Chapter 8

Extending Standard Quantum Mechanics for Background Independent Theories

- The Problem of Time.
- Covariant quantum mechanics.
- Multiple-Event Probability.
- General boundaries formulation of quantum mechanics.
- Emergence of Temporal Phenomena.
- Consistent discrete quantum gravity.

8.1 Introduction

The general-relativistic revolution of our understanding of space and time has proven extremely effective empirically. Conventional textbook quantum mechanics (QM), and conventional quantum field theory (QFT), however, are formulated in a language which is incompatible with the general relativistic notions of space and, especially, time. Is there a formulation of QM compatible with these notions? Such a formulation should be required, in particular, in order to provide a clear interpretative framework to any attempt to formulate a quantum theory of gravity in a form consistent with the general-relativistic understanding of space and time [??].

It is often claimed that either GR or quantum mechanics most go through a profound, radical change if we are to have a viable theory of quantum gravity. There is tension between the quantum mechanics that was formulated in the non-relativistic setting, this does not necessarily mean that there is tension between the underlying principles of general relativity and quantum mechanics.

Finally, in the opinion of Rovelli and collaberators, the idea that quantum mechanics admits a very simple and straightforward generalization which is fully consistent with general relativity. And therefore that the contradictions between quantum theory and general relativity might be only apparent.

Problems:

• describes mechanics in terms of evolution of observables and states in time.

• time is used to order a sequence of measurements.

• should we basis notions from non-relativist quantum mechanics that don't sit well with special relativistic mechanics and aren't general enough to emcumpass GR.

The fundamental theory is described by an extended Heisenberg-picture canonical quantum mechanics, equipped with the standard probabilistic interpretation.

8.2 The Problem of Time

The label of "the problem of time" is often given to a number of related, but slightly different isues.

Briefly put, the problem of time is as follows: how is one to apply quantum mechanics to general relativity in which a classical non-dynamical background time is missing?

8.3 Rovelli's Formulation of Classical Mechanics

We denote its boundary value $\varphi(s)$

$$\Phi(x(s)) =: \varphi(s). \tag{8.1}$$

The dynamics are governed by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi + \frac{1}{2} m^2 \Phi^2 \tag{8.2}$$

with boundary conditions on Σ.

8.4 Covaraint Quantum Mechanics

A general formulism which gives no privilege to the time variable.

8.4.1 States and Observables

does the measurement of a quantum system necessarily break Lorentz invariance? conventional notions of position and time do not exist.

The basic notions used in the formulation of quantum mechanics were developed for application in nonrelativistic context. They are not easily extendible to relativistic theories. Is there a version of the notions of "state" and "observer" that can be formulated in a naturally relativistic manner?

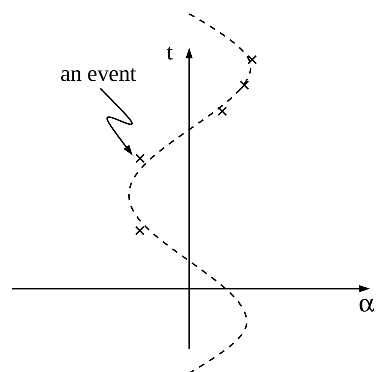

Figure 8.1: Coordinatized by the partial variables α and t. Rovelli calls **useful** measurement is an **event**. Rovelli calls this the **event space** or **relativistic configuration space**.

the motion as a relation between partial observables

$$f(\alpha, t) = \alpha - A\sin(\omega t + \phi) = 0 \tag{8.3}$$

Exact Lorentz invariant detector.

8.4.2 Spacetime States

In a sense, Heisenberg states have be thought of as representative of the history of the system.

In GR an instant in "time" is an abstract 3d spacial surface defined in terms of coordinates. We need to generalise to arbitrary surfaces bounding a region of spacetime.

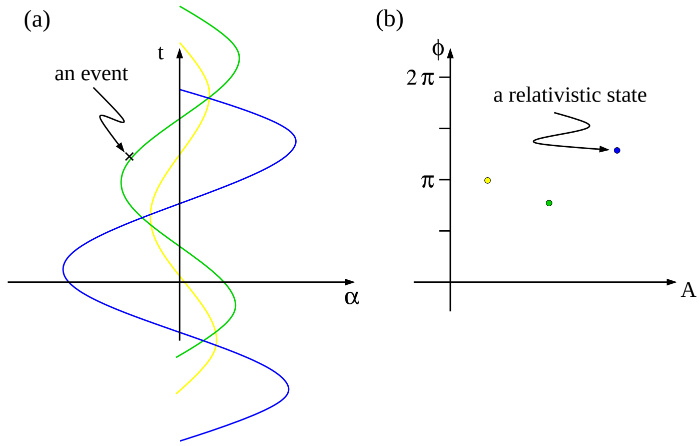

Figure 8.2: space of physical motions or relativistic phase space.

Given any compact support complex function $f(X,T)$

$$|f> = \int dXdT\ f(X,T)\ |X;T> \tag{8.4}$$

These states generalize the conventional wavefunctions for which $f(X,T) = \Psi(X)\delta(T)$. Conventional wavefunctions correspond to results of an instantaneous position measurement.

$$\begin{aligned} W(X,T;X',T') &= <X,T|X',T'> \\ &= \int \frac{d^3p}{2\pi\hbar}\ dE\ e^{(i/\hbar)[p(X-X')-E(T-T')]}\ \delta(E-p^2/2m) \\ &= \int \frac{d^3p}{2\pi\hbar}\ e^{(i/\hbar)[p(X-X')-p^2/2m(T-T')]} \\ &= \left(\frac{2\pi m}{i\hbar(T-T')}\right)^{\frac{1}{2}}\ \exp\left\{-\frac{m(X-X')^2}{2i\hbar(T-T')}\right\} \end{aligned} \tag{8.5}$$

When viewed as a function of X and T, with X' and T' fixed, this is a solution of Schrödinger's equation which at time $T=T'$ is a delta function in $X-X'$. Each function $\Psi(X)$ determines a solution of Schrödinger's equation by

$$\Psi(X,T) = \int dX'\ W(X,T;X',0)\ \Psi(X). \tag{8.6}$$

$$(i\hbar\frac{\partial}{\partial T} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2})W(X,T;X',T') = \delta(X-X')\delta(T-T') \tag{8.7}$$

$$\begin{aligned}\left(i\hbar\frac{\partial}{\partial T}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)\psi(X,T) &= \int dX'\left(i\hbar\frac{\partial}{\partial T}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)W(X,T;X',0)\,\Psi(X')\\ &= \int dX'\delta(X-X')\delta(T)\,\Psi(X')\\ &= 0. \end{aligned} \tag{8.8}$$

As is a function $f(X,T)$ with compact support there exists T_0, T_1 such that for $T > T_1$ and $T < T_0$ $f(X,T) = 0$

$$\begin{aligned}\left(i\hbar\frac{\partial}{\partial T}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)\psi_f(X,T) &= \int dX'dT'\left(i\hbar\frac{\partial}{\partial T}+\frac{\hbar^2}{2m}\frac{\partial^2}{\partial X^2}\right)W(X,T;X',T')\,f(X',T')\\ &= \int dX'\int_{T_0}^{T_1}dT'\,\delta(X-X')\delta(T-T')\,f(X',T')\\ &= 0 \text{ for } T > T_1 \end{aligned} \tag{8.9}$$

8.4.3 Review of General Relativistic Classical Mechanics

A Complete set of commuting observables

The outcomes of the measutement of a complete set of commuting observables charaterizes the state.

