Chapter 8

Covariant LQG and Spinfoams

8.1 Geometry

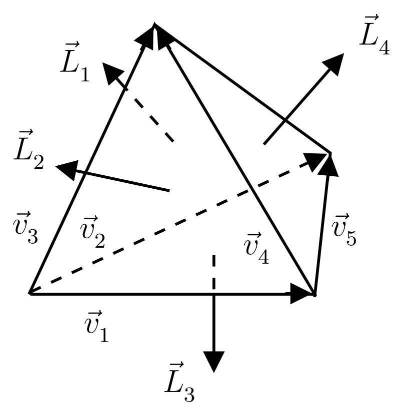


Figure 8.1: Four normals $\vec{L}_a$ to the faces.

The vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ define the tetrahedron. The vectors $\vec{v}_4$ and $\vec{v}_5$ can be given by

$$\begin{aligned} \vec{v}_4 &= \vec{v}_3 - \vec{v}_1 \\ \vec{v}_5 &= \vec{v}_2 - \vec{v}_1 \end{aligned} \tag{8.1}$$

$$
\begin{aligned}
\vec{L}_1 &= -\frac{1}{2}\vec{v}_2 \times \vec{v}_3 \\
\vec{L}_2 &= -\frac{1}{2}\vec{v}_3 \times \vec{v}_1 \\
\vec{L}_3 &= -\frac{1}{2}\vec{v}_1 \times \vec{v}_2 \\
\vec{L}_4 &= -\frac{1}{2}\vec{v}_4 \times \vec{v}_5 \\
&= -\frac{1}{2}(\vec{v}_3 - \vec{v}_1) \times (\vec{v}_2 - \vec{v}_1)
\end{aligned} \tag{8.2}
$$

(a) Closure.

$$
\begin{aligned}
\vec{C} &:= \sum_{a=1}^{4} \vec{L}_a \\
&= -\frac{1}{2}\Big(\vec{v}_2 \times \vec{v}_3 + \vec{v}_3 \times \vec{v}_1 + \vec{v}_1 \times \vec{v}_2 \\
&\qquad + \vec{v}_3 \times \vec{v}_2 - \vec{v}_3 \times \vec{v}_1 - \vec{v}_1 \times \vec{v}_2\Big) \\
&= 0.
\end{aligned} \tag{8.3}
$$

(b) (i) Areas.

The area of a triangle is

$$A = \frac{1}{2}|\vec{A} \times \vec{B}|$$

so the area of the $i-th$ face is

$$A_i = |\vec{L}_i|.$$

(ii) Volume.

In terms of $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ the volume is

$$V = \frac{1}{6}\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \tag{8.4}$$

Consider the cross product

$$
\begin{aligned}
\vec{L}_1 \times \vec{L}_2 &= \frac{1}{4}(\vec{v}_2 \times \vec{v}_3) \times (\vec{v}_3 \times \vec{v}_1) \\
&= \frac{1}{4}[\vec{v}_3(\vec{v}_2 \times \vec{v}_3) \cdot \vec{v}_1 - \vec{v}_1(\vec{v}_2 \times \vec{v}_3) \cdot \vec{v}_3] \\
&= -\frac{3}{2}V\vec{v}_3
\end{aligned} \tag{8.5}
$$

where we used the identity $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{C}(\vec{A} \cdot \vec{B}) - \vec{B}(\vec{A} \cdot \vec{C})$ and (8.4). Now

$$
\begin{aligned}
(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 &= \frac{3}{4}V\vec{v}_3 \cdot (\vec{v}_1 \times \vec{v}_2) \\
&= \frac{9}{2}V\frac{1}{6}\vec{v}_1 \cdot (\vec{v}_2 \times \vec{v}_3) \\
&= \frac{9}{2}V^2.
\end{aligned} \tag{8.6}
$$

So that

$$
V^2 = \frac{2}{9}(\vec{L}_1 \times \vec{L}_2) \cdot \vec{L}_3 = \frac{2}{9}\epsilon_{ijk}L_1^i L_2^j L_3^k = \frac{2}{9}\det L. \tag{8.7}
$$

(iii) Angles between edges.

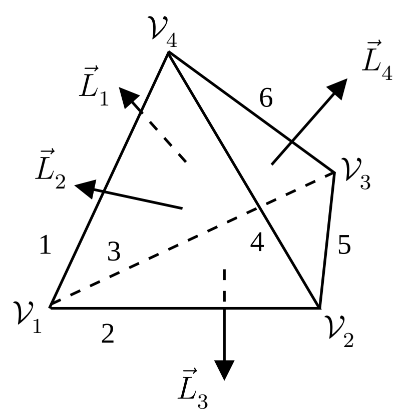


Figure 8.2: Edges.

As with (8.5) we can derive

$$
\begin{aligned}
\vec{L}_1 \times \vec{L}_2 &= -\frac{3}{2}V\vec{v}_3 \\
\vec{L}_3 \times \vec{L}_1 &= -\frac{3}{2}V\vec{v}_2 \\
\vec{L}_2 \times \vec{L}_3 &= -\frac{3}{2}V\vec{v}_1
\end{aligned} \tag{8.8}
$$

Therefore we can write for the angles at vertex $\mathcal{V}_1$

$$
\begin{aligned}
\cos\theta_{12} &= \frac{\vec{v}_1 \cdot \vec{v}_2}{|\vec{v}_1||\vec{v}_2|} = \frac{(\vec{L}_2 \times \vec{L}_3) \cdot (\vec{L}_3 \times \vec{L}_1)}{|\vec{L}_2 \times \vec{L}_3||\vec{L}_3 \times \vec{L}_1|} \\
\cos\theta_{23} &= \frac{\vec{v}_2 \cdot \vec{v}_3}{|\vec{v}_2||\vec{v}_3|} = \frac{(\vec{L}_3 \times \vec{L}_1) \cdot (\vec{L}_1 \times \vec{L}_2)}{|\vec{L}_3 \times \vec{L}_1||\vec{L}_1 \times \vec{L}_2|} \\
\cos\theta_{31} &= \frac{\vec{v}_3 \cdot \vec{v}_1}{|\vec{v}_3||\vec{v}_1|} = \frac{(\vec{L}_1 \times \vec{L}_2) \cdot (\vec{L}_2 \times \vec{L}_3)}{|\vec{L}_1 \times \vec{L}_2||\vec{L}_2 \times \vec{L}_3|}
\end{aligned} \tag{8.9}
$$

By symmetry at vertex $\mathcal{V}_2$ ($123 \rightarrow 254$) we have

$$
\begin{aligned}
\cos\theta_{25} &= \frac{(\vec{L}_5 \times \vec{L}_4) \cdot (\vec{L}_4 \times \vec{L}_2)}{|\vec{L}_5 \times \vec{L}_4||\vec{L}_4 \times \vec{L}_2|} \\
\cos\theta_{54} &= \frac{(\vec{L}_4 \times \vec{L}_2) \cdot (\vec{L}_2 \times \vec{L}_5)}{|\vec{L}_4 \times \vec{L}_2||\vec{L}_2 \times \vec{L}_5|} \\
\cos\theta_{42} &= \frac{(\vec{L}_2 \times \vec{L}_5) \cdot (\vec{L}_5 \times \vec{L}_4)}{|\vec{L}_2 \times \vec{L}_5||\vec{L}_5 \times \vec{L}_4|}
\end{aligned} \tag{8.10}
$$

Similar results hold for vertices $\mathcal{V}_3$ and $\mathcal{V}_4$.

8.2 Three dimensional Quantum Gravity

8.3 Simplices in 2,3,4 Dimensions

We start with a point - this is a zero-simplex. A edge is a 1-simplex. A triangle is a 2-simplex. A tetrahedra is a 4-simplex.

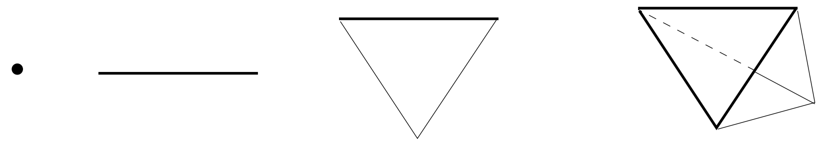


Figure 8.3: 0-simplex, 1-simplex, 2-simplex and 3-simplex.

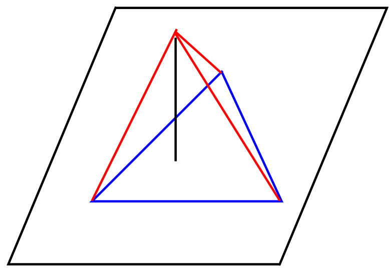


Figure 8.4: We cunstruct a 3-simplex by lifting the center of the 2-simplex.

8.3.1 A Crash Course in Simplexes and Complexes

Dimension of simplex	0	1	2	3	4
Number of vertices	1	2	3	4	5
Number of edges	0	1	3	6	10
Number of triangles	0	0	1	4	10
Number of tetrahedra	0	0	0	1	5

We cunstruct a 3-simplex by lifting the center of the 2-simplex - see (8.4). This implies that the number of edges of the n-simplex, $\#e_n$, goes as $0+1+\cdots+(n-2)+(n-1)+n$ so is $(n+1)n/2$ or

$$\#e_n = \frac{(n+1)n}{2}. \tag{8.11}$$

There are $n+1$ choices for the start point of an edge and n choices for the end point. Multiplying these numbers gives $(n+1)n$, but since this counts each edge twice, the total number of edges is $(n+1)n/2$, just as we found above.

Let us denote the number of triangles in an $n-$simplex as $\#t_n$. Then it is fairly easy to see that $\#t_n = \#t_{n-1}+$ [number of edges of the $(n-1)-$ simplex] or $\#t_n = \#t_{n-1} + \frac{n(n-1)}{2}$.So we need to solve:

$$\#t_n - \#t_{n-1} = \frac{n(n-1)}{2}.$$

We can use the trial solution $\#t_n = A(n+1)n(n-1)$,

$$\begin{aligned} A(n+1)n(n-1) - An(n-1)(n-2) &= An(n-1)[(n+1)-(n-2)] \\ &= 3An(n-1), \end{aligned}$$

so works if we choose $A = 1/6$. The number of triangles is

$$\#t_n = \frac{(n+1)n(n-1)}{6}, \tag{8.12}$$

(note this obviously works for $n = 0$ and $n = 1$).

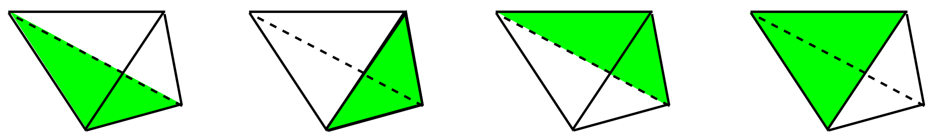


Figure 8.5: Structure of interaction vertex.

