Quantum Electrodynamics

Contents

List of Figures

Part I

Introduction and Background

Chapter 1

Introduction

Feynman derived his rules in a non-rigorous fashion but it still incorporated all QED processes. These rules were shown to follow from a systematic treatment within the framework of quantum field theory. Here we follow the route taken by Feynmann.

1.1 The Electromagnetic Field and the Photon

Light behaves as a wave as it demonstrates interference and diffraction. Maxwell's theory seemed to confirm the wave theory of light.

But then the development following the discovery of the photoelectric effect led to the realisation that sometimes light behaves like particles.

1.2 Review of Special Relativistic Notation

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.1}$$

$$\eta^{\mu\nu} = (\eta^{-1})_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.2}$$

The world four vector

$$x^\mu = \{x^0, x^1, x^2, x^3\} = \{t, x, y, z\} \tag{1.3}$$

describes the spacetime coordinates. The covariant four vector is

$$x_\mu = \eta_{\mu\nu} x^\nu = \{t, -x, -y, -z\} = \{x_0, x_1, x_2, x_3\} \tag{1.4}$$

We have

$$\begin{aligned} x \cdot x &= x^\mu x_\mu & (1.5) \\ &= t^2 - x^2 - y^2 - z^2. & (1.6) \end{aligned}$$

The definition of four-momentum vector is analogous,

$$p^\mu = \{E, p_x, p_y, p_z\}, \tag{1.7}$$

and the scalar product $p_1 \cdot p_2$ is

$$p_1 \cdot p_2 = p_1^\mu p_{2\mu} = E_1 E_2 - \mathbf{p}_1 \cdot \mathbf{p}_2 \tag{1.8}$$

and the scalar product $x \cdot p$ is

$$x \cdot p = x^\mu p_\mu = x_\mu p^\mu = Et - \mathbf{x} \cdot \mathbf{p}. \tag{1.9}$$

We use the general notions for four vectors

$$a = \{a_0, a_1, a_2, a_3\}. \tag{1.10}$$

We denote three-vectors by bold type as in

$$\mathbf{a} = \{a_1, a_2, a_3\}. \tag{1.11}$$

The components

$$a^\mu = \{a^0, a^1, a^2, a^3\}$$

1.3 Maxwell's Equations

Classical electromagnetism is described by Maxwell's equations. In the presence of a charge density $\rho(\mathbf{x}, t)$ and current density $\mathbf{j}(\mathbf{x}, t)$, the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ satify the equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \rho && (1.12)\\
\nabla \times \mathbf{B} &= \mathbf{j} + \frac{\partial \mathbf{E}}{\partial t} && (1.13)\\
\nabla \cdot \mathbf{B} &= 0 && (1.14)\\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && (1.15)
\end{aligned}$$

From the second pair of Maxwell's equations follows the existence of scalar and vector $\phi(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ potentials defined by

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \cdot \phi - \frac{\partial \mathbf{A}}{\partial t} \tag{1.16}$$

These equations do not determine the potential uniquely, since for an aribtrary function $\Lambda(\mathbf{x}, t)$ the transformation

$$\phi \to \phi' = \phi + \frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A}' = \mathbf{A} - \nabla \Lambda \tag{1.17}$$

Expressed in terms of potentials the first Maxwell equation becomes

$$-\nabla^2 \phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = \Box \phi - \frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \rho \tag{1.18}$$

For the second we need $\nabla \times (\nabla \times \mathbf{A})$

$$\begin{aligned}
[\nabla \times (\nabla \times \mathbf{A})]_i &= \epsilon_{ijk} \partial_j (\epsilon_{kj'k'} \partial_{j'} A_{k'}) \\
&= \epsilon_{ijk} \epsilon_{ki'j'} \partial_j \partial_{i'} A_{j'} \\
&= (\delta_{ii'} \delta_{jj'} - \delta_{ij'} \delta_{ji'}) \partial_j \partial_{i'} A_{j'} \\
&= \partial_j \partial_{i'} A_j - \nabla^2 A_i && (1.19)
\end{aligned}$$

$$\Box \mathbf{A} + \nabla \left(\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A}\right) = \mathbf{j} \tag{1.20}$$

In four-vector notation the gauge transformations (1.17) read

$$A^\mu \to A^\mu + \partial^\mu \Lambda(x).$$

The first (1.18) and second (1.20) Maxwell equations can be combined into one equation

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \tag{1.21}$$

1.4 Transverse Gauge field

It is always possible to find a function $\Lambda(x)$ such that the transformed potential satisfies the Lorentz gauge

$$\partial_\mu A^\mu = 0. \tag{1.22}$$

Only in this gauge does the wave equation have the simple form

$$\Box A^\mu = 0. \tag{1.23}$$

For

$$k^\mu k_\mu = 0 \tag{1.24}$$

its solutions are plane waves

$$A^\mu(x,k) = \epsilon^\mu N_k (e^{-ik\cdot x} + e^{ik\cdot x}) \tag{1.25}$$

In the Lorentz gauge, we can make futher gauge transformations $A^\mu \to A^\mu + \partial^\mu \Lambda(x)$ provided Λ satisfies

$$\Box \Lambda(x) = 0.$$

Such regauging does obviously does not change the Lorentz condition. The radiation gauge

$$A^0 = 0, \quad \nabla \cdot \mathbf{A} = 0. \tag{1.26}$$

can be chosen. To see this consider aribtrary $\mathbf{A}'(x)$, and postulate $\Lambda(x)$ such that

$$\nabla \cdot \mathbf{A}(x) = \nabla \cdot \mathbf{A}'(x) - \nabla^2 \Lambda(x) = 0 \tag{1.27}$$

We obviously need to solve the equation

$$-\nabla^2 \Lambda(x) = \nabla \cdot \mathbf{A}'(x) \tag{1.28}$$

Notice this is just the equation

$$-\nabla^2 \Lambda(x) = f(x) \tag{1.29}$$

which we know has the solution

$$\Lambda(x) = \int d^3r' \frac{f(r')}{4\pi |r' - r|} \tag{1.30}$$

Therefore this gauge can always be choosen. In this gauge the timelike component of ϵ^μ vanishes. ϵ satisfies

$$\epsilon \cdot \mathbf{k} = 0 \tag{1.31}$$

and normalised such that

$$\epsilon \cdot \epsilon = 1 \tag{1.32}$$

$$\begin{aligned} \epsilon^\mu k_\mu &= 0 \\ \epsilon^\mu \epsilon_\mu &= -1 \end{aligned} \tag{1.33}$$

1.5 Blackbody Radiation, the Photoelectric Effect and the Compton Effect

$$E = hf \tag{1.34}$$

where f is the frequency and h is Plank's constant.

1.5.1 Blackbody Radiation

Boltzmann statistics for a gas of free particles is

$$p(\vec{v}) = Ne^{-E/kT}$$

Classical physics can be used to derive an equation which describes the intensity of the blackbody radiation as a function of frequency for a fixed temperature - the result is known as the Rayleigh-Jeans law. Although the Rayleigh-Jeans law works for low frequencies, it diverges as f^2; this divergence for high frequencies is called the ultraviolet catastrophe.

Planck's law states that

$$I(f,T) = \frac{2hf^3}{c^2}\frac{1}{e^{\frac{hf}{kT}}-1} \tag{1.35}$$

We analyse using Bose-Einstein statistics: a "gas" of photons. There is no constrainton the number of photons. We find that for a system of photons, that the number of photons $n(\epsilon)d\epsilon$ in a (small) energy range ϵ to $\epsilon + d\epsilon$ is given by

Consider an energy level ϵ_i with degeneracy g_i, containing n_i bosons. This can be represented by $g_i - 1$ lines, and the bosons by n_i circles. The number of distinct orderings of lines and circles is

$$\frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \tag{1.36}$$

The total number of microstates for a given distribution is

$$\prod_i \frac{(n_i + g_i - 1)!}{n_i!(g_i - 1)!} \tag{1.37}$$

For $g_i >> 1$ this can be replaced by

$$t(\{n_i\}) = \prod_i \frac{(n_i + g_i)!}{n_i!g_i!}. \tag{1.38}$$

If we assume that both g_i and n_i are large enough for Stirling's approximation to hold for $\ln g_i!$ and $\ln n_i!$, we find that $\ln t$ is given by

$$\ln t \approx \sum_i [(n_i + g_i)\ln(n_i + g_i) - g_i \ln g_i - n_i \ln n_i]dn_i \tag{1.39}$$

We want to maximise $\ln t$ subject to the constraint

$$\sum_i \epsilon_i n_i = U. \tag{1.40}$$

If $\ln t$ were maximal the change in $\ln t$ resulting from changes dn_i in each of the n_i's would zanish:

$$d \ln t \approx \sum_i [\ln(n_i + g_i) - \ln n_i] dn_i = 0. \tag{1.41}$$

If all the dn_i's were independent from each other, each coefficient in (1.41) would have to zanish. However because of the constraint (1.40) it no longer follows that all the dn_i's are independent from each other as they have to satisfy

$$dU = \sum_i \epsilon_i dn_i = 0. \tag{1.42}$$

Adding this multiplied by the Lagrange multiplier β to (1.41) we obtain

$$\sum_i [\ln(n_i + g_i) - \ln n_i + \beta \epsilon_i] dn_i = 0 \tag{1.43}$$

which is a condition for a maximal value for $\ln t$ subject to the constraint (1.40). For appropriate value of β we can consider all dn_i independent from each other. Therefore, each coefficient must zanish separately:

$$\ln \left(\frac{n_i + g_i}{n_i} \right) + \beta \epsilon_i = 0. \tag{1.44}$$

We then find that the most probable distribution is

$$n_i = \frac{g_i}{e^{-\beta \epsilon_i} - 1}. \tag{1.45}$$

This is the Bose-Einstein distribution. β is related to the temperature, so the Bose-Einstein distribution takes the form.

$$n_i = \frac{g_i}{e^{\frac{\epsilon_i}{kT}} - 1}. \tag{1.46}$$

We consider a “gas” of photons.

$$n(\epsilon)d\epsilon = \frac{g(\epsilon)d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{1.47}$$

where $g(\epsilon)$ are the density of states. We first derive the density of states as a function of the wavevector $\vec{k}$.

In order to determine the available wavevectors, we ask what standing waves can propagate within the box subject to boundary condition that the amplitude is zero at the boundaries. Using a cube with a side of length L, we see that there must be an integer number of half wavelengths in L for each of the directions. Hence if the vector is $\vec{k}$, with Cartesian components (k_x, k_y, k_z), we must have

$$k_x = \frac{n_x \pi}{L}, \quad k_y = \frac{n_y \pi}{L}, \quad k_z = \frac{n_z \pi}{L} \tag{1.48}$$

where n_x, n_y and n_z are from the set of positive non-zero integers - these define"elementary cells". The total number of states with wavwenumber, $|\vec{k}|$, less than some value k is found by counting the number of triples (n_x, n_y, n_z) such that

$$\frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} < k \tag{1.49}$$

We can find this by considering the octant of a sphere of radius k in $\vec{k}-$space. The volume of the sphere in $\vec{k}-$space is

$$\frac{4}{3}\pi k^3$$

and the volume of an elementary cell is

$$\left(\frac{\pi}{L}\right)^3$$

therefore the number of states satisfying (1.49) is 1/8 the volume of the sphere divided by the volume of an elementary cell,

$$N(k) = \frac{1}{8}\frac{4}{3}\pi k^3 \left(\frac{L}{\pi}\right)^3 = \frac{1}{3}4\pi \frac{V}{(2\pi)^3} k^3$$

where V is the volume of the container. We must multiply this by a factor of 2 from the fact that there are two polarisations for the photons. Using $g(k) = 2 \times dN(k)/dk$ we obtain for waves in a box

$$g(k)dk = 2 \times 4\pi \frac{V}{(2\pi)^3} k^2 dk. \tag{1.50}$$

Now we use the quantum relationship between photon energy and wave number:

$$\epsilon = hf = \frac{hc}{\lambda} = hc\frac{k}{2\pi} = \hbar ck. \tag{1.51}$$

$$g(\epsilon)d\epsilon = 8\pi \frac{V}{(2\pi)^3} \frac{\epsilon^2}{(\hbar c)^3} d\epsilon \tag{1.52}$$

Thus, the number of photons $n(\epsilon)d\epsilon$ in the energy range from ϵ to $\epsilon + d\epsilon$ is

$$n(\epsilon)d\epsilon = \frac{8\pi V}{(hc)^3} \frac{\epsilon^2 d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{1.53}$$

The energy $u(\epsilon)$ in the range ϵ and $\epsilon + d\epsilon$ is given by:

$$u(\epsilon)d\epsilon = n(\epsilon)\epsilon d\epsilon = \frac{8\pi V}{(hc)^3} \frac{\epsilon^3 d\epsilon}{e^{\frac{\epsilon}{kT}} - 1} \tag{1.54}$$

Since the energyand frequency are related by the quantum formula,

$$\epsilon = hf,$$

we find that the density is:

$$u(f)df = \frac{8\pi V h}{c^3} \frac{f^3 df}{e^{\frac{hf}{kT}} - 1} \tag{1.55}$$

The total energy, U. This the area under the graph of the energy spectrum

$$U = \int_0^\infty u(f)df = \frac{8\pi V}{h^2 c^3} \int_0^\infty \frac{(hf)^3 df}{e^{\frac{hf}{kT}} - 1} \tag{1.56}$$

Define

$$y = \frac{hf}{kT}$$

in terms of which, the expression for the total energy becomes

$$U = \int_0^\infty u(f)df = \frac{8\pi V}{(hc)^3}(kT)^4 \int_0^\infty \frac{y^3 dy}{e^y - 1}. \tag{1.57}$$

The integral is $\pi^4/15$, hence

$$\frac{U}{V} = \frac{8\pi^5}{15(hc)^3}(kT)^4. \tag{1.58}$$

1.5.2 The Photoelectric Effect

The photoelectric effect is the ejection of electrons from a metal surface exposed to electromagnetic radiation. The energy of the emitted electrons is given by the frequency of the irradiating light.

An increase in the intensity of the radiation leads to the emision of more electrons, but does not change their energy. This clearly contradict the view of Maxwell's wave theory where the energy of a wave is given by its intensity.

There is no smaller quantity of energy in radiation of a certain frequency f than the energy of a single photon. The radiation is regarded as a stream of photons, each having an energ hf.

1.5.3 The Compton Effect

The successes of blackbody radiation and the photoelectric effect were not sufficient to convince all scientists of the idea that radiation is quanatised. Further evidence for the photon concept came from the so-called Compton effect.

In 1923 Compton was studying the the scattering of x-rays off graphite. Classically, the charges should oscillate at the same frequency as of the incoming radiation and then give off radiation of the same frequency. However he found that radiation was being emmitted at a longer wavelength. This is called the Compton effect. Specifically, if the incoming radiation is scattered by an angle θ and if λ and λ' are wavelengths of the incident and scattered radiation, respectively, we find that

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta). \tag{1.59}$$

where m_0 is the rest mass of the electron. Thomson scattering, the classical theory of an electromagnetic wave scattered by charged particles, cannot explain the results of the experiment and demonstrates that light cannot be explained purely as a wave phenomenon.

The photon idea provides a clear explanation and provides additional direct confirmation of the quantum nature of radiation. The results can be analyzed in terms of a collision between a photon and an electron (in the experiment the energy of the photon was very much larger than the binding energy of the electron and could therefore be considered as a free electron). The incident photon collides with an at rest electron, which then recoils as a result of the impact, the scattered photon has less energy, smaller frwquency, and longer wavelength than the incident photon.

In fact we can derive (1.59) by a simple calculation. Classically we know from the equation $E^2 - c^2p^2 = m_0^2c^4$ that for a photon implies $p = E/c$. Since the energy of a photon is hf, its momentum is

$$p = \frac{hf}{c} = \frac{h}{\lambda}$$

Part of the energy of the radiation is transferred to the recoiling electron, we have

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + E_{kin} \tag{1.60}$$

where λ' is the wavelength after scattering and $E_{kin} = (\gamma - 1)m_0c^2$ is the relativistic kinematic energy of the recoiling electron. We consider the collision in the $x - y$ plane, where the incoming photon is scattered by an angle θ and the electron, initially at rest, is deflected by an angle ϕ. Conservation of momentum in the x and y directions respectively gives

$$\begin{aligned} \frac{h}{\lambda} &= \frac{h}{\lambda'}\cos\theta + p_e\cos\phi \\ 0 &= \frac{h}{\lambda'}\sin\theta - p_e\sin\phi \end{aligned} \tag{1.61}$$

where $p_e = mv = \gamma m_0 v$. By noting from (1.61) that

$$p_e^2\sin^2\phi^2 = \frac{h^2}{\lambda'}\sin^2\theta, \qquad p_e^2\cos^2\phi^2 = h^2\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\cos\theta\right)^2$$

and adding these together we can eliminate ϕ and after some manipulation obtain

$$p_e^2 = \frac{h^2}{\lambda^2} + \frac{h^2}{\lambda'^2} - \frac{2h^2}{\lambda\lambda'}\cos\theta.$$

We can obtain another expression for p_e^2 by using $E^2 = p_e^2c^2 + m_0^2c^4 = (E_{kin} + m_0c^2)^2$ and (1.60)

$$\begin{aligned} p_e^2 &= \left(\frac{E_{kin}}{c} + m_0 c\right)^2 - m_0^2c^2 \\ &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'} + m_0 c\right)^2 - m_0^2c^2 \\ &= \left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right)^2 + 2\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) m_0 c. \end{aligned}$$

Equating these two expressions for p_e^2 we obtain after cancellation of terms

$$-\frac{2h^2}{\lambda\lambda'}\cos\theta = -\frac{2h^2}{\lambda\lambda'} + 2\left(\frac{h}{\lambda} - \frac{h}{\lambda'}\right) m_0 c^2$$

which after simplifying gives the final result

$$\lambda' - \lambda = \frac{h}{m_0 c}(1 - \cos\theta).$$

the quantity $h/m_0c = 2.43 \times 10^{-12} m$ is called the Compton wavelength. The wavelength shift $\lambda' - \lambda$ is at most twice the Compton wavelength (for $\theta = 180^0$).

1.6 Photons

The formula

$$E = hf$$

means that the energy E carried by a photon and the frequency of the photon's electromagnetic vibration are directly proportional, the constant of proportionality being Planck's constant, h.

We find the energy of the electromagnetic field of a plane wave

$$\vec{\epsilon} N_k() \tag{1.62}$$

by using

$$E_{photon} = \frac{1}{8\pi} \int d^3x < \mathbf{E}^2 + \mathbf{B}^2 >= \frac{1}{4\pi} \int d^3x < \mathbf{B}^2 > \tag{1.63}$$

and substituiting in

$$\mathbf{B} = \nabla \times \mathbf{A} = iN_k \, \mathbf{k} \times \epsilon \, (e^{-ik\cdot x} - e^{ik\cdot x}) = 2N_k \, \mathbf{k} \times \epsilon \, \sin(k \cdot x) \tag{1.64}$$

and using

$$\begin{aligned}
(\mathbf{k} \times \epsilon) \cdot (\mathbf{k} \times \epsilon) &= \epsilon \cdot \epsilon \, \mathbf{k} \cdot \mathbf{k} - (\mathbf{k} \cdot \epsilon)^2 \\
&= (\epsilon_0^2 - \epsilon \cdot \epsilon)\mathbf{k}^2 - (k_0 \epsilon_0 - k \cdot \epsilon)^2 \\
&= \epsilon_0^2 \mathbf{k}^2 + \mathbf{k}^2 - k_0^2 \epsilon_0^2 \\
&= \mathbf{k}^2 = \omega^2.
\end{aligned} \tag{1.65}$$

We find the energy to be

$$E_{photon} = \frac{4\omega^2}{4\pi} N_k^2 \int d^3x < \sin^2(\omega t - \mathbf{k} \cdot \mathbf{x}) >= \frac{2\omega^2}{4\pi} N_k^2 V \tag{1.66}$$

where V the volume of the box. The condition $E_{photon} = \hbar\omega$ (where $\omega = 2\pi f$ is the angular frequency and $\hbar = h/2\pi$) leads to the normalisation constant

$$N_k = \sqrt{\frac{4\pi}{2\omega V}}. \tag{1.67}$$

We write

$$A_\mu(x, k) = \sqrt{\frac{4\pi}{2\omega V}} \, \epsilon_\mu(k, \lambda)(e^{-ik\cdot x} + e^{ik\cdot x}). \tag{1.68}$$

Part II

Relativistic Quantum Mechanics

Chapter 2

Dirac Spinors

As a precursor to the Dirac equation, we introduce the Klein-Gordon equation which describes relativistic scalar particles.