We can define an orthonormal basis in this Hilbert space by diagonalizing a completeset of commuting self-adjoint operators. Just as in the hydrogen atom where we use the self-adjoint operators J, j_z to form an orthonormal basis.

In a system with a finite number n of degrees of freedom, we choose $n+1$ partial observables (typically the n lagrangian variables plus the time variable), and form the $n+1$ dimensional extended configuration space, or event space, $\mathcal{C}$. The extended configuration space of a relativistic particle is the Minkowski space. The extended configuration space of a homogenous and isotropic cosmological model where a is the volume of the universe and ϕ is the matter density, is coordinatized by a and ϕ. Points in $\mathcal{C}$ are called events and denoted $s, s', s'', \ldots$: for instance, a point in Minkowski space $s = x^\mu = (\tilde{\mathbf{x}}, t)$, or a given value of radius of the universe and matter density $s = (a, \phi)$, define an event. Measuring a complete set of partial observables, that is, determining a point in $\mathcal{C}$ is to detect the happening of a certain event. For instance: a particle is detected in a point of Minkowski space, a certain value of radius of the universe and average energy density are measured, and so on. Each such detection describes an interaction of the system with another system, playing the role of observer.

8.4.4 Conditional Probability Interpretation

The definition of probability does not require any notion of time. The normalization is a prior requirement that follows from the very definition of probability. Similarly, total probability doesn't have to be conserved in time if there is no evaluation in time. Rather, conditional probability has to satisfy some requirement that follow from its physical interpretation, and it can be shown that these requirements reduce to unitary evolution if one is considering the "evolution" of the observables with respect to partial observables (clocks) that have suitable properties, may not exists in a system, exist only in some states, or exist but have these properties only within some approximation (see Peres 1980, Page and Wootters 1983).

Given a complete commuting set of partial variables, we ask if some subset takes the values such and such, then what are the probabilties that the remaining partial variables take the permissible values, say A, B or C? The definition of probability requires the probabilties of measuring A, B or C should add up to 1.

This requirement reduces to the usual one in the case where one of the partial observables forms an ideal clock. The associated transition amplitude A is given by $A =< \eta|U|\psi >$, where $U : \mathcal{H}_1 \to \mathcal{H}_2$ is the time-evolution operator of the system, evolving from t_1 to t_2. The simplest interpretation of $P = |A|^2$ is as expressing the probability of finding the state η at time t_2 given that the state ψ was prepared at time t_1. Thus, we are dealing with a conditional probability.

$$\mathcal{P}_{RR'} = \left| \frac{W(R,R')}{\sqrt{W(R,R)}\sqrt{W(R',R')}} \right|^2 . \tag{8.10}$$

8.4.5 Time of Arrival in Quantum Mechanics

8.5 Boundary Hilbert Space

$$\mathcal{K}_{t_1,t_2} = \mathcal{H}^*_{t_1} \otimes \mathcal{H}_{t_2} \tag{8.11}$$

The tensor product of two quantum state spaces describes the ensemble of the measurements described by the two factors. Therefore $\mathcal{K}_{\Sigma_T}$ is the space of the possible results of all measurements performed at $t = 0$ and at $t = T$ [**?**] Observations at two different times are correlated by the dynamics. Hence $\mathcal{K}_{\Sigma_T}$ is a kinematical state space in the sense that it describes more outcomes than the physically realizable ones. Dynamics is then a restriction on the possible outcome of observations []. It expresses the fact that measurement outcomes are correlated. The state $< 0_{\Sigma_T}|$, seen as a linear functional on $\mathcal{K}_{\Sigma_T}$, assigns an amplitude to any outcome of observations. This amplitude gives the correlation between outcomes at $t = 0$ and outcomes at $t = T$. Therefore the theory can be represented as follows. The Hilbert space $\mathcal{K}_{\Sigma_T}$ describes all possible outcomes of measurements made on σ_T. Dynamics is given by the single linear functional

$$\rho : \mathcal{K}_{\Sigma_T} \to \mathbb{C} \tag{8.12}$$

$$|\Psi > \mapsto < 0_{\Sigma_T}|\Psi > . \tag{8.13}$$

For a given collection of measurement outcomes described by a state $|\Psi >$, the quantity $< 0_{\Sigma_T}|\Psi >$ gives the correlation probability amplitude between these measurements.

8.5.1 General Relativistic Quantum Mechanics

The kernel of H, formed by the (possibly generalized) states in $\mathcal{K}$ satisfying the "Wheeler-DeWitt" equation

$$H\psi = 0 \tag{8.14}$$

is called the physical state space H. $\mathbb{P}$ is the linear (self-adjoint) operator $P : \mathcal{K} \to \mathcal{H}$, given by:

$$\mathbb{P}\psi = \int dn\, e^{-inH}\, \psi \tag{8.15}$$

If zero is in the discrete (resp. continuous) spectrum of H, then $\mathcal{H}$ is a proper (resp. generalized) eigenspace of $\mathcal{K}$. On the linear space $\mathcal{H}$ we consider the Hilbert structure

$$< \mathbb{P}s'|\mathbb{P}s >_{\mathcal{H}} := < s'|\mathbb{P}|s > \tag{8.16}$$

which is well defined when H is a generalized eigenspace. We also write $< s'|s >_{\mathcal{H}} =< \mathbb{P}s'|\mathbb{P}s >_{\mathcal{H}}$. Remark that since in general $\mathbb{P}$ is not a true projector, $< s'|\mathbb{P}|s >$ may very well be different from $< \mathbb{P}s'|\mathbb{P}s >$. In particular, this last quantity is in general divergent in the case in which H is a generalized eigenspace.

8.5.2 Single Event Probability

The probability $\mathcal{P}_{s \Rightarrow s'}$ of observing the event s_2 if the event s was observed (we shall write $\mathcal{P}_{s \Rightarrow s'}$ simply as $\mathcal{P}_{s'}$ when there is no need of indicating the initial state) is given by the modulus square of the amplitude

$$A_{s \Rightarrow s'} = < s|\mathbb{P}|s >, \tag{8.17}$$

where the states are normalized in $\mathcal{H}$, not in $\mathcal{K}$, that is,

$$< \mathbb{P}s|\mathbb{P}s >=< \mathbb{P}s'|\mathbb{P}s' >= 1. \tag{8.18}$$

$$\mathcal{P}_{s \Rightarrow s'} = \frac{| < s'|s >_{\mathcal{H}} |^2}{< s'|s' >_{\mathcal{H}} < s|s >_{\mathcal{H}}} = \frac{| < \mathbb{P}s'|\mathbb{P}s >_{\mathcal{H}} |^2}{< \mathbb{P}s'|\mathbb{P}s' >_{\mathcal{H}} < \mathbb{P}s|\mathbb{P}s >_{\mathcal{H}}} \tag{8.19}$$

Notice that this probability is a standard quantum mechanical probability computed in the physical Hilbert space $\mathcal{H}$, in the following sense. The states $|s>$ and $|s'>$ in $\mathcal{K}$ "project" down to physical states $|\mathbb{P}s>$ and $|\mathbb{P}s'>$ in $\mathcal{H}$. The probability (??) is then simply the standard probability amplitude of measuring the physical state $|\mathbb{P}s'>$ if the physical state $|\mathbb{P}s>$ was measured.