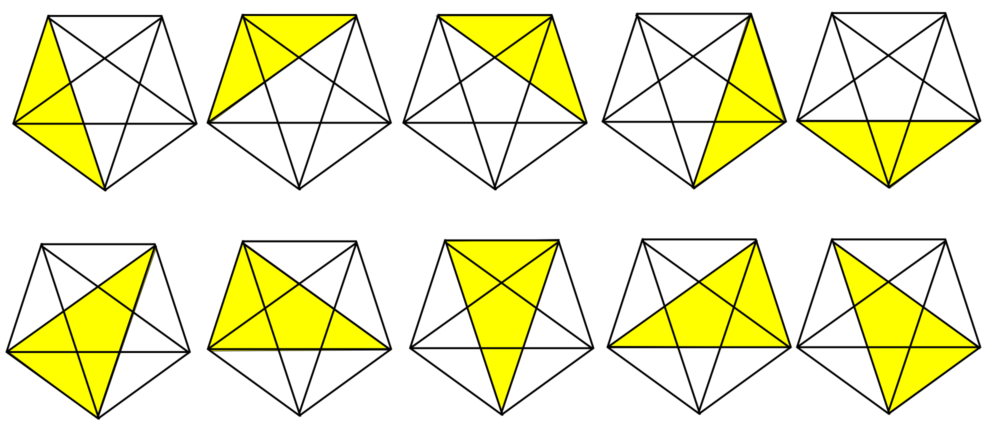


Figure 8.6: 4-simplex. All 10 faces of a 4-simplex.

Let us denote the number of tetrahedra in an $n-$simplex as $\#tet_n$. Then it is fairly easy to see that $\#tet_n = \#tet_{n-1}+$ [number of triangles of the $(n-1)-$ simplex] or $\#tet_n = \#tet_{n-1} + \frac{n(n-1)(n-2)}{6}$. So we need to solve:

$$\#tet_n - \#tet_{n-1} = \frac{n(n-1)(n-2)}{6}.$$

We can use the trial solution $\#t_n = A(n+1)n(n-1)(n-2)$,

$$
\begin{aligned}
A(n+1)n(n-1)(n-2) - An(n-1)(n-2)(n-3) \\
&= An(n-1)(n-2)[(n+1)-(n-3)] \\
&= 4An(n-1)(n-2),
\end{aligned}
$$

so works if we choose $A = 1/24$. The number of tetrahedra is

$$
\#tet_n = \frac{(n+1)n(n-1)(n-2)}{24}, \tag{8.13}
$$

(note this obviously works for $n = 0$, $n = 1$ and $n = 1$).

In an n-dimensional simplex there are

$$
C(n,k) = \frac{(n+1)!}{k!(n+1-k)!} \tag{8.14}
$$

$k-$simplicies.

8.3.2 Geometry of Simplicies

A 4-simplex in $\mathbb{R}^4$ is characterized by four vectors. These can be the vector pointing form one of the vertices. Let us denote these by e^I_a, where $I = 1, \ldots, 4$ is an index for a vector in $\mathbb{R}^4$, and $a = 1, \ldots, 4$ indecates a vertex at which the vector is directed. Thus, e^I_1 is a vector pointing from the vertex (0) to the vertex (1).

Consider a tetrahedron. Its geometry is determined by the 3 displacement vectors from one vertex. Alternatively, it can be determined by a set of 4 bivectors E_i satifying the *closure constraint*

$$
E_0 + E_1 + E_2 + E_4 = 0 \tag{8.15}
$$

These can be obtained for each triangle by taking the wedge product of the displacement vectors of the edges normal to then. The constraint says that the triangles close to form a tetrahedron.

Instead of vectors it is sometimes more convenuient to use the so-called bivectors E^{IJ} obtained by from two vectors $E^{IJ}_{ab} = e^{[I}_a e^{J]}_b$ where the brackets denote the antisymmetrization of the indices. These bivectors also characterize a 4-simplex and are obviously in one-to-one correspondance with faces of the 4-simplex h. For example, the bivector E^{IJ}_{12} corresponds to the face whose vertices are $(0), (1), (2)$. For the 4 vectors there correspond by 6 bivectors. The norm of each bivector is proportional to the squared area of the corresponding face

$$E^{IJ}_{ab}E_{abIJ} = 2A^2_{ab} \tag{8.16}$$

where A_{ab} is the area of the face $(0),(a),(b)$ with no summation over a,b assumed. This is a norm with the geometric interpretation of the area of the face

$$\begin{aligned}
|E_{ab}|^2 &= \frac{1}{2}(e^I_a e^J_b - e^J_a e^I_b)\frac{1}{2}(e_{aI}e_{bJ} - e_{aJ}e_{bI}) \\
&= \frac{1}{4}(e^I_a e_{aI}\, e^J_b e_{bJ} - e^I_a e_{bI}\, e^J_b e_{aJ} - e^J_a e_{bJ}\, e^I_b e_{aI} + e^J_a e_{aJ}\, e^I_b e_{bI}) \\
&= \frac{1}{2}(g_{aa}g_{bb} - g_{ab}g_{ab})
\end{aligned} \tag{8.17}$$

$$|E_{ab}|^2 = g_{aa}g_{bb} - g_{ab}g_{ab} \equiv 2A^2_{ab} \tag{8.18}$$

The volume of h can be obtained by wedging two bivectors that correspond to faces that do not share an edge. This is given by

$$V_h = \frac{1}{4!}\epsilon_{IJKL}E^{IJ}_{12}E^{KL}_{34}. \tag{8.19}$$

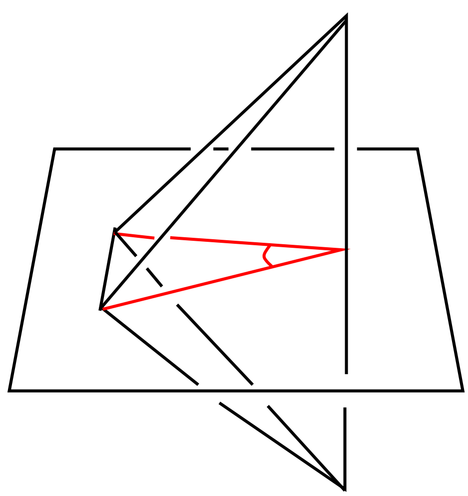


Figure 8.7: dihedralRov. The dihedral angle at the centre of one edge of a three-simplex.

3d Dihedral angles

$$E_{ab} \cdot E_{ac} = g_{aa}g_{bc} - g_{ab}g_{ac} = 2J_{aabc} \tag{8.20}$$

The angle between the infintesimal surface elements $dx^a dy^a$ and $dx^a dz^c$, if we take the scalar product of the normals of these two surface elements (in the 3-space they span),

$$A_{ab}A_{ac}\cos\theta_{aa\,bc} = g^{ef}(\epsilon_{egh}\delta^g_a\delta^h_b)(\epsilon_{fgh}\delta^g_a\delta^h_c) = g_{aa}g_{bc} - g_{ab}g_{ac} = 2J_{aabc} \tag{8.21}$$

8.3.3 The Dual Complex

In two dimensions the dual of a triangle is a point, and the dual of a edge is an edge.

In three dimension the dual of a tetrahedron is a point, the dual of a triangle is an edge, the dual of an edge is a triangle, and the dual of a point is a tetrahedron.

In four dimensions the dual of a 4-simplex is a point, the dual of a tetdrahedron is an edge, the dual of a face is a face, the dual of an edge is a tetdredron, and the dual of point is a 4-simplex.

Generally, in $d-$dimensions the dual of a $k-$simplex is a $(d-k)-$simplex.

The centres of the 4-simplices become the vertices of the dual complex. Connect the centres with an interval - these are the edges of the dual complex. The dual edges are in one-to-one correspondence with the tetrahedron t of the original simplicial decomposition.

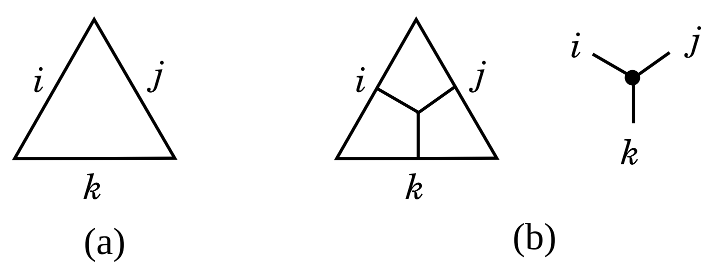


Figure 8.8: $1\triangle$

For a manifold $\mathcal{M}$ with an arbitrary cellular decomposition $\triangle$. There is a notion of the associated dual 2-complex of $\triangle$ denoted by $\triangle^*$. The dual 2-complex $\triangle^*$ is defined by a set of vertices $v \in \triangle^*$ (dual to 3-cells in $\triangle$) edges $e \in \triangle^*$ (dual to 2-cells in $\triangle$) and faces $f \in \triangle^*$ (dual to 1-cells in $\triangle$).

cellular decomposition $\triangle$ and associated dual 2-complex $\mathcal{F}_\triangle$.

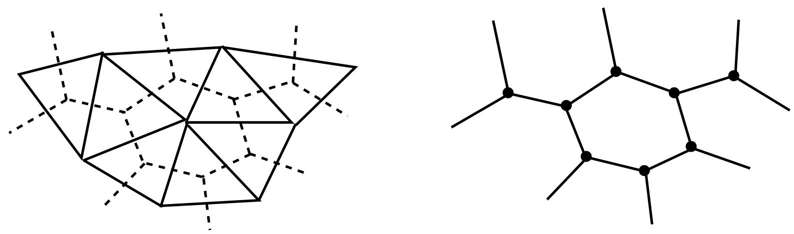


Figure 8.9: 2$\triangle$

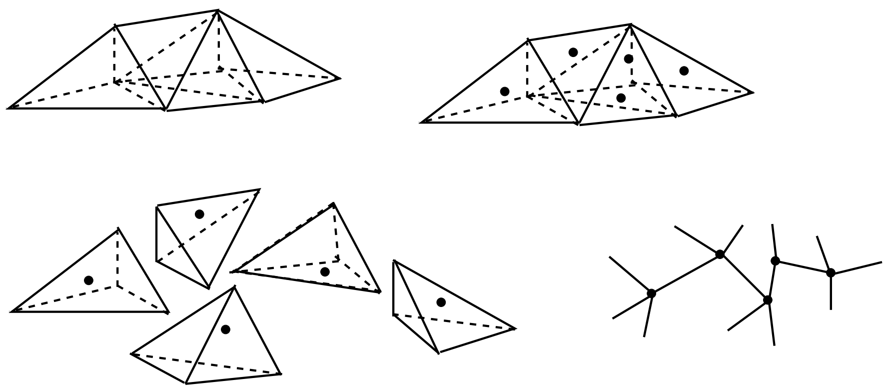


Figure 8.10: Part of a four-valent spin network corresponds to five tetrahedra of a cellular decomposition of 3-dimensional space.

vertices $v \in \mathcal{F}_\triangle$ (dual to 3-cells in $\triangle$)
edges $e \in \mathcal{F}_\triangle$ (dual to 2-cells in $\triangle$)
and faces $f \in \mathcal{F}_\triangle$ (dual to 1-cells in $\triangle$)

Each dual face f is goes inside, is surrounded by the triangles which all share the segment of $\triangle$ as a component of their boundary.

8.4 Barrett-Crane Model

Quantization is determined by the quantization of the topological field theory. As the constraints are non-derivative the gravitaional field has the same commutaion relations as the topological theory.

$$SO(4, \mathbb{C}) \tag{8.22}$$

Topological field theory:

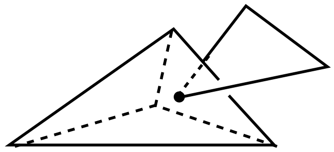


Figure 8.11: Structurex.