2.1 Klein-Gordon Equation

From quantum mechanics we know about the correspodance between Schrodinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = \left[-\frac{\hbar^2}{2m_0}\nabla^2 + V(x)\right]\psi(\mathbf{x},t) \tag{2.1}$$

and the non-relativistic energy relation,

$$E = \frac{\mathbf{p}^2}{2m_0} + V(\mathbf{x}). \tag{2.2}$$

The former can be obtained from the latter via the substitutions

$$E \to \hat{E} \;=\; i\hbar\frac{\partial}{\partial t} \tag{2.3}$$

$$\mathbf{p} \to \hat{\mathbf{p}} \;=\; -i\hbar\nabla. \tag{2.4}$$

Now consider the classical relativistic equation

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{2.5}$$

Make the same substitutions as before (that is 2.4 and 2.4). In terms of these operators the Einstein relation between energy, momentum, and mass can be written as

$$-\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -\hbar^2 \nabla^2 \phi + m_0^2 \phi \tag{2.6}$$

2.1.1 Current density

Multiply Schrodinger's equation from the left by ψ^* and its conjugate by the left ψ then substract. One obtains

$$i\hbar \frac{\partial |\psi|^2}{\partial t} = -\frac{\hbar^2}{2m_0} \left[\psi^*(\mathbf{x}, t) \nabla^2 \psi(\mathbf{x}, t) - \psi(\mathbf{x}, t) \nabla^2 \psi^*(\mathbf{x}, t) \right] \tag{2.7}$$

This is the continuity equation in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{2.8}$$

where

$$\rho = |\psi|^2 \tag{2.9}$$

is the probability density and

$$\mathbf{j} = -\frac{i\hbar}{2m_0} \left[\psi^*(\mathbf{x}, t) \nabla^2 \psi - \psi(\mathbf{x}, t) \nabla^2 \psi^* \right] \tag{2.10}$$

is the current density.

$$\int_V \frac{\partial \rho}{\partial t} d^3x = \frac{\partial}{\partial t} \int_V \rho d^3x = -\int_V \nabla \cdot \mathbf{j} d^3x = \int_S \mathbf{j} \cdot d\mathbf{S} = 0. \tag{2.11}$$

Hence,

$$\int_V \frac{\partial \rho}{\partial t} d^3x = Const. \tag{2.12}$$

By similar reasoning (see section) obtain for the Klein-Gordon equation the four-current density

$$j_\mu = \frac{i\hbar}{2m_0}(\phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^*) \tag{2.13}$$

The probability density is

$$\rho = j_0 = \frac{i\hbar}{2m_0}(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}) \tag{2.14}$$

At a given time both ϕ and $\partial\phi/\partial t$ may have arbitrary values and so ρ could be negative!

2.2 Dirac's Equation

Dirac wanted to construct a Hamiltonian that is linear in spatial derivatives, not just in time derivatives so that time and space are put of the same footing. He postulated an equation of the form

$$i\hbar\frac{\partial \psi}{\partial t} = [-i\hbar(\hat{\alpha}_1 \frac{\partial}{\partial x^1} + \hat{\alpha}_2 \frac{\partial}{\partial x^2} + \hat{\alpha}_3 \frac{\partial}{\partial x^3}) + \hat{\beta} m_0]\psi \tag{2.15}$$

where $\hat{\alpha}_,\hat{\beta}$ are $N \times N$ matrices and ψ is a column vector

$$\begin{pmatrix} \psi_1(\mathbf{x}, t) \\ \psi_2(\mathbf{x}, t) \\ \vdots \\ \psi_N(\mathbf{x}, t) \end{pmatrix} \tag{2.16}$$

To find the concrete form of this equation we follow the natural requirements:

• Energy-momentum relation for relativistic free particle

$$E^2 = \mathbf{p}^2 + m_0^2, \tag{2.17}$$

• continuity equation for the density

• Lorentz covariance

2.2.1 Energy-momentum relation for relativistic free particle

Every component ψ_σ of the spinor must satify the Klein-gordon equation

$$-\hbar^2 \frac{\partial^2 \psi_\sigma}{\partial t^2} = \left(-\hbar^2 \nabla^2 + m_0^2\right) \psi_\sigma \tag{2.18}$$

$$\begin{aligned}
-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} &= i\hbar \frac{\partial}{\partial t}(i\hbar \frac{\partial}{\partial t}\psi) = i\hbar \frac{\partial}{\partial t}\hat{\mathcal{H}}\psi = \hat{\mathcal{H}}^2 \psi \\
&= \left[-i\hbar \hat{\alpha}_i \frac{\partial}{\partial x^i} + \hat{\beta} m_0\right] \left[-i\hbar \hat{\alpha}_j \frac{\partial}{\partial x^j} + \hat{\beta} m_0\right] \psi \\
&= -\hbar^2 \sum_{i,j=1}^{3} \frac{\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} - i\hbar m_0 \sum_{i=1}^{3} (\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i) \frac{\partial \psi}{\partial x^i} + \hat{\beta}^2 m_0^2 \psi
\end{aligned} \tag{2.19}$$

Comparison with (2.18) implies the following requirements

$$\begin{aligned}
\hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= 2\delta_{ij} 1, \\
\hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= 0, \\
\hat{\alpha}_i^2 = \hat{\beta}^2 &= 1.
\end{aligned} \tag{2.20}$$

For the Hamiltonian to be Hermitian, the matrices $\hat{\alpha}_i, \hat{\beta}$ have to be Hermitian

$$\hat{\alpha}_i^\dagger = \hat{\alpha}_i, \quad \hat{\beta}^\dagger = \hat{\beta}. \tag{2.21}$$

Therefore the eigenvalues are real. Since $\hat{\alpha}_i^2 = 1$ and $\hat{\beta}^2 = 1$, it follows that the eigenvalues can only take the values ± 1. The eigenvalues are independent of the representation. Consider the diagonal representation of $\hat{\alpha}_i$, for example, we have

$$\hat{\alpha}_i = \begin{pmatrix} A_1 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ 0 & 0 & A_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0\cdots & 0 & A_N \end{pmatrix} \tag{2.22}$$

with eigenvalues $A_1, \ldots, A_N$, and $\hat{\alpha}_i^2 = 1$ yields

$$\hat{\alpha}_i^2 = I = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0\cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} A_1^2 & 0 & 0 & \cdots & 0 \\ 0 & A_2^2 & 0 & \cdots & 0 \\ 0 & 0 & A_3^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0\cdots & 0 & A_N^2 \end{pmatrix} \tag{2.23}$$

from which

$$A_k^2 = 1, \quad i.e. \quad A_k = \pm 1. \tag{2.24}$$

Now from the anticommutation relations we have

$$\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta}$$

and the identity

$$Tr\hat{A}\hat{B} = Tr\hat{B}\hat{A}$$

we conclude

$$Tr\hat{\alpha}_i = -Tr\hat{\beta}\hat{\alpha}_i\hat{\beta} = -Tr\hat{\alpha}_i\hat{\beta}^2 = -Tr\hat{\alpha}_i = 0. \tag{2.25}$$

We see that the matrices $\hat{\alpha}_i, \beta$ must even dimensional.

The smallest even dimension for which the (2.20) can be fulfilled is $N = 4$. In fact it is easily shown that the following is a representation

$$\hat{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{\alpha}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$
$$\hat{\alpha}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.26}$$

To see this note that

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \tag{2.27}$$

where $\hat{\sigma}_i$ are Pauli's 2×2 matrices which satify the relation

$$\hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i = 2\delta_{ij} \mathbf{1}, \tag{2.28}$$

as this means

$$\begin{aligned} \hat{\alpha}_i \hat{\alpha}_j + \hat{\alpha}_j \hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_j \\ \hat{\sigma}_j & 0 \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix} \\ &= \begin{pmatrix} \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i & 0 \\ 0 & \hat{\sigma}_i \hat{\sigma}_j + \hat{\sigma}_j \hat{\sigma}_i \end{pmatrix} \\ &= 2\delta_{ij} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \end{aligned} \tag{2.29}$$

and also

$$\begin{aligned} \hat{\alpha}_i \hat{\beta} + \hat{\beta} \hat{\alpha}_i &= \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} + \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} + \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix} = 0. \end{aligned} \tag{2.30}$$

2.2.2 Continuity equation for the density

We need to construct the four-current density and the equation of continuity. Let us multiply (2.15) from the left by $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

$$i\hbar \psi^\dagger \frac{\partial}{\partial t} \psi = -i\hbar \sum_{k=1}^{3} \psi^\dagger \hat{\alpha}_k \frac{\partial}{\partial x^k} \psi + m_0 \psi^\dagger \hat{\beta} \psi \tag{2.31}$$

Take the Hermitian conjugate of (2.15)

$$i\hbar \frac{\partial \psi^\dagger}{\partial t} = i\hbar \sum_{k=1}^{3} \frac{\partial \psi^\dagger}{\partial x^k} \hat{\alpha}_k^\dagger + m_0 \psi^\dagger \hat{\beta}^\dagger \tag{2.32}$$

and multiply from the right by ψ, taking into account the Hermiticty of $\hat{\alpha}_i, \hat{\beta}$, we get

$$-i\hbar \frac{\partial \psi^\dagger}{\partial t} \psi = i\hbar \sum_{k=1}^{3} \frac{\partial \psi^\dagger}{\partial x^k} \hat{\alpha}_k \psi + m_0 \psi^\dagger \hat{\beta} \psi \tag{2.33}$$

Then, subtraction of (2.33) from (2.31) yields

$$i\hbar \frac{\partial}{\partial t} (\psi^\dagger \psi) = -i\hbar \sum_{k=1}^{3} \frac{\partial}{\partial x^k} (\psi^\dagger \hat{\alpha}_k \psi) \tag{2.34}$$

which can be seen as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{2.35}$$

where

$$\rho = \psi^\dagger \psi \tag{2.36}$$

is the positive definite density and

$$j^k = \psi^\dagger \hat{\alpha}^k \psi \tag{2.37}$$

We still have to show that $\rho(x)$ is the temporal component of a four-vector so that the spatial integral $\int \rho d^3x$ becomes constant in time. Only the the probability interpretation of $\rho(x)$ ensured.

2.2.3 Lorentz covariance

We require a the form invaraince of the Dirac equation (2.15) with respect to Lorentz transformations is still to be shown,before we can regard the Dirac equation as an acceptable relativistic wave equation. We come back to Lorentz invariance in another section

We also note that (2.26) is a special representation. The choice of the matrices (2.26) is not the only possible represntation. It is easy seen that each unitary transformation $\hat{S}$ yields matrices

$$\hat{\alpha}'_i = \hat{S} \hat{\alpha}_i \hat{S}^{-1}, \qquad \hat{\beta}' = \hat{S} \hat{\beta} \hat{S}^{-1} \tag{2.38}$$

which also satisfy the algebra (2.20). First we take $\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}I$

$$\begin{aligned}
\hat{\alpha}'_i\hat{\alpha}'_j + \hat{\alpha}'_j\hat{\alpha}'_i &= \hat{S}\hat{\alpha}_i\hat{S}^{-1}\hat{S}\hat{\alpha}_j\hat{S}^{-1} + \hat{S}\hat{\alpha}_j\hat{S}^{-1}\hat{S}\hat{\alpha}_i\hat{S}^{-1} \\
&= \hat{S}\hat{\alpha}_i\hat{\alpha}_j\hat{S}^{-1} + \hat{S}\hat{\alpha}_j\hat{\alpha}_i\hat{S}^{-1} \\
&= \hat{S}(\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i)\hat{S}^{-1} \\
&= 2\delta_{ij}\hat{S}I\hat{S}^{-1} \\
&= 2\delta_{ij}I. \qquad (2.39)
\end{aligned}$$

Then take $\hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0$

$$\begin{aligned}
\hat{\alpha}'_i\hat{\beta}' + \hat{\beta}'\hat{\alpha}'_i &= \hat{S}\hat{\alpha}_i\hat{S}^{-1}\hat{S}\hat{\beta}\hat{S}^{-1} + \hat{S}\hat{\beta}\hat{S}^{-1}\hat{S}\hat{\alpha}_i\hat{S}^{-1} \\
&= \hat{S}(\hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i)\hat{S}^{-1} \\
&= 0. \qquad (2.40)
\end{aligned}$$

Then take $\hat{\alpha}_i^2 = I$

$$\begin{aligned}
\hat{\alpha}_i'^2 &= \hat{S}\hat{\alpha}_i\hat{S}^{-1}\hat{S}\hat{\alpha}_i\hat{S}^{-1} \\
&= \hat{S}\hat{\alpha}_i^2\hat{S}^{-1} \\
&= \hat{S}I\hat{S}^{-1} \\
&= I. \qquad (2.41)
\end{aligned}$$

Similarly we get $\hat{\beta}'^2 = I$. They also satify $\hat{\alpha}_i{}'\dagger = \hat{\alpha}'_i$ and $\hat{\beta}'^\dagger = \hat{\beta}'$:

$$\begin{aligned}
(\hat{\alpha}')^\dagger &= (\hat{S}\hat{\alpha}_i\hat{S}^{-1})^\dagger \\
&= (\hat{S}^{-1})^\dagger\hat{\alpha}_i^\dagger\hat{S}^\dagger \\
&= (\hat{S}^\dagger)^\dagger\hat{\alpha}_i^\dagger\hat{S}^\dagger \\
&= \hat{S}\hat{\alpha}_i^\dagger\hat{S}^\dagger \\
&= \hat{\alpha}' \qquad (2.42)
\end{aligned}$$

where we used the that $\hat{S}$ is unitary. Similarly we get $(\hat{\beta}')^\dagger = \hat{\beta}'$.

2.3 Free Motion of a Dirac Particle

$$i\hbar\frac{\partial\psi}{\partial t}=\hat{\mathcal{H}}\psi=\left(\hat{\alpha}\cdot\hat{\mathbf{p}}+m_0\hat{\beta}\right)\psi \tag{2.43}$$

Stationary states can be found by substituting

$$\psi(\mathbf{x},t)=\psi(\mathbf{x})\exp[-(i/\hbar)\epsilon t] \tag{2.44}$$

into the Dirac equation. We get

$$\epsilon\psi(\mathbf{x})=\hat{\mathcal{H}}\psi(\mathbf{x}) \tag{2.45}$$

Split the four-component spinor into two two-component spinors ϕ and χ, i.e.

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=\begin{pmatrix}\varphi\\ \chi\end{pmatrix} \tag{2.46}$$

$$\varphi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}\quad\text{and}\quad\chi=\begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}. \tag{2.47}$$

or

$$\begin{aligned}\epsilon\psi&=\hat{\alpha}\chi\cdot\hat{\mathbf{p}}+m_0\varphi\\ \epsilon\chi&=\hat{\alpha}\varphi\cdot\hat{\mathbf{p}}-m_0\chi\end{aligned} \tag{2.48}$$

The states

$$\begin{pmatrix}\varphi\\ \chi\end{pmatrix}=\begin{pmatrix}\varphi_0\\ \chi_0\end{pmatrix}\exp[(i/\hbar)\mathbf{p}\cdot\mathbf{x}] \tag{2.49}$$

This results in

$$\begin{aligned}(\epsilon-m_0)I\varphi_0-\hat{\sigma}\cdot\hat{\mathbf{p}}\chi_0&=0,\\ -\hat{\sigma}\cdot\hat{\mathbf{p}}\;\varphi_0+(\epsilon+m_0)I\chi_0&=0.\end{aligned} \tag{2.50}$$

This linear homogeneous system of equations for φ_0 and χ_0 has notrivial solutions only in the case of a vanishing determinant of the coefficients, that is

$$\begin{vmatrix} (\epsilon - m_0)I & -\hat{\sigma} \cdot \hat{\mathbf{p}} \\ -\hat{\sigma} \cdot \hat{\mathbf{p}} & (\epsilon + m_0)I \end{vmatrix} = 0. \tag{2.51}$$

Using the relation

$$(\hat{\sigma} \cdot \mathbf{A})(\hat{\sigma} \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} I + i\hat{\sigma} \cdot (\mathbf{A} \times \mathbf{B}) \tag{2.52}$$

(2.51) becomes

$$\begin{aligned} (\epsilon^2 - m_0^2)I - (\hat{\sigma} \cdot \hat{\mathbf{p}})(\hat{\sigma} \cdot \hat{\mathbf{p}}) &= (\epsilon^2 - m_0^2)I - \mathbf{p} \cdot \mathbf{p} I - i\hat{\sigma} \cdot (\mathbf{p} \times \mathbf{p}) \\ &= (\epsilon^2 - m_0^2)I - \mathbf{p} \cdot \mathbf{p} I = 0 \end{aligned} \tag{2.53}$$

or

$$\epsilon^2 = m_0^2 + \mathbf{p}^2 \tag{2.54}$$

from which it follows

$$\epsilon = \pm E_p, \quad E_p = \sqrt{\mathbf{p}^2 + m_0^2} \tag{2.55}$$

2.4 Positive and Negative Energy Eigenvectors

with solutions

$$\begin{aligned} \varphi(t) &= \varphi(0)e^{-im_0 t} \\ \chi(t) &= \chi(0)e^{-im_0 t} \end{aligned} \tag{2.56}$$

φ represents a particle, while χ represents an antiparticle.

$$\chi_0 = \frac{(\hat{\sigma} \cdot \hat{\mathbf{p}})}{m_0 + \epsilon} \varphi_0. \tag{2.57}$$

Let us denote the two-spinor φ_0 in the form

$$\varphi_0 = U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} \tag{2.58}$$

with the normalisation

$$U^\dagger U = U_1^* U_1 + U_2^* U_2 = 1, \tag{2.59}$$

where U_1, U_2 are complex numbers.

2.5 Helicity

There is another quantum number, the helicity, can be used to classify the free one-particle states. Its operator should commute with the operators whose eigenvalues have already been introduced to label our free solutions.

$$\hat{\mathbf{S}} = \frac{\hbar}{2}\hat{\mathbf{\Sigma}} = \begin{pmatrix} \hat{\sigma} & 0 \\ 0 & \hat{\sigma} \end{pmatrix} \tag{2.60}$$

The helicity commutes with the Hamiltonian

$$\begin{aligned}
[\hat{\mathcal{H}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}] &= \left[\hat{\alpha} \cdot \hat{\mathbf{p}} + m_0 \hat{\beta}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] \\
&= \begin{pmatrix} 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \\ \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \end{pmatrix} \begin{pmatrix} \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix} - \begin{pmatrix} \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \\ 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \end{pmatrix} \begin{pmatrix} 0 & \hat{\sigma} \cdot \hat{\mathbf{p}} \\ \hat{\sigma} \cdot \hat{\mathbf{p}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 \\ (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 & 0 \end{pmatrix} - \begin{pmatrix} 0 & (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 \\ (\hat{\sigma} \cdot \hat{\mathbf{p}})^2 & 0 \end{pmatrix} \\
&= 0
\end{aligned}$$

Hence

$$\left[\hat{\mathcal{H}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] = 0 \tag{2.61}$$

and obviously we have

$$\left[\hat{\mathbf{p}}, \hat{\mathbf{\Sigma}} \cdot \hat{\mathbf{p}}\right] = 0 \tag{2.62}$$

the helicity operator

$$\hat{\Lambda}_S = \frac{\hbar}{2}\hat{\mathbf{\Sigma}} \cdot \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} = \hat{\mathbf{S}} \cdot \frac{\hat{\mathbf{p}}}{|\mathbf{p}|} \tag{2.63}$$

Helicity is the projection of the spin onto the direction of momentum.

If the electron wave propagates into th edirection of the $z-$axis, we have

$$\mathbf{p} = \{0, 0, p\}$$

and because of (2.63),

$$\hat{\Lambda}_S = \hat{S}_z = \frac{\hbar}{2}\hat{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.64}$$

with eigenvalues $\pm\hbar/2$. Clearly, the eigenvectors of $\hat{\Lambda}_S$ are

$$\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ u_1 \end{pmatrix} \tag{2.65}$$

with

$$u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad u_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.66}$$

If in any particular direction a quantum state take one of two values it is likely to do with spin half particles. In fact Lorentz-covariance of Dirac's equation will imply that these two-state systmes transform under rotations as two-spinors.

2.6 Lorentz Covariance of the Dirac Equation

A theory to be Lorentz-covaraint if its form is invariant under a transition from one inertial system to another.

$$\left(i\hbar\gamma^\mu \frac{\partial}{\partial x^\mu} - m_0 c\right)\psi(x) \qquad \text{and} \qquad \left(i\hbar{\gamma'}^\mu \frac{\partial}{\partial x^\mu} - m_0 c\right)\psi'(x') \tag{2.67}$$

$$\frac{\hat{\beta}}{c}\left(i\hbar\frac{\partial}{\partial t}+i\hbar c\sum_{k=1}^{3}\hat{\alpha}_k\frac{\partial}{\partial x^k}-\hat{\beta}m_0 c\right)\psi(x)=0$$

or

$$\left(\hat{\beta}i\hbar\frac{\partial}{\partial ct}+i\hbar\sum_{k=1}^{3}\hat{\beta}\hat{\alpha}_k\frac{\partial}{\partial x^k}-m_0 c\right)\psi(x)=0$$

With the definitions

$$\gamma^0, \quad \gamma^i=\hat{\beta}\hat{\alpha}_i, \quad i=1,2,3 \tag{2.68}$$

this can be finally written in the form

$$i\hbar\left(\gamma^0\frac{\partial}{\partial x^0}+\gamma^1\frac{\partial}{\partial x^1}+\gamma^2\frac{\partial}{\partial x^2}+\gamma^3\frac{\partial}{\partial x^3}+-m_0 c\right)\psi(x)=0. \tag{2.69}$$

A more elegant formulation of the anti-commutation relations (2.20) is possible with the γ matrices

$$\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=2g^{\mu\nu}I \tag{2.70}$$

I being the unit 4×4 unit matrix.