This definition of probability reduces to the conventional one in the non-relativistic case. In the spin system considered above, for instance, the states $\mathbb{P}|S, t>$, are normalized and the amplitude for measuring the spin S' at time t' if S at time t was measured is

$$A_{St \Rightarrow S't'} =< S', t'|\mathbb{P}|S, t >=< S'|e^{-iH_0(t'-t)}|S > \tag{8.20}$$

in agreement with conventional QM. More in general, in the case in which $H = p_t + H_0$, the definitions above reduce to the standard QM postulates regarding states, observables and probability.

8.6 Multiple-Event Probability

This section is mostly based on [298].

Time ordering does not appear to be a fundamental structure required for the definition of quantum theory and the calculation of its probability amplitudes. This can be seen as reinforcing the hypothesis that the fundamental theory of nature can be formulated in a timeless language [287], and that temporal phenomena could be emergent [288], [289], [290].

We state the problem and we point out a certain number of diffculties that emerge in trying to assign a probability to sets of events in a general relativistic context. In particular, we discuss the diffculties of two apparently "natural" solutions. The first is a direct generalization of the single event probability postulate: the probability of an ensemble of events is determined by the projection on the physical Hilbert space of the subspace of the kinematical Hilbert space associated to this ensemble of events. We show that this postulate is not viable because it does not reduce to the standard QM probabilities in the nonrelativistic case. The second is the use of conditional probability, widely discussed in the literature. We point out certain diffculties with the operational definition of this probability.

We indicate a possible general way for solving the problem. This is based on the observation that a multiple-event probability, such as $\mathcal{P}_{\psi \Rightarrow \psi' \psi''}$ can always be reinterpreted as a single-event probability, once the dynamics and the quantum nature of the apparatus making the measurements are taken into account. If we do so, the time order gets naturally coded into the dynamics of the system. This strategy provides a general way for dealing with multiple-event probabilities in general relativistic quantum mechanics.

We find that the general theory does not say what happens at different times: for every physical situation it gives the probability distribution for all the events, including those that we may wish to view as records of previous events.

We comment on the meaning of the notion of probability in the timeless case. In particular, we clarify the apparent difficulty presented by the fact that probabilities assigned to the possible values of a variable may not sum up to one. In the following section we summarize the results and discuss the issues that remain open.

A typical example is the following.

8.6.1 Multiple-Event Probability

Consider a partial observable A in $\mathcal{K}$ and let a be one of its eigenvalues. If a is nondegenerate, and $|s' >$ is the corresponding eigenstate, then (?) provides the probability amplitude of measuring a. What happens if a is degenerate?

Let us say for simplicity that a is doubly degenerate, and that $|s' >$ and $|s'' >$ are two orthogonal eigenstates having eigenvalue a, that is, they span the a-eigenspace $\mathcal{K}_a$. Then, to measure the eigenvalue a, or, equivalently, to measure its associated projection operator

$$\pi_a = |s' >< s'| + |s'' >< s''|,$$

means that we have a measuring apparatus that gives us a Yes answer if either the event s' or the event s'' happen (Yes answer corresponds to the eigenvalue 1 of a). In order to compute the probability of having a Yes answer, we need therefore the probability $\mathcal{P}_{s'\ OR\ s''}$ that the event s' OR the event s' happens.

Alternatively, suppose that we have a measuring apparatus that gives us a Yes answer if both the event s' and the event s'' happen. In order to compute the probability of having a Yes answer, we need therefore the probability $\mathcal{P}_{s'\ AND\ s''}$ AND s'' that the event s' AND the event s'' happen. The solution of one case gives immediately the solution of the other since, clearly

$$\mathcal{P}_{s'\ OR\ s''} = \mathcal{P}_{s'} + \mathcal{P}_{s''} - \mathcal{P}_{s'\ AND\ s''} \tag{8.21}$$

There are two possibilities: either $\mathcal{P}_{s'\ AND\ s''}$ is always zero, or not. We consider the two cases separately.

(i) **Mutually exclusive events.** If $\mathcal{P}_{s\Rightarrow s'\ AND\ s''} = 0$ for any s, then s' and s'' are alternative events that cannot both happen. That is, if one happens, the probability that the other happens is zero. By (?) and the given interpretation, this is equivalent to

$$< s'|\mathbb{P}|s'' > = 0 \tag{8.22}$$

In this case, (??) gives

$$\mathcal{P}_{s'\ OR\ s''} = \mathcal{P}_{s'} + \mathcal{P}_{s''}.$$

That is, the probability of s' OR s'' is simply the sum of the probabilities of s' and s''. Observe that this can be written generalizing (??) to

$$\mathcal{P}_{s\Rightarrow a} = < s|\Pi_a|s > \tag{8.23}$$

A typical example is the following. In the two-state spin system considered in the previous section, let $|s' >= |\uparrow, t >$ and $|s'' >= |\downarrow, t >$. In this case, $< s'|P|s'' >=<\uparrow |U(0)| \downarrow>= 0$. The two events are mutually exclusive. Therefore $\mathcal{P}_{s'\ ANDs''} = 0$. The projector on the a-eigenspace $\mathcal{K}_a$ is

$$\pi_a = |\uparrow, t ><\uparrow, t| + |\downarrow, t ><\downarrow, t|. \tag{8.24}$$

The projection $\mathcal{H}_a$ of $\mathcal{K}_a$ in $\mathcal{H}$ is spanned by the two orthogonal states $\mathbb{P}|\uparrow, t >= U^\dagger(t - t_0)|\uparrow>$ and $\mathbb{P}|\downarrow, t >= U^\dagger(t - t_0)|\downarrow>$, therefore

$$\Pi_a = U^\dagger(t - t_0)\big(|\uparrow><\uparrow| + |\downarrow><\downarrow|\big)U(t - t_0). \tag{8.25}$$

In this two-state system, $\Pi_a = \mathbf{1}$ and the corresponding probability is $\mathcal{P}_a = 1$. Not so, of course, in general.

(ii) **None exclusive events.** The interesting case is when

Let

$$|s' >= |\uparrow, t' > \text{ and } |s'' >= |\leftarrow, t'' >= \frac{|\uparrow, t'' > + |\downarrow, t'' >}{\sqrt{2}}.$$

In this case,

$$< s''|\mathbb{P}|s' > = <\leftarrow |U(t''-t')|\uparrow> \neq 0,$$

in general. The question we are asking is: what is the probability of detecting the spin $\uparrow$ at time t' AND the spin $\leftarrow$ at t''? The problem is of course well posed: if a particle is in a certain initial state at t, what is the probability of finding it with a certain spin at time t' AND with another spin at a later time t''? This can be measured by measuring the fraction of a beam that passes through a sequence of two Stern-Gerlach apparatuses.