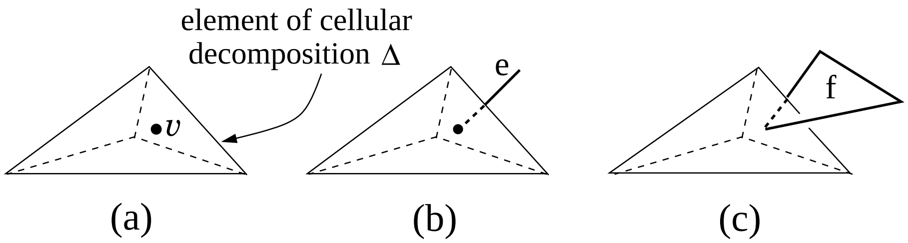


Figure 8.12: (a) $v \in \mathcal{F}_\triangle$ - dual to 3-cells in $\triangle$; (b) $e \in \mathcal{F}_\triangle$ - dual to 2-cells in $\triangle$; (c) faces $f \in \mathcal{F}_\triangle$ - dual to 1-cells in $\triangle$

$$S^{BF} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_i. \tag{8.23}$$

No local degrees of freedom:

$$F^i = -\Lambda B^i, \qquad \mathcal{D} \wedge B^i = 0 \tag{8.24}$$

We add a quadratic constraint

$$B^{(i} \wedge B^{j)} = \frac{1}{3}\delta^{ij} B^k \wedge B_k \tag{8.25}$$

The result is general relativity:

$$S_{Plebanski} = \int B^i \wedge F_i + \frac{\Lambda}{2} B^i \wedge B_I - \frac{1}{2}\phi_{ij} B^i \wedge B^j \tag{8.26}$$

One way to think of the affect of including the quadratic constraint is that it breaks the topological invariance.

Consider the action

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2) \tag{8.27}$$

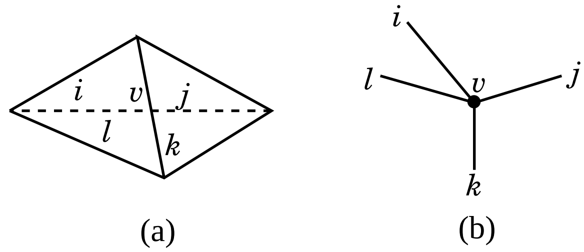


Figure 8.13: △dual3

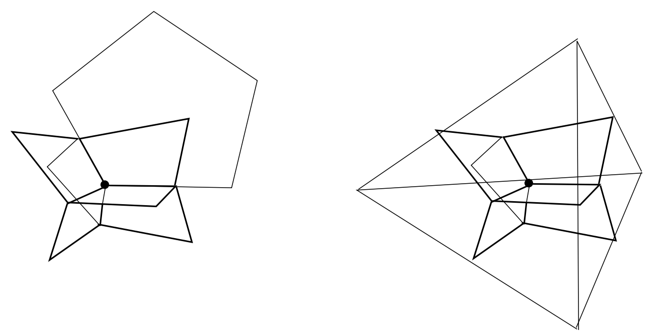


Figure 8.14: wedge.

which has two constrained degrees of freedom (q_1, q_2), which we assume to live on a circle, with conjugate momentum (p_1, p_2). This theory is completely constrained because both q_1 and q_2 must be zero. There are no degrees of freedom. Now let us impose another constraint with corresponding Lagrange multipier ξ. The action principle becomes

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 q_1 + \lambda_2 q_2 + \xi(\lambda_1 - \lambda_2)) \tag{8.28}$$

The two original Lagrange multipliers are constrained and we now have one degree of freedom: λ_1 has to be equal to λ_2 and thus only $q_1 + q_2$ has to be zero whereas the difference is free,

$$S = \int dt(\dot{q}_1 p_1 + \dot{q}_2 p_2 + \lambda_1 (q_1 + q_2)). \tag{8.29}$$

This mimics the transition from BF-theory to gravity where additional constraints (the simplicity constraints) reduce the freedom the original Lagrange multipliers of the BF-theory and thereby introduce local degrees of freedom.

The $SO(4)$ (or $SO(3,1)$) BF theory

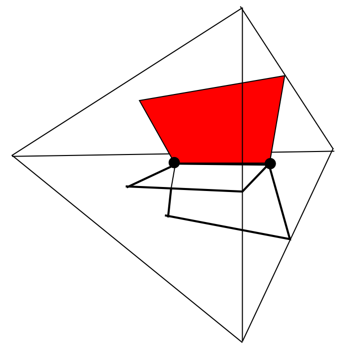


Figure 8.15: dualface3.

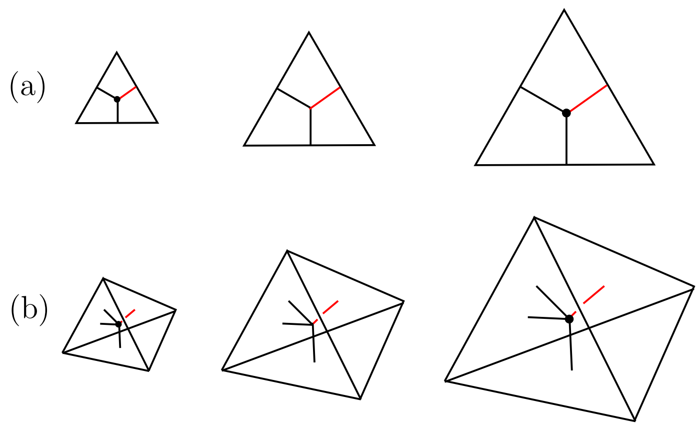


Figure 8.16: dualface3to4. (a) One dimension suppressed. Varing the normal coordinate sweeps out the 2-dimensional wedge in fig 8.15. (b) One dimension suppressed. Varing the normal coordinate sweeps out a 2-dimesional dual face corresponding to the dual edge from the centre of the 4-simplex to the centre of this boundary tetrahedron. There are 4 dual faces to the dual edge.

$$\int B_{\mu\nu}^{IJ} F(A)_{IJ\rho\sigma} \epsilon^{\mu\nu\rho\sigma} d^4x \tag{8.30}$$

where $F(A)_{IJ}^{\mu\nu}$ is the curvature of an $SO(4)$ connection and $B_{\mu\nu}^{IJ}$ is a Lie-algebra two form field. The simplicity constraints

$$\epsilon_{IJKL} B_{\mu\nu}^{IJ} B_{\nu\rho}^{KL} \propto \epsilon_{\mu\nu\rho\sigma} \tag{8.31}$$

The simplicty constraints ensures B comes from a tetrad field.

Sector I:

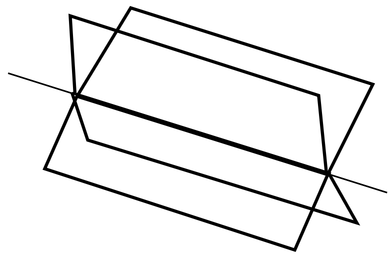


Figure 8.17: duale4duale. For a 4-simplex a dual edge e has four dual faces meeting at it.

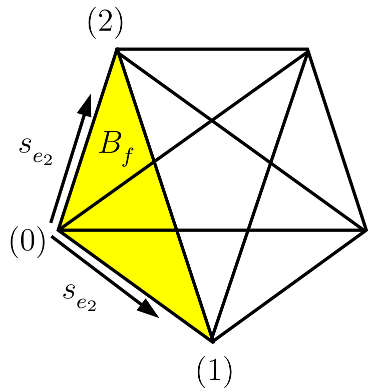


Figure 8.18: BCSim1. In the discretization of BF we assign a variable B_f to each triangle f of the triangulation. B_f can be taken to be the surface integral of B on f. We can discretize the gravitational field e, by assigning a variable e_s to each segment of the triangle. e_s can be taken to be the line integral of e along the segment s of the triangulation

$$B^{IJ}_{\mu\nu} = \pm e^I_\mu \wedge e^J_\nu \tag{8.32}$$

or

Sector II:

$$B^{IJ}_{\mu\nu} = \pm \epsilon^{IJ}{}_{KL} e^K_\mu \wedge e^L_\nu. \tag{8.33}$$

The case of sector II we obtain Einstein's theory and is referred to as the gravitational sector. In the case of sector I the corresponding thoery has no local degrees of freedom, this is the so-called topological sector.

8.4.1 The Partition Function

We approximate the B field with a distributional filed $B(t)$ with non-zero values on the triangles of the original triangulation. This gives an exact theory for a topological field theory like the BF one as there are no local degrees of freedom, but is only an approximation for gravity. However, this approximation would become better and better for more refined triangulations.

$$Z = \int \prod_t dg_t \prod_f \delta\left(\prod g_t\right), \tag{8.34}$$

The $\delta-$functions impose the constraint that the holonomy around each dual face is trivial - flatness.

$$\delta\left(g\right) = \sum_{j \in irredreps} d_j tr[\rho_j(g)] \tag{8.35}$$

where $tr[\rho_j(g)]$ is the character of g in the irreducible representation $\rho_j(g)$.

Simplicity constraints for 4-simplicies

Now we look into the details of the simplicity constraints for the 4-simplices which are the building elements of spinfoams.

(1) **Simplicity**. For each triangle Δ, the bivector B_Δ must satisfy

$$\epsilon_{IJKL} B^{IJ}_{\Delta} B^{KL}_{\Delta} = 0, \tag{8.36}$$

i.e. it is simple.

(2) **Cross simplicity**. when the triangles Δ and Δ' belong to the same tetrahedron, that is, when they share a common edge, we have

$$\epsilon_{IJKL} B^{IJ}_{\Delta} B^{KL}_{\Delta'} = 0.$$

Which implies that the sum $B_\Delta + B_{\Delta'}$ of two bivectors is again simple

$$\epsilon_{IJKL}(B^{IJ}_{\Delta} + B^{IJ}_{\Delta'})(B^{KL}_{\Delta} + B^{KL}_{\Delta'}) = 0. \tag{8.37}$$

(3)

Now we need to realize these conditions at the quantum level where the bivectors are represented by operators.

8.4.2 Hilbert Space of a Simplex

Quantizing a tetrahedron

$$
\begin{aligned}
\hat{E}^I_0 &= J^I \otimes 1 \otimes 1 \otimes 1 \\
\hat{E}^I_1 &= 1 \otimes J^I \otimes 1 \otimes 1 \\
\hat{E}^I_1 &= 1 \otimes 1 \otimes J^I \otimes 1 \\
\hat{E}^I_1 &= 1 \otimes 1 \otimes 1 \otimes J^I
\end{aligned}
\tag{8.38}
$$

with $I = 1, 2, 3$, and the closure constraint

$$\sum_i \hat{E}^I_i \psi = 0 \text{ for all } \psi \in \mathcal{H}^{\otimes 4}. \tag{8.39}$$

This is a quantization of the condition that the bivectors associated to the faces of a tetrahedron must sum to zero.

Infact the closure constraint maintains the $SU(2)$ invariance of the state ψ: the tensor product the four spaces carries a reducible representation of $SU(2)$, that can be decomposed into its irreducible components. The spin zero component is the $SU(2)$ invariant part of the tensor product. The linear space of such states, satisfying the closure constraint, form the Hilbert space of a quantum tetrahedron denoted

$$\mathcal{H}_0 := Inv(j_0 \otimes j_1 \otimes j_2 \otimes j_3) \tag{8.40}$$

This constraint ensures that the norms of these bivectors can be interpreted as areas of the faces of a geometric tetrahedron.

We can think of a spin network living in the dual complex of the triangulation. Each edge intersects each individual traingle of the tetrahedron labelled by an irreducible representation of $SU(2)$.

Thus a spin network completely characterizes a state of the quantum tetrahedron.

Quantizing a 4-simplex

In the quantum theory the variables B^{IJ}_Δ by $so(4)$ generators J^{IJ}_Δ.

