\sqrt{|\gamma|\hbar\pi}}\int_{-\infty}^{\infty}(x-<x>)^2\exp\left[-\frac{1}{|\gamma|\hbar}(x-<x>)^2\right]dx \\
&= \frac{1}{\sqrt{|\gamma|\hbar\pi}}\int_{-\infty}^{\infty}x^2\exp\left[-\frac{1}{|\gamma|\hbar}x^2\right]dx \\
&= \frac{1}{\sqrt{|\gamma|\hbar\pi}}\frac{1}{2}\sqrt{(|\gamma|\hbar)^3\pi} \\
&= \frac{1}{2}|\gamma|\hbar. \qquad (7.141)
\end{aligned}$$

So the wavefunction is

$$\psi_0(x) = [2\pi(\Delta x)^2]^{-\frac{1}{4}}\exp\left[-\frac{(x-<x>)^2}{4(\Delta x)^2} + \frac{i<p>x}{\hbar}\right] \qquad (7.142)$$

As the uncertainty priniple is saturated we have equality: $(\Delta x)(\Delta p) = \hbar/2$. As ω is defined by $\omega = \Delta p/\Delta x$,

$$\left(\Delta_{\Psi_z}(\hat{X})\right)^2 = \frac{\hbar}{2\omega}, \qquad \left(\Delta_{\Psi_z}(\hat{P})\right)^2 = \frac{\hbar\omega}{2}. \qquad (7.143)$$

and the wavefunction can be written

$$\psi_0(x) = \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{\omega}{2\hbar}(x-<x>)^2 + \frac{i<p>x}{\hbar}\right] \qquad (7.144)$$

We now transform to the momentum representation

$$\begin{aligned}
\psi_0(p) &:= \frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\psi_0(x)dx \\
&= \left(\frac{\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty}\exp\left[-\frac{\omega}{2\hbar}(x-<x>)^2 - \frac{i(p-<p>)x}{\hbar}\right]dx \\
&= \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}}\frac{1}{\hbar}\int_{-\infty}^{\infty}\exp\left[-\frac{\omega}{2\hbar}\left\{(x-<x>)^2 + \frac{i2(p-<p>)x}{\omega}\right\}\right]dx
\end{aligned} \qquad (7.145)$$

The contents of the curly brackets rearranged becomes

$$
\begin{aligned}
&(x- < x >)^2 + \frac{i2(p- < p >)x}{\omega} \\
&= x^2 + 2\frac{- < x > \omega + i(p- < p >)}{\omega} x + < x >^2 \\
&= \left(x + \frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 - \left(\frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 + < x >^2 \\
&\mapsto x^2 - \left(\frac{- < x > \omega + i(p- < p >)}{\omega}\right)^2 + < x >^2 \\
&= x^2 + \frac{(p- < p >)^2}{\omega^2} + 2\frac{i(p- < p >)}{\omega} < x >
\end{aligned}
\tag{7.146}
$$

Therefore

$$
\begin{aligned}
&\psi_0(p) \\
= \;& \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}\left\{x^2 + \frac{(p- < p >)^2}{\omega^2} + 2\frac{i(p- < p >)}{\omega} < x >\right\}\right] dx \\
= \;& e^{\frac{i<x><p>}{\hbar}} \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp\left[-\frac{\omega}{2\hbar}\frac{(p- < p >)^2}{\omega^2} - \frac{ip < x >}{\hbar}\right] \int_{-\infty}^{\infty} \exp\left[-\frac{\omega}{2\hbar}x^2\right] dx \\
= \;& e^{\frac{i<x><p>}{\hbar}} \left(\frac{\omega}{4\pi^3\hbar}\right)^{\frac{1}{4}} \frac{1}{\hbar} \sqrt{\left(\frac{2\pi\hbar}{\omega}\right)} \exp\left[-\frac{1}{2\hbar\omega}(p- < p >)^2 - \frac{ip < x >}{\hbar}\right] \\
= \;& e^{\frac{i<x><p>}{\hbar}} \left(\frac{\hbar}{\pi\omega}\right)^{\frac{1}{4}} \frac{1}{\hbar} \exp\left[-\frac{\hbar}{2\omega}\frac{(p- < p >)^2}{\hbar^2} - \frac{ip < x >}{\hbar}\right]
\end{aligned}
\tag{7.147}
$$