Now, in ordinary quantum mechanics, these probabilities depends on the time ordering between the events. For instance, let the initial state $|s>$ be the state $|\rightarrow>= \frac{|\uparrow>-|\downarrow>}{\sqrt{2}}$ at time $t=0$, and let $U(t) = \mathbf{1}$ for all t. Then

$$\mathcal{P}_{s\Rightarrow(s'\ AND\ s'')} \begin{cases} \frac{1}{4} \text{ if } t' < t'' \\ 0 \text{ if } t'' < t' \end{cases}. \tag{8.26}$$

Because the sequence

$$|\rightarrow> \quad \Rightarrow \quad |\uparrow> \quad \Rightarrow \quad |\leftarrow> \tag{8.27}$$

has probability 1/4; while the sequence

$$|\rightarrow> \quad \Rightarrow \quad |\leftarrow> \quad \Rightarrow \quad |\uparrow> \tag{8.28}$$

cannot happen. The standard way of obtaining these probabilities in conventional quantum mechanics is via the projection postulate. For instance, say $t' < t''$, that is, case (22). We have:

(i) at time t_2 the spin $\uparrow$ is measured with probability $|<\uparrow|\leftarrow>|^2 = 1/2$;

(ii) the state is hence projected to $|\uparrow>$;

(iii) at time t'' the spin $\leftarrow$ is measured with probability $|<\leftarrow|\uparrow>|^2 = 1/2$, giving total probability $1/2 \times 1/2 = 1/4$. Standard QM gives also, easily

$$\mathcal{P}_{s\Rightarrow(s'\ OR\ s'')} \begin{cases} \frac{3}{4} \text{ if } t' < t'' \\ \frac{1}{2} \text{ if } t'' < t' \end{cases}. \tag{8.29}$$

Comparing with (??), notice that the probabilities $\mathcal{P}_{s\Rightarrow s'}$ and $\mathcal{P}_{s\Rightarrow s''}$ relevant here (with two detectors) are different from the probabilities $\mathcal{P}_{s\Rightarrow s'}$ and $\mathcal{P}_{s\Rightarrow s''}$ relevant when only one detector is present. For instance, in the first case, we have $\mathcal{P}_{s\Rightarrow s''} = |<\rightarrow|\uparrow><\uparrow|\leftarrow>|^2 + |<\rightarrow|\downarrow><\downarrow|\leftarrow>|^2 = 1/2$, because of the presence of a detector in s'; while in the absence of this, we would clearly have $\mathcal{P}_{s\Rightarrow s''} = |<\rightarrow|\leftarrow>|^2 = 0$. This well known fact will play an important role below.

How do we recover these probabilities in relativistic quantum mechanics, where we do not have a notion of time ordering in t?

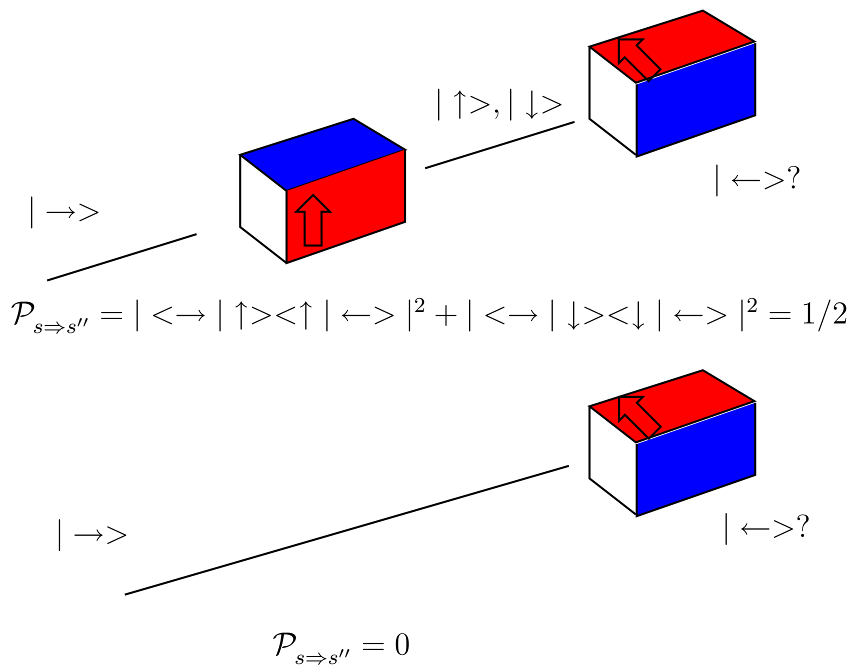

Figure 8.3: multprob2F. The probability $\mathcal{P}_{s\Rightarrow s''}$ is different depending on whever a detector is present or absent in s'. This well known fact plays an important role in the discussion below.

8.6.2 Multi-event probability from the coupling of an apparatus

Consider the experimental question of what is the probability for a spin system to have spin $\uparrow$ at time t' AND spin $\leftarrow$ at a later time t''. This is a statement that gets a precise meaning in an appropriate measurement context. In conventional QM, it can be dealt with by separating the two measurements in time, and using the collapse algorithm to compute joint probabilities.

Can obtain the same probabilities without invoking the collapse postulate? The answer is yes, and follows from an analysis of the experimental situation in volved in the experiment in which we measure the two spins at different times. The key is to bring the appartus that makes the measurement in to the picture.

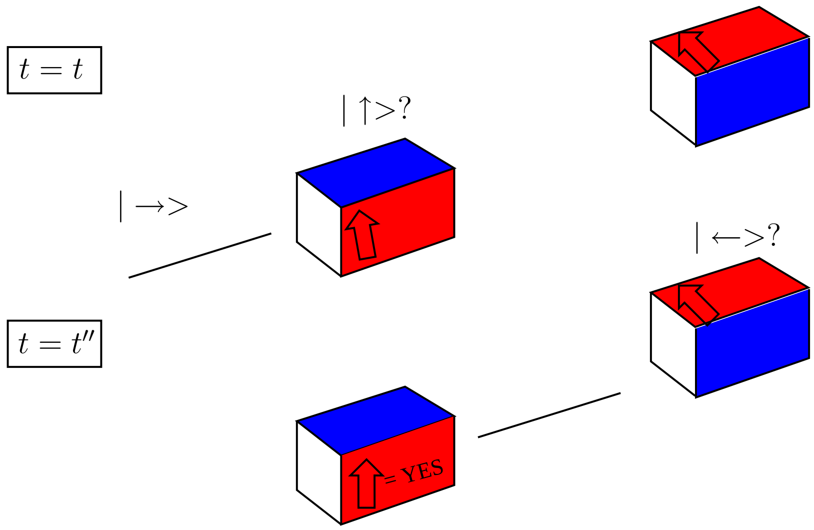

Figure 8.4: multprob3F.