The sum of four bivectors corresponding to the faces of each a tetrahedron of the 4-simplex is zero. This is imposed by summing over simple representations

$$\sum_{j\in irredreps} d_j tr[\rho_j(g)] \;\rightarrow\; \sum_{simple\ reps} d_j tr[\rho_j(g)] \tag{8.41}$$

8.4.3 Simple Representations

A general element of the $so(4)$ algebra can be expressed by

$$\hat{X} = X^{IJ}\hat{J}_{IJ}, \tag{8.42}$$

where $\hat{J}_{IJ}$ are generators of $so(4)$.

$$[J_{IJ}, J_{KL}] = i\delta_{IK}J_{JL} - i\delta_{JK}J_{IL} - i\delta_{IL}J_{JK} + i\delta_{JL}J_{IK}. \tag{8.43}$$

There is a basis of generators, J^+_i, J^-_j, which form two commuting copies of $su(2)$,

$$[J^\pm_i, J^\pm_j] = i\epsilon_{ij}{}^k J^\pm_k, \quad [J^+_i, J^-_j] = 0. \tag{8.44}$$

We pick a fixed 4-vector $n_0 = (1,0,0,0)$ and break X^{IJ} into "time-like", X^{0i}, and spacial parts, $X^i = (1/2)\epsilon^i{}_{jk}X^{ij}$. We can then define self dual and anti-self dual parts.

Given J_{IJ} we can form dual and self-dual parts

$$J^+_i = \frac{1}{2}\Big(\frac{1}{2}\epsilon_i{}^{jk}J_{jk} + J_{0i}\Big)\ , \quad J^-_i = \frac{1}{2}\Big(\frac{1}{2}\epsilon_i{}^{jk}J_{jk} - J_{0i}\Big) \tag{8.45}$$

First we find the commutators for J_i's and J_{0i}

$$\begin{aligned}
[J_i, J_{i'}] &= \frac{1}{4}\epsilon_i{}^{jk}\epsilon_{i'}{}^{j'k'}[J_{jk}, J_{j'k'}] \\
&= i\frac{1}{4}\epsilon_i{}^{jk}\epsilon_{i'}{}^{j'k'}(\delta_{jj'}J_{kk'} - \delta_{kj'}J_{jk'} - \delta_{jk'}J_{kj'} + \delta_{kk'}J_{jj'}) \\
&= i\frac{1}{4}\epsilon_i{}^{jk}\epsilon_{i'j}{}^{k'}J_{kk'} - \dots \\
&= iJ_{i'i}
\end{aligned} \tag{8.46}$$

$$[J_i, J_{0i'}] = \frac{1}{2}\epsilon_i{}^{jk}[J_{jk}, J_{0i'}] = i\epsilon_{ii'}{}^k J_{k0}, \quad \text{and} \quad [J_{0i}, J_{0i'}] = iJ_{ii'} \tag{8.47}$$

$$\begin{aligned}
[J_i^+, J_{i'}^+] &= \frac{1}{4}[J_i + J_{0i}, J_{i'} + J_{0i'}] \\
&= \frac{1}{4}([J_i, J_{i'}] + [J_i, J_{0i'}] + [J_{0i}, J_{i'}] + [J_{0i}, J_{0i'}]) \\
&= i\frac{1}{4}(J_{ii'} - J_{i'i} + 2i\epsilon_{ii'}{}^k J_{k0}) \\
&= i\frac{1}{2}\big(\frac{1}{2}\epsilon_{ii'}{}^k \epsilon_k{}^{lm} J_{lm} + i\epsilon_{ii'}{}^k J_{k0}\big) \\
&= i\frac{1}{2}\epsilon_{ii'}{}^k (J_k + J_{0k}) \\
&= \epsilon_{ii'}{}^k J_k^+
\end{aligned} \tag{8.48}$$

Similarly we get for $J_i^- = (J_i - J_{0i})/2$

$$[J_i^-, J_j^-] = i\epsilon_{ij}{}^k J_k^-, \tag{8.49}$$

Given a J_i^+ and J_i^- such that $[J_i^\pm, J_j^\pm] = i\epsilon_{ij}{}^k J_k^\pm$ and $[J_i^\pm, J_j^\pm] = 0$ then it can be verified that

$$J_{ij} = \epsilon_{ij}{}^k (J_k^+ + J_k^-), \qquad J_{0i} = J_i^+ - J_i^-. \tag{8.50}$$

satisfies eq. (8.43). Thus there is a one to one relation between J_{IJ} and J_i^+, J_i^-. Thus the irreducible representations of $SO(4)$ have a one to one correspondance to two copies of $SU(2)$. (j^+, j^-). $so(4) \simeq su(2) \oplus su(2)$.

The number of generators of $so(4)$ is six as $J_{IJ} = -J_{JI}$. J_+ and J_- each span a three dimensional subalgebra of $so(4)$

$$\begin{aligned}
X^{IJ}J_{IJ} &= X^{k0}J_{k0} + X^{0k}J_{0k} + X^{ij}J_{ij} \\
&= (X^{k0} - X^{0k})(J_k^+ - J_k^-) + X^{ij}\epsilon_{ij}{}^k (J_k^+ + J_k^-) \\
&= (X^{k0} - X^{0k} + X^{ij}\epsilon_{ij}{}^k)J_k^+ + (X^{0k} - X^{k0} + X^{ij}\epsilon_{ij}{}^k)J_k^- \\
&= X^{+k}J_k^+ + X^{-k}J_k^-
\end{aligned} \tag{8.51}$$

A group element U of $SO(4)$ can be written

$$
\begin{aligned}
U &= \exp(X^{IJ}J_{IJ}) \\
&= \exp(X^{+k}J_k^+ + X^{-k}J_k^-) \\
&= \exp(X^{+k}J_k^+)\exp(X^{-k}J_k^-) \\
&= U_+U_-
\end{aligned} \tag{8.52}
$$

$$
\begin{aligned}
UU' &= \exp(X^{IJ}J_{IJ})\exp(Y^{IJ}J_{IJ}) \\
&= \exp(X^{+k}J_k^+ + X^{-k}J_k^-)\ \exp(X'^{+k}J_k^+ + X'^{-k}J_k^-) \\
&= [\exp(X^{+k}J_k^+)\exp(X'^{+k}J_k^+)]\ [\exp(X^{-k}J_k^-)\exp(X'^{-k}J_k^-)] \\
&= (U_+U'_+)(U_-U'_-)
\end{aligned} \tag{8.53}
$$

We see that $SO(4)$ can be put in the form of the direct product of two copies of $SU(2)$, $SO(4) \simeq SU(2) \otimes SU(2)$. In component form where indices in an $SO(4)$ representation are given by couples of indices in an $SU(2)$ representation

$$
U^{(aa')(bb')} = U_+^{ab}\ U_-^{a'b'} \tag{8.54}
$$

are composite indices, with matrix multiplication

$$
U^{(aa')(bb')}U'^{(bb')(cc')} = (U_+^{ab}U_+'^{bc})(U_-^{a'b'}U_-'^{b'c'}) \tag{8.55}
$$

that is

$$
UU' = U_+U'_+ \otimes U_-U'_- \tag{8.56}
$$

acting on a vector in components

$$
U^{(aa')(bb')}x_{(bb')} = (U_+^{ab}u_b)(U_-^{a'b'}\ v_{b'}) \tag{8.57}
$$

and so

$$
U|x> = U(|u> \otimes |v>)\ = U_+|u> \otimes U_-\ |v> . \tag{8.58}
$$

We introduce the notation

$$
U = \exp(X^{ij}J_{ij}) = \exp\left(X^{ij}\epsilon_{ij}{}^k(J_k^+ + J_k^-)\right) = (g_+, g_-) \tag{8.59}
$$

$$(g_+, g_-)(g'_+, g'_-) = (g_+ g'_+, g_- g_-) \tag{8.60}$$

For the case of $SO(4)$ we use the notation $\mathbf{g} = (g_+, g_-)$ for an $SO(4)$ group element, $\mathbf{j} = (j_+, j_-)$ for an $SO(4)$ irreducible representation, $d_{\mathbf{j}} = d_{j_+} d_{j_-}, d_j = 2j + 1$ for the corresponding dimension and $tr_{\mathbf{j}_f}(\mathbf{g}) = tr_{j_f^+}(g^+) tr_{j_f^-}(g^-)$ for the characters.

The simple representations

Casimir invariants:

The Lie group $SO(4)$ has two Casimir invariants: the scalar Casimir

$$C = J_{IJ} J^{IJ} = |J|^2 \tag{8.61}$$

and the pseudo-scalar Casimir

$$\tilde{C} = \epsilon_{IJKL} J^{IJ} J^{KL}. \tag{8.62}$$

Now say we impose the condition

$$\epsilon^{IJKL} J_{IJ} J_{KL} = 0 \tag{8.63}$$

$$\begin{aligned} \epsilon^{IJKL} J_{IJ} J_{KL} &= \epsilon^{0JKL} J_{0J} J_{KL} + \epsilon^{iJKL} J_{iJ} J_{KL} \\ &= \epsilon^{0jkl} J_{0j} J_{kl} + \epsilon^{iJ0L} J_{iJ} J_{0L} + \epsilon^{iJkL} J_{iJ} J_{kL} \\ &= \epsilon^{jkl} J_{0j} J_{kl} + \epsilon^{ij0k} J_{ij} J_{0k} + \epsilon^{iJjk0} J_{ij} J_{k0} + \epsilon^{i0kl} J_{i0} J_{kl} \end{aligned} \tag{8.64}$$

with $\epsilon^{ijk} := \epsilon^{0IJK}$

$$\begin{aligned} \epsilon^{ijk} J_{0i} J_{jk} &= \epsilon^{ijk} (J_i^+ - J_i^-) \epsilon_{jk}{}^l (J_l^+ + J_l^-) \\ &= \delta^{il} (J_i^+ - J_i^-)(J_l^+ + J_l^-) \\ &= J^{+i} J_i^+ - J^{-i} J_i^- = 0 \end{aligned} \tag{8.65}$$

in terms of the Casimir invariants of $SU(2)$ the condition (8.63) becomes

$$(J^+)^2 - (J^-)^2 = 0 \tag{8.66}$$

This corresponds to a special representation of particular importance for which $j = j^+ = j^-$, this representation is called simple. Its algebra is that of the *diagonal* elements of $so(4)$ algebra: the ones of the form

$$X^i J_i^+ + X^i J_i^-. \tag{8.67}$$

Exponentiating these we get the *diagonal* elements of the $SO(4)$ group for which we write (g, g). These digonal elements form an $SU(2)$ subgroup of $SO(4)$, which depends on n. It is the subgroup of $SO(4)$ that leaves the vector n invaraint.