In the representation we have been using so far

$$\gamma^i=\begin{pmatrix}0 & \hat{\sigma}^i\\ -\hat{\sigma}^i & 0\end{pmatrix}, \quad \gamma^0=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}. \tag{2.71}$$

2.6.1 Coupling of Dirac spinors to the Electromagnetic Field

A short hand notation is often conveient: the so-called Dirac/Feynman-daggr notation, for example

$$A\!\!\!/ := \gamma^\mu A_\mu = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^0 A^0 - \sum_{i=1}^{3}\gamma^i A^i = \gamma^0 A^0 - \gamma\cdot\mathbf{A}. \tag{2.72}$$

Another example is the nabla dagger notation:

$$\not{\nabla} := \gamma^\mu \frac{\partial}{\partial x^\mu} = \gamma^0 \frac{\partial}{\partial ct} + \sum_{i=1}^{3} \gamma^i \frac{\partial}{\partial x^i} = \frac{\gamma^0}{c}\frac{\partial}{\partial t} + \gamma \cdot \nabla. \tag{2.73}$$

with this the Dirac equation (2.69) can be written in the concise form

$$(i\hbar\not{\nabla} - m_0 c)\psi = 0, \tag{2.74}$$

or with $\hat{p}_\mu = i\hbar\partial/\partial x^\mu$,

$$(\not{p} - m_0 c)\psi = 0, \tag{2.75}$$

Under a gauge transformation the wavefunction $\psi(x)$ transforms as

$$\psi(x) \rightarrow e^{i\Lambda(x)}\psi(x) \tag{2.76}$$

and

$$A^\mu(x) \rightarrow A^\mu(x) - \frac{1}{e}\partial^\mu \Lambda(x) \tag{2.77}$$

thus

$$\left(i\hbar\frac{\partial}{\partial x^\mu} - eA^\mu\right) \tag{2.78}$$

is gauge covariant. The minimal coupling prescription

$$i\hbar\frac{\partial}{\partial x^\mu} \rightarrow \left(i\hbar\frac{\partial}{\partial x^\mu} - eA^\mu\right) \tag{2.79}$$

results in the equation

$$(i\hbar\gamma^\mu\frac{\partial}{\partial x^\mu} - eA_\mu\gamma^\mu - m_0)\psi = 0. \tag{2.80}$$

Both $\hat{p}^\mu$ and A^μ are four-vectors; hence the difference $\hat{p}^\mu - (e/c)A^\mu$ is a four-vector too. While discussing the covariance in the following section we can confine ourselves to the free equations (2.74) and (2.75).

2.7 Lorentz Transformations for Dirac Spinors

How do we construct the wave function $\psi'(x')$ in one inertial frame if we know the wave function $\psi(x)$ in another frame, where the two frames are related by the Lorentz transformation a^{ν}_{μ}? Here we construct the Lorentz transformation law between $\psi(x)$ and $\psi'(x')$.

We start from the principle of special relativity which states that the laws of physics should take the same form in all inertial systems, $\psi'(x')$ must be a solution of a Dirac equation which has the form

$$\left(i\hbar\gamma^{\mu'}\frac{\partial}{\partial x'^{\mu}} - m_0\right)\psi'(x') = 0 \tag{2.81}$$

in the primed system, where $\gamma^{\mu'}$ satisfy the same anti-commutation relations as γ^{μ}:

$$\gamma^{\mu'}\gamma^{\nu'} + \gamma^{\nu'}\gamma^{\mu'} = 2\eta^{\mu\nu}I \tag{2.82}$$

and

$$\gamma'^{0\dagger} = \gamma'^{0} \tag{2.83}$$

$$\gamma'^{i\dagger} = -\gamma'^{i} \quad i = 1, 2, 3. \tag{2.84}$$

It can be shown that $\gamma^{\mu'}$ that satify the above relations are identical to γ^{μ} up to a unitary transformation $\hat{U}$, i.e.

$$\gamma^{\mu'} = \hat{U}^{\dagger}\gamma^{\mu}\hat{U}, \quad \hat{U}^{\dagger} = \hat{U}^{-1}. \tag{2.85}$$

Since unitary transformations do not change the physics, we may use the same γ matrices in both Lorentz systems. From now on we just take $\gamma^{\mu} = \gamma^{\mu'}$.

$$\left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi(x) = 0 \quad \text{and} \quad \left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x'^{\mu}} - m_0\right)\psi'(x') = 0$$

Let $\hat{a}$ denote the matrix of the Lorentz transformation a^{ν}_{μ}. We write

$$\psi'(x') = \psi'(\hat{a}x') = \hat{S}(\hat{a})\psi(x) = \hat{S}(\hat{a}^{-1}x') \tag{2.86}$$

We must have an inverse transformation

$$\psi(x) = \hat{S}^{-1}(\hat{a})\psi'(x') = \hat{S}^{-1}(\hat{a})\psi'(\hat{a}x)$$

Start with Dirac equation

$$\left(i\hbar\gamma^{\mu}\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi(x) = 0$$

expressing $\psi(x)$ by $\hat{S}^{-1}(\hat{a})\psi'(x')$ yields

$$\left(i\hbar\gamma^{\mu}\hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^{\mu}} - m_0\hat{S}^{-1}(\hat{a})\right)\psi'(x') = 0.$$

We the multiply by $\hat{S}(\hat{a})$

$$\left(i\hbar\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})\frac{\partial}{\partial x^{\mu}} - m_0\right)\psi'(x') = 0 \tag{2.87}$$

Now we transform $\partial/\partial x^{\mu}$ to x' coordinates. Using

$$x'^{\nu} = a^{\nu}{}_{\mu}x^{\mu} \qquad \Rightarrow \qquad \frac{\partial x'^{\nu}}{\partial x^{\mu}} = a^{\nu}{}_{\mu}$$

the transformation for the partial derivative is given by

$$\frac{\partial}{\partial x^{\mu}} = \frac{\partial x'^{\nu}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\nu}} = a^{\nu}{}_{\mu}\frac{\partial}{\partial x'^{\nu}} \tag{2.88}$$

So that (2.87) becomes

$$\left(i\hbar(\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})a^{\nu}{}_{\mu})\frac{\partial}{\partial x'^{\nu}} - m_0\right)\psi'(x') = 0. \tag{2.89}$$

Comparing this to Dirac's equation in the x' coordinates we see that $\hat{S}(\hat{a})$ must satisfy

$$\hat{S}(\hat{a})\gamma^{\mu}\hat{S}^{-1}(\hat{a})a^{\nu}{}_{\mu} = \gamma^{\nu} \tag{2.90}$$

or equivalently

$$\hat{S}(\hat{a})\gamma^{\nu}\hat{S}^{-1}(\hat{a}) = a_{\mu}{}^{\nu}\gamma^{\mu} \tag{2.91}$$

2.8 Infintesimal Generating Technique for Lorentz Transformations

We first give the idea of the infintesimal generating technique with a couple of simple examples.

Example 2: Lorentz transformation in x_1−direction for $2d$-spacetime

We derive the Lorentz transformation formula for boosts in the x_1−direction. Consider two inertia frames, the 'primed' frame one moving away from the 'unprimed' frame at an infintesimal velocity δv along the $x_1$direction. For an infintesimal relative velocity the spacetime transformation is Galilean:

$$x'_1 = x_1 - \delta v t. \tag{2.92}$$

How is special relativity brought into the calculation? This is done by requiring that

$$x_1^2 - t^2 = x_1'^2 - t'^2. \tag{2.93}$$

From this we see that $t' \neq t$, and so t should transform some way as well. Let us write

$$t' = t + a\delta v x_1. \tag{2.94}$$

Using this in (2.93) we find $a = -1$. The two transformation equations can be combined in the matrix equation

$$\begin{aligned} \begin{pmatrix} t' \\ x' \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta v \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\right] \begin{pmatrix} t \\ x \end{pmatrix} \\ &= (\mathbf{1} + \delta v \hat{I}_2) \begin{pmatrix} t \\ x \end{pmatrix} \end{aligned} \tag{2.95}$$

where $\hat{I}_x = -\mathbf{1}$. Now we repeat the transformation N times to generate a finite transformation with velocity pararmeter $\theta = N\delta v$. Then

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \left(\mathbf{1} + \frac{\theta}{N}\hat{I}_x\right)^N \begin{pmatrix} t \\ x \end{pmatrix} \tag{2.96}$$

In the limit $N \to \infty$,

$$\lim_{N\to\infty}\left(\mathbf{1}+\frac{\theta}{N}\hat{I}_x\right)^N = \exp(\theta\hat{I}_2). \tag{2.97}$$

Noting $\hat{I}_x^2 = \mathbf{1}$, we expand the expontential

$$\begin{aligned}
\exp(\theta\hat{I}_x) &= \mathbf{1}+\theta\hat{I}_x+\frac{\theta^2\hat{I}_x^2}{2!}+\frac{\theta^3\hat{I}_x^3}{3!}+\ldots \\
&= \mathbf{1}[1+\frac{1}{2!}\theta^2+\ldots]+\hat{I}_x[\theta-\frac{\theta^3}{3!}+\ldots] \\
&= \\
&= \begin{pmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{pmatrix}
\end{aligned} \tag{2.98}$$

$\cosh\theta$ and $\sinh\theta$ can be identified by considering the origin of the primed coordinate system, $x' = 0$, or $x = vt$. Substituiting this into () we have

$$0 = x\cosh\theta - t\sinh\theta. \tag{2.99}$$

So

$$\tan\theta = v$$

Using $1-\tanh^2\theta = (\cosh^2\theta)^{-1}$,

$$\cosh\theta = \frac{1}{(1-v^2)^{1/2}}, \qquad \sinh\theta = \frac{v}{(1-v^2)^{1/2}}. \tag{2.100}$$

We finally obtain the known Lorentz transformations

$$t' = \frac{t-vx}{(1-v^2)^{1/2}}, \qquad x' = \frac{x-vt}{(1-v^2)^{1/2}} \tag{2.101}$$

□

This result can easily be generalised to 4-minkowski space time. We just use the generator

$$I_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.102}$$

We end up with the answer

$$t' = \frac{t - vx}{(1-v^2)^{1/2}}, \qquad x' = \frac{x - vt}{(1-v^2)^{1/2}}, \qquad y' = y, \qquad z' = z. \tag{2.103}$$

If we had wanted to do boost in the direction given by the unit vector

$$\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma) \tag{2.104}$$

we would use the generator

$$I_n = \begin{pmatrix} 0 & -\cos\alpha & -\cos\beta & -\cos\gamma \\ -\cos\alpha & 0 & 0 & 0 \\ -\cos\beta & 0 & 0 & 0 \\ -\cos\gamma & 0 & 0 & 0 \end{pmatrix}. \tag{2.105}$$

Note that, to first order, $t^2 - \vec{r}^2 = (t')^2 - \vec{r}'^2$ is satified for an infintesimal relative velocity.

The reader is invited to do the full calculation and derive the Lorentz transformation formula.

Example 2: Rotations about the $z-$direction for $3d$-spacetime

It is easily seen, drawing a diagram, that under an infintesimal rotation $\delta\phi$ around the $z-$axis results in

$$x' = x + y\delta\phi, \qquad y' = y - x\delta\phi \tag{2.106}$$

The two transformation equations can be combined in the matrix equation

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\phi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \begin{pmatrix} x \\ y \end{pmatrix} \\ &= (\mathbf{1} + \delta\phi i\hat{\sigma}_2) \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned} \tag{2.107}$$

where $\hat{\sigma}_2$ is a Pauli matrix. Let us now do the expontentiation using $\hat{\sigma}_2^2 = 1$:

$$
\begin{aligned}
\exp(\phi i\hat{\sigma}_2) &= \mathbf{1} + \phi i\hat{\sigma}_2 - \frac{\phi^2}{2!}\hat{\sigma}_2^2 - i\frac{\phi^3}{3!}\hat{\sigma}_2^3 \\
&= \mathbf{1}(1 - \frac{\phi^2}{2!} + \dots) + i\hat{\sigma}_2(\phi - \frac{\phi^3}{3!} + \dots) \\
&= \mathbf{1}\cos\phi + i\hat{\sigma}_2\sin\phi \\
&= \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}
\end{aligned}
\tag{2.108}
$$

So that

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\tag{2.109}
$$

$\square$

In 4d-minkowski space time we use the generator

$$
I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\tag{2.110}
$$

We end up with the answer

$$
t' = t, \qquad x' = x\cos\phi + y\sin\phi, \qquad y' = -x\sin\phi + y\cos\phi, \qquad z' = z.
\tag{2.111}
$$

Proper Lorentz Transformations

$$
a^{\nu}{}_{\mu} = \delta^{\nu}_{\mu} + \Delta\omega^{\nu}{}_{\mu}
\tag{2.112}
$$

We denote the inverse Lorentz Transformation as $a_{\nu}{}^{\sigma}$. Then, neglecting terms quadtratic in $\Delta\omega$,

$$
\begin{aligned}
a^{\mu}{}_{\nu}a_{\mu}{}^{\sigma} &= \delta^{\sigma}_{\nu} = (\delta^{\mu}_{\nu} + \Delta\omega^{\mu}{}_{\nu})(\delta^{\sigma}_{\mu} + \Delta\omega_{\mu}{}^{\sigma}) \\
&\approx \delta^{\mu}_{\nu}\delta^{\sigma}_{\mu} + \delta^{\mu}_{\nu}\Delta\omega_{\mu}{}^{\sigma} + \delta^{\sigma}_{\mu}\Delta\omega^{\mu}{}_{\nu} \\
&= \delta^{\sigma}_{\nu} + \Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu}
\end{aligned}
\tag{2.113}
$$

Hence,

$$\Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu} = 0$$

or

$$\eta^{\mu\nu}\left(\Delta\omega_{\nu}{}^{\sigma} + \Delta\omega^{\sigma}{}_{\nu}\right) = 0 = \Delta\omega^{\mu\sigma} + \Delta\omega^{\sigma\mu}.$$

so we must have

$$\Delta\omega^{\mu\nu} = -\Delta\omega^{\nu\mu} \tag{2.114}$$

Consequently, there are six independent non-vanishing parameters $\Delta\omega^{\mu\nu}$.

2.9 The $\hat{S}$ Operator for Infintesimal Lorentz Transformations

We aim to determine the operator $\hat{S}$ by assertaining its infintesimal form by finding its expansion to linear order in the generators $\Delta\omega^{\mu\nu}$. We write

$$\hat{S}(\Delta\omega^{\mu\nu}) \;=\; \mathbf{1} - \frac{i}{4}\hat{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \tag{2.115}$$

where $\sigma_{\alpha\beta} = -\sigma_{\beta\alpha}$. The inverse operator being

$$\hat{S}^{-1}(\Delta\omega^{\mu\nu}) \;=\; \mathbf{1} + \frac{i}{4}\hat{\sigma}_{\mu\nu}\Delta\omega^{\mu\nu} \tag{2.116}$$

By finding $\sigma_{\alpha\beta}$ we can find $\hat{S}$. By substituting (2.115) and (2.116) into the defing equation for $\hat{S}$:

$$(\delta^{\nu}_{\mu} + \Delta\omega_{\mu}{}^{\nu})\gamma^{\mu} = \hat{S}(\Delta\omega^{\mu\nu})\gamma^{\nu}\hat{S}^{-1}(\Delta\omega^{\mu\nu})$$

we can find an equation that determines $\sigma_{\alpha\beta}$.

$$(\delta^{\nu}_{\mu} + \Delta\omega_{\mu}{}^{\nu})\gamma^{\mu} = \left(\mathbf{1} - \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)\gamma^{\nu}\left(\mathbf{1} + \frac{i}{4}\hat{\sigma}_{\alpha\beta}\Delta\omega^{\alpha\beta}\right)$$

or omitting the quadratic terms in $\Delta\omega_{\mu}{}^{\nu}$,

$$\Delta\omega_{\mu}{}^{\nu}\gamma^{\mu} = -\frac{i}{4}\Delta\omega^{\alpha\beta}(\hat{\sigma}_{\alpha\beta}\gamma^{\nu} - \gamma^{\nu}\hat{\sigma}_{\alpha\beta}) \tag{2.117}$$

Using the antisymmetry of $\Delta\omega_{\mu}{}^{\nu}$, the LHS becomes

$$\begin{aligned}
\Delta\omega_{\mu}{}^{\nu}\gamma^{\mu} &= \eta^{\nu}{}_{\sigma}\Delta\omega_{\mu}{}^{\sigma}\gamma^{\mu} \\
&= \Delta\omega_{\beta}{}^{\alpha}(\eta^{\nu}{}_{\alpha}\gamma^{\beta}) \\
&= -\Delta\omega^{\alpha\beta}(\eta^{\nu}{}_{\alpha}\gamma_{\beta}) \\
&= -\frac{1}{2}\Delta\omega^{\alpha\beta}(\eta^{\nu}{}_{\alpha}\gamma_{\beta} - \eta^{\nu}{}_{\beta}\gamma_{\alpha})
\end{aligned} \tag{2.118}$$

Comparing this with (2.117), we end up with the relation

$$-2i(\eta_{\nu}{}^{\alpha}\gamma_{\beta} - \eta_{\nu}{}^{\beta}\gamma_{\alpha}) = [\hat{\sigma}_{\alpha\beta}, \gamma^{\nu}] \tag{2.119}$$

It is shown in section A.6 that this is solved by

$$\hat{\sigma}_{\alpha\beta} = \frac{i}{2}[\gamma_{\alpha}, \gamma_{\beta}].$$

The operator $\hat{S}(\Delta\omega^{\mu\nu})$ is now

$$\hat{S}(\Delta\omega^{\mu\nu}) = \mathbf{1} + \frac{1}{8}[\gamma_{\mu}, \gamma_{\nu}]\Delta\omega^{\mu\nu} \tag{2.120}$$

The problem of finding $\hat{S}$ for finite proper Lorentz transformations has now been essentially solved! To construct $\hat{S}$ we now sucessively apply the infintesimal operators (2.120). In order to do this we write

$$\Delta\omega^{\nu}{}_{\mu} = \Delta\omega(\hat{I}_{\mathbf{n}})^{\nu}{}_{\mu} \tag{2.121}$$

Here $\Delta\omega$ is an inintesimal parameter of the Lorentz group around an axis in the $\mathbf{n}$ direction.

2.10 The $\hat{S}$ Operator for Proper Lorentz Transformations

$$
\begin{aligned}
\psi'(x') = \hat{S}(\hat{a})\psi(x) &= \lim_{N\to\infty}\left(\mathbf{1} - \frac{i}{4}\frac{\omega}{N}\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}\right)^N \psi(x) \\
&= e^{-(i/4)\omega\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}}\psi(x)
\end{aligned}
\tag{2.122}
$$

2.10.1 Example: Lorentz boost along x-axis

$$
(I_x)^0{}_1 = (I_x)^1{}_0 = -(I_x)^{01} = +(I_x)^{10} = -1.
$$

So (2.122) becomes

$$
\begin{aligned}
\psi'(x') &= \exp\left\{-\frac{i}{4}\omega\left[\hat{\sigma}_{01}(I_x)^{01} + \hat{\sigma}_{10}(I_x)^{10}\right]\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{4}\omega\left[\hat{\sigma}_{01}(+1) + \hat{\sigma}_{10}(-1)\right]\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{2}\omega\hat{\sigma}_{01}\right\}\psi(x)
\end{aligned}
\tag{2.123}
$$

□

2.10.2 Example: Rotation around z-axis

Recall that

$$
\Delta\omega^\nu{}_\mu = \delta\phi(\hat{I}_3)^\nu{}_\mu,
\tag{2.124}
$$

where $\hat{I}_3$ is given by (). Thus only the elements $(\hat{I}_3)^{12} = -(\hat{I}_3)^{21}$ are non-zero, and we get

$$
\begin{aligned}
\psi'(x') &= \exp\left\{-\frac{i}{4}\phi\hat{\sigma}_{\mu\nu}(\hat{I}_3)^{\mu\nu}\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{4}\phi\left[\hat{\sigma}_{12}(I_x)^{12}+\hat{\sigma}_{21}(I_3)^{21}\right]\right\}\psi(x) \\
&= \exp\left\{-\frac{i}{4}\phi\left[\hat{\sigma}_{12}(-1)+\hat{\sigma}_{21}(+1)\right]\right\}\psi(x) \\
&= \exp\left\{\frac{i}{2}\phi\hat{\sigma}_{12}\right\}\psi(x) = \exp\left\{\frac{i}{2}\phi\hat{\sigma}^{12}\right\}\psi(x) \qquad (2.125)
\end{aligned}
$$

□

2.10.3 Spinor for spatial rotations

2.11 The Four-Current Density

$$
j^\mu(x) = \psi^\dagger(x)\gamma^0\gamma^\mu\psi(x). \qquad (2.126)
$$

This current density transforms under the Lorentz transformation as

$$
\begin{aligned}
j'^\mu(x') &= \psi^{\dagger\prime}(x')\gamma^0\gamma^\mu\psi'(x') \\
&= \psi^\dagger(x)\hat{S}^\dagger\gamma^0\gamma^\mu\hat{S}\psi(x) \\
&= \psi^\dagger(x)\gamma^0(\gamma^0\hat{S}^\dagger\gamma^0)\gamma^\mu\hat{S}\psi(x) \\
&= \psi^\dagger(x)\gamma^0\hat{S}^{-1}\gamma^\mu\hat{S}\psi(x) \\
&= \psi^\dagger(x)\gamma^0(a^\mu{}_\nu\gamma^\nu)\psi(x) \\
&= a^\mu{}_\nu j^\nu(x) \qquad (2.127)
\end{aligned}
$$

and as such is identified as a four-vector.