8.6.3 Multi-event probability from the single-event one in non-relativistic QM

Consider an initial state $|\psi>$ at time t. What is the probability of detecting the state $|\psi'>$ at time t' AND the state $|\psi''>$ at time $t''>t'$? This can be computed by projecting on $|\psi'>$ at time t', which gives

$$\mathcal{P}_{\psi\Rightarrow\psi'\psi''}=|<\psi''|U(t''-t')\Pi_{\psi'}U(t'-t)|\psi>|^2. \tag{8.30}$$

where $\prod_{\psi'}=|\psi'><\psi'|$ and all states are normailzed $<\psi|\psi>=1$ .Now, describe the apparatus measuring $|\psi'>$ as a two-state system which is initially in a state $|No>$, which interacts with the system at the time t', jumping to the state $|Yes>$ if and only if the state of the system is in the state $|\psi'>$. The interacting dynamics are given by the unitary evolution operator $U_{\psi',t'}$, which is defined in three parts. Say that evolution begins at t and ends at t''. Evolution which occurs before the interaction, i.e., $t'',t<t'$ and evolution which occurs after the interaction, i.e., $t'',t>t'$ is given by the same unitary operator

$$<\psi'',A''|U_{\psi',t'}|\psi,A>=<\psi''|U_{\psi',t'}|\psi>\delta_{A''A}. \tag{8.31}$$

Now, consider evolution which starts before the interaction at t' and ends after it, i.e., $t''<t'<t'$. Say $A=A''$,

$$<\psi''|U(t''-t')(1-\Pi_{\psi'})U(t'-t)|\psi> \tag{8.32}$$

is the amplitude for the system to evolve up t' at which point and a state is measured which is not ψ', after which the system evolves to the state ψ''. Now, say $A \neq A''$,

$$< \psi''|U(t''-t')\Pi_{\psi'}U(t'-t)|\psi > \tag{8.33}$$

is the amplitude for the system evolves up t' at which point the state ψ' is measured, after which the system evolves to the state ψ''. This can be expressed as

$$< \psi'', A''|U_{\psi',t'}(t''-t)|\psi, A > \;=\; < \psi''|U(t''-t')\big(\Pi_{\psi'}\mathcal{I}_{A''A} + (1-\Pi_{\psi'})\delta_{A''A}\big)U(t'-t)|\psi > \tag{8.34}$$

where

$$\mathcal{I}_{AA'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{8.35}$$

is a matrix that flips the apparatus state. Equations (8.31) and (8.34) together define the unitary operator $U_{\psi',t'}$.

Then the question we are interested in can be rephrased as follows: "Given an initial state $|\psi, No>$ at time t, what is the probability of measuring the state $|\psi'', Yes>$ at time t''?" Notice that "the system state is $|\psi''>$" and "the apparatus state is $|Yes>$", are compatible statements in quantum theory: they refer to orthogonal observables, both at time t'', that are, and can be, measured together! In other words, the question can be captured by the single-event probability amplitude

$$\mathcal{P}_{\psi\Rightarrow\psi'\psi''} = |< \psi'', Yes|U_{\psi',t'}(t''-t)|\psi, No>|^2. \tag{8.36}$$

8.6.4 The Problem of the "Frozen-time" Formalism

In the generally covariant context, dynamics can be entirely expressed in terms of Dirac observables. Indeed, notice that the probability of a sequence of measurements can be written as in equation (??), namely as the expectation value of the projection operator Π_s defined in (?), or (??). This operator is a Dirac observable of the extended system that includes the measuring devices.

In the present context, this is the answer to the long-standing problem of the description of dynamics in the "frozen-time" formalism; namely in the Dirac's quantization of a system whose dynamics is expressed by constraints [41], [42]. Dynamics is coded into (non-commuting) Dirac observables defined in terms of sets of interactions between (what we call) the system and (what we call) the measuring devices.

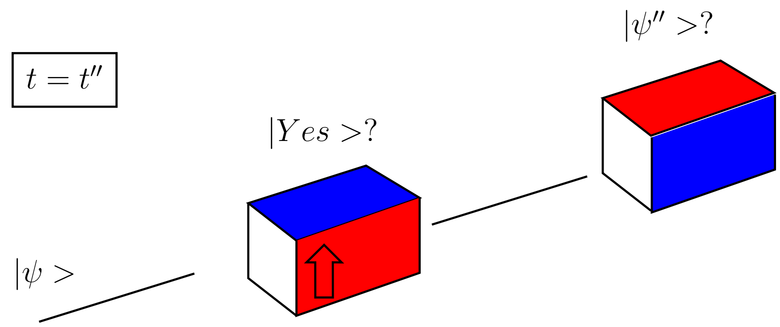

Figure 8.5: multprob4F. “the apparatus state is Yes” while “the system is ψ''” are compatible statements in quantum theory: they refer to orthogonal observables, both at time t''. By moving the quantum/classical boundary, a question such as in fig (8.4) is captured by a single-event probability amplitude.

8.6.5 Future Developments of Multiple-Event Probabilities

The extension of these ideas to field theory, and in particular the connection between this formalism and the boundary formalism [8, 18] which is presently used [19] to compute probability amplitudes in background independent quantum gravity, in the context of loop quantum gravity (next chapter)[8, 20, 21].

8.7 General Boundaries Formulation

In quantum field theory (QFT) on Minkowski space, we can use the Schrödinger picture and have states associated to flat spacelike (hyper-)surfaces. The transition amplitude between an initial state and a final state is obtained by acting with the unitary evolution operator on the former and taking the inner product with the latter. The possibility of a Schrodinger picture has also been considered in QFT on curved spacetime [?]. In this case, states live on arbitrary spacelike Cauchy surfaces forming a foliation of spacetime. Evolution along these surfaces is non-unitary in general, as it does not correspond to a symmetry of the metric.

This restriction is sufficient for ordinary QFT on Minkowski space, but has no meaning in a generally covariant context. It is necessary to consider arbitrary boundary surfaces: How would one encode the fact that initial and final states refer to certain physical times or have a certain physical time interval between them? An element of $\mathcal{H}_{kin}$ or $\mathcal{H}_{diff}$ contains information about quantum 3-geometries, but it does not specify the proper time that is associated to a transition process from one 3-geometry to another 3-geomery.

It has been proposed [??, ??] that this missing information could be encoded by using closed boundaries, in place of the $3d$ hypersurfaces of a foliation (see Fig. M.-19). On a closed boundary, the configuration consists of spatial 3-geometries g_f, g_i, and a timelike

3-geometry g_t. Along the time-like part, we can impose conditions on the proper distance between the space-like surfaces Σ_i and Σ_f. This distance corresponds to the proper time that is measured by clocks with world lines on Σ_t.