Adjoint action of $SO(4)$ - inducing two $SU(2)$ transformations

$$\begin{aligned}
\mathbf{X}' &= X^{IJ}\, \mathbf{g} J_{IJ} \mathbf{g}^{-1} \\
&= \exp(iY^{KL} J_{KL})\; X^{IJ} J_{IJ}\; \exp(-iY^{IJ} J_{KL}) \\
&= \exp(iY^{+k} J_k^+ + iY^{-k} J_k^-)\; (X^i J_i^+ + X^i J_i^-)\; \exp(-iY^{+k} J_k^+ - iY^{-k} J_k^-) \\
&= X^i \exp(iY^{+k} J_k^+) J_i^+ \exp(-iY^{+k} J_k^+) \\
&\qquad\quad + X^i \exp(iY^{-k} J_k^-) J_i^- \exp(-iY^{-k} J_k^-) \\
&= g^+ X^+ (g^+)^{-1} + g^- X^- (g^-)^{-1}
\end{aligned} \tag{8.68}$$

So under the adjoint action of $SO(4)$ the self-dual and anti-self-dual elements transform as $SU(2)$ Lie algebra elements:

$$\mathbf{g}\mathbf{X}\mathbf{g}^{-1} = g^+ X^+ (g^+)^{-1} + g^- X^- (g^-)^{-1}. \tag{8.69}$$

8.4.4 Quantizing a 4-Simplex

is of the form

$$\epsilon_{IJKL} J_\triangle^{IJ} J_\triangle^{KL} = (\vec{J}_\triangle^+)^2 - (\vec{J}_\triangle^-)^2 = 0. \tag{8.70}$$

This means that the $so(4)$ representation (j^+, j^-) associated with each triangle $\triangle$ must carry the same spin on its self-dual and anti self-dual part,

$$j^+ = j^-.$$

The closure relation for each tetrahedron

see fig (8.19)

$$B_{AB} + B_{AC} + B_{AD} + B_{AE} = 0 \tag{8.71}$$

for tetrahedron A, and so on. This is the discrete version of the Guass law ensuring the $SO(4)$ gauge invariance.

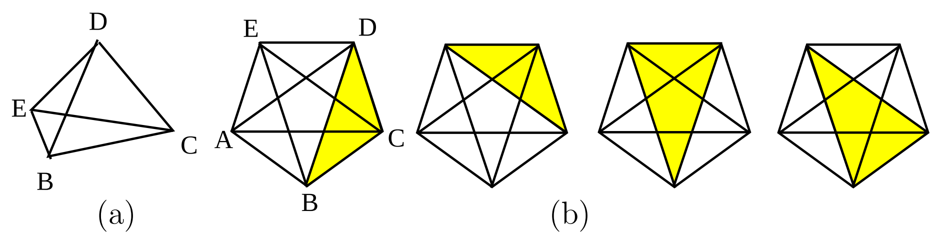


Figure 8.19: 4simclose. The closure condition for the tetrahedron A.

8.4.5 The Barrett-Crane Model

A tentative quantization of 4d Riemmanian GR on a fixed two-complex.

8.4.6 State Sum

How 6-j Symbols Appears:

Since every edge bounds three faces, each integral of the form

$$\int dg_{\sigma t}\, \rho^{j_1}(g_{\sigma t})^{\alpha}_{\alpha'}\, \rho^{j_2}(g_{\sigma t})^{\beta}_{\beta'}\, \rho^{j_3}(g_{\sigma t})^{\gamma}_{\gamma'} = v^{\alpha\beta\gamma\delta}\; v_{\alpha'\beta'\gamma'}. \tag{8.72}$$

The closure condition $\sum_a J_a = 0$ means we are restricted to $so(4)$-invariant states in the tensor product $\mathcal{H} = \otimes_a \mathcal{H}_{(j_a,j_a)}$,...

We use the standard recoupling basis of interwiners,

only simple representations will be including in the state sum

The interwiners are constrained by the simplicity constraints eq (??)-(??). Strongly imposing the cross-simplicity conditions $\epsilon_{IJKL} J_1 J_2 = 0$ forces the recoupled representation

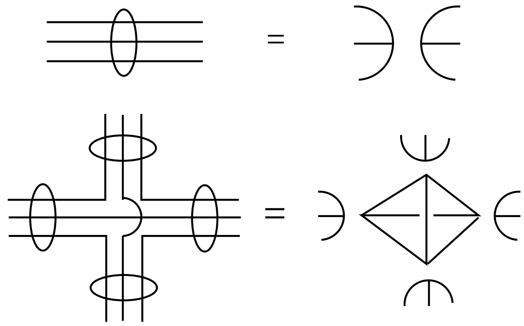


Figure 8.20: 6jappearsChD. How 6-j Symbols Appears.

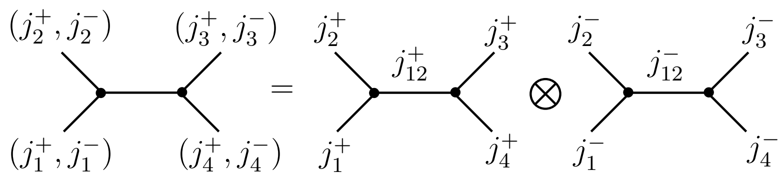


Figure 8.21: basisinter. The interwiner.

to be simple, $j_{12}^+ = j_{12}^-$. Imposing the othe cross-simplicity constraints $\epsilon_{IJKL} J_1 J_3 = 0$ and $\epsilon_{IJKL} J_1 J_4 = 0$ leads to a unqiue choice of interwiner, the Barrett-Crane interwiner.

$$i_{BC}^{(aa')(bb')(cc')(dd')} = \sum_j (2j+1)\; v^{abf} v^{fcd} v^{a'b'f'} v^{f'c'd'}, \tag{8.73}$$

where indices in an $SO(4)$ representation are given by couples of indices in an $SU(2)$ representation, and the indices f and f' are in the representation j. This is the Barrett-Crane interwiner.

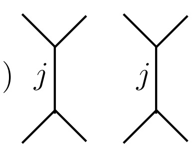
$i_{BC} = \sum_j (2j+1)$ 

Figure 8.22: BCintdecom. The BC interwiner.

$$Z_{BC} = \sum_{simple\ j_f} \prod_f \dim(j_f) \prod_v \{10j\}. \tag{8.74}$$

The set of two-complexes summed over is formed by a single two-complex. This is chosen to be the two-skeleton of the dual of a 4d triangulation.

$$Z_{BC} = \sum_{simple\ j_f} \prod_f \dim(j_f) \ \prod_v$$

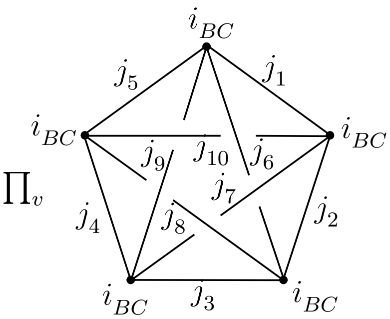


Figure 8.23: tenjsymbBC. Barrett-Crane partition function.

$$A(j_1 \dots j_{10}) = \sum_{i_1 \dots i_5}$$

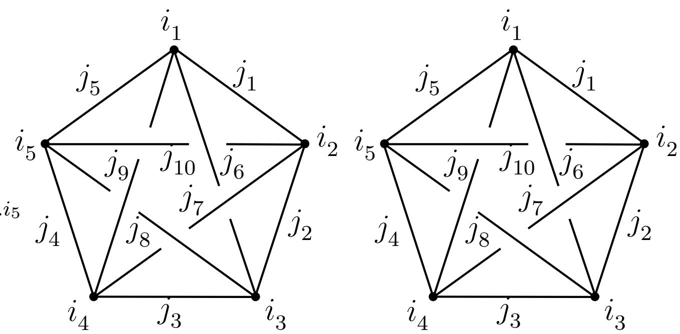


Figure 8.24: tenjsymAmpl. This depends on ten spins and is called the $10j$ symbol. The vertex amplitude for the BC model.

$$1_{\mathcal{H}_0} = \sum |j_1, \dots j_F, i_1 \dots i_{F-3} >< j_1, \dots j_F, i_1 \dots i_{F-3}|. \tag{8.75}$$

8.5 Lorentzian Barrett-Crane Model

opened up the way to the Lorentzian generalization [197] of the original Barrett-Crane model in Euclidean signature.

8.6 The Difficulties with Barrett-Crane Type Models

cast doubt on the physical correctness of the Barrett-Crane model.

8.6.1 Discretization Dependence

at odds with background independence

8.6.2 Boundary State Space

The Barrett-Crane model impliments the the constraints strongly at the quantum level identifying the boundary states. Although this bounadry state space is similar to, it does not exactly match, that of LQG. The Hilbert space of the boundary states arising from the Barrett-Crane model are too poor to allow the right tensorial structure for the graviton propagator in the semiclassical limit (see chapter 9).

leads to an over constrained Hilbert space, with not enough degrees of freedom to describe the 3-geometry. It is more natural to impose these constraints more weakly, using for instance coherent states.

8.6.3 Link to Canonical Approach

BC model imposes the constraints too, strongly and there is not a proper match with the states of the canonical theory (LQG) living on the boundary.

8.7 New Spinfoam Models for Quantum Gravity

The difficulties are related to the fact that in the BC model the interwiner quantum numbers are fully constrained. This follows from the fact that the simplicity constraints are imposed as strong operator equations,

$$C_n \psi = 0. \tag{8.76}$$

However, these constraints are second class and the imposing of second class constraints strongly may lead to the incorrect elimination of physical degrees of freedom [Dirac]. As we will see the simplicity constraints can be imposed weakly,

$$< \phi C_n , \psi > = 0. \tag{8.77}$$

i.e., the simplicity constraints hold at the level of expectation values. A familiar example is the Gupta-Bleuer quantization of Maxwell's field (see []??). If the Lorentz constraint $\partial_\mu A^\mu(x) = 0$ strongly as a quantum constraint

$$\partial_\mu \hat{A}^\mu | \psi > = 0$$

leads to the elimination of phsical degrees of freedom. However if only the positive frequency part is imposed strongly

$$\partial_\mu \hat{A}^{\mu+}|\psi>=0$$

We take this as the restriction on states which are allowed by the theory. This condition implies in its adjoint form

$$<\psi|\partial_\mu \hat{A}^{\mu-}=0$$

so that the Lorentz constraint is imposed weakly

$$<\phi|\partial_\mu(\hat{A}^{\mu+}+\hat{A}^{\mu+})|\psi>=<\phi|\partial_\mu \hat{A}^{\mu}|\psi>=0.$$

This ensures that the Lorenz condition and hence Maxwell's equations hold in the classical limit of the theory.

8.8 EPR

Acheivments of this approach

(i) the geometric interpretation for all the variables become fully transparent;

(ii) the boundary states fully capture the gravitational field variables;

and

(iii) correspond precisely to the spin network states of $SO(3)$ LQG. The boundary states of the model are precisely the eigenstates of the same quantities as the corresponding LQG states. It provides a novel independent derivation of LQG kinematics, in particular of the quantization of area and volume.

(iv) the vertex of this theory is similar to the BC vertex, but the corresponding dynamics may have a better low-energy behaviour and yield the correct low-enery (graviton) $n-$point functions (see chapter 9).

8.8.1 Outline of Derivation

We discretize GR via Regge and quantize.

Constraints on the Hilbert space associated to a bounadry of the triagulation.

Dynamics by giving the amplitude associated to each 4-simplex.