2.12 Plane Waves in Arbitrary Directions

Free solutions have the form

$$
\psi^r = \omega^r(0)e^{-\epsilon_r(m_0/\hbar)t} \qquad (2.128)
$$

We have

$$\omega^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \omega^3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \omega^4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.129}$$

$$\omega^r(p) = \hat{S}(-\mathbf{v})\omega^r(0) = e^{-(\omega/2)\alpha\cdot\mathbf{v}/v}\omega^r(0) \tag{2.130}$$

$$\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu} = 2\big(\hat{\sigma}_{01}(\hat{I}_{\mathbf{n}})^{01} + (\hat{I}_{\mathbf{n}})^{02} + (\hat{I}_{\mathbf{n}})^{03}\big) \tag{2.131}$$

$$\frac{\mathbf{v}}{v} = (\cos\alpha, \cos\beta, \cos\gamma) \tag{2.132}$$

Also

$$\begin{aligned} \hat{\sigma}_{0i} &= \frac{i}{2}(\gamma_0\gamma_i - \gamma_i\gamma_0) \\ &= i\gamma_0\gamma_i \\ &= -i\gamma^0\gamma^i = -i\gamma^0\gamma^0\alpha_i = -i\alpha_i \end{aligned} \tag{2.133}$$

With this the spinor transformation for Lorentz transformations to interial systems with direction of velocity $\mathbf{v}/v$ now becomes

$$\hat{S}(-\mathbf{v}) = \hat{S}\left(-\frac{\mathbf{p}}{E}\right) = e^{-(\omega/2)\hat{\alpha}\cdot\mathbf{v}/v} \tag{2.134}$$

When we expand $\hat{S}$ we will need the following

$$\begin{aligned} (\hat{\alpha}\cdot\mathbf{v})^2 &= \hat{\alpha}^i\hat{\alpha}^j v_i v_j \\ &= \gamma^0\gamma^i\gamma^0\gamma^j v_i v_j \\ &= -\gamma^i\gamma^j v_i v_j \\ &= -\frac{1}{2}(\gamma^i\gamma^j + \gamma^j\gamma^i)v_i v_j \\ &= -\frac{1}{2}2\eta^{ij}\mathbf{1}v_i v_j = +v^2\mathbf{1} \end{aligned} \tag{2.135}$$

We expand $\hat{S}$

$$\begin{aligned}\hat{S}(-\mathbf{v}) &= \mathbf{1} - \frac{\omega}{2}\frac{\hat{\alpha}\cdot\mathbf{v}}{v} + \frac{1}{2!}\frac{\omega^2}{4v^2}(\hat{\alpha}\cdot\mathbf{v})^2 - \frac{1}{3!}\frac{\omega^3}{8v^3}(\hat{\alpha}\cdot\mathbf{v})^3 + \dots \\ &= \mathbf{1}(1+\frac{1}{2}\frac{\omega^2}{4}) - \frac{\hat{\alpha}\cdot\mathbf{v}}{v}(\frac{\omega}{2}+) \\ &= \mathbf{1}\cosh\frac{\omega}{2} - \frac{\hat{\alpha}\cdot\mathbf{v}}{v}\sinh\frac{\omega}{2} \end{aligned} \tag{2.136}$$

The matrix written out $\hat{\alpha}\cdot\mathbf{v}/v$

$$\begin{aligned}\frac{\hat{\alpha}\cdot\mathbf{v}}{v} &= \hat{\alpha}_x\frac{v_x}{v} + \hat{\alpha}_y\frac{v_y}{v} + \hat{\alpha}_z\frac{v_z}{v} \\ &= \frac{p_x}{p}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + \frac{ip_y}{p}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ & \quad +\frac{p_z}{p}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\ &= \frac{1}{p}\begin{pmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{pmatrix} \end{aligned} \tag{2.137}$$

where $p_\pm = p_x \pm ip_y$. We obtain

$$\hat{S}(-\mathbf{v}) = \cosh\frac{\omega}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \cosh\frac{\omega}{2}\frac{\tanh\frac{\omega}{2}}{p}\begin{pmatrix} 0 & 0 & p_z & p_- \\ 0 & 0 & p_+ & -p_z \\ p_z & p_- & 0 & 0 \\ p_+ & -p_z & 0 & 0 \end{pmatrix} \tag{2.138}$$

To find expressions for $\cosh\frac{\omega}{2}$ and $\tanh\frac{\omega}{2}$ we consider only motion in the x direction. We convert the rotation angle ω with the aid of

$$-v_x = \tanh\omega, \tag{2.139}$$

or

$$\omega = \tanh^{-1}(-v_x) = -\tanh^{-1}(v_x) \tag{2.140}$$

We need the equations

$$\begin{aligned}
\cosh\frac{\omega}{2}\sinh\frac{\omega}{2} &= \frac{1}{2}\sinh\omega, \\
\cosh\frac{\omega}{2}\cosh\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega + 1), \\
\sinh\frac{\omega}{2}\sinh\frac{\omega}{2} &= \frac{1}{2}(\cosh\omega - 1).
\end{aligned} \tag{2.141}$$

Therefore

$$\tanh\frac{x}{2} = \frac{\sinh x}{\cosh x + 1} = \frac{\tanh x}{1 + 1/\cosh x} = \frac{\tanh x}{1 + \sqrt{1 - \tanh^2 x}} \tag{2.142}$$

With (2.140)

$$\begin{aligned}
-\tanh\frac{\omega}{2} &= \frac{-\tanh\omega}{1 + \sqrt{1 - \tanh^2\omega}} \\
&= \frac{v_x}{1 + \sqrt{1 - v_x^2}} \\
&= \frac{m_0/\sqrt{1 - v_x^2}}{m_0/\sqrt{1 - v_x^2}}\frac{v_x}{1 + \sqrt{1 - v_x^2}} \\
&= \frac{p_x}{E + m_0}
\end{aligned} \tag{2.143}$$

taking into account that we are considering only motion in the x direction, we may write

$$-\tanh\frac{\omega}{2} = \frac{p}{E + m_0}. \tag{2.144}$$

And

$$
\begin{aligned}
\cosh\frac{\omega}{2} &= \frac{1}{\sqrt{1-\tanh\frac{\omega}{2}}} \\
&= \frac{1}{\sqrt{1-[\tanh\omega/(1+\sqrt{1-\tanh^2\omega})^2]}} \\
&= \frac{1}{\sqrt{1-[v_x/(1+\sqrt{1-v_x^2})^2]}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{(1+\sqrt{1-v_x^2})^2+v_x^2}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{(1+2\sqrt{1-v_x^2}+1-v_x^2)-v_x^2}} \\
&= \frac{1+\sqrt{1-v_x^2}}{\sqrt{2}\sqrt{1-v_x^2+\sqrt{1-v_x^2}}} \\
&= \frac{[1/\sqrt{1-v_x^2}+1]m_0}{\sqrt{1+[1/\sqrt{1-v_x^2}]}\sqrt{2}m_0} \\
&= \frac{E+m_0}{\sqrt{m_0+E}\sqrt{2m_0}} \\
&= \sqrt{\frac{E+m_0}{2m_0}} \qquad (2.145)
\end{aligned}
$$

substituting this result and(2.144) into (2.138) we obtain

$$
\begin{aligned}
\hat{S}(-\mathbf{v}) &= \sqrt{\frac{E+m_0}{2m_0}}\begin{bmatrix} 1 & 0 & \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} \\ 0 & 1 & \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} \\ \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} & 1 & 0 \\ \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} & 0 & 1 \end{bmatrix} \\
&= [\omega^1(\mathbf{p}),\omega^2(\mathbf{p}),\omega^3(\mathbf{p}),\omega^4(\mathbf{p})]. \qquad (2.146)
\end{aligned}
$$

2.13 Bilinear Covariants

2.13.1 Linear independence

$$\begin{aligned} \hat{\Gamma}^S &= \mathbf{I}, \quad \hat{\Gamma}^V_\mu = \gamma_\mu, \quad \hat{\Gamma}^T_{\mu\nu} = \hat{\sigma}_{\mu\nu} = -\hat{\sigma}_{\nu\mu} \\ \hat{\Gamma}^P &= i\gamma^0\gamma^1\gamma^2\gamma^3 \equiv \gamma^5, \quad \hat{\Gamma}^a_\mu = \gamma^5\gamma_\mu \end{aligned} \tag{2.147}$$

i) $(\hat{\Gamma}^n)^2 = \pm\mathbf{I}$.

Proof: Proved explicitly.

ii) To each $\hat{\Gamma}^n$ except $\hat{\Gamma}^S$ there exists at least one $\hat{\Gamma}^m$ such that

$$\hat{\Gamma}^n\hat{\Gamma}^m = -\hat{\Gamma}^m\hat{\Gamma}^n \tag{2.148}$$

Proof: Proved explicitly.

iii) $Tr(\hat{\Gamma}^n) = 0$

By (2.148)

$$-\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m = +\hat{\Gamma}^n(\hat{\Gamma}^m)^2$$

As $(\hat{\Gamma}^m)^2 = \pm\mathbf{I}$

$$\pm Tr(\hat{\Gamma}^n) = Tr(\hat{\Gamma}^n(\hat{\Gamma}^m)^2) = -Tr(\hat{\Gamma}^m\hat{\Gamma}^n\hat{\Gamma}^m) = -Tr(\hat{\Gamma}^n\hat{\Gamma}^m\hat{\Gamma}^m) = 0.$$

iv) For given $\hat{\Gamma}^a$ and $\hat{\Gamma}^b$ $(a \neq b)$ there exists a $\hat{\Gamma}^n \neq \hat{\Gamma}^S$ such that

$$\hat{\Gamma}^a\hat{\Gamma}^b = f^n_{ab}\hat{\Gamma}^n. \tag{2.149}$$

Proof: Proved explicitly.

v) The $\hat{\Gamma}^n$ are linearly independent. Suppose

$$\sum_n a_n\hat{\Gamma}^n = 0. \tag{2.150}$$

Multiply from the right by $\hat{\Gamma}^m \neq \hat{\Gamma}^S$ we get

$$
\begin{aligned}
0 &= \sum_n a_n Tr(\hat{\Gamma}^n \hat{\Gamma}^m) \\
&= a_m (\hat{\Gamma}^m)^2 + \sum_{n \neq m} a_n Tr(\hat{\Gamma}^n \hat{\Gamma}^m) \\
&= a_m (\hat{\Gamma}^m)^2 + \sum_{n \neq m} a_n Tr(f_{nm}^{\nu} \hat{\Gamma}^{\nu}) \\
&= \pm 4 a_m. \qquad (2.151)
\end{aligned}
$$

Thus $a_m = 0$ for all $m \neq S$. Now in the case of $\hat{\Gamma}^m = \hat{\Gamma}^S$

$$
0 = Tr\left(\sum_n a_n \hat{\Gamma}^S \hat{\Gamma}^n\right) = a_S Tr(\mathbf{I}) + \sum_{n \neq S} a_n Tr(\hat{\Gamma}^n) = 0, \qquad (2.152)
$$

i.e. $a_S = 0$.

$\square$

2.13.2 Lorentz transformations

Under Lorentz transformations

$$
\psi \to \hat{S}\psi, \qquad \overline{\psi} \to \overline{\psi}\hat{S}^{-1} \qquad (2.153)
$$

This is proved by

$$
\begin{aligned}
\overline{\psi}'(x') &= \psi'^{\dagger}(x')\gamma^0 \\
&= \psi^{\dagger}(x)\hat{S}^{\dagger}\gamma^0 \\
&= \psi^{\dagger}(x)\gamma^0(\gamma^0\hat{S}^{\dagger}\gamma^0) \\
&= \overline{\psi}(x)\hat{S}^{-1}. \qquad (2.154)
\end{aligned}
$$

$\square$

We now prove

$$
\gamma_5 \hat{S} = det|a|\ \hat{S}\gamma_5. \qquad (2.155)
$$

This is easily proven that for proper Lorentz transformations (here $det|a| = 1$), first

$$[\gamma_5, \hat{\sigma}_{\mu\nu}] = \frac{1}{2}(\gamma_5(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) - (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\gamma_5) = 0$$

where we have used $\gamma^\mu\gamma_5 + \gamma_5\gamma^\mu = 0$. So from the formula for proper Lorentz transformations

$$\hat{S}(\hat{a}) = \exp\left(-\frac{i}{4}\hat{\sigma}_{\mu\nu}(I_{\mathbf{n}})^{\mu\nu}\right),$$

we have

$$[\hat{S}(\hat{a}), \gamma_5] = 0. \tag{2.156}$$

We know prove (2.155) for spacial reflections, which are given by

$$\mathbf{x}' = -\mathbf{x}, \qquad t' = t, \tag{2.157}$$

with corresponding transformation matrix

$$a^\nu{}_\mu = \eta^{\mu\nu}. \tag{2.158}$$

The relation

$$\hat{S}^{-1}\gamma^\mu\hat{S} = a^\mu{}_\nu\gamma^\nu$$

holds for improper Lorentz transformations as well. Let us dente the parity operator by $\hat{P}$. We can then write

$$a^\nu{}_\mu\gamma^\mu = \hat{P}\gamma^\nu\hat{P}^{-1}$$

or

$$a^\sigma{}_\nu a^\nu{}_\mu\gamma^\mu = \hat{P}a^\sigma{}_\nu\gamma^\nu\hat{P}^{-1}$$

This is equivalent to

$$\delta^{\sigma}_{\mu}\gamma^{\mu} = \hat{P}(\sum_{\nu=0}^{3}\eta^{\sigma\nu}\gamma^{\nu})\hat{P}^{-1} \tag{2.159}$$

which in turn is equivalent to

$$\hat{P}^{-1}\gamma^{\sigma}\hat{P} = \eta^{\sigma\sigma}\gamma^{\sigma}. \tag{2.160}$$

This has the simple solution

$$\hat{P} = e^{i\varphi}\gamma^{0}, \qquad \hat{P}^{-1} = e^{-i\varphi}\gamma^{0} \tag{2.161}$$

For this operator we easily have

$$\hat{P}\gamma_5 = -\gamma_5\hat{P}. \tag{2.162}$$

$\square$

i) $\overline{\psi}\psi$ is a scalar:

$$\begin{aligned} \overline{\psi}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\hat{S}\psi \\ &= \overline{\psi}\psi \end{aligned}$$

ii) $\overline{\psi}\gamma_5\psi$ is a pseudoscalar:

$$\begin{aligned} \overline{\psi}\gamma_5\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}\psi \\ &= det|a|\,\overline{\psi}\hat{S}^{-1}\hat{S}\gamma_5\psi \\ &= det|a|\,\overline{\psi}\gamma_5\psi \end{aligned}$$

iii) $\overline{\psi}\gamma^{\mu}\psi$ is a vector:

$$\begin{aligned} \overline{\psi}\gamma^{\mu}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma^{\mu}\hat{S}\psi \\ &= a^{\mu}{}_{\nu}\overline{\psi}\gamma^{\nu}\psi \end{aligned}$$

iv) $\overline{\psi}\gamma_5\gamma^{\mu}\psi$ is a pseudovector:

$$
\begin{aligned}
\overline{\psi}\gamma_5\gamma^\mu\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\gamma_5\gamma^\mu\hat{S}\psi \\
&= \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}(\hat{S}^{-1}\gamma^\mu\hat{S})\psi \\
&= \overline{\psi}\hat{S}^{-1}\gamma_5\hat{S}(a^\mu{}_\nu\gamma^\nu)\psi \\
&= det|a|a^\mu{}_\nu\,\overline{\psi}\gamma^\nu\psi \\
&= det|a|a^\mu{}_\nu\,\overline{\psi}\gamma^\nu\psi
\end{aligned}
$$

v) $\overline{\psi}\hat{\sigma}^{\mu\nu}\psi$ is a pseudovector:

$$
\begin{aligned}
\overline{\psi}\hat{\sigma}^{\mu\nu}\psi &\rightarrow \overline{\psi}\hat{S}^{-1}\hat{\sigma}^{\mu\nu}\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\hat{S}^{-1}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\hat{S}^{-1}(\gamma^\mu\hat{S}\hat{S}^{-1}\gamma^\nu - \gamma^\nu\hat{S}\hat{S}^{-1}\gamma^\mu)\hat{S}\psi \\
&= \frac{i}{2}\overline{\psi}\{(a^\mu{}_\rho\gamma^\rho)(a^\nu{}_\tau\gamma^\tau) - (a^\mu{}_\rho\gamma^\rho)(a^\nu{}_\tau\gamma^\tau)\}\psi \\
&= a^\mu{}_\rho a^\nu{}_\tau\,\overline{\psi}\frac{i}{2}(\gamma^\rho\gamma^\tau - \gamma^\tau\gamma^\rho)\psi \\
&= a^\mu{}_\rho a^\nu{}_\tau\,\overline{\psi}\hat{\sigma}^{\rho\tau}\psi.
\end{aligned}
$$

□

2.14 Properties of Free Solutions

$$
(p_\mu\gamma^\mu - \epsilon_r m_0)\omega^r(\mathbf{p}) = 0 \tag{2.163}
$$

$$
\begin{aligned}
[(p_\mu\gamma^\mu - \epsilon_r m_0)\omega^r(\mathbf{p})]^\dagger &= 0 = \omega^{r\dagger}(\mathbf{p})(p_\mu\gamma^\mu - \epsilon_r m_0)^\dagger \\
&= \omega^{r\dagger}(\mathbf{p})(p_\mu\gamma^{\mu\dagger} - \epsilon_r m_0) \\
&= \omega^{r\dagger}(\mathbf{p})(p_0\gamma^0 - p_k\gamma^k - \epsilon_r m_0)
\end{aligned}
$$

Multiplication from the right by γ^0 yields

$$
\begin{aligned}
\omega^{r\dagger}(\mathbf{p})(p_\mu\gamma^{\mu\dagger} - \epsilon_r m_0)\gamma^0 &= 0 = \omega^{r\dagger}(\mathbf{p})\gamma^0(p_0\gamma^0 + p_k\gamma^k - \epsilon_r m_0) \\
&= \overline{\omega}^r(\mathbf{p})(p_\mu\gamma^\mu - \epsilon_r m_0)
\end{aligned}
$$

2.14.1 The normalisation condition

The quantity $\overline{\omega}^r(\mathbf{p})\omega^{r'}(\mathbf{p})$ is a Lorentz scalar and hence

$$\overline{\omega}^r(\mathbf{p})\omega^{r'}(\mathbf{p}) = \overline{\omega}^r(0)\omega^{r'}(0) = \omega^{r\dagger}(0)\gamma^0\omega^{r'}(0) = \delta_{rr'}\epsilon_r. \tag{2.164}$$

2.14.2 The completenes relation

We have

$$\omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \delta_{rr'}(E/m_0)$$

This is proved in section A.8.