In background independent quantum gravity, on the other hand, there is no fixed space-time geometry; in this case, states live on arbitrary Cauchy surfaces and the requirement that the surface is spacelike is encoded in the state itself, which represents a quantum state of a spacelike geometry (see e.g. [??, ??]).

$$A = < \Psi_f | e^{-i\mathcal{H}(t_f - t_i)} | \Psi_i > . \tag{8.37}$$

$$A = \int \mathcal{D}\varphi_f \int \mathcal{D}\varphi_i \; \Psi_f^*[\varphi_f] W[\varphi_f, t_f; \varphi_i, t_i] \Psi_i[\varphi_i] \tag{8.38}$$

8.7.1 Preparation - Measurement

We prepare a state ψ at t_1, wait for a time Δt, then measure if we obtain the state η at t_2. The probability for this depends on Δt:

$$P = | < \eta | U(\Delta t) | \psi > |^2$$

Recall properties of U:

(i) Composition: $U(\Delta t)U(\Delta t') = U(\Delta t + \Delta t')$

(ii) Unitarity: $U^\dagger = U^{-1}$

8.7.2 Basic Idea

8.7.3 Basic structures

To prepare the ground for the general boundary formulation let us think of the standard formulation. There are geometric structures: points in time and intervals of time. There are algebraic structures: states and transition amplitudes. To each point $t \in \mathbb{R}$ in time we associate a Hilbert space $\mathcal{H}_t$ of states. These state spaces are simply copies of the usual state space $\mathcal{H}$ and the labeling by a time is only a formality at this point. To each time interval $[t_1, t_2] \subset \mathbb{R}$ we associate a linear map $\rho_{[t_1,t_2]} : \mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \to \mathbb{C}$, called amplitude map. This sends a pair of an initial and a final state (ψ, η) to the transition amplitude $\rho_{[t1,t2]}(\psi \otimes \eta) =< \eta | U(t_1, t_2) | \psi >$.

Basic algebraic structures:

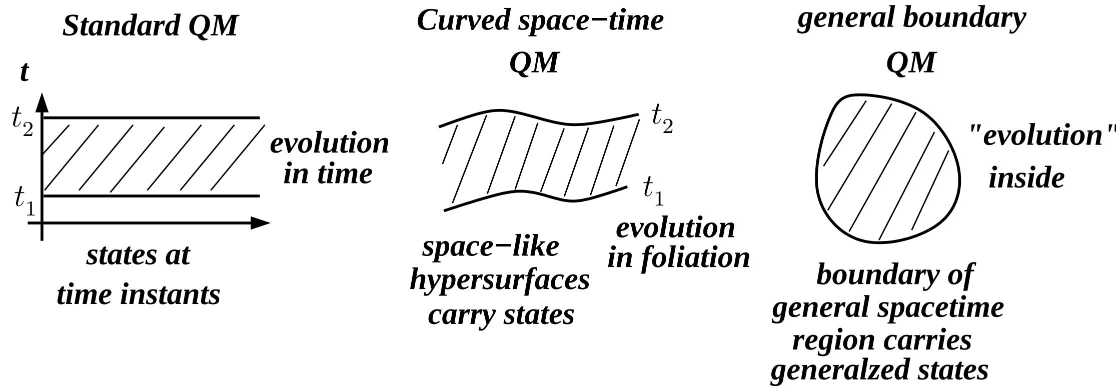

Figure 8.6: GenBoundaF. Basic idea.

- To each hypersurface Σ associate a Hilbert space $\mathcal{H}_\Sigma$ of states.
- To each region $\mathcal{M}$ with boundary Σ associate a linear amplitude map $\rho_\mathcal{M} : \mathcal{H}_\Sigma \to \mathbb{C}$.

The structures are subject to a number of rules. For example:

- $\overline{\Sigma}$ is Σ with the opposite orientation. Then $\mathcal{H}_{\overline{\Sigma}} = \mathcal{H}^*_\Sigma$.
- $\Sigma = \Sigma_1 \cup \Sigma_2$ is a disjoint union of hypersurfaces. Then $\mathcal{H} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$.

Recovering standard quantum mechanics

Consider the geometry of a stanard transtion amplitude.

- region: $\mathcal{M} = [t_1, t_2] \times \mathbb{R}^3$
- boundary: $\partial\mathcal{M} = \Sigma_1 \cup \overline{\Sigma}_2$
- state space: $\mathcal{H}_{\partial\mathcal{M}} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\overline{\Sigma}_2} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}^*_{\Sigma_2}$

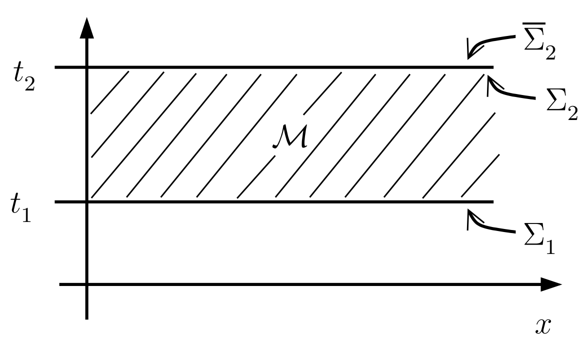

Figure 8.7: GenBoundcF. .

(i) Via time-tranlation symmetry identify $\mathcal{H}_{\Sigma_1} \cong \mathcal{H}_{\Sigma_2} \cong \mathcal{H}$, where $\mathcal{H}$ is the state space of standard quantum mechanics.

(ii) The amplitude map $\rho_{\mathcal{M}} : \mathcal{H} \otimes \mathcal{H}^* \to \mathbb{C}$.

(iii) The relation to the standard amplitude is:

$$\rho_{\mathcal{M}}(\psi \otimes \eta) =< \eta | U(t_2 - t_1) | \psi > \tag{8.39}$$

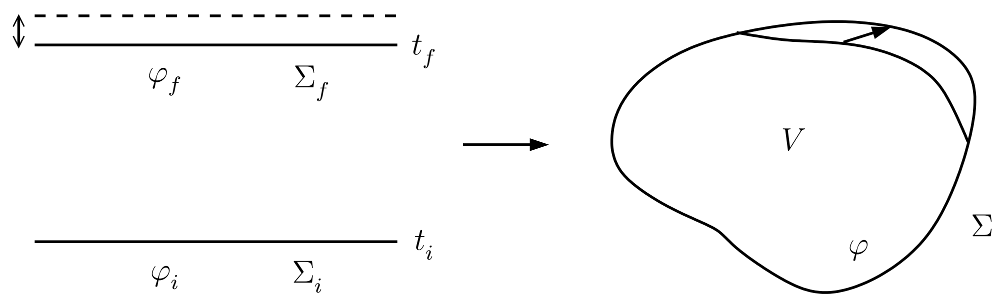

Figure 8.8: GenBound1F.

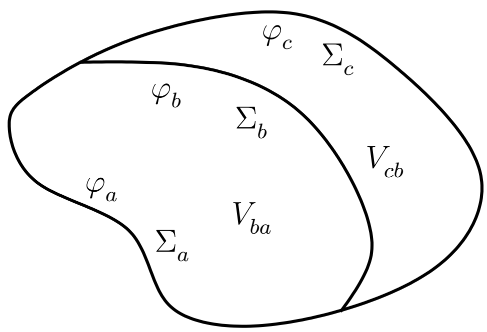

Figure 8.9: GenBound2F. Splitting of V.