8.8.2 Area

$$A_f^2 = \frac{1}{2} B_f^{ij}\ B_{f\ ij} \tag{8.78}$$

Writing this in terms of the momenta J gives

$$\begin{aligned}
A^2 &= \frac{1}{2} B_f^{ij}\ B_{f\ ij} \\
&= \frac{8\pi\hbar G\gamma}{\gamma^2-1}(J^{ij} - \frac{1}{\gamma}\ {}^*J^{ij})\ \frac{8\pi\hbar G\gamma}{\gamma^2-1}(J_{ij} - \frac{1}{\gamma}\ {}^*J_{ij}) \\
&= \left(\frac{8\pi\hbar G\gamma}{\gamma^2-1}\right)^2 \left(\frac{1}{2}C_1 - \left(1-\frac{1}{\gamma^2}\right)L^2 - \frac{1}{2\gamma}C_2\right) \\
&= \left(\frac{8\pi\hbar G\gamma}{\gamma^2-1}\right)^2 \left(\frac{1}{2}\frac{\gamma}{2}\left(1+\frac{1}{\gamma^2}\right) - \left(1-\frac{1}{\gamma^2}\right) - \frac{1}{\gamma}\right)L^2 \\
&= (8\pi\hbar G\gamma)^2\gamma^2 L^2
\end{aligned} \tag{8.79}$$

That is

$$A = 8\pi\hbar G\gamma\sqrt{k(k+1)} \tag{8.80}$$

8.9 Freidel, Krasnov

8.9.1 New Geometric Interpretation of Simplicity Constraints

Lemma 8.9.1 *A bivector X^{IJ} in $\mathbb{R}^4$ is an anti-symmetrized product of two vectors if and only if there exists a vector n^I such that $X^{IJ}n_J = 0$.*

Proof. First we prove that if a bivector is simple it defines a two-plane. To do this, pick a direction such that X^{IJ} in its orthogonal basis vectors components of X^{IJ} in are is X^{0i} then in this subspace the condition $\epsilon_{IJKL}X^{IJ}X^{KL} = 0$ becomes

$$X^{0j}(\epsilon_{jkl}X^{kl}) \quad = \quad X^{0j}\tilde{X}_j = 0, \tag{8.81}$$

A rotation about the vector X^{0i} produces vectors spanning a two-plane.

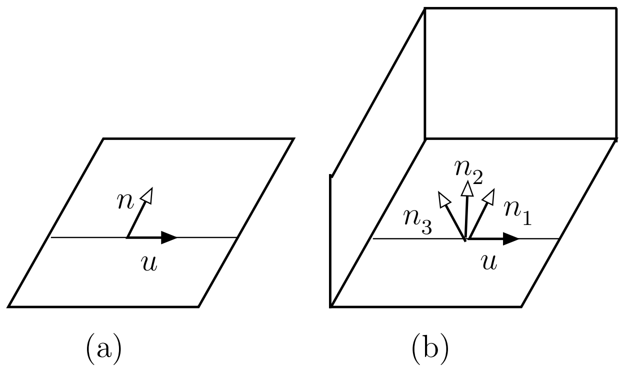


Figure 8.25: newgeom4. (a) In two dimensions there is only one independent norm $\vec{n}$ to the line with tangent $\vec{u}$. (b) In three dimensions there are many normals to $\vec{u}$ spaning a plane.

Now we move on to proving the lemma. A simple bivector defines a two-plane in $\mathbb{R}^4$. In $\mathbb{R}^4$ there is more than one normal vector - see fig.(8.25). Taking n^I to be any of the vectors orthogonal to this plane proves the only if case.

Now assume there exists a vector n^I such that $X^{IJ}n_J = 0$. Take the vector $n_0 = (1, 0, 0, 0)$ then $X^{0i} = 0$. The spacial part of the bivector, $X_i = \frac{1}{2}\epsilon_{ijk}X^{ij}$, is orthogonal to n^0. Now any vector in three dimensions can be written as the vector product of two 3-vectors $\vec{u}$ and $\vec{v}$, $X_i = \epsilon_{ijk}u_j v_k$. Contracting this with ϵ_{ijk} gives

$$X_{jk} = u_{[j}v_{k]}. \tag{8.82}$$

Defining the 4-vectors $u = (0, \vec{u})$ and $v = (0, \vec{v})$, eq.(8.82) and $X^{i0} = 0$ can be written

$$X^{IJ} = u^{[I}v^{J]}. \tag{8.83}$$

As this it is expressed as a 4-tensor equation it holds in general. This proves the if case.

An alternative proof that X^{IJ} is simple is based on the following. In three diemsions it is a straightforward fact that any bivector can always be written in the form $u^{[i}v^{j]}$ where u and v are vectors, visualized as a parallelogram. In dimensions greater than three the sum of two bivectors is not always expressible in the form $u^{[I}v^{J]}$. In general they are expressed as

$$u_1^{[I}v_1^{J]} + u_2^{[I}v_2^{J]}, \tag{8.84}$$

which can be visualised a pair of parallelograms. Moreover, there always exists four vectors u_1, u_2, v_1, v_2 such that a bivector can be written in this form and u_1, u_2, v_1, v_2 span $\mathbb{R}^4$ forming a vector basis.

So assume that X^{IJ} is not simple. It can always be expressed as in (8.84) where u_1, u_2, v_1, v_2 can be taken as a basis. The vector n^I can be represented as a linear combination these basis vectors. It is then easy to see that the condition $X^{IJ}n_j = 0$ implies the four vectors u_1, u_2, v_1, v_2 are linearly dependent contradicting the assumption that X^{IJ} is not simple.

□

Lemma 8.9.2 *Two simple bivectors X_1^{IJ} and X_2^{IJ} span three-dimensional subspace of $\mathbb{R}^4$ if and only if there exists a vector n^I such that $X_1^{IJ}n_J = 0$ and $X_2^{IJ}n_J = 0$.*

Proof. If two bivectors are simple and the two-planes defined by them span a three-dimensional subspace, then n^I can be chosen to be a vector orthogonal to this subspace. This proves the only if case.

□

Lemma 8.9.3 *A bivector X_1^{IJ} in $\mathbb{R}^4$ is an antisymmetrized product of two vectors if and only if there exists a vector such that $\tilde{X}^{IJ}n_J = 0$.*

Proof. We need to show a bivector X^{IJ} is simple if and only if its dual bivector $\tilde{X}^{IJ}$ is simple. We know

$$4!\delta_{I_1}^{[J_1}\delta_{I_2}^{J_2}\delta_{I_3}^{J_3}\delta_{I_4}^{J_4]} = \epsilon^{J_1J_2J_3J_4}\epsilon_{I_1I_2I_3I_4}.$$

We use this to write

$$\begin{aligned} n_{[I_1}n_{I_2}n_{I_3}n_{I_4]} &= n_{[J_1}n_{J_2}n_{J_3}n_{J_4]}\delta_{I_1}^{[J_1}\cdots\delta_{I_4}^{J_4]} \\ &= \left(\frac{1}{4!}n_{[J_1}n_{J_2}n_{J_3}n_{J_4]}\epsilon^{J_1J_2J_3J_4}\right)\epsilon_{I_1I_2I_3I_4} \\ &= \alpha\epsilon_{I_1I_2I_3I_4}. \end{aligned} \tag{8.85}$$

Now say $X^{IJ}n_J = 0$ then

$$\tilde{X}_{IJ}n^J = \frac{1}{2}\epsilon_{IJKL}X^{KL}n^J = \frac{1}{2\alpha}n_{[I}n_Jn_Kn_{L]}X^{KL}n^J = 0.$$

As the dual of $\tilde{X}^{IJ}$ is X^{IJ},

$$
\begin{aligned}
\tilde{\tilde{X}}^{IJ} &= \frac{1}{2}\epsilon^{IJ}{}_{KL}\tilde{X}^{KL} \\
&= \frac{1}{2}\epsilon^{IJ}{}_{KL}(\frac{1}{2}\epsilon^{KL}{}_{MN}X^{MN}) \\
&= \frac{1}{2}(\delta^I_M\delta^J_N - \delta^I_N\delta^J_M)X^{MN} \\
&= X^{IJ},
\end{aligned} \tag{8.86}
$$

the converse is also true. Then this lemma follows from lemma 8.9.1.

□

Lemma 8.9.4 *Two simple bivectors X_1^{IJ} and X_2^{IJ} span a three-dimensional subspace of $\mathbb{R}^4$ if and only if there exists a vector such that $\tilde{X}_1^{IJ}n_J = 0$ and $\tilde{X}_2^{IJ}n_J = 0$.*

Proof. Two simple bivectors $X_{1,2}^{IJ}$ span a three-dimensional subspace if and only if their duals $\tilde{X}_{1,2}^{IJ}$ do, so this lemma follows from lemma.8.9.2.

□

Self- and anti-self-dual decomposition

$$
X^{\pm J}(X) = \frac{1}{2}\left(n_I\tilde{X}^{IJ} \pm \sqrt{\sigma}N_I X^{IJ}\right). \tag{8.87}
$$

$$
\begin{aligned}
X^{IJ}n_J = 0 &\quad\Leftrightarrow\quad X^{+J} = X^{-J} = n_I\tilde{X}^{IJ} \\
\epsilon^{IJKL}n_J X^{KL} = 0 &\quad\Leftrightarrow\quad X^{+J} = -X^{-J} = n_I X^{IJ}.
\end{aligned} \tag{8.88}
$$

The graviataional sector the X_f^{IJ} is the dual of the area bivectors A_f^{IJ}

$$
X_f^{IJ} = \frac{1}{2}\epsilon^{IJ}{}_{KL}A_f^{IJ}. \tag{8.89}
$$

there exists a unit vector n_t such that $n_{t1}\tilde{X}_f^{IJ} = 0$ There exists an $SO(4)$ transformation that brings the vector n_t^I

$$
(X_f, -X_f) \to (g^+X_f(g^+)^{-1}, -g^-X_f(g^-)^{-1}). \tag{8.90}
$$

the simplicity-intersection constraints on X_f for the gravitational sector can be expressed as

$$\mathbf{X}_f = (X_f, -n_t^{-1} X_f n_t). \tag{8.91}$$

8.9.2 Coherent States in the Spin Foam Model

8.9.3 Coherent States

Coherent states for $SU(2)$

$$1_j = \sum_m |j, m >< j, m|, \tag{8.92}$$

$$\delta_{mm'} = d_j \int_{SU(2)} dg\; t^j_{mj}(g) \overline{t^j_{m'j}(g)} \tag{8.93}$$

$t^j_{mj}(g)$ and $t^j_{mj}(gh)$ differ only by a phase for any group element h from the $U(1)$ subgroup of $SU(2)$. The $U(1)$ subgroup being of the form

$$\begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}. \tag{8.94}$$

$$1_j = d_j \int_{G/H} dn |j, n >< j, n|. \tag{8.95}$$

$$< j, n|\hat{J}^i|j, n > \sigma_i = j n \sigma_3 n^{-1} \equiv j n^i \sigma_i \tag{8.96}$$

or

$$< j, \hat{n}|\hat{J}^i|j, \hat{n} > = < j, j|\hat{J}^{i\prime}|j, j > \tag{8.97}$$

where $J^{i\prime} = g(\hat{n})^{-1} J^i g(\hat{n})$ is the rotated generator.

Thus the state $|j, n >$ describes a vector in $\mathbb{R}^3$ of length j and of direction...

$$\Delta J^2 = j + j^2 - m^2 \tag{8.98}$$

$|j,\hat{n}>=g(\hat{n})|j,j>$, where $\hat{n}$ is a unit vector defining a direction on the sphere S^2 and $g(\hat{n})$ an $SU(2)$ group element rotating the direction $\hat{z}\equiv(0,0,1)$ into the direction $\hat{n}$.

Just as $|j,j>$ has direction z with minimal uncertainty, $|j,\hat{n}>$ has direction $\hat{n}$ with minimal uncertainty.