2.14.3 The closure relation

In the rest frame of the electron we have

$$\sum_{r=1}^{4}\epsilon_r\omega^r(0)\overline{w}^r_\beta(0) = \delta_{\alpha\beta} \tag{2.165}$$

We know that

$$\omega^r(p) = \hat{S}\left(\frac{-\mathbf{p}}{E}\right)w^r(0)$$

and so

$$\begin{aligned}
\overline{\omega}^r(p) &= \omega^{r\dagger}(p)\gamma^0 \\
&= \left(\hat{S}\left(\frac{-\mathbf{p}}{E}\right)\omega^r(0)\right)^\dagger\gamma^0 \\
&= w^{r\dagger}(0)\gamma^0\gamma^0\hat{S}^\dagger\left(\frac{-\mathbf{p}}{E}\right)\gamma^0 \\
&= \overline{\omega}^r(0)\hat{S}^{-1}\left(\frac{-\mathbf{p}}{E}\right).
\end{aligned} \tag{2.166}$$

where we have used

$$\hat{S}^\dagger = \gamma^0 \hat{S}^{-1} \gamma^0.$$

Using these we find

$$\begin{aligned}
\sum_{r=1}^{4} \epsilon_r \omega_\alpha^r(p) \overline{\omega}_\beta^r(p) &= \sum_{r=1}^{4} \sum_{\gamma,\lambda=1}^{4} \epsilon_r \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E} \right) \omega_\gamma^r(0) \, \overline{\omega}_\lambda^r(0) \hat{S}_{\lambda\beta}^{-1} \left(\frac{-\mathbf{p}}{E} \right) \\
&= \sum_{\gamma,\lambda=1}^{4} \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E} \right) \hat{S}_{\lambda\beta} \left(\frac{-\mathbf{p}}{E} \right) \sum_{r=1}^{4} \epsilon_r \omega_\gamma^r(0) \overline{\omega}_\lambda^r(0) \\
&= \sum_{\gamma,\lambda=1}^{4} \hat{S}_{\alpha\gamma} \left(\frac{-\mathbf{p}}{E} \right) \hat{S}_{\lambda\beta} \left(\frac{-\mathbf{p}}{E} \right) \delta_{\gamma\lambda} \\
&= \delta_{\alpha\beta} \qquad (2.167)
\end{aligned}$$

Therefore we have the closure relation

$$\sum_{r=1}^{4} \epsilon_r \omega_\alpha^r(p) \overline{\omega}_\beta^r(p) = \delta_{\alpha\beta}. \qquad (2.168)$$

2.15 Projection Operators for Energy and Spin

2.15.1 Projection Operators for Energy

Recall

$$(p_\mu \gamma^\mu - \epsilon_r m_0 c) w^r(\mathbf{p}) \quad \text{which implies} \quad \epsilon_r p_\mu \gamma^\mu w^r(\mathbf{p}) = m_0 c w^r(\mathbf{p})$$

We immediately see that the projection operator for eigenstates with positive or negative energy is given by

$$\hat{\Lambda}_r(p) = \frac{\epsilon_r p_\mu \gamma^\mu + m_0}{2m_0} \qquad (2.169)$$

and that it is Lorentz covariant. We check that it has all the properties of a projection operator. Obviously

$$\hat{\Lambda}_+(p) + \hat{\Lambda}_-(p) = \frac{+p_\mu\gamma^\mu + m_0}{2m_0} + \frac{-p_\mu\gamma^\mu + m_0}{2m_0 c} = 1$$

We now establish $(\hat{\Lambda}_+)^2 = \hat{\Lambda}_+$, $(\hat{\Lambda}_-)^2 = \hat{\Lambda}_-$, and $\hat{\Lambda}_+\hat{\Lambda}_- = 0$. This is done with the help of

$$\begin{aligned} p^\mu\gamma_\mu p^\nu\gamma_\nu &= \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)p^\mu p^\nu \\ &= \eta_{\mu\nu}p^\mu p^\nu \\ &= E^2 - \mathbf{p}^2 \\ &= (m_0^2 + \mathbf{p}^2) - \mathbf{p}^2 = m_0^2. \end{aligned} \tag{2.170}$$

Now

$$\begin{aligned} \hat{\Lambda}_r(p)\hat{\Lambda}_{r'}(p) &= \frac{(\epsilon_r p^\mu\gamma_\mu + m_0)(\epsilon_{r'}p^\mu\gamma_\mu + m_0)}{4m_0^2} \\ &= \frac{(\epsilon_r\epsilon_{r'}p^\mu p^\nu\gamma_\mu\gamma_\nu + m_0^2 + (\epsilon_r + \epsilon_{r'})m_0 p^\mu\gamma_\mu}{4m_0^2} \\ &= \frac{m_0^2(1 + \epsilon_r\epsilon_{r'}) + m_0 p^\mu\gamma_\mu\epsilon_r(1 + \epsilon_r\epsilon_{r'})}{4m_0^2} \\ &= \frac{1 + \epsilon_r\epsilon_{r'}}{2}\frac{\epsilon_r p_\mu\gamma^\mu + m_0}{2m_0} = \frac{1 + \epsilon_r\epsilon_{r'}}{2}\hat{\Lambda}_r(p). \end{aligned} \tag{2.171}$$

2.15.2 Projection Operators for Spin

In the non-relativistic limit the operator for "spin up" or "spin down"

$$\hat{P}_\pm = \frac{1 \pm \hat{\sigma}_3}{2}$$

We can generalise this to a spin-projection operator in an arbitrary direction

$$\hat{P}(\mathbf{u}) = \frac{1 + \hat{\sigma}\cdot\mathbf{u}}{2} \tag{2.172}$$

where $\mathbf{u}$ is a unit vector. We need the relativistic generalisation of this. To that end introduce the four-vector

$$u_z^\nu \tag{2.173}$$

which in the rest system of the electron is

$$(u_z^\nu)_{R.S.} = (0,0,0,1) = (0,0,0,\mathbf{u}_z) \tag{2.174}$$

$$\begin{aligned}
\gamma_5\gamma_3(u_z^3)_{R.S.} &= \gamma_5\gamma_3 \\
&= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma_3 \\
&= i\gamma^0\gamma^1\gamma^2 \\
&= i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}
\end{aligned} \tag{2.175}$$

In the rest frame we have for positive-energy $\omega^{1,2}(0)$

$$\begin{aligned}
\hat{\Sigma}(u_z^3)\omega^{1,2}(0) &= \frac{1+\gamma_5\gamma^3(u_z^3)_{R.S.}}{2}\omega^{1,2}(0) \\
&= \frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & I-\sigma_3 \end{pmatrix}\omega^{1,2}(0) \\
&= \frac{1}{2}\begin{cases} 1\cdot\omega^1(0) \\ 0\cdot\omega^2(0) \end{cases} .
\end{aligned} \tag{2.176}$$

In the rest frame we have for negitive-energy

$$\begin{aligned}
\hat{\Sigma}(u_z^3)\omega^{3,4}(0) &= \frac{1+\gamma_5\gamma^3(u_z^3)_{R.S.}}{2}\omega^{3,4}(0) \\
&= \frac{1}{2}\begin{pmatrix} I+\sigma_3 & 0 \\ 0 & I-\sigma_3 \end{pmatrix}\omega^{3,4}(0) \\
&= \frac{1}{2}\begin{cases} 0\cdot\omega^3(0) \\ 1\cdot\omega^4(0) \end{cases}
\end{aligned} \tag{2.177}$$

The projection of negative-energy states arre opposite to those of positive-energy states. The opposite occurs because the spin of the missing particle of spin $\uparrow$ corresponds to a particle of spin $\downarrow$.

We generalise the spin projection operator for an arbitrary spin vector s^μ with $s^\mu p_\mu = 0$:

$$\hat{\Sigma}(s) = \frac{1}{2}(1+\gamma_5 s^\mu\gamma_\mu) \tag{2.178}$$

We show that it is a true projection operator. We have

$$\hat{\Sigma}(s) + \hat{\Sigma}(-s) = 1 \tag{2.179}$$

$$\begin{aligned}
\hat{\Sigma}^2(s) &= \frac{1}{4}(1 + \gamma_5 s^\mu \gamma_\mu)(1 + \gamma_5 s^\nu \gamma_\nu) \\
&= \frac{1}{4}(1 + 2\gamma_5 s^\mu \gamma_\mu + s^\mu s^\nu \gamma_5 \gamma_\mu \gamma_5 \gamma_\nu) \\
&= \frac{1}{4}(1 + 2\gamma_5 s^\mu \gamma_\mu - s^\mu s^\nu \gamma_5^2 \gamma_\mu \gamma_\nu) \\
&= \frac{1}{4}(1 + 2\gamma_5 s^\mu \gamma_\mu - s^\mu s^\nu \gamma_5^2 \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu)) \\
&= \frac{1}{4}(1 + 2\gamma_5 s^\mu \gamma_\mu - s \cdot s) \\
&= \frac{1}{2}(1 + \gamma_5 s^\mu \gamma_\mu) = \hat{\Sigma}(s)
\end{aligned} \tag{2.180}$$

Similarly $\hat{\Sigma}^2(-s) = \hat{\Sigma}(-s)$.

$$\begin{aligned}
\hat{\Sigma}(s)\hat{\Sigma}(-s) &= \frac{1}{4}(1 + \gamma_5 s^\mu \gamma_\mu)(1 - \gamma_5 s^\nu \gamma_\nu) \\
&= \frac{1}{4}(1 + s \cdot s) = 0.
\end{aligned} \tag{2.181}$$

2.15.3 Simjultaneous Projection Operators for Energy and Spin

$$\begin{aligned}
p_\mu \gamma^\mu \gamma_5 \gamma^\nu s_\nu &= p_\mu \gamma^\mu i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\nu s_\nu \\
&= -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 \gamma^\mu \gamma^\nu s_\nu p_\mu \\
&= -\gamma_5 (2\eta^{\mu\nu} - \gamma^\nu \gamma^\mu) s_\nu p_\mu \\
&= s^\mu p_\mu \gamma_5 + s_\nu p_\mu \gamma_5 \gamma^\nu \gamma^\mu \\
&= \gamma_5 s_\nu \gamma^\nu p_\mu \gamma^\mu
\end{aligned} \tag{2.182}$$

implies

$$\left[\hat{\Sigma}(s), \hat{\Lambda}_\pm(p)\right] = 0 \tag{2.183}$$

$$\begin{aligned}\hat{P}_1 &= \hat{\Lambda}_+(p)\hat{\Sigma}(u_z)\\ \hat{P}_2 &= \hat{\Lambda}_+(p)\hat{\Sigma}(-u_z)\\ \hat{P}_3 &= \hat{\Lambda}_-(p)\hat{\Sigma}(u_z)\\ \hat{P}_4 &= \hat{\Lambda}_-(p)\hat{\Sigma}(-u_z)\end{aligned} \tag{2.184}$$

2.16 Summary

Maxwell's equations with source are

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = j^\mu \tag{2.185}$$

where we are free to perform gauge transformations

$$A^\mu \to A^{\mu'} = A^\mu + \partial^\mu \Lambda. \tag{2.186}$$

The free Dirac equation can be written

$$i\hbar\frac{\partial\psi(x)}{\partial t} = [\alpha\cdot(-i\hbar\nabla) + \beta m_0]\psi(x) \tag{2.187}$$

where $= (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3)$ and $\hat{\beta}$ are 4×4 Hermitian matrices satisfying

$$\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}, \quad \hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0, \quad \beta^2 = 1, \quad i = 1,2,3. \tag{2.188}$$

With

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha_i \tag{2.189}$$

Dirac's equation becomes

$$i\hbar\gamma^\mu\frac{\partial\psi(x)}{\partial x^\mu} - m_0\psi(x) = 0 \tag{2.190}$$

with the 4×4 matrices γ^μ, $\mu = 0,\ldots,3$, satifying the anticommutation relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu} \tag{2.191}$$

and Hermiticity conditions

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{2.192}$$

Coupling of a Spinor to the electromagnetic field is given by

$$(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu(x) - m_0)\psi(x) = 0. \tag{2.193}$$

The four current density of the dirac field is given by

$$j^\mu = \overline{\psi}\gamma^\mu\psi \tag{2.194}$$

The plane wave for a photon is

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{2.195}$$

Plane waves of electrons: incoming

$$\psi(x) = \sqrt{\frac{m_0}{EV}}\, u(p,s)e^{-ip\cdot x} \tag{2.196}$$

and outgoing

$$\overline{\psi}(x) = \sqrt{\frac{m_0}{EV}}\, \overline{u}(p,s)e^{ip\cdot x}. \tag{2.197}$$

The general free solution has the form

$$\psi^r(x) = \omega^r(p)e^{-i\epsilon_r p_\mu x^\mu} \tag{2.198}$$

where

$$[\omega^1(p), \omega^2(p), \omega^3(p), \omega^4(p)] = \sqrt{\frac{E+m_0}{2m_0}}\begin{bmatrix} 1 & 0 & \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} \\ 0 & 1 & \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} \\ \frac{p_z}{E+m_0} & \frac{p_-}{E+m_0} & 1 & 0 \\ \frac{p_+}{E+m_0} & \frac{-p_z}{E+m_0} & 0 & 1 \end{bmatrix} \tag{2.199}$$

The orthogonality condition for spinors

$$\omega^{r\dagger}(\epsilon_r p)\omega^{r'}(\epsilon_r{}' p) = \frac{E_p}{m_0}\delta_{rr'} \tag{2.200}$$

The completeness relation for spinors

$$\sum_{r=1}^{4} \epsilon_r \omega^r(p)\overline{\omega}^r_\alpha(p) = \delta_{\alpha\beta} \tag{2.201}$$

There are projection operators for Energy

$$\hat{\Lambda}_\pm(p) = \frac{\pm p_\mu \gamma^\mu + m_0}{2m_0} \tag{2.202}$$

and spin

$$\hat{\Sigma} = \frac{1}{2}(1 + \gamma_5 s_\mu \gamma^\mu) \tag{2.203}$$

such that

$$\hat{\Sigma}u(p,+s) = u(p,+s), \quad \hat{\Sigma}u(p,-s) = 0. \tag{2.204}$$

The basic bilinear covariants of Dirac theory are

$$\begin{array}{rl}
\overline{\psi}\psi & \text{scalar} \\
\overline{\psi}\gamma^\mu\psi & \text{vector} \\
\overline{\psi}\sigma^{\mu\nu}\psi & \text{antisymmetric second-rank tensor} \\
\overline{\psi}\gamma^5\gamma^\mu\psi & \text{pseudo-vector} \\
\overline{\psi}\gamma^5\psi & \text{pseudo-scalar}
\end{array} \tag{2.205}$$

Part III

Perturbation Theory and Propagator Methods

Chapter 3

Perturbation Theory

3.1 Non-Relativistic Green's Function

Given Schrodinger's equation

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}_0(x) - V(x)\right)\psi(x) = 0 \tag{3.1}$$

The retarded Green's function is defined by the differential equation

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0(x') - V(x')\right)G^+(x';x) = \delta^4(x'-x) \tag{3.2}$$

and the boundary condition

$$G^+(x';x) = 0 \quad \text{for } t' < t. \tag{3.3}$$

3.1.1 Free Green's function in momentum space

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}_0(x')\right)G^+(x';x) = \delta^4(x'-x) \tag{3.4}$$

where

$$\hat{H}_0 = -\frac{\hbar^2}{2m}\nabla'^2 \tag{3.5}$$

As the above differential equation can be turned into an algebraic equation in energy-momentum space, we write

$$G_0^+(x'-x) = \int \frac{d^3pdE}{(2\pi\hbar)^4} \exp\left[-\frac{i}{\hbar}E(t'-t)\right] \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] G_0^+(p;E) \tag{3.6}$$

and apply

$$\begin{aligned}
&\left(i\hbar\frac{\partial}{\partial t'} + \frac{\hbar^2}{2m}\nabla'^2\right) G_0^+(x'-x) \\
= \; & \int \frac{d^3pdE}{(2\pi\hbar)^4} \left\{\left(E - \frac{\mathbf{p}^2}{2m}\right) G_0^+(p;E)\right\} \exp\left[-\frac{i}{\hbar}E(t'-t)\right] \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] \\
= \; & \hbar\delta^4(x'-x)
\end{aligned} \tag{3.7}$$

We will recall the δ-function integral representation

$$\int \frac{d^3pdE}{(2\pi\hbar)^4} \exp\left[-\frac{i}{\hbar}E(t'-t)\right] \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] = \delta^4(x'-x)$$

Therefore for $E \neq \mathbf{p}^2/2m$ we obtain

$$G_0^+(\mathbf{p},E) = \frac{\hbar}{E - \frac{\mathbf{p}^2}{2m}} \tag{3.8}$$

How do we deal with the singularity when we do the inverse Fourier transformation? The clue how to proceed comes from the integral representation of the step function:

$$\Theta(\tau) = -\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon}. \tag{3.9}$$

By adding a small imaginary part $i\epsilon$ to the energy one will obtain the retardation condition (3.3), while the resulting Green's function still satisfies the Green's function differential equation (3.4) in the limit $\epsilon \to 0$. Write

$$G_0^+(x'-x) = \hbar \int \frac{d^3p}{(2\pi\hbar)^3} \exp\left[\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})\right] \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \frac{\exp\left[-iE(t'-t)/\hbar\right]}{E - \frac{\mathbf{p}^2}{2m} + i\epsilon} \tag{3.10}$$

With the substitution $E' = E - \mathbf{p}^2/2m$ the last integral becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dE'}{2\pi\hbar} \frac{\exp[-i(E' + \mathbf{p}^2/2m)(t'-t)/\hbar]}{E' + i\epsilon} \\
&= \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t'-t)\right] \int_{-\infty}^{\infty} \frac{dE'}{2\pi\hbar} \frac{\exp[-iE'(t'-t)/\hbar]}{E' + i\epsilon} \\
&= \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t'-t)\right] \frac{-i}{\hbar} \cdot \frac{-1}{2\pi i} \int_{-\infty}^{\infty} dE' \frac{\exp[-iE'(t'-t)/\hbar]}{E' + i\epsilon} \\
&= \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t'-t)\right] \left[-\frac{i}{\hbar}\Theta\left(\frac{t'-t}{\hbar}\right)\right] \\
&= -\frac{i}{\hbar} \exp\left[-\frac{i}{\hbar}\frac{\mathbf{p}^2}{2m}(t'-t)\right] \Theta(t'-t) \qquad (3.11)
\end{aligned}
$$

We do indeed recover the retardation condition. Now (3.10) becomes

$$
G_0^+(x'-x) = -i\Theta(t'-t) \int \frac{d^3p}{(2\pi\hbar)^3} \exp\left\{\frac{i}{\hbar}\left[\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x}) - \frac{\mathbf{p}^2}{2m}(t'-t)\right]\right\} \qquad (3.12)
$$

This can be expressed in terms of plane waves of the free Schrodinger's equation. The δ-function normalised plane waves are

$$
\begin{aligned}
\phi_p(\mathbf{x}, t) &= \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{i}{\hbar}\left((\mathbf{p}\cdot\mathbf{x} - \frac{\mathbf{p}^2}{2m}t\right)\right] \\
&= \frac{1}{\sqrt{2\pi\hbar}} \exp[i(\mathbf{k}\cdot\mathbf{x} - \omega t)] \qquad (3.13)
\end{aligned}
$$

where

$$
\hbar\omega = \frac{\mathbf{p}^2}{2m}, \quad \hbar\mathbf{k} = \mathbf{p}.
$$

$$
G_0^+(x'-x) = -i\Theta(t'-t) \int d^3p \; \phi_p(\mathbf{x}', t')\phi_p^*(\mathbf{x}, t) \qquad (3.14)
$$

3.1.2 Full Green's function in terms of plane waves

We give a proof that the Green's function can be written as

$$
G^+(x'; x) = -i\Theta(t'-t) \sum_n \psi_n^*(x)\psi_n(x'). \qquad (3.15)
$$

where $\psi_n(\mathbf{x}, t)$ are a complete set of eigenfunctions of Schrodinger's equation. We do this using the closure relation

$$\sum_n \psi_n^*(\mathbf{x}, t)\psi_n(\mathbf{x}', t) = \delta^3(\mathbf{x}' - \mathbf{x}). \tag{3.16}$$

for the eigenfunctions of Schrodinger's equation

$$\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)\psi_n(x') = 0. \tag{3.17}$$

Using this we have

$$\begin{aligned}
\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right) G^+(x'; x) &= \hbar\delta(t' - t)\sum_n \psi_n^*(\mathbf{x}, t)\psi_n(\mathbf{x}', t) \\
&\quad -i\Theta(t' - t)\sum_n \left[\left(i\hbar\frac{\partial}{\partial t'} - \hat{H}(x')\right)\psi_n(x')\right]\psi_n^*(x) \\
&= \hbar\delta(t' - t)\delta^3(\mathbf{x}' - \mathbf{x}) \\
&= \hbar\delta^4(x' - x).
\end{aligned} \tag{3.18}$$

We have the relations describing the evolution of solutions of Schrodinger's equation:

$$\begin{aligned}
i\int d^3x G^+(x'; x)\psi_n(x) &= \Theta(t' - t)\sum_m \psi_m(x')\int d^3x \psi_m^*(x)\psi_n(x) \\
&= \Theta(t' - t)\psi_n(x')
\end{aligned} \tag{3.19}$$

and on the other hand

$$\begin{aligned}
i\int d^3x \psi_n^*(x')G^+(x'; x) &= \Theta(t' - t)\sum_m \int d^3x' \psi_m(x')\psi_n^*(x')\ \psi_m^*(x) \\
&= \Theta(t' - t)\psi_n^*(x)
\end{aligned} \tag{3.20}$$

The first of these relations expresses the propagation of $\psi_n(x)$ forward in time and the second corresponding backward propagation of $\psi_n^*(x')$.

3.1.3 Perturbation theory

$$\left(i\hbar\frac{\partial}{\partial t'}-\hat{H}_0\right)G^+(x';x)=\delta^4(x'-x)+V(x')G^+(x';x) \tag{3.21}$$

The RHS can be interpreted as the source term in an inhomogeneous Schrodinger equation:

$$\left(i\hbar\frac{\partial}{\partial t'}-\hat{H}_0(x')\right)G^+(x';x)=\rho(x';x) \tag{3.22}$$

Using the free Green's function G_0 the solution is

$$G^+(x';x)=\int d^4x_1 G_0^+(x';x_1)\rho(x_1;x) \tag{3.23}$$

This leads to the following integral equation for the integrating Green's function

$$\begin{aligned}
G^+(x';x) &= \int d^4x_1 G_0^+(x';x_1)\big(\delta^4(x_1-x)+V(x_1)G^+(x_1;x)\big) \\
&= G_0^+(x';x)+\int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x)
\end{aligned} \tag{3.24}$$

Repeatedly substituting this equation into itself we obtain perturbative series

$$\begin{aligned}
G^+(x';x) &= G_0^+(x';x)+\int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x) \\
&= G_0^+(x';x)+\int d^4x_1 G_0^+(x';x_1)V(x_1)G_0^+(x_1;x) \\
&\quad +\int d^4x_1 d^4x_2 G_0^+(x';x_1)V(x_1)G_0^+(x_1;x_2)V(x_2)G_0^+(x_2;x) \\
&\quad +\ldots \quad .
\end{aligned} \tag{3.25}$$

3.1.4 Boundary condition

$$\begin{aligned}
\psi(x') &= \lim_{t\to-\infty} i\int d^3x G^+(x';x)\phi(x) \\
&= \lim_{t\to-\infty} i\int d^3x \left(G_0^+(x';x) + \int d^4x_1 G_0^+(x';x_1)V(x_1)G^+(x_1;x)\right)\phi(x) \\
&= \phi(x') + \lim_{t\to-\infty}\int d^4x_1 G_0^+(x';x_1)V(x_1) i\int d^3x G^+(x_1;x)\phi(x) \\
&= \phi(x') + \lim_{t\to-\infty}\int d^4x_1 G_0^+(x';x_1)V(x_1)\psi(x_1) \qquad (3.26)
\end{aligned}$$

The second the on the RHS is the scattered wave.