The norm in the momentum representation is defined by

$$\int_{-\infty}^{\infty} |\psi_0(p)|^2 \frac{dp}{\hbar}$$

Normalisation is guaranted by

$$
\begin{aligned}
1=\int_{-\infty}^{\infty}|\psi_0(x)|^2dx &= \int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{2\pi}\hbar}\int_{-\infty}^{\infty}e^{ipx/\hbar}\psi_0(p)dp\right|^2 dx \\
&= \frac{1}{2\pi\hbar^2}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}e^{-ipx/\hbar}\psi_0^*(p)dp\right)\left(\int_{-\infty}^{\infty}e^{i\tilde{p}x/\hbar}\psi_0(\tilde{p})d\tilde{p}\right)dx \\
&= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\psi_0^*(p)\psi_0(\tilde{p})\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(\tilde{p}-p)x/\hbar}\frac{dx}{\hbar}\right)\right]dpd\tilde{p} \\
&= \frac{1}{\hbar}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi_0^*(p)\psi_0(\tilde{p})\delta(p-\tilde{p})dpd\tilde{p} \\
&= \int_{-\infty}^{\infty}|\psi_0(p)|^2\frac{dp}{\hbar}.
\end{aligned}
$$

$\square$

Overcompletenes

Let

$$
\hat{X}'=\sqrt{\frac{\omega}{2\hbar}} \qquad \text{and} \qquad \hat{P}'=\frac{1}{\sqrt{2\hbar\omega}}\hat{P}
$$

Prove

$$
z=\frac{1}{\sqrt{2}}(X'+iP'), \quad \overline{z}=\frac{1}{\sqrt{2}}(X'-iP')
$$

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z>e^{-\frac{1}{2}(X'^2+P'^2)}<z^*|=1.
$$

Proof:

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z>e^{-z\overline{z}}<z^*| &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}e^{-z\overline{z}}\sum_{n,n'}\frac{(z\hat{a}^\dagger)^n}{n!}|0><0|\frac{(\overline{z}\hat{a})^{n'}}{n'!} \\
&= \sum_{n,n'}(\hat{a}^\dagger)^n|0><0|(\hat{a})^{n'}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}e^{-z\overline{z}}\frac{z^n\overline{z}^{n'}}{n!n'!}
\end{aligned}
$$

The integrals are easily performed by introducing "polar coordinates"

$$X' = \sqrt{2\rho}\cos\theta, \qquad P' = \sqrt{2\rho}\sin\theta$$

We use the volume element,

$$dX'dP' = \sqrt{2\rho}d\theta d\sqrt{2\rho} = d\theta d\rho$$

(see diagram) in the above integral. we have

$$z = \sqrt{\rho}e^{i\theta}, \qquad \overline{z} = \sqrt{\rho}e^{-i\theta}.$$

The integral

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}e^{-z\overline{z}}\frac{z^n\overline{z}^{n'}}{n!n'!}$$

becomes

$$\int_0^{\infty}\int_0^{2\pi}d\rho\frac{d\theta}{2\pi}e^{i(n-n')\theta}e^{-\rho}\frac{(\sqrt{\rho})^{n+n'}}{n!n'!}.$$

We have

$$\int_0^{2\pi}\frac{d\theta}{2\pi}e^{i(n-n')\theta} = \delta_{nn'}$$

and

$$\begin{aligned}
\int_0^{\infty}d\rho\rho^n e^{-\rho} &= (-1)^n\frac{\partial^n}{\partial\alpha^n}\int_0^{\infty}d\rho e^{-\alpha\rho}\bigg|_{\alpha=1} \\
&= (-1)^n\frac{\partial^n}{\partial\alpha^n}\left(\frac{1}{\alpha}\right)\bigg|_{\alpha=1} \\
&= n!.
\end{aligned}$$