Thus, the highest and lowest states $m=\pm j$ minimize the uncertainty relation correspond to cohernt states.

$$|j,j> \quad \text{and} \quad |j,-j>$$

Coherent states for $SO(4)$

$$|j,j>\otimes|j,j> \tag{8.99}$$

for the gravitational sector

$$|j,g>\otimes\overline{|j,g>} \tag{8.100}$$

8.9.4 Partition Function

$$Z=\sum_{j_f}\prod_f d_{j_f}\int\prod_{(t,\sigma)}dg_{t\sigma}\prod_f\left(\prod_\sigma g_{\sigma t}g_{t\sigma'}\right). \tag{8.101}$$

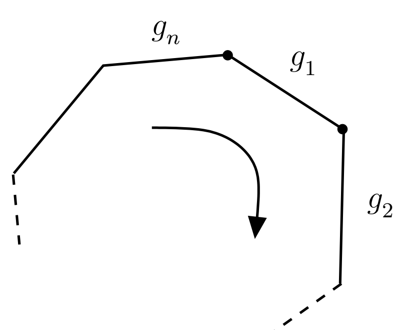


Figure 8.26: dualfacehol. .

We insert the decomposition of unity

$$1_j=d_j\int dn|j,n><j,n|, \quad |j,n>\equiv|j_+,n_->\otimes|j_-,n_->. \tag{8.102}$$

For example (see fig 8.30) one term is

$$\int dn_{tf} \int dn_{t'f} \ldots |j_f, n_{tf} >< j_f, n_{tf}| \; (g_{\sigma t})^{-1} g_{t\sigma'} \; |j_f, n_{t'f} >< j_f, n_{t'f}| \ldots \tag{8.103}$$

The partition function becomes

$$Z = \sum_{j_f} \prod_f d_{j_f} \int \prod_{(t,\sigma)} dg_{t\sigma} \prod_{t,f} d_{j_f} dn_{tf} \prod_{\sigma,f} < j_f, n_{tf}| \; (g_{\sigma t})^{-1} g_{\sigma t'} \; |j_f, n_{t'f}| > . \tag{8.104}$$

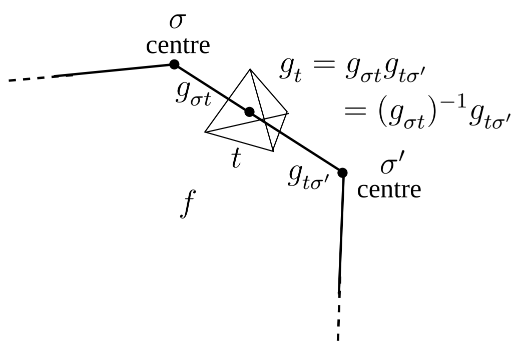


Figure 8.27: coherIncer. Schematic. .

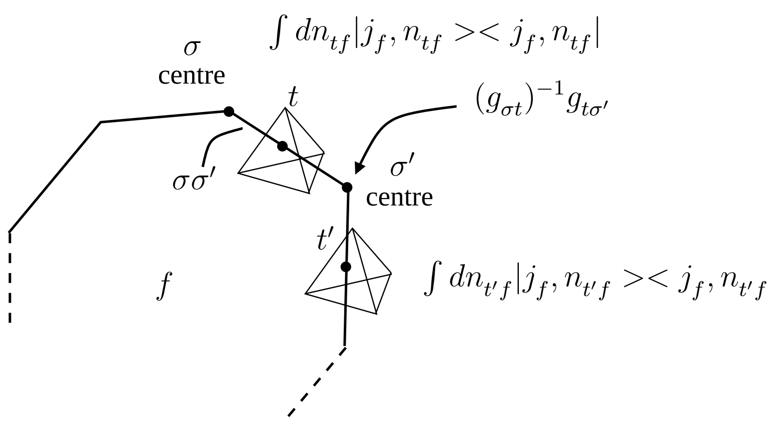


Figure 8.28: coherIncer2. Schematic. Have an edge $\sigma\sigma'$ in one-to-one correspondence with the common tetrahedra t. There is an insertion of a complete set of coherent states for each tetrahedron.

8.9.5 Imposing the Simplicty Constraints

it is possible to associate to each state $|\mathbf{j}_f, \mathbf{n}_{tf} >$ a bivector

$$\mathbf{X}_{(\mathbf{j}_f,\mathbf{n}_{tf})} = <\mathbf{j}_f,\mathbf{n}_{tf}|\hat{\mathbf{X}}|\mathbf{j}_f,\mathbf{n}_{tf}> \tag{8.105}$$

where $\hat{\mathbf{X}}$ denotes an $SO(4)$ Lie algebra element.

demanding that these bivectors satisfying the geometrical simplicity and cross simplicity conditions is equivalent to the requirement that there exists an $SU(2)$ rgoup element u_t such that

$$\mathbf{X}_{(\mathbf{j}_f,\mathbf{n}_{tf})} = (X_f, -u_t X_f u_t^{-1}). \tag{8.106}$$

rewrite the simplicity constraints in terms of the spins j_f and the $SU(2)$ elements n_f as:

$$j_f^+ = j_f^-, \qquad (n_{tf}^+, n_{tf}^-) = (n_{tf} h_{\phi tf}, u_t n_{tf} h_{\phi tf}^{-1} \epsilon) \tag{8.107}$$

where $\phi_{tf} \in [0, 2\pi]$.

8.9.6 The Error in the way the Constraints were Imposed in the BC Model

For a triangulation of a 4-dimensional manifold there are exactly 4 dual faces that share each paricular dual edge (see fig 8.16), and thus each group element $g_{t\sigma}$ enters into 4 characters. Integrating over these holonomies one produces a product of $15j$ symbols one for each 4-simplex

$$\int dg_{\sigma t}\, \rho^{j_1}(g_{\sigma t})^{\alpha}_{\alpha'}\, \rho^{j_2}(g_{\sigma t})^{\beta}_{\beta'}\, \rho^{j_3}(g_{\sigma t})^{\gamma}_{\gamma'}\, \rho^{j_4}(g_{\sigma t})^{\sigma}_{\sigma'} = \sum_i \nu_i^{\alpha\beta\gamma\delta}\, \nu^i_{\alpha'\beta'\gamma'\delta'}. \tag{8.108}$$

The edge amplitutdes require constraints are not imposed at each 4-simplex individually since a tetrahedron is shared by two 4-simplices.

If one ignores this requirement and impose the constraints at the level of each 4-simplex, the vertex amplitude reduces to that of the Barrett-Crane model. The error of imposing the constraints in the BG model comes from ignoring the fact that the face normals n_{tf} in two neighbouring 4-simplices should be the same.

8.9.7 The New Model

One imposes the following two constraints on the partition function of the BF theory:

(i) only sum over the simple representations be included in the state sum

and

(ii) instead of integrating over all $SO(4)$ group elements n_{tf} we integrate over the one having the form

$$\mathbf{n}_{tf} = (n_{tf} h_{\phi_t f}, u_t n_{tf} \epsilon h_{\phi_t f}^{-1}).$$

8.9.8 Boundary State Space

$$\mathcal{H}_{(j_1 j_1)\dots(j_4,j_4)} \tag{8.109}$$

8.9.9 Cubulations

8.10 Linear 2-Cell Spin Foam Models

Standard spin-foam models as formulated up to now are piecewise flat geometries defined on piecewise linear manifolds. For LQG canonical theory to match these models it would have to be restricted to the category of piecewise linear manifolds and piecewise linear spin-networks.

Can a spin foam model be formulated that can match the diffeomorphism invariant framework of loop quantum gravity?

The aim would be to generalise the spin-foam framework so that it can be used to define a spin foam history of an arbitrary spin-network state of LQG. This is achieved by defining spin foams on arbitrary linear 2-cell complexes instead of on just 2-simplicial complexes.

It also allows for a notion of embedded spin foams in which we can consider knotting or linking spin-foam histories.

The main tools of spin foams as described in the previous section can be successfully generalised to this new framework.

8.10.1 Foams

A 2-cell is a convex compact polygon.

Here we redefine a foam as an oriented linear 2-cell complex. Briefly, each foam κ consists of 2-cells (faces), 1-cell (edges), and 0-cells (vertices).

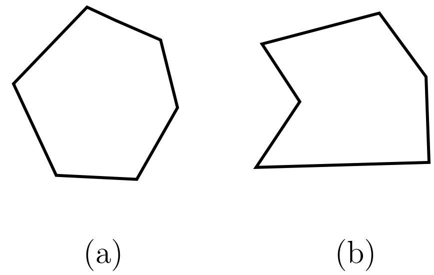


Figure 8.29: (a) A 2-cell (b) Not a 2-cell.

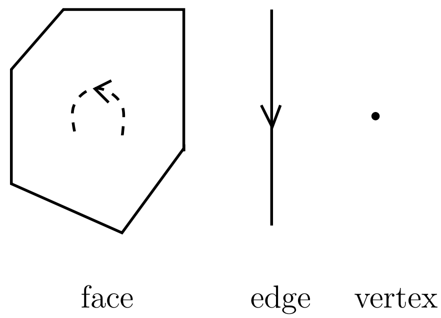


Figure 8.30: .

The faces are polygons.

Obviously triangulated manifolds qualify.

8.11 Group-Field-Theory

A spin foam model can be recast in the form of a rather peculiar field theory over the cartesian product of a group [].

given any spin foam model, there is an algorithm for constructing a field theory whose Feynmann expansion gives back the spin foam model.

$$W[s] = \int \mathcal{D}\Phi \; f_s[\Phi] \; e^{\int \Phi^2 - \lambda \int \Phi^5}. \tag{8.110}$$

8.12 Semi-Classical Limit

8.13 Reduced Phase Space Path Integral

8.13.1 Introduction

Path Integrals: Covariant formulation of quantum theory expressed as a sum over paths

Formally the path integral can be written as

$$\int \mathcal{D}q \exp^{iS[q]/\hbar} \qquad \text{or} \qquad \int \mathcal{D}q\mathcal{D}p \exp^{iS[q,p]/\hbar} \tag{8.111}$$

To define a path integral directly involves defining the components of these formal expressions:

1. What is the space of paths $q(t)$ over which we integrate?

2. $\mathcal{D}$: What is the measure on this space of paths?

3. $S[q]$: What is the phase associated with each path.

We hope that each of these components can be determined by constructing the path integral from the canonical theory.

We outline the standard construction of a path integral representation for Non-Relativistic Quantum Mechanics with a polynomial Hamiltonian, $H(q,p)$.

We want path integral representation of the propagator

$$< x'|e^{-iH\Delta t/\hbar}|x> = < x', t'|x, t> \tag{8.112}$$

Split the exponential into a product of N identical terms.

$$< x'|\prod_{n=1}^{N} e^{-i\hat{H}\epsilon/\hbar}|x> \qquad \text{where} \qquad \epsilon = \frac{\Delta t}{N} \tag{8.113}$$

Insert a complete basis in x between each exponential.

$$< x'|e^{-iH/\hbar}\int dx_{n-1}|x_{n-1}>< x_{n-1}|e^{-iH/\hbar}\ldots|x> \tag{8.114}$$

Step 2

By defining $x_N = x'$ and $x_0 = x$ this can be written in a simple form.

$$< x'|e^{-iH\Delta t/\hbar}|x> = \prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N} < x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}> \tag{8.115}$$

In the limit $N \to \infty$ $(\epsilon \to 0)$ we can expand each term in the 2nd product of () in ϵ.

$$< x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}> =< x_n|1 - i\hat{H}\epsilon/\hbar|x_{n-1}> +\mathcal{O}(\epsilon^2) \tag{8.116}$$

This can be written as an integral over momentum p_n.

$$< x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}> \;=\; \frac{1}{2\pi\hbar}\int dp_n e^{ip_n(x_n - x_{n-1})/\hbar}[1 - i\epsilon H(p_n x_{n-1}, x_n)/\hbar + \mathcal{O}(\epsilon^2)] \tag{8.117}$$

Step 3

Inserting the matrix elements $< x_n|e^{-i\hat{H}\epsilon/\hbar}|x_{n-1}>$ into the expression for the full matrix element and simplifying.

$$\begin{aligned} < x_n|e^{-i\hat{H}\Delta t/\hbar}|x_{n-1}> \;=\; & \lim_{N\to\infty}\prod_{m=1}^{N-1}\left[\int dx_m\right]\prod_{n=1}^{N}\left[\frac{1}{2\pi\hbar}\int dp_n\right] \\ & e^{i\epsilon\sum_{n=1}^{N}(p_n(x_n x_{n-1})/\epsilon\hbar)}\prod_{n=1}^{N}[1 - i\epsilon H(p_n, x_n, x_{n-1})/\hbar + \mathcal{O}(\epsilon^2)] \end{aligned} \tag{8.118}$$

The limit $N \to \infty$ defines the measure of the phase space path integral integral as the integral over the position at each time between t and t' and the integral over all momenta.