We consider a scattering problem where no interaction occurs in the distant past and future:

$$V(\mathbf{x},t) \to 0 \quad \text{for} \quad t \to \mp\infty \qquad (3.27)$$

The initial wave ϕ is therefore a solution of the Schrodinger equation for free particles, which fulfills the initial conditions of the experiment. The exact wavefunction $\psi(\mathbf{x},t)$ then approaches the incoming wave $\phi(\mathbf{x},t)$ in the limit $t \to -\infty$:

$$\psi(\mathbf{x},t) \to \phi(\mathbf{x},t). \qquad (3.28)$$

3.1.5 The scattering matrix

Let $\phi_i(x)$ and $\phi_f(x)$ denote the intial and final free wave with quantum numbers i and f that are emitted, observed at the begining, end of the scattering process respectively. The full wavefunction $\psi_i(x)$ is given in terms the integral equation,

$$\psi_i(x) = \phi_i(x) + \int d^4x_1 G_0^+(x,x_1)V(x_1)\psi_i(x_1) \qquad (3.29)$$

The wavefunction $\psi_i(x)$ satifies the boundary condition $\psi_i(\mathbf{x},t) \to \phi_i(\mathbf{x},t)$ for $t \to -\infty$. The scattering matrix results from the projection of $\psi_i(x)$ on the final state $\phi_f(\mathbf{x},t)$

$$S_{fi} = \lim_{t\to+\infty} \left\langle \phi_f(x) \middle| \psi_i(x) \right\rangle \qquad (3.30)$$

$$
\begin{aligned}
S_{fi} &= \lim_{t\to+\infty} \left\langle \phi_f(x) \middle| \phi_i(x) + \int d^4x_1 G_0^+(x,x_1)V(x_1)\psi(x_1) \right\rangle \\
&= \delta_{fi} + \lim_{t\to+\infty} \int d^3x \phi_f^*(x) \int d^4x_1 G_0^+(x,x_1)V(x_1)\psi(x_1) \\
&= \delta_{fi} + \lim_{t\to+\infty} \int d^4x_1 \left(\int d^3x \phi_f^*(x) G_0^+(x,x_1) \right) V(x_1)\psi(x_1) \qquad (3.31)
\end{aligned}
$$

Using the equations (3.20) for free particles we obtain

$$
\int d^3x \phi_f^*(x) G_0^+(x';x_1) = -i\phi_f(x_1) \quad \text{for } t' > t_1, \qquad (3.32)
$$

so the x integral can be carried out resulting in

$$
S_{fi} = \delta_{fi} - i \lim_{t\to+\infty} \int d^4x_1 \phi_f^*(x_1)V(x_1)\psi(x_1) \qquad (3.33)
$$

Now repeatedly substituting (3.29) into this we obtain

$$
\begin{aligned}
S_{fi} &= \delta_{fi} - i\int d^4x_1 \phi_f^*(x_1)V(x_1)\phi_i(x_1) \\
&\quad -i\int d^4x_1 d^4x_2 \phi_f^*(x_1)V(x_1)G_0^+(x_1;x_2)V(x_2)\phi_i(x_2) \\
&\quad -i\int d^4x_1 d^4x_2 \phi_f^*(x_1)V(x_1)G_0^+(x_1;x_2)V(x_2)G_0^+(x_2;x_3)V(x_3)\phi_i(x_3) \\
&\quad + \ldots \quad . \qquad (3.34)
\end{aligned}
$$

Each line represents a free Green's function $G_0^+(x_i;x_{i-1})$, i.e. the amplitude that a particle wave originating at the spacetime point x_{i-1} and propagates freely to the spacetime point x_i. At the point x_i the particle wave is scattered with probability amplitude $V(x_i)$ per unit spacetime volume. Such points are called interaction vertices and are denoted by filled-in circles. The resulting scattered wave then again propagates freely foward in time (recall $G_0^+(x_{i+1};x_i) = 0$ for $t_{i+1} < t_i$) from the spacetime point x_i towards the point x_{i+1} with the amplitude $G_0^+(x_{i+1};x_i)$ where the next interaction takes place, and so on.

3.2 The Electron and Positron Propagator

3.2.1 Differential equation for relativistic propagator

Let us introduce the relativistic propagator

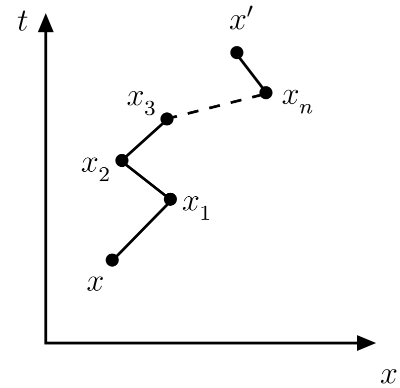


Figure 3.1: nth-order Green's function as the probability amplitude for multiple scattering.

$$S_F(x', x; A) \tag{3.35}$$

in analogy to the nonrelativistic propagator, by satisfying the following differential equation

$$\left[\gamma_\mu \left(i\hbar \frac{\partial}{\partial x'_\mu} - \frac{e}{c} A^\mu(x') \right) - m_0 \right] S_F(x', x; A) = \hbar \delta^4(x' - x)\mathbb{1}. \tag{3.36}$$

We from now on use natural units

$$\frac{e}{\hbar c} \to e, \quad \frac{m_0 c}{\hbar} \to m_0. \tag{3.37}$$

Thus we write

$$\left[\gamma_\mu \left(i\partial'^\mu - eA^\mu(x') \right) - m_0 \right] S_F(x', x; A) = \delta^4(x' - x)\mathbb{1}. \tag{3.38}$$

where we have suppressed the unit matrix, however, it must be kept in mind that we are dealing with a matrix equation.

The free-particle propagator satisfies (3.38) with the interaction term $\gamma_\mu A^\mu(x')$ absent, i.e.

$$(i\gamma_\mu \partial'^\mu - m_0) S_F(x', x) = \delta^4(x' - x)\mathbb{1}. \tag{3.39}$$

As in the non-relativistic case we calculate $S_F(x', x)$ in momentum space.

3.2.2 Non-interacting propagator in momentum space

$$S_F(x', x) = S_F(x' - x) = \int \frac{d^4p}{(2\pi)^4} \exp[-ip \cdot (x' - x)]\, S_F(p) \tag{3.40}$$

which implies that

$$(p^\mu \gamma_\mu + m_0) S_F(p) = \mathbb{1} \tag{3.41}$$

This can be solved for $S_F(p)$ by multiplying by $(p^\mu \gamma_\mu + m_0)$ from the left

$$(p^\mu \gamma_\mu + m_0)(p^\nu \gamma_\nu - m_0) S_F(p) = (p^\mu \gamma_\mu + m_0) \tag{3.42}$$

Since

$$p^\mu \gamma_\mu p^\nu \gamma_\nu = \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) p^\mu p^\nu = \eta_{\mu\nu} p^\mu p^\nu = p_\mu p^\mu = p^2$$

we then have

$$(p^2 - m_0^2) S_F(p) = p^\mu \gamma_\mu + m_0$$

so

$$S_F(p) = \frac{p^\mu \gamma_\mu + m_0}{p^2 - m_0^2} \quad \text{for } p^2 \neq m_0^2 \tag{3.43}$$

Let us consider the inverse Fourier transformation.

$$\begin{aligned} S_F(x' - x) &= \int \frac{d^4}{(2\pi)^4} S_F(p) \exp[-ip \cdot (x' - x)] \\ &= \int \frac{d^4}{(2\pi)^4} S_F(p) \exp\{[-ip_0 \cdot (t' - t) - \mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})]\} \\ &= \int \frac{d^3}{(2\pi)^3} S_F(p) \exp[i\mathbf{p} \cdot (\mathbf{x}' - \mathbf{x})] \times \int_C \frac{dp_0}{2\pi} \frac{\exp[-ip_0 \cdot (t' - t)]}{p^2 - m_0^2} \end{aligned} \tag{3.44}$$

where C is contour of integration choosen to avoid the singularities of $S_F(p)$. As we know from the nonrelativistic case the choice of contour encodes the boundary conditions imposed on $S_F(x' - x)$.

3.2.3 Propagator describing positive-energy particle waves

Considering the particle's propagation forward in time implies that $t' - t$ is positive so that the p_0 integration must be performed along the contour closed in the lower half plane as this gives zanishing contribution. Then the only pole is at

$$p_0 = +E_p = +\sqrt{\mathbf{p}^2 + m_0^2}.$$

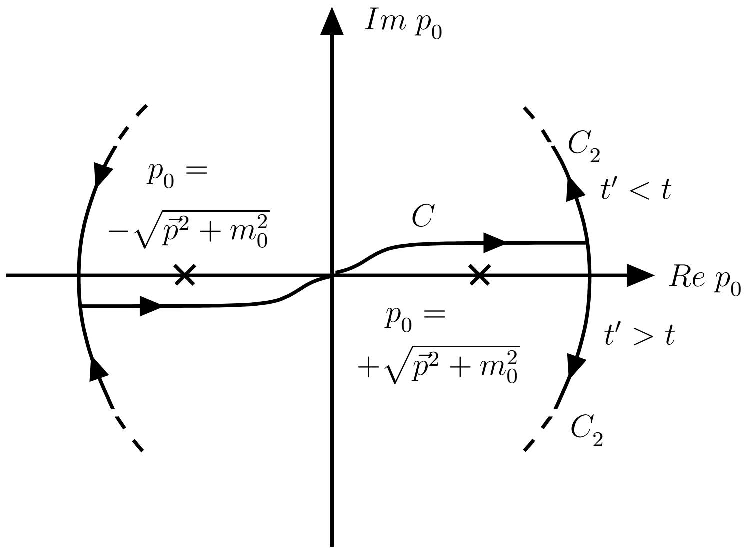


Figure 3.2:

The propagator is then

$$\begin{aligned} S_F^{(t'>t)}(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3}\exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\exp[-iE_p(t'-t)] \\ &\quad \times\frac{(E_p\gamma^0 - \mathbf{p}\cdot\gamma + m_0)}{2E_p} \qquad \text{for} \quad t' > t. \end{aligned} \tag{3.45}$$

Instead of deforming the contour as in fig (3.2.3), we can move the poles an infintesimal distance η off the real axis, as shown in fig (), and perform the p_0 integration alng the whole real axis

3.2.4 Propagator describing negative-energy particle waves

On the other hand, considering the particle's propagation backward in time implies that $t' - t$ is negative so that the p_0 integration must be performed along the contour closed in the upper half plane as this gives zanishing contribution. Then the only pole is at

$$p_0 = -E_p = -\sqrt{\mathbf{p}^2 + m_0^2}.$$

$$\begin{aligned} S_F^{(t>t')}(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3} \exp[i\mathbf{p}\cdot(\mathbf{x}'-\mathbf{x})]\exp[+iE_p(t'-t)] \\ &\quad \times \frac{(-E_p\gamma^0 - \mathbf{p}\cdot\gamma + m_0)}{2E_p} \qquad \text{for} \quad t' < t. \end{aligned} \tag{3.46}$$

3.3 Propagating Positive and Negative Particles

We combine the two propagators describing positive-energy particle waves and negative-energy particle waves moving forward and backward in time, respectively.

$$S_F(x'-x) = S_F^{(t'>t)}(x'-x) + S_F^{(t>t')}(x'-x) \tag{3.47}$$

$$\begin{aligned} S_F(x'-x) &= -i\int \frac{d^3p}{(2\pi)^3} \\ &\quad \Big\{ \exp[-i(+E_p)(t'-t)]\exp[+i\vec{p}\cdot(\vec{x}'-\vec{x})]\frac{(+E_p\gamma^0 + p_i\gamma^i + m_0)}{2E_p}\,\Theta(t'-t) \\ &\quad + \exp[-i(-E_p)(t'-t)]\exp[-i\vec{p}\cdot(\vec{x}'-\vec{x})]\frac{(-E_p\gamma^0 - p_i\gamma^i + m_0)}{2E_p}\Theta(t-t')\Big\} \\ &= -i\int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\Big\{\frac{p_\mu\gamma^\mu + m_0}{2m_0}\exp[-ip\cdot(x'-x)]\Theta(t'-t) \\ &\qquad + \frac{-p_\mu\gamma^\mu + m_0}{2m_0}\exp[ip\cdot(x'-x)]\Theta(t'-t)) \\ &= -i\int \frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}(\hat{\Lambda}_+(p)\exp[-ip\cdot(x'-x)]\Theta(t-t') \\ &\qquad + \hat{\Lambda}_-(p)\exp[ip\cdot(x'-x)]\Theta(t'-t)) \end{aligned} \tag{3.48}$$

3.3.1 Free propagator in terms of plane waves

This can also by written in terms of the normalised Dirac plane waves

$$
\begin{aligned}
S_F(x'-x) &= -i\Theta(t'-t)\int d^3p \sum_{r=1}^{2} \psi_p^r(x')\overline{\psi}_p^r(x) \\
&\quad +i\Theta(t-t')\int d^3p \sum_{r=3}^{4} \psi_p^r(x')\overline{\psi}_p^r(x)
\end{aligned} \tag{3.49}
$$

Proof:

$$
\begin{aligned}
\sum_{r=1}^{2} \psi_p^r(x')\overline{\psi}_p^r(x) &= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p) \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\underbrace{\sum_{r=1}^{4}\epsilon_r\omega^r(p)\overline{\omega}^r(p)}_{=1}\frac{p_\mu\gamma^\mu+m_0}{2m_0} \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\,\frac{p_\mu\gamma^\mu+m_0}{2m_0} \\
&= \frac{1}{(2\pi)^2}\frac{m_0}{E_p}\exp[-ip\cdot(x'-x)]\,\hat{\Lambda}_+(p)
\end{aligned}
$$

A similary calulation for the secod part gives

$$
\sum_{r=3}^{4} \psi_p^r(x')\overline{\psi}_p^r(x) = -\frac{1}{(2\pi)^3}\frac{m_0}{E_p}\exp[ip\cdot(x'-x)]\,\hat{\Lambda}_-(p) \tag{3.50}
$$

$\square$

Using this we easily verify:

$$
\Theta(t'-t)\psi^{(+E)}(x') = i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x), \tag{3.51}
$$

$$
\Theta(t-t')\psi^{(-E)}(x') = -i\int d^3x S_F(x'-x)\gamma_0\psi^{(-E)}(x), \tag{3.52}
$$

(see section D.2). Equation (3.51) explicitly expresses the interpretation of electrons in terms of positive-energy solutions propagating forward in time and equation (3.52) the interpretation of positrons in terms of negative-energy solutions moving backward in time.

The reader should be warnerd no to take this pictorial description of the mathematics as a literal process in space and time. For example, for x and x' with space-like separation,

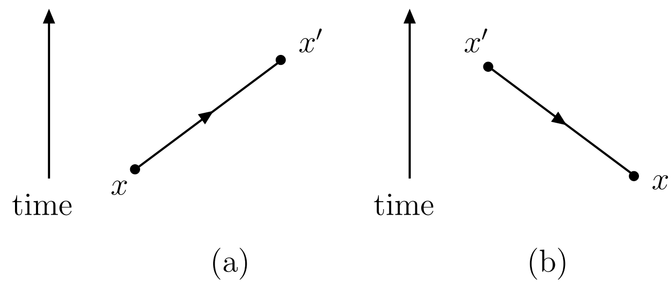


Figure 3.3: (a) $t < t'$: an electron propagated from x to x'. (b) $t > t'$: a positron propagated from x' to x.

our naive interptetation of propagation would imply the electron/positron travels between the two points with speed greater than the speed of light.

3.4 Perturbation Expansion for the Stuckelberg-Feynmann Propagator

Equations () or () determine the free-particle propagator of the electron-positron theory. Here we develop a perturbative expansion for how this is modified to the exact propagator in the presence of an electromagnetic potential, the so-called Stuckelberg-Feynmann propagator, $S_F(x', x; A)$.

$$(i\gamma_\mu \partial^\mu - m_0) S_F(x', x; A) = \delta^4(x' - x) + eA_\mu(x')\gamma^\mu S_F(x', x; A) \tag{3.53}$$

This can be viewed as in inhomogeneous Dirac equation of the form

$$(i\gamma_\mu \partial^\mu - m_0)\Psi(x) = \rho(x) \tag{3.54}$$

which is solved by

$$\Psi(x) = \Psi_0(x) + \int d^4y S_F(x - y)\rho(y) \tag{3.55}$$

As is easily seen:

$$\begin{aligned}
(i\gamma_\mu \partial_x^\mu - m_0)\Psi(x) &= (i\gamma_\mu \partial_x^\mu - m_0)\Psi_0(x) + \int d^4y (i\gamma_\mu \partial_x^\mu - m_0) S_F(x-y)\rho(y) \\
&= \int d^4y \delta^4(x-y)\rho(y) \\
&= \rho(x) \qquad (3.56)
\end{aligned}$$

In this way we obtain an integral equation for $S_F(x', x; A)$

$$\begin{aligned}
S_F(x', x; A) &= \int d^4y S_F(x'-y) \, \left[\delta^4(y-x) + eA_\mu(x')\gamma^\mu S_F(y, x; A)\right] \\
&= S_F(x'-y) + e\int d^4y S_F(x'-y) A_\mu(x')\gamma^\mu S_F(y, x; A). \qquad (3.57)
\end{aligned}$$

Repeatedly substituting this equation into itself we obtain

$$\begin{aligned}
S_F(x', x; A) &= \int d^4y S_F(x'-y) + e\int d^4x_1 S_F(x'-x_1) A_\mu(x_1)\gamma^\mu S_F(x_1 - x) \\
&+ e^2 \int d^4x_1 d^4x_2 S_F(x'-x_1) A_\mu(x_1)\gamma^\mu S_F(x_1 - x_2) A_\nu(x_2)\gamma^\nu S_F(x_2 - x) \\
&+ \ \ldots \ . \qquad (3.58)
\end{aligned}$$

3.4.1 Boundary condition of Feynman and Stuckelberg

$$\Psi(x) = \psi(x) + \int S_F(x-y) e\gamma_\mu A^\mu(y)\Psi(y) \qquad (3.59)$$

The second term on the RHS represents the scattered wave.

Now by () $t \equiv x^0 \to +\infty$

$$S_F(x-y) \to -i\int d^3p \sum_{r=1}^{2} \psi_p^r(x)\overline{\psi}_p^r(y)$$

and $t \equiv x^0 \to -\infty$

$$S_F(x-y) \to +i\int d^3p \sum_{r=3}^{4} \psi_p^r(x)\overline{\psi}_p^r(y)$$

So that

$$\Psi(x)-\psi(x) \to \int d^3p \sum_{r=1}^{2} \psi_p^r(x) \left(-ie \int d^4y\, \overline{\psi}_p^r(x) A_\mu(y)\gamma^\mu \Psi(y)\right) \quad \text{for} \quad t \to +\infty \tag{3.60}$$

and

$$\Psi(x)-\psi(x) \to \int d^3p \sum_{r=3}^{4} \psi_p^r(x) \left(+ie \int d^4y\, \overline{\psi}_p^r(x) A_\mu(y)\gamma^\mu \Psi(y)\right) \quad \text{for} \quad t \to -\infty \tag{3.61}$$

Therefore the scattered wave contains only positive frequencies

3.5 The $S-$Matrix Elements

The S-matrix elements are defined in the same manner as in the nonrelativistic case.

Let $\psi_f(x)$ denote the final free wave with quantum numbers f that is observed at the end of the scattering process.

$$\begin{aligned} S_{fi} &= \lim_{t\to\pm\infty} < \psi_f(x)|\Psi_i(x) > \\ &= \lim_{t\to\pm\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4y S_F(x-y) e A_\mu(y)\gamma^\mu \Psi_i(x) \right\rangle \end{aligned} \tag{3.62}$$

There are four basic processes to consider: (a) electron scattering; (b) positron scattering; (c) electron-positron pair creation; (d) pair annihilation.

We will need the following relations for adjoint spinors (proven in section D.2).

$$\Theta(t-t')\overline{\psi}^{(+E)}(x') = i \int d^3x \overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \tag{3.63}$$

$$\Theta(t'-t)\overline{\psi}^{(-E)}(x') = -i \int d^3x \overline{\psi}^{(-E)}(x)\gamma_0 S_F(x'-x). \tag{3.64}$$

These are the adjoint spinor versions of equations (3.51) and (3.52).