8.8 The Vacuum State

Consider a real massive scalar field $\phi(x)$ on Minkowski space. To start with assume it is a free field. We write $x = (t, \vec{x})$. Denote by $\varphi(\vec{x})$ the classical field configuration at time zero: $\varphi(\vec{x}) = \phi(\vec{x}, t = 0)$. The state space at time zero, $H_{t=0}$, is Fock space, where the (distributional) field operator $\varphi(\vec{x})$ and the hamiltonian H are defined. The lowest eigenstate of H is the vacuum state $|0_M >$, and its energy E_0 is zero. Fock space admits countable bases. Choose a basis $|n >$ of eigenstates of H with eigenvalues

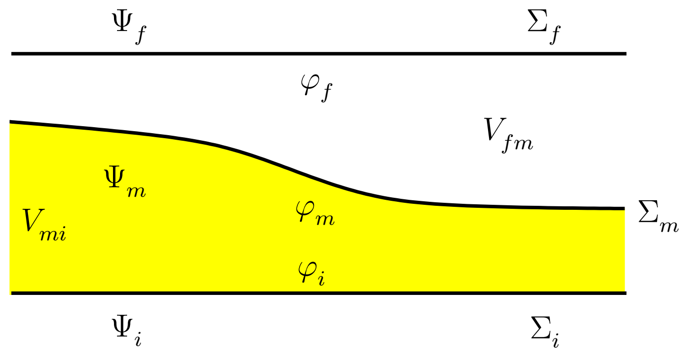

Figure 8.10: GenBound3F. Evolution to a closed surface Σ_m.

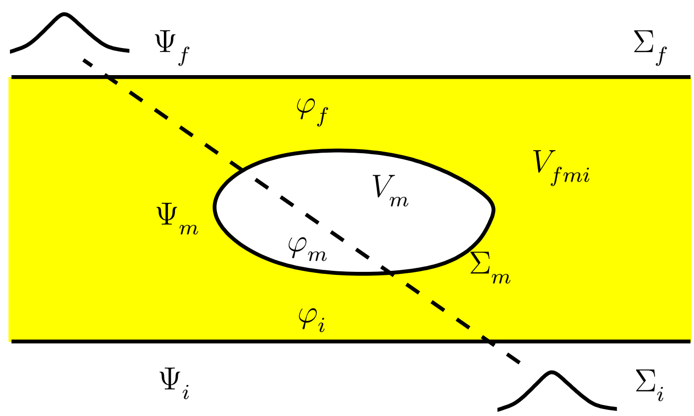

Figure 8.11: GenBound4F. Splitting of V_{fi}

$$W(T) = \sum_n e^{-TE_n} |n >< n|. \tag{8.40}$$

In the large T limit, this becomes the projector on the

$$\lim_{T\to\infty} W(T) = |0_M >< 0_M|. \tag{8.41}$$

8.8.1 The Vacuum State

The quantum state $|0_M>$ that describes the Minkowski vacuum is not singled out by the dynamics alone in quantum gravity. Rather, it is singled out as the lowest eigenstate of an energy H_T which is the variable canonically conjugate to a nonlocal function T of the gravitational

This situation has an analogy in the simple quantum system formed by a single a relativistic particle. In the Hilbert space of such a system there is no preferred vacuum state. But we can choose a preferred Lorentz frame, and therefore a preferred Lorentz time x_0. The conjugate variable to x_0 is the momentum p_0, and there is a (generalized) state of minimum p_0

To find the Minkowski vacuum state, we can repeat the very same procedure used above. The only difference is that the bulk functional integral is not over the bulk matter fields, but also over the bulk metric. This difference has no bearing on the above formulas, which regard the boundary metric, which, in the two cases, is an independent variable.

As a first example, a boundary metric can be defined as follows. Consider a three-sphere formed by two polar in and out regions and one equatorial side region. Let the matter+gravity field on the three-sphere be split as

$$\varphi = (\varphi_{out}, \varphi_{in}, \varphi_{side}). \tag{8.42}$$

Fix the equatorial field φ_{side} to take the special value φ_{RT} defined as follows. Consider a cylindrical surface Σ_{RT} of radius R and height T in $\mathbb{R}^4$, as defined above. Let Σ_{in} in (and Σ_{out}) be a ($3d$) disk located within the lower (and upper) basis of Σ_{RT}, and let φ_{side} side the part of Σ_{RT} outside these disks, so that

$$\Sigma_{RT} = \Sigma_{in} \cup \Sigma_{out} \cup \Sigma_{insid}. \tag{8.43}$$

Let g_{RT} be the metric of side and let $\varphi_{RT} = (g_{RT}, 0)$ be the boundary field on side determined by the metric being g_{RT} and all other fields being zero.

Given arbitrary values φ_{out} and φ_{in} of all the fields, included the metric, in the two disks, consider $W[(\varphi_{out}, \varphi_{in}, \varphi_{RT})]$. In writing the boundary field as composed by three parts as $\varphi(\varphi_{out}, \varphi_{in}, \varphi_{side})$ we are in fact splitting $\mathcal{K}$ as

$$\mathcal{K} = H_{out} \otimes H^*_{in} \otimes H_{side} \tag{8.44}$$

Fixing $\varphi_{side} = \varphi_{RT}$ means contracting the covariant vacuum state $|0_\Sigma>$ in $\mathcal{K}$ with the bra state $<\varphi_{RT}|$ in $\mathcal{H}_{side}$. For large enough R and T, we expect the resulting state in $\mathcal{H}_{out} \otimes \mathcal{H}^*_{in}$ in to reduce to the Minkowski vacuum. That is

$$\lim_{R,T\to\infty} < \varphi_{RT}|0_\Sigma> = |0_M> \otimes < 0_M|. \tag{8.45}$$

Therefore for a generic in configuration, and up to normalization

$$\Psi_M[\varphi] = \lim_{R,T\to\infty} W[(\varphi, \varphi_{in}, \varphi_{RT}]. \tag{8.46}$$

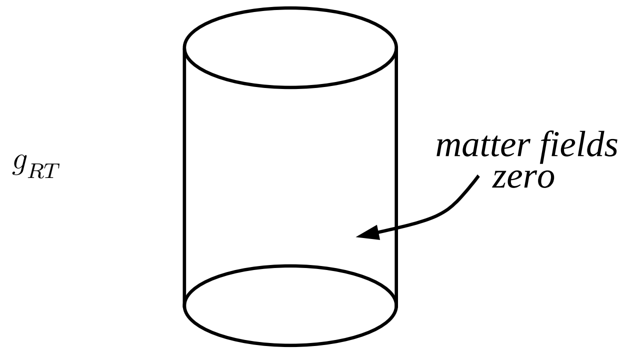

Figure 8.12: VacuumFig.

gives the vacuum functional for large R and T. (Below we shall use simpler geometry for the boundary).

These formulas allow us to extract the Minkowski vacuum state from a euclidean gravitational functional integral. n-particle scattering states can then be obtained by generalizations of the flat space formalism, and, if it is well defined, by analytic continuation in the single variable T. Notice that we are precisely in the case of time independent φ_{side}, where analytical continuation may be well defined.

8.9 Emergence of Temporal Phenomena

The hypothesis that the fundamental theory of nature can be formulated in a timeless language [287], and that temporal phenomena could be emergent [288], [289], [290].