Almost in path integral form, except the final product. This can can be replaced according to

$$\lim_{N\to\infty}\prod_{n=1}^{N}[1 - i\epsilon H(p_n, x_n, x_{n-1})/\hbar + \mathcal{O}(\epsilon)] = \exp\left[-i/\hbar \lim_{N\to\infty}\sum_{n=1}^{N}\epsilon H(p_n, x_n, x_{n-1})\right] \tag{8.119}$$

The final result is

$$< x'|e^{-iH\Delta t/\hbar}|x> = \lim_{N\to\infty} \prod_{m=1}^{N-1} \left[\int dx_m\right] \prod_{n=1}^{N} \left[\frac{1}{2\pi\hbar} \int dxp_n\right] \exp\left(i/\hbar S[x,p,N]\right) \quad (8.120)$$

Where we recognise S as the discretised action.

$$S[x,p,N] = \sum_{n=1}^{N} \epsilon(p_n(x_n - x_{n-1})/\epsilon - H[x_n, x_{n-1}, p_n]). \quad (8.121)$$

In the limit $N \to \infty$ this is the action

$$S[x,p] = \int dt(p\dot{x} - H(x,p)). \quad (8.122)$$

8.13.2 Derivation of the Reduced Phase Space Path Integral

The reduced phase space path integral is formally derived as the generating functional of n-point functions. The path-integral is an integral over the reduced pahse space.

n-point functions

Consider the Heisenberg picture

$$Q_A(t) = e^{-iHt/\hbar} Q^A e^{iHt/\hbar}.$$

$$\tau^{A_1A_2}(t_1,t_2) :=< \Omega, T\{Q^{A_1}(t_1)Q^{A_2}(t_2)\}\Omega >$$

This can be written as

$$\tau^{A_1A_2}(t_1,t_2) = [\theta(t_1 - t_2)]W^{A_1A_2}(t_1,t_2) + [\theta(t_2 - t_1)]W^{A_2A_1}(t_2,t_1)$$

where we have defined the unordered Wightman function

$$W^{A_1A_2}(t_1,t_2) := < \Omega, Q^{A_1}(t_1)Q^{A_2}(t_2)\Omega >$$

$$
\begin{aligned}
\tau^{A_1A_2A_3}(t_1,t_2,t_3) &= [\theta(t_1-t_2)][\theta(t_2-t_3)]W^{A_1A_2A_3}(t_1,t_2,t_3) \\
&+[\theta(t_1-t_3)][\theta(t_3-t_2)]W^{A_1A_3A_2}(t_1,t_3,t_2) \\
&+[\theta(t_2-t_1)][\theta(t_1-t_3)]W^{A_2A_1A_3}(t_2,,t_1,t_3) \\
&+\cdots
\end{aligned}
$$

In general we have

$$
\tau^{A_1\ldots A_n}(t_1,\ldots,t_n) = \sum_{\pi\in S_n}\prod_{k=1}^{n-1}[\theta(t_{\pi(k)}-t_{\pi(k+1)}]W^{A_{\pi(1)}\cdots A_{\pi(n)}}(t_1,\ldots,t_n) \tag{8.123}
$$

where

$$
W^{A_1\ldots A_n}(t_1,\ldots,t_n) := <\Omega|Q^{A_1}(t_1)\cdots Q^{A_n}(t_n)\Omega> \tag{8.124}
$$

As $H\Omega = 0$ we have

$$
e^{-it_+H/\hbar}\Omega = e^{-it_-H/\hbar}\Omega = \Omega
$$

for any $t_\pm$. Using this we may write

$$
W^{A_1\ldots A_n}(t_1,\ldots,t_n) :=<\Omega|e^{iH(t_+-t_1)/\hbar}Q^{A_1}e^{iH(t_1-t_2)/\hbar}Q^{A_2}\cdots Q^{A_n}e^{iH(t_n-t_-)/\hbar}\Omega> \tag{8.125}
$$

for any $t_\pm$.

Path-integral for matrix elements $<\psi_f|U(t_f-t_i)|\psi_i>$

$$
<\psi_f|U(t_f-t_i)|\psi_i> \;=\; \int\{\prod_{n=0}^{N}[dQ_n]\}<\psi_f|Q_n><Q_n|Q_{n-1}>\cdots<Q_1|Q_0><Q_0|\psi_i>
$$

write $<\psi_f|Q_n>=\overline{\psi_f(Q_n)}$ and $<Q_0|\psi_i>=\psi_i(Q_0)$. Introduce a complete set of intermediate momentum eigenstates

$$
\begin{aligned}
<Q_n|Q_{n-1}> &= <Q(t_i+n\epsilon)|Q(t_i+(n-1)\epsilon)> \\
&= \int dP_n<Q(t_i+n\epsilon)|P_n><P_n|Q(t_i+(n-1)\epsilon)> \qquad (8.126)
\end{aligned}
$$

then

$$
\begin{aligned}
& < \psi_f | U(t_f - t_i) | \psi_i > \\
= & \int \{ \prod_{n=0}^{N} [dQ_n] \} \overline{\psi_f(Q_n)} \psi_i(Q_0) (\int dP_n < Q_n | P_n >< P_n | Q_{n-1} >) \cdots \\
& \cdots (\int dP_1 < Q_1 | P_1 >< P_1 | Q_0 >) \\
= & \int \{ \prod_{n=0}^{N} [dQ_n] \} \overline{\psi_f(Q_n)} \psi_i(Q_0)
\end{aligned} \tag{8.127}
$$

Trotter product formula - when A and B are self-adjoint operators

$$
e^{t(A+B)} = \lim_{n \to \infty} (e^{tA/n} e^{tB/n})^n
$$

It allows us to replace $\exp[\lambda(A+B)]$ by $\exp(\lambda A)\exp(\lambda B)$ when λ is small.

$$
\begin{aligned}
& e^{t(A+B)} - (e^{tA/n} e^{tB/n})^n \\
= & e^{t(A+B)} - e^{tA/n} e^{tB/n} \ldots e^{tA/n} e^{tB/n} \\
= & e^{t(A+B)} - e^{tA/n} e^{tB/n} e^{t(A+B)} (e^{tA/n} e^{tB/n})^{n-2} + \\
& e^{tA/n} e^{tB/n} e^{t(A+B)} (e^{tA/n} e^{tB/n})^{n-2} - (e^{tA/n} e^{tB/n})^2 e^{t(A+B)} (e^{tA/n} e^{tB/n})^{n-3} + \\
& (e^{tA/n} e^{tB/n})^{n-1} e^{t(A+B)} - (e^{tA/n} e^{tB/n})^n
\end{aligned} \tag{8.128}
$$

The Baker-Campbell-Hausdorff says

$$
\begin{aligned}
\exp^{(A+B)/n} &= \exp^{A/n} \exp^{A/n} \exp^{[A,B]/2n^2} \\
&= \exp^{A/n} \exp^{A/n} + \mathcal{O}(1/n^2)
\end{aligned}
$$

which implies

$$
\exp^{(A+B)} = (\exp^{A/n} \exp^{A/n})^n + \mathcal{O}(1/n^2)
$$

$$
< Q | P > = \prod_A \frac{\exp(-iQ^A P_A/\hbar)}{\sqrt{2\pi}} \tag{8.129}
$$

we obtain the formal result

$$< \psi_f | U(t_f - t_i) | \psi_i > \;=\; \int \{\prod_{n=0}^{N} [dQ_n]\} \, \{\prod_{n=0}^{N} [d(P_n/\sqrt{2\pi})]\} \overline{\psi_f(Q_n)} \psi_i(Q_0)$$
$$\exp(-i\frac{\epsilon}{\hbar} \sum_{n=1}^{N} \{[\sum_A \frac{Q_n^A - Q_{n-1}^A}{\epsilon} P_{A_n}] - H(Q_n, P_n)\}). \tag{8.130}$$

One now takes $N \to \infty$ and formally obtains

$$< \psi_f | U(t_f - t_i) | \psi_i > \;=\; \int [DQ][DP/\sqrt{2\pi}] \overline{\psi_f(Q(t_f))} \psi_i(Q(t_i)) \exp(-i\frac{\epsilon}{\hbar}[\sum_A \dot{Q}^A P_A] - H(Q, P)) \tag{8.131}$$

Path-integral for n-point functions

Note

$$W^{A_1 \ldots A_n}(t_1, \ldots, t_n) \;=\; \prod_{k=1}^{n} \int [dQ_k] \;\; < \Omega | U(t_+ - t_1) | Q_1 > Q_1^{A_1} \;\; \times$$
$$[\prod_{k=1}^{n-1} < Q_k | U(t_k - t_{k+1}) | Q_{k+1} > Q_{k+1}^{A_k + 1}] \;\; < Q_n | U(t_n - t_-) | \Omega > \tag{8.132}$$

$$W^{A_1 \ldots A_n}(t_1, \ldots, t_n) \;=\; \int [\mathcal{D}Q][\mathcal{D}P/\sqrt{2\pi}] \overline{\Omega(Q(t_+))} \Omega(Q(t_-)) \;\; \times$$
$$\exp(-\frac{i}{\hbar} \int_{t_-}^{t_+} dt \{[\dot{Q}^A P_A] - H(Q, P)\})) \; \prod_{k=1}^{n} Q_k^{A_k}(t_k) \tag{8.133}$$

where

$$[\mathcal{D}Q] = \prod_{t \in [t_-, t_+]} \prod_A dQ^A(t) \tag{8.134}$$

and similarly for $[\mathcal{D}P]$.

8.13.3 Canonical Path Integral Measures for Holst and Plebanski Gravity

8.13.4 Corrections to the Measure of the "New Spin Foam Models"

We have calculated the appropriate formal path integrals for Holst gravity and Plebanski gravity with Immiriz parameter determined by the canonical analysis.

The main difference between the formal path integral expression for Holst gravity derived above and the "new spin foam Models" that are supposed to be quantisation of Holst gravity are:

1) the appearance of the local measure,

2) the continuum rather than discrete formulation (triangulation)

3) lack of manifest spacetime covariance (if the continuum limit of spin foam models, if one could take it, should be spacetime covariance).

The next step is to propose a discretisation of the path integral derived in this section which takes into account the proper measure factor.

8.14 Covariant Loop Quantum Gravity

Covariant Loop Quantum Gravity (CLQG) is a program to quantize gravity employing loops within a Lorentz covariant canonical formulation [204].