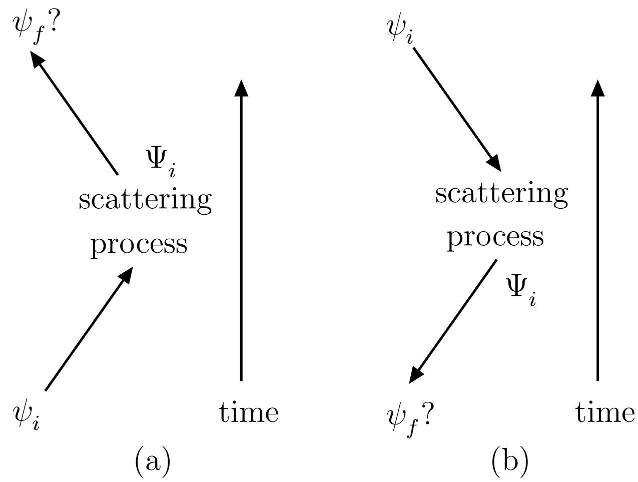


Figure 3.4: $\Psi_i(x)$ stands for the incoming wave, which either reduces at $y_0 \to -\infty$ to an incident positive energy wave $\psi_i(x)$ or at $y_0 \to +\infty$ to an incident negative energy wave $\psi_i(x)$. (a) ψ_f describes an electron in the limit $t \to +\infty$. (b) ψ_f describes a positron in the limit $t \to -\infty$.

Using (3.63), for electron scattering we have

$$
\begin{aligned}
S_{fi} &= \lim_{t\to+\infty} \left\langle \psi_f(x) \middle| \psi_i(x) + \int d^4y S_F(x-y) e A_\mu(y)\gamma^\mu \Psi_i(x) \right\rangle \\
&= \delta_{fi} + e \lim_{t\to+\infty} \int d^3x \psi_f^*(x) \int d^4y S_F(x-y) A_\mu(y)\gamma^\mu \Psi_i(x) \\
&= \delta_{fi} - ie \lim_{t\to+\infty} \int d^4y \left(i \int d^3x \overline{\psi}_f(x)\gamma^0 S_F(x-y) \right) A_\mu(y)\gamma^\mu \Psi_i(x) \\
&= \delta_{fi} - ie \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x)
\end{aligned}
$$

while, using (3.64) similarly, positron scattering is described by

$$S_{fi} = \delta_{fi} + ie \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x)$$

Both results can be combined by writing

$$S_{fi} = \delta_{fi} - ie\varepsilon_f \int d^4y \overline{\psi}_f(y) A_\mu(y)\gamma^\mu \Psi_i(x) \tag{3.65}$$

where $\varepsilon_f = +1$ for positive energy waves in the future and $\varepsilon_f = -1$ for energy waves in the past. $\Psi_i(x)$ stands for the incoming wave

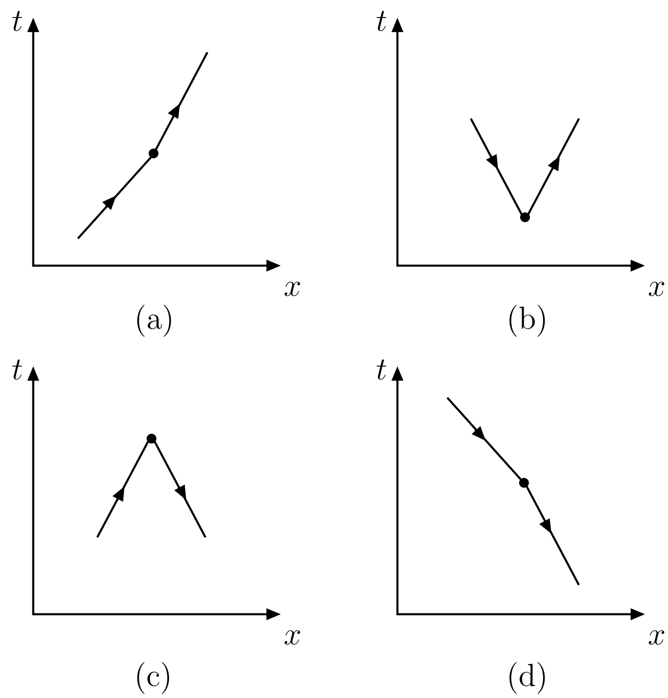


Figure 3.5: (a) electron scattering; (b) electron-positron pair creation; (c) pair annihilation (d) positron scattering.

Repeated substitution of (3.59)

$$\begin{aligned}
S_{fi} &= \delta_{fi} - e\varepsilon_f \int d^4y \overline{\psi}_f(y) A_\mu \gamma^\mu(y) \Psi_i(y) \\
&= \delta_{fi} - e\varepsilon_f \left[\int d^4y_1 \overline{\psi}_f(y_1) A_\mu(y_1) \gamma^\mu \psi_i(y_1) \right. \\
&\quad \left. + \int d^4y_1 \int d^4y_2 \overline{\psi}_f(y_2) A_{\mu_2}(y_2) \gamma^{\mu_2} S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i(y_1) \right] \\
&\quad + \ldots \quad . \\
&= \delta_{fi} + \sum_{n=1}^{\infty} S_{fi}^{(n)}
\end{aligned} \tag{3.66}$$

where

$$\begin{aligned}
S_{fi}^{(n)} &= -ie^n \varepsilon_f \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\
&\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i(y_1)
\end{aligned} \tag{3.67}$$

3.5.1 "Ordinary" scattering of electrons

• $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with positive energy.

In this case

$$\Psi_i(y) \to \psi_i^{(+E)}(y) = \sqrt{\frac{m_0}{E_-}} \frac{1}{(2\pi)^{3/2}} u(p_-, s_-) \exp(-ip_- \cdot x) \quad \text{as} \quad y_0 \to -\infty \tag{3.68}$$

an incoming electron with positive energy E_- and momentum p_- and spin s_-

$$\begin{aligned} S_{fi}^{(n)} &= -ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(+E)}(y_n) A_{\mu_n}(y_n) \gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1}) \gamma^{\mu_{n-1}} \ldots \\ &\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1) \gamma^{\mu_1} \psi_i^{(+E)}(y_1) \end{aligned} \tag{3.69}$$

In addition to ordinary scattering intermediate pair creation and pair annihilation are included in the series.

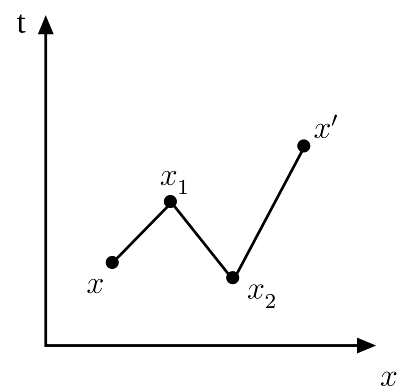


Figure 3.6: The electron at x_1 propagates backward in time from x_1 to x_2. Physically a positron-electron pair is created at x_2, the positron propagates forward in time where it anihilates with the intial electron at x_1.

3.5.2 Pair production processes

• $\Psi_i(y)$ in this case at $y_0 \to +\infty$ reduces to a plane wave with negative energy.

The positron state at $t \to +\infty$ is described by a plane wave of negative energy. We use the notation

By hole theory a positron is an electron with negative energy, negative momentum and negative spin.

Now we need the plane wave propagating backward in time. There will be an exponential with a positive sign in the exponent

$$e^{(i+p_+\cdot y)}$$

expressing the property that it has negative energy and momentum. It also will involve

$$v(p_+,+1/2)=\omega^4(p_+) \quad \text{and} \quad v(p_+,-1/2)=\omega^3(p_+)$$

where ω^4 is the spinor corresponding to a negative energy electron with spin up and ω^3 a negative energy electron with spin down. By using the spinors $v(p,s)$ we take care of the fact that the spin of electrons withe negative enery is $-s$. Here s is the spin of the positron.

$$\psi_i^{(electron)}(-p_f,-s_f)=Const.\ v(p_f,s_f)\exp(+ip_f\cdot x) \tag{3.70}$$

$$\Psi_i(x)\rightarrow\sqrt{\frac{m_0}{E_-}}\frac{1}{(2\pi)^{3/2}}v(p_+,s_+)e^{i+p_+\cdot y)} \quad \text{as} \quad y_0\rightarrow\infty \tag{3.71}$$

$$\begin{aligned} S_{fi}^{(n)} \;\; = \;\; & -ie^n\int d^4y_1\ldots\int d^4y_n\overline{\psi}_f^{(+E)}(y_n)A_{\mu_n}(y_n)\gamma^{\mu_n}S_F(y_n-y_{n-1})A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}}\ldots \\ & \times S_F(y_2-y_1)A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i(-E)(y_1) \end{aligned} \tag{3.72}$$

3.5.3 Pair annihilation processes

- $\Psi_i(y)$ in this case at $y_0\rightarrow-\infty$ reduces to a plane wave with negative energy.

$$\begin{aligned} S_{fi}^{(n)} \;\; = \;\; & +ie^n\int d^4y_1\ldots\int d^4y_n\overline{\psi}_f^{(-E)}(y_n)A_{\mu_n}(y_n)\gamma^{\mu_n}S_F(y_n-y_{n-1})A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}}\ldots \\ & \times S_F(y_2-y_1)A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i^{(+E)}(y_1) \end{aligned} \tag{3.73}$$

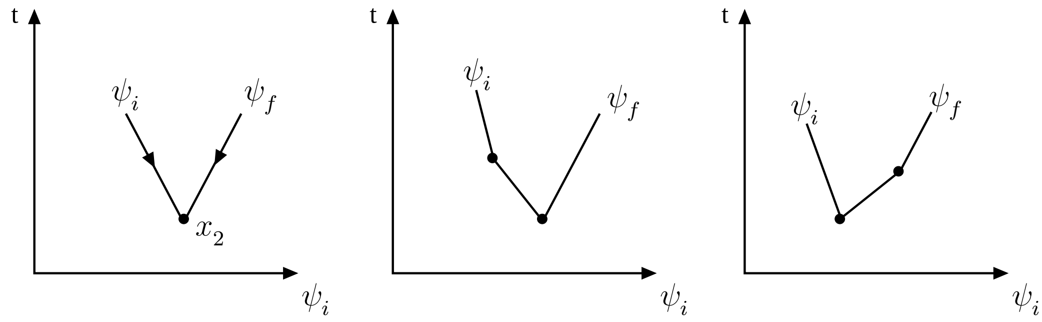


Figure 3.7:

3.5.4 Scattering of positrons

- $\Psi_i(y)$ in this case at $y_0 \to -\infty$ reduces to a plane wave with negative energy.

$$
\begin{aligned}
S_{fi}^{(n)} &= +ie^n \int d^4y_1 \ldots \int d^4y_n \overline{\psi}_f^{(-E)}(y_n) A_{\mu_n}(y_n)\gamma^{\mu_n} S_F(y_n - y_{n-1}) A_{\mu_{n-1}}(y_{n-1})\gamma^{\mu_{n-1}} \ldots \\
&\quad \times S_F(y_2 - y_1) A_{\mu_1}(y_1)\gamma^{\mu_1}\psi_i^{(-E)}(y_1) \qquad (3.74)
\end{aligned}
$$

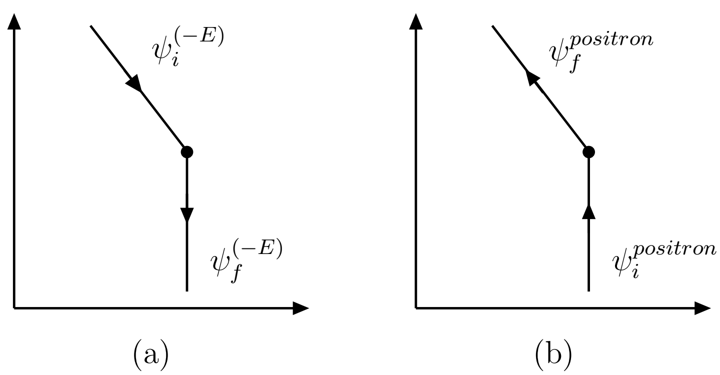


Figure 3.8: Lowest order positron scattering. (a) incoming negative energy electron $\psi_i^{(-E)}$ is scattered into an outgoing negative energy electron $\psi_f^{(-E)}$. (b) This corresponds to an incident positron $\psi_f^{positron}$ and emerging positron $\psi_f^{positron}$. This is the link between the calculational technique and the real physical picture of positron scattering.

Part IV

Quantum Electrodynamical Processes

Chapter 4

Scattering of an Electron off a Coulomb Potential

4.1 The Scattering Amplitude

We calculate the Rutherford scattering of an electron at a fixed Coulomb potential to lowest order of perturbation theory. The appropriate S-Matrix element is the first order term of (3.69)

$$S_{fi} = -ie \int d^4x \, \overline{\psi}_f(x) A_\mu \gamma^\mu(x) \psi_i(x) \tag{4.1}$$

$\psi_i(x)$ is given by the incoming plane wave of an electron with momentum p_i and s_i:

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} \, u(p_i, s_i) e^{-ip_i \cdot x} \tag{4.2}$$

$\overline{\psi}_f(x)$ is given by

$$\overline{\psi}_f(x) = \sqrt{\frac{m_0}{E_f V}} \, \overline{u}(p_f, s_f) e^{ip_f \cdot x}. \tag{4.3}$$

Recall

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial \mathbf{t}}, \quad \mathbf{B} = -\nabla \times \mathbf{A}$$

Choosing

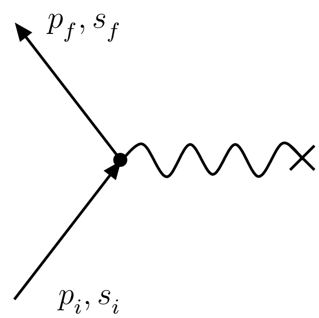


Figure 4.1:

$$A_0(x) = A_0(\mathbf{x}) = -\frac{Ze}{|\mathbf{x}|}, \quad \mathbf{A}(x) = 0. \tag{4.4}$$

corresponds to a Coulomb force generated by a static charge $-Ze$. With these assumptions the S-Matrix element becomes

$$S_{fi} = iZe^2 \frac{1}{V}\sqrt{\frac{m_0^2}{E_f E_i}}\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i) \int d^4x \; e^{i(p_f - p_i)\cdot x}\frac{1}{|\mathbf{x}|}. \tag{4.5}$$

The integral over the time coordinate can be integrated

$$\int_{-\infty}^{\infty} dx_0 \; e^{i(E_f - E_i)t} = 2\pi\delta(E_f - E_i) \tag{4.6}$$

The remaining integral is

$$A_0(\mathbf{x}) = -Ze \int d^3x \frac{1}{|x|} e^{-i\mathbf{q}\cdot\mathbf{x}}$$

where $\mathbf{q}$ is the momentum transfer i.e. $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$. This can be evaluated using integration by parts of Poisson's formula $\Delta(1/|\mathbf{x}|) = -4\pi\delta^3(\mathbf{x})$:

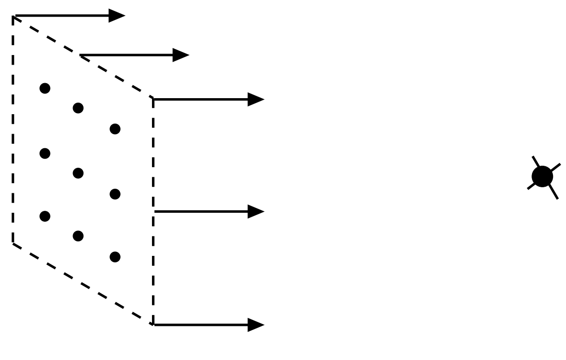
$$\begin{aligned}
\int d^3x \frac{1}{|\mathbf{x}|} e^{-i\mathbf{q}\cdot\mathbf{x}} &= -\frac{1}{\mathbf{q}^2}\int d^3x \frac{1}{|\mathbf{x}|}\Delta e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= -\frac{1}{\mathbf{q}^2}\int d^3x \Delta\left(\frac{1}{|\mathbf{x}|}\right) e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= -\frac{1}{\mathbf{q}^2}\int d^3x (-4\pi\delta^3(\mathbf{x})) e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= \frac{4\pi}{\mathbf{q}^2}
\end{aligned} \tag{4.7}$$

Thus the $S-$matrix element becomes

$$S_{fi} = iZe^2 \frac{1}{V} \sqrt{\frac{m_0^2}{E_f E_i}}\, \overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\, \frac{4\pi}{q^2}\, 2\pi\delta(E_f - E_i). \tag{4.8}$$

4.2 The Cross Section

A differential cross section σ is defined by the effective area of target particles.



target

Figure 4.2: .

$$\text{Chance of hitting a Coulomb potential} = \frac{N_T \sigma}{A} \tag{4.9}$$

Let us say that there are N_I incoming particles. The number of scattering events is then

$$\text{number of events} = N_I \frac{N_T \sigma}{A} \tag{4.10}$$

so that the cross section is then expressed as

$$\sigma = \left(\frac{\text{number of events}}{N_I N_T} \right) A. \tag{4.11}$$

We wish the express the cross section in terms of the flux of the incoming beam,

$$\text{flux} = \rho v$$

where v is the velocity of the beam moving toward the stationary target. The number of particles in the beam N_I is equal to the density of the beam ρ times the volume of the beam, vtA. The cross section can therefore be written as

$$
\begin{aligned}
\sigma &= \frac{\text{number of events}/t}{(\rho v t A)N_T/t}A \\
&= \frac{\text{number of events}/t}{\rho v} \cdot \frac{1}{N_T} \\
&= \frac{\text{transition rate}}{\text{flux}} \cdot \frac{1}{N_T} \qquad (4.12)
\end{aligned}
$$

$$
dW = \frac{|S_{fi}|^2 dN_f}{T}\frac{1}{J} \qquad (4.13)
$$

dN_f is now determined.

4.3 Transition Probability Per Particle into Final States

Standing waves in a cubical box of volume $V = L^3$ require

$$
\begin{aligned}
p_x L &= n_x 2\pi, \\
p_y L &= n_y 2\pi, \\
p_z L &= n_z 2\pi, \qquad (4.14)
\end{aligned}
$$

with integer number n_x, n_y, n_z. For large L the discrete set of $\mathbf{p}-$values approaches a continuum. The number of states is

$$
\begin{aligned}
dN &= dn_x dn_y dn_z \\
&= \frac{1}{(2\pi)^3} L^3 dp_x dp_y dp_z \\
&= \frac{V}{(2\pi)^3} d^3p. \qquad (4.15)
\end{aligned}
$$

The transition probability per particle into these final states is

$$
\begin{aligned}
dW &= |S_{fi}|^2 \frac{V d^3 p_f}{(2\pi)^3} \\
&= \frac{Z^2 (4\pi\alpha)^2 m_0^2}{E_i V} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{d^3 p_f}{(2\pi)^3 E_f} (2\pi\delta(E_f - E_i))^2
\end{aligned}
\tag{4.16}
$$

4.4 Transition Probability Per Particle, Per Unit Time

We smear out the $\delta-$function $2\pi\delta(E_f - E_i)$:

$$
\begin{aligned}
\int_{-T/2}^{T/2} dx_0 e^{i(E_f - E_i)x_0} &= \left[\frac{1}{i(E_f - E_i)} e^{i(E_f - E_i)x_0} \right]_{-T/2}^{T/2} \\
&= \frac{2\sin(E_f - E_i)T/2}{E_f - E_i}
\end{aligned}
\tag{4.17}
$$

Thus we replace the square of the $\delta-$function is replaced by

$$
(2\pi\delta(E_f - E_i))^2 \;\Rightarrow\; 4\frac{\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2}
\tag{4.18}
$$

The area of this function is

$$
\int_{-\infty}^{\infty} 4\frac{\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2} dE_f = 2\pi T.
\tag{4.19}
$$

Knowing that the "area" under the square of the δ is $\lim_{T\to\infty} 2\pi T$, we make the replacement

$$
(2\pi\delta(E_f - E_i))^2 \to 2\pi T\delta(E_f - E_i).
\tag{4.20}
$$

Denote the rate R

$$
dR = \frac{dW}{T} = \frac{Z^2\alpha^2 m_0^2}{E_i V} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{d^3 p_f}{E_f} \delta(E_f - E_i)
\tag{4.21}
$$

4.5 Formula for Differential Cross Section

The scattering cross section can be defined as the transition probabilty per particle and per unit time divided by the incoming current of particles

$$J^a_{inc}(x) = \overline{\psi}_i(x)\gamma^a\psi_i(x) \tag{4.22}$$

Taking the spinors with spin polarisation in the z-direction we determine the current

$$\begin{aligned}
J^a_{inc} &= \overline{\psi}_i(x)\gamma^a\psi_i(x) \\
&= \frac{m_0}{E_i V}\,\overline{u}(p_i, s_i)\gamma^3 u(p_i, s_i) \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\begin{pmatrix} 1 \; 0 \; \dfrac{p_i}{E_i + m_0} \; 0 \end{pmatrix}\,\gamma^0\gamma^3\,\begin{pmatrix} 1 \\ 0 \\ \frac{p_i}{E_i+m_0} \\ 0 \end{pmatrix} \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\begin{pmatrix} 1 \; 0 \; \dfrac{p_i}{E_i + m_0} \; 0 \end{pmatrix}\,\begin{pmatrix} 1 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix}\,\begin{pmatrix} 1 \\ 0 \\ \frac{p_i}{E_i+m_0} \\ 0 \end{pmatrix} \\
&= \frac{m_0}{E_i V}\,\frac{(E_i + m_0)}{2m_0}\,\begin{pmatrix} 1 \; 0 \; \dfrac{p_i}{E_i + m_0} \; 0 \end{pmatrix}\,\begin{pmatrix} \frac{p_i}{E_i+m_0} \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{p_i}{E_i}\frac{1}{V}.
\end{aligned} \tag{4.23}$$

$$|\mathbf{J}_{inc}| = \frac{|\mathbf{v}_i|}{V}. \tag{4.24}$$

The differential cross section can now be determined

$$d\sigma = \frac{dR}{|\mathbf{J}_{inc}|} = \frac{4Z^2\alpha^2 m_0^2}{E_i V \frac{|v_i|}{V}} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4}\,\frac{d\mathbf{p}_f^3}{E_f}\,\delta(E_f - E_i) \tag{4.25}$$

Use

$$d^3p_f = \mathbf{p}_f^2 d|\mathbf{p}_f| d\Omega_f \tag{4.26}$$

Then the differential cross section becomes

$$d\sigma = \frac{4Z^2\alpha^2 m_0^2}{E_i V \frac{|v_i|}{V}} \frac{|\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2}{|\mathbf{q}|^4} \frac{\mathbf{p}_f^2 d|\mathbf{p}_f|}{E_f} d\Omega_f \; \delta(E_f - E_i) \tag{4.27}$$

4.6 Averaging Over Spin

The differential cross section above can be applied to calculate the scattering of a particle with initial polarisation (s_i) to final polarisation (s_f).