Generally covariant theories and The problem of time

see week 41 beaz

Rovelli wants to use thermodynamics to **define** what we call time as we usualy mean. does this as follows. Given a mixed state with density matrix D, find some operator H such that D is the Gibbs state $\exp(-H/kT)$. In lots of cases this isn't hard; it basically amounts to

$$H = -kT \ln D \tag{8.47}$$

Of course, H will depend on T, but this is really is just saying that fixing your temperature fixes your units of time!

Operator theorists have pondered this notoin very carefully for a long time and generalised it into the Tomita-Takesaki theorem. This gives a very general way of finding a Hamiltonian (hence a notion of time evolution) from a state of a quantum system! For example, one can use this trick to start with a Robertson-Walkerr universe full of blackbody radiation, and recover a notion of "time".

8.9.1 Gibb's distribution

The quantum state is then given by the Gibbs density matrix

$$\omega = Ne^{\beta H} \tag{8.48}$$

where H is the hamiltonian, defined on a Hilbert space $\mathcal{H}$, and $N = tr[e^{\beta H}]$.

The KMS condition

$$\omega() = \omega() \tag{8.49}$$

8.10 Consistent Discretization of Classical and Quantum Gravity

Explicitly working at the physical level.

[325]

Kinematic variables do not have a well defined action as quantum operators on states that are annihilated by the constraints.

Worse, such states are expected to have a distributional nature as a subset of the full space of states. This implies that they do not really admit a probabilistic interpretation (kuchar) [41]

???

[325]

"..it is the presence of the Hamiltonian constraint in general relativity what really complicates the application of this [Page-Wootters] proposal. In the consistent discrete canonical formulation of general relativity the constraints are not present. Therefore it opens the possibility to revisit the Page and Wootters proposal."

As mentioned in [77],

"...One way out could be to look at constraint quantization from an entirely new point of view [here he refers to consistent discretization] which proves useful also in the discrete formulations of classical GR, that is numerical GR. While being a fascinating possibility, such a procedure would be rather drastic step in the sense that it would render most results of LQG obtained so far obsolete."

8.10.1 The problem of Time:

There is no external purely classical time parameter - reference bodies are necessarily part of the system under consideration. They are necessarily a closed systems where everything behaves quantum mechanically. Page and Wootters proposed to treat all variables quantum mechanically and use one of the variables as a clock, as long as it behaves semiclassically.

The clock variable must change during evolution, therefore it cannot commute with the constraints. This implies that it will not be well defined on the physical space of states annihilated by all the constraints.

If one tries to work in the kinematical Hilbert space, these wave functions are distributional and cannot be used to construct a probabilistic interpretation.

Since their discrete theory in constraint free, one can follow the Page Wootters procedure.

$$P_{sim}(\Delta P^{\phi}, \Delta A) = \lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{P^{\phi}_{(1)},A_{(1)}}^{P^{\phi}_{(2)},A_{(2)}} \Psi^2[A, P^{\phi}, n] dP^{\phi} dA. \tag{8.50}$$

$$P_{cond}(\Delta P^{\phi}, \Delta A) = \frac{\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{P^{\phi}_{(1)},A_{(1)}}^{P^{\phi}_{(2)},A_{(2)}} \Psi^2[A, P^{\phi}, n] dP^{\phi} dA}{\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N} \int_{-\infty,A_{(1)}}^{\infty,A_{(2)}} \Psi^2[A, P^{\phi}, n] dP^{\phi} dA}. \tag{8.51}$$

They have studied the parameterized non relativistic particle with this method: [325] R. Gambini, R. Porto and J. Pulliin gr-qc/0302064

$$S = \int \left[\dot{q} + p_0 \dot{q}^0 - N(p_0 + \frac{p^2}{2m} + \lambda q) \right] d\tau, \tag{8.52}$$

$$L(n, n+1) = p^n (q_{n+1} - q_n) + p_0^n (q_{n+1} - q_n^0) - N_n (p_0^n + \frac{p_n^2}{2m} \lambda q_n). \tag{8.53}$$

the conditional probability to obtain $q = x$ given $q^0 = t$

$$(q = x|q^0 = t) = \frac{\sum_{n=-\infty}^{\infty} |\Psi(x,t,n)|^2}{\sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} dx |\Psi(x,t,n)|^2} \tag{8.54}$$

One can show that this relational description recovers usual quantum mechanics when the discrete approximation approaches the continuum limit and when the clock variable behaves sufficiently to a classical clock.

The procedure can be extended to situations where the simultaneity surfaces are not transverse to the classical orbits of the system.

8.10.2 Realistic clocks, universal decoherence and the black hole information paradox

The system has an unitary evolution in n. At t cannot be perfectly correlated with n, even in the semi-classical regime of the clock, the evolution in t is not perfectly unitary. In fact one can show that the density matrix evolves according to

$$\frac{\partial}{\partial t}\rho_2 = -i[\mathcal{H}_2, \rho_2(t)] - \sigma[\mathcal{H}_2, [\mathcal{H}_2, \rho_2(t)]]. \tag{8.55}$$

This equation was first proposed by Milburn based on phenomenological arguments, and is a particular type of non-unitary evolutions considered by Lindblad.

Our derivation allows to estimate σ that is of order of the Planck time.

$$\rho_{2_{nm}}(t) = \rho_{2_{nm}}(0) e^{-i\omega_{nm}t} e^{(-\sigma(\omega_{nm})^2)t} \tag{8.56}$$

This equation does not violate the conservation of energy like Hawking proposal for information loss. One could expect to confirm this type of equation by studying some mesesoscopic quantum systems.

Information loss problem in Black Holes

It provides a new and very effective mechanism for treating the information loss problem in Black Holes.

They have shown that for any Black hole bigger than 600 Plank masses the information loss induced by their equation is enough to dissipate all the black hole information before to its evaporation.

For very small black holes, Hawking's semi-classical analysis not valid.

8.10.3 Quantum Cosmology:

8.11 Bearing of Matters of Quantum Gravity on Interpretations of Quantum Mechanics

In some interpretations of quantum mechanics, the wave function is considered a real entity that evolves unitarily, except at measurement time, when it undergoes a sudden change. In particular, some interpretations make the hypothesis that this collapse is a real physical phenomenon whose peculiar nonlocal dynamics is not yet understood. If this is the case, the full freedom of moving the quantum/classical boundary is broken, because once the collapse has happened no more interference between the two branches of a measurement outcome is possible, even in principle. If this is the case, the strategy adopted here is not viable in general, because it assumes, instead, that no true physical collapse happens at anytime.

In some others interpretations, the wave function, or the "quantum state", is not considered as a real entity. Rather, only quantum events are considered real, and probabilities like $| < s'|s > |^2$ are directly interpreted as conditional probabilities for these events to happen. In particular, in [371], [378] these quantum events are assumed to happen at interactions between systems, and to be real only with respect to the interacting systems themselves. From this perspective, there is no specific physical phenomenon corresponding to a quantum collapse, and the strategy considered here is viable. With respect to an external system, what happens at the interaction between system and apparatus is not a sudden change in a hypothetical real state , but simply an entanglement between the probabilities of various outcomes of observations on the system or the apparatus. We refer to [371], [378], and Section 5.6 of [20] for a discussion of this point of view.