Hence

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dX'dP'}{2\pi}|z>e^{-z\overline{z}}<z^*| &= \sum_n \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0><0|\frac{\hat{a}^n}{\sqrt{n!}} \\
&= \sum_n |n><n|,
\end{aligned}
$$

□

Poisson summation theorem

Prove that if

$$
\psi(y) = \sum_{\infty}^{\infty} f(y+ns)
$$

then

$$
\sum_{n=-\infty}^{\infty} f(y+ns) = \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{s}\right)\exp(2\pi iyn/s)
$$

and

$$
\sum_{\infty}^{\infty} f(ns) = \frac{2\pi}{s}\sum_{\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{s}\right)
$$

where $\tilde{f}(k) := \int_{\mathbb{R}} \frac{dx}{2\pi} e^{-ik\cdot x} f(x)$.

Proof

For any integer m we have

$$
\begin{aligned}
\psi(y+ms) &= \sum_{n=-\infty}^{\infty} f(y+(n+m)s) \\
&= \sum_{n=\infty}^{\infty} f(y+ns) \\
&= \psi(y)
\end{aligned}
$$

so that $\psi(y)$ is periodic with period s. As such $\psi(y)$ can be written

$$\sum_{n=-\infty}^{\infty} f(y+ns) = \sum_{m=\infty}^{\infty} \alpha_m e^{im(2\pi/s)y}$$

where

$$\alpha_m = \frac{1}{s}\int_0^s \left(\sum_{n=-\infty}^{\infty} f(\tau+ns)\right) e^{-im(2\pi/s)\tau} d\tau$$

so that

$$\begin{aligned}
\sum_{n=\infty}^{\infty} f(y+ns) &= \sum_{m=\infty}^{\infty} \frac{1}{s}\int_0^s \left(\sum_{n=-\infty}^{\infty} f(\tau+ns)\right) e^{-im(2\pi/s)\tau} d\tau \; e^{im(2\pi/s)y} \\
&= \frac{1}{s}\sum_{m=-\infty}^{\infty} e^{im(2\pi/s)y} \sum_{n=-\infty}^{\infty} \int_0^s f(\tau+ns) e^{-im(2\pi/s)\tau} d\tau.
\end{aligned}$$

The sum over n may be rewritten as follows

$$\begin{aligned}
& \sum_{n=-\infty}^{\infty} \int_0^s f(\tau+ns) e^{-im(2\pi/s)\tau} d\tau \\
= & \cdots + \int_0^s f(\tau-s) e^{-im(2\pi/s)\tau} d\tau + \int_0^s f(\tau) e^{-im(2\pi/s)\tau} d\tau + \int_0^s f(\tau+s) e^{-im(2\pi/s)\tau} d\tau + \ldots \\
= & \cdots + \int_{-s}^0 f(\tau) e^{-im(2\pi/s)(\tau+s)} d\tau + \int_0^s f(\tau) e^{-im(2\pi/s)\tau} d\tau + \int_s^{2s} f(\tau) e^{-im(2\pi/s)(\tau-s)} d\tau + \ldots \\
= & \sum_{n=-\infty}^{\infty} \int_{sn}^{s(n+1)} f(\tau) e^{-im(2\pi/s)(\tau-ns)} d\tau \\
= & \sum_{n=-\infty}^{\infty} \int_{sn}^{s(n+1)} f(\tau) e^{-im(2\pi/s)\tau} d\tau \\
= & \int_{-\infty}^{\infty} f(\tau) e^{-im(2\pi/s)\tau} d\tau.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} f(y+ns) &= \frac{1}{s}\sum_{n=-\infty}^{\infty} e^{in(2\pi/s)y}\int_{-\infty}^{\infty} f(\tau)e^{-in(2\pi/s)\tau}d\tau \\
&= \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} e^{in(2\pi/s)y}\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\tau)e^{-in(2\pi/s)\tau}d\tau \\
&= \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{s}\right)\exp(2\pi i y n/s)
\end{aligned}$$

If, finally, we set $y = 0$, we obtain

$$\sum_{n=-\infty}^{\infty} f(ns) = \frac{2\pi}{s}\sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\tilde{f}\left(\frac{2\pi n}{s}\right).$$

□