First we give a simple example. From the relation

$$\overline{w}^r(p_\mu \gamma^\mu - \epsilon_r m_0) = 0.$$

we see that

$$\overline{w}^r_\gamma(p_i)\Lambda_{\gamma\delta}(p) = 0$$

for $r = 3, 4$, where

$$\Lambda_{\gamma\delta}(p) = \frac{-p_\mu \gamma^\mu + m_0}{2m_0}$$

$$\begin{aligned}
\sum_{s_i} u_\beta(p_i, s_i)\overline{u}_\beta(p_i, s_i) &= u_\beta(p_i, \uparrow)\overline{u}_\beta(p_i, \uparrow) + u_\beta(p_i, \downarrow)\overline{u}_\beta(p_i, \downarrow) \\
&= \sum_{r=1}^{2} w^r_\beta(p_i)\overline{w}^r_\delta(p_i) \\
&= \sum_{r=1}^{2} w^r_\beta(p_i) \sum_{\gamma=1}^{4} \overline{w}^r_\gamma(p_i)\Lambda_{\gamma\delta}(p) \\
&= \sum_{\gamma, r=1}^{4} \epsilon_r w^r_\beta(p_i)\overline{w}^r_\gamma(p_i)(\Lambda(p))_{\gamma\delta} \\
&= (\Lambda(p))_{\beta\delta}
\end{aligned} \tag{4.28}$$

where we used the completeness relation

$$\sum_{r=1}^{4} \epsilon_r w^r_\beta(p_i)\overline{w}^r_\gamma(p_i) = \delta_{\beta\gamma}.$$

Using this result and similar considerations we now calculate the spin sum.

$$\begin{aligned}
& \sum_{\alpha,\sigma,\beta,\delta} \sum_{s_f} \overline{u}_\alpha(p_f, s_f)\gamma^0_{\alpha\beta} \left(\sum_{s_f} u_\alpha(p_i, s_i)\overline{u}_\delta(p_i, s_i) \right) \gamma^0_{\delta\sigma} u_\sigma(p_f, s_f) \\
= & \sum_{\alpha,\sigma} \sum_{s_f} \overline{u}_\alpha(p_f, s_f) \left(\gamma^0 \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma^0 \right)_{\alpha\sigma} u_\sigma(p_f, s_f) \\
= & \sum_{\alpha,\sigma} \sum_{r=1}^{2} \overline{w}^r_\alpha(p_f) \left(\gamma^0 \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma^0 \right)_{\alpha\sigma} w^r_\sigma(p_f) \\
= & \sum_{\alpha,\sigma} \sum_{r=1}^{4} \epsilon_r \overline{w}^r_\alpha(p_f) \left(\gamma^0 \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma^0 \right)_{\alpha\sigma} \sum_{\tau=1}^{4} \left(\frac{(p_f)_\nu \gamma^\nu + m_0}{2m_0} \right)_{\sigma\tau} w^r_\tau(p_f) \\
= & \sum_{\alpha,\sigma} \left(\gamma^0 \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma^0 \right)_{\alpha\sigma} \left(\frac{(p_f)_\nu \gamma^\nu + m_0}{2m_0} \right)_{\sigma\alpha} \\
= & \; Tr \left[\gamma^0 \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma^0 \frac{(p_f)_\nu \gamma^\nu + m_0}{2m_0} \right] \qquad (4.29)
\end{aligned}$$

Using this the differential cross section can be written

$$\frac{d\overline{\sigma}}{d\Omega_f} = \frac{4Z^2\alpha^2 m_0^2}{2|\mathbf{q}|^4} Tr \left[\gamma^0 \frac{(p_i)_\mu \gamma^\mu + m_0}{2m_0} \gamma^0 \frac{(p_f)_\nu \gamma^\nu + m_0}{2m_0} \right]. \qquad (4.30)$$

4.7 Taking the Trace of the Product of Gamma Matrices in the Differential Cross Section

We first prove that the trace of an odd number of $\gamma-$matrices vanishes. To do this we use $(\gamma^5)^2 = I$ and $\gamma_\mu\gamma^5 + \gamma^5\gamma_\mu = 0$.

$$\begin{aligned}
Tr\gamma_\mu \dots \gamma_\nu &= Tr\gamma_\mu \dots \gamma_\nu \gamma^5\gamma^5 \\
&= (-1)^n Tr\gamma^5\gamma_\mu \dots \gamma_\nu \gamma^5 \\
&= (-1)^n Tr\gamma_\mu \dots \gamma_\nu \gamma^5\gamma^5 \\
&= (-1)^n Tr\gamma_\mu \dots \gamma_\nu \qquad (4.31)
\end{aligned}$$

where in the third line we used the cyclic permutation of the trace. With this () reduces to

$$\frac{d\overline{\sigma}}{d\Omega_f} = \frac{4Z^2\alpha^2 m_0^2}{2|\mathbf{q}|^4}\left[Tr(\gamma^0 (p_i)_\mu \gamma^\mu \gamma^0 (p_f)_\nu \gamma^\nu) + m_0^2 Tr(\gamma^0)^2\right]. \tag{4.32}$$

We have $Tr(\gamma^0)^2 = Tr\mathbb{1} = 4$. To evaluate the first trace we derive a couple of results: Firstly

$$\begin{aligned} a_\mu b_\nu Tr\gamma^\mu\gamma^\nu &= a_\mu b_\nu \frac{1}{2} Tr(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) \\ &= a_\mu b_\nu \eta^{\mu\nu} Tr\mathbb{1} \\ &= 4a\cdot b. \end{aligned} \tag{4.33}$$

where we have used μ and ν are dummy variables in the first line. Secondly, starting with

$$\begin{aligned} a_\mu b_\nu c_\gamma d_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta &= 2a\cdot b\, c_\gamma d_\delta \, Tr\gamma^\gamma\gamma^\delta - b_\mu a_\nu c_\gamma d_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta \\ &= 2a\cdot b\, c_\gamma d_\delta \, Tr\gamma^\gamma\gamma^\delta - 2a\cdot c b_\mu d_\delta \, Tr\gamma^\mu\gamma^\delta \\ &\quad + b_\mu c_\nu a_\gamma d_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta \\ &= 2a\cdot b\, c_\gamma d_\delta \, Tr\gamma^\gamma\gamma^\delta - 2a\cdot c\, b_\mu d_\delta \, Tr\gamma^\mu\gamma^\delta \\ &\quad + 2a\cdot d\, b_\mu c_\nu \, Tr\gamma^\mu\gamma^\nu - b_\mu c_\nu d_\gamma a_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta \end{aligned}$$

then using the cyclic property of the trace we find

$$\begin{aligned} a_\mu b_\nu c_\gamma d_\delta \, Tr\gamma^\mu\gamma^\nu\gamma^\gamma\gamma^\delta &= a\cdot b\, c_\mu d_\nu \, Tr\gamma^\mu\gamma^\nu - a\cdot c\, b_\mu d_\nu \, Tr\gamma^\mu\gamma^\nu \\ &\quad + a\cdot d\, b_\mu c_\nu \, Tr\gamma^\mu\gamma^\nu \end{aligned} \tag{4.34}$$

Using the second result first with $a = c = (1,0,0,0)$ we get

$$(p_i)_\mu (p_f)_\nu \, Tr(\gamma^0\gamma^\mu\gamma^0\gamma^\nu) = a\cdot p_i \, Tr - a\cdot a \, Tr + a\cdot p_f \, Tr$$

Now using the first result we have

$$\begin{aligned} (p_i)_\mu (p_f)_\nu \, Tr(\gamma^0\gamma^\mu\gamma^0\gamma^\nu) &= 4(a\cdot p_i)(a\cdot p_f) - (a\cdot a)4(p_i\cdot p_f) + 4(a\cdot p_f)(a\cdot p_i) \\ &= 4E_iE_f - 4(E_iE_f - \vec{p}_i\cdot\vec{p}_f) + 4E_iE_f \end{aligned} \tag{4.35}$$

4.8 Mott Scattering Formula

The $\delta(E_i - E_f)$ function of the cross section ensures energy conservation $E_i = E_f$, thus $E_i^2 = E_f^2$:

$$m_0^2 + \vec{p}_i^2 = m_0^2 + \vec{p}_f^2$$

implying

$$|\vec{p}_i| = |\vec{p}_f| = |\vec{p}|$$

The scalar product of initial and final momentum is the following function of the scattering angle θ

$$\begin{aligned} p_i \cdot p_f &= |\mathbf{p}|^2 \cos\theta \\ &= |\mathbf{p}|^2 \left(1 - 2\sin^2\frac{\theta}{2}\right) \\ &= \beta^2 E^2 \left(1 - 2\sin^2\frac{\theta}{2}\right) \end{aligned} \tag{4.36}$$

From this we have for the momentum transfer

$$\begin{aligned} |q| &= |p_f - p_i| \\ &= \sqrt{|p_f|^2 + |p_i|^2 - p_f \cdot p_i} \\ &= \sqrt{2p^2 - |p|\cos\theta} \\ &= 2|p|\sin\frac{\theta}{2} \end{aligned} \tag{4.37}$$

A simple exercise, left to the reader, gives us the well known Mott scattering formula

$$\frac{d\overline{\sigma}}{d\Omega_f} = \frac{Z^2\alpha^2(1 - \beta^2\sin^2\frac{\theta}{2})}{4\beta^2|\mathbf{p}|^2\sin^4\frac{\theta}{2}} \tag{4.38}$$

In the limit $\beta \to 0$ (small velocities) reduces to Rutherford's scattering formula

$$\frac{d\overline{\sigma}}{d\Omega_f} = \frac{Z^2\alpha^2}{4\beta^2|\mathbf{p}|^2\sin^4\frac{\theta}{2}} \tag{4.39}$$

Chapter 5

Scattering of an Electron off a Free Proton

In the last example we considered scattering off a central potential. Now we make our first step toward the derivation of Feynmann's rules by considering scattering off two free particles.

5.1 Inhomogeneous Wave Equation and Photon Proporgator

In electromagnetism the invariance in A_μ comes about because the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is left unchanged by the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)$$

We wish to calculate the four-potential $A_\mu(x)$ produced by a source current $J^\nu(x)$ term,

$$\Box A^\nu(x) - \partial^\mu \partial_\mu A^\nu(x) = 4\pi J^\nu(x) \tag{5.1}$$

We are free to choose the most convenient gauge for the calculation intened to make.

We will choose the Lorentz gauge

$$\partial_\mu A^\mu(x) = 0$$

In momentum space this reads

$$k^\mu A^\mu(k) = 0.$$

$$\Box A^\nu(x) = 4\pi J^\nu(x) \tag{5.2}$$

The solution of the above equation may be systematically formulated using the appropriate Green's function which we call $D_F(x-y)$, the propogator for electromagnetism.

$$\Box D_F(x-y)(x) = 4\pi\delta^4(x-y). \tag{5.3}$$

The Fourier-transformed proporgator is defined by

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp[-iq\cdot(x-y)] D_F(q) \tag{5.4}$$

Using

$$\delta^4(x-y) = \int \frac{d^4p}{(2\pi)^4} \exp[-iq\cdot(x-y)] \tag{5.5}$$

and making comparison we get

$$D_F(q) = -\frac{4\pi}{q^2} \tag{5.6}$$

The four-potential $A^\mu(x)$ solving (5.2) is

$$A^\mu(x) = \int d^4y D_F(x-y) J^\mu(y). \tag{5.7}$$

5.2 Potential of Proton Current

$$S_{fi} = -ie \int d^4x \overline{\psi}_f(x)\gamma_\mu A^\mu(x)\Psi_i(x) \tag{5.8}$$

At first order the four-potential $A^\mu(x)$ is the field produced by the proton to lowest order in α.

$$S_{fi} = -i\int d^4x d^4y \left[e\overline{\psi}_f(x)\gamma_\mu\psi_i(x)\right] D_F(x-y)J^\mu(y). \tag{5.9}$$

The term in the brackets represents the current of the electron. As the electron and proton play equivalent roles in the scattering process, the proton's current should be of the same form as the electronic current. Therefore we make the replacement

$$J^{\mu}(y) \to e_p \overline{\psi}^p_f(y)\gamma^{\mu}\psi^p_i(y) \tag{5.10}$$

where $\overline{\psi}^p_f(y)$ and $\psi^p_i(y)$ have the same for as the electron wavefunctions

$$\begin{aligned} \psi^p_i(y) &= \sqrt{\frac{M_0}{E^p_i V}} u(P_i, S_i) \exp(-iP_i \cdot y) \\ \psi^p_f(y) &= \sqrt{\frac{M_0}{E^p_f V}} u(P_f, S_f) \exp(-iP_f \cdot y) \end{aligned} \tag{5.11}$$

where P_i and P_f denote the four-momentum of the proton, S_i, S_f and E^p_i, E^p_f denote its spin and energy respectively. M_0 is the proton's rest mass. The proton's current is then

$$J^{\mu}_{fi}(y) = -\sqrt{\frac{M_0^2}{E^p_f E^p_i}} \frac{e}{V} \exp[i(P_f - P_i) \cdot y]\, \overline{u}(P_f, S_f)\gamma^{\mu} u(P_i, S_i). \tag{5.12}$$

Inserting this into the expression for the S-matrix gives

$$\begin{aligned} S_{fi} &= +i\frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E^p_f E^p_i}}\, [\overline{u}(p_f, s_f)\gamma_{\mu} u(p_i, s_i)] \\ &\times \int d^4x d^4y \frac{d^4q}{(2\pi)^4} \exp[-iq \cdot (x-y)] \exp[i(p_f - p_i) \cdot x] \exp[i(P_f - P_i) \cdot x] \\ &\times \left(\frac{-4\pi}{q^2 + i\epsilon}\right) [\overline{u}(P_f, S_f)\gamma_{\mu} u(P_i, S_i)] \end{aligned} \tag{5.13}$$

5.3 Conservation of Four-Momentum

The $x-$ and $y-$integrations give

$$\begin{aligned} \int d^4x \exp(i(p_f - p_i - q) \cdot x) &= (2\pi)^4 \delta^4(p_f - p_i - q) \\ \int d^4y \exp(i(p_f - p_i - q) \cdot y) &= (2\pi)^4 \delta^4(P_f - P_i - q) \end{aligned} \tag{5.14}$$

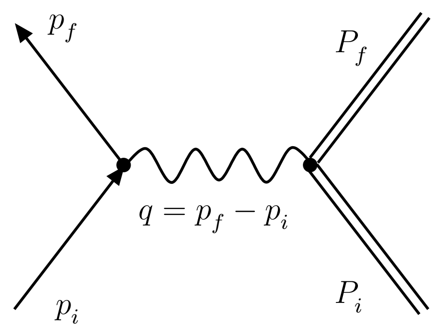


Figure 5.1: Lowest order electron-proton scattering.

The integration over q is then readily done:

$$\begin{aligned}&\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_f - p_i - q)(2\pi)^4\delta^4(P_f - P_i - q)\left[-\frac{4\pi}{q^2 + i\epsilon}\right]\\&= (2\pi)^4\delta^4(P_f - P_i + p_f - p_i)\left[-\frac{4\pi}{(p_f - p_i)^2 + i\epsilon}\right]\end{aligned} \tag{5.15}$$

5.4 Remarks on the Form of the S-matrix Element

Here we display properties of S-matrix element that are a first step toward "deriving" the Feynmann rules for QED. The total S-matrix element the reads

$$S_{fi} = i(2\pi)^4\delta^4(P_f - P_i + p_f - p_i)\ M_{fi}\ \frac{1}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E_f^p E_i^p}} \tag{5.16}$$

where

$$M_{fi} = [\overline{u}(p_f, s_f)(-ie\gamma_\mu)u(p_i, s_i)]\frac{-4\pi\ \eta^{\mu\nu}}{(p_f - p_i)^2 + i\epsilon}[\overline{u}(P_f, S_f)(-ie_p\gamma_\mu)u(P_i, S_i)] \tag{5.17}$$

This describes the lowest order contribution. This is put in diagrammatic form in fig (**??**). The wavy line represents the virtual photon being exchanged between the electron and proton. The four momentum of the photon is

$$q = p_f - p_i = P_f - P_i \tag{5.18}$$

• The following factor represents the amplitude for the propagation of a photon with momentum q:

$$\frac{-4\pi\ \eta^{\mu\nu}}{q^2 + i\epsilon} \tag{5.19}$$

• There is a factor of $-ie\gamma_\mu$ for every vertex.

• These act between spinors $u(p,s)$ describing the free ingoing and outgoing Dirac particles.

• There is a four dimensional $\delta-$function, ensuring conservation of total energy and momentum in the scattering process.

5.5 The Scattering Cross Section

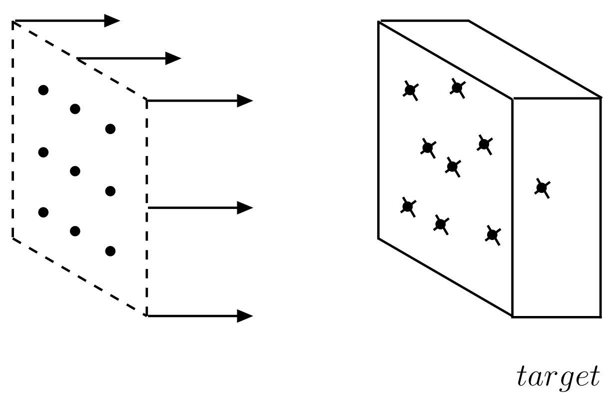


Figure 5.2: .

We divide $|S_{fi}|^2$ by the time interval and the space volume of the reaction (Dirac waves normalised so that there is one particle per unit volume)

$$W_{fi} = \frac{|S_{fi}|^2}{VT} \tag{5.20}$$

Now we come to calculating the cross section. As in section we have to consider the square of the δ^4-function

$$(2\pi)^4\delta^4\left(p_f + P_f - p_i - P_i\right) = \lim_{T\to\infty, V\to\infty} \int_{-T/2}^{T/2}\int_V d^3\mathbf{x} \exp\left[ix\cdot\left(p_f + P_f - p_i - P_i\right)\right] \tag{5.21}$$

$$\left[(2\pi)^4\delta^4\left(p_f + P_f - p_i - P_i\right)\right]^2 \rightarrow TV(2\pi)^4\delta^4\left(p_f + P_f - p_i - P_i\right) \tag{5.22}$$

To obtain the transition rate to a group of final states with momenta in the intervals $f = 1, 2$, we multiply by the number of these states which is

$$\frac{Vd^3\mathbf{P}_1}{(2\pi)^3}\frac{Vd^3\mathbf{P}_2}{(2\pi)^3} \tag{5.23}$$

$$\text{number of target particles per unit volume} = 1/V$$

Combining these results with (), we obtain the required formula for the differential cross section

$$\begin{aligned} d\sigma &= V^2\frac{d^3\mathbf{p}_f}{(2\pi)^3}\frac{d^3\mathbf{P}_f}{(2\pi)^3}\frac{1}{|\mathbf{J_{inc}}|}\frac{1}{1/V}W_{fi} \\ &= \end{aligned} \tag{5.24}$$

5.6 Lorentz Invariance

Each particle leaving the scattering process contributes a factor

$$\frac{m_0}{E_f}\frac{d^3p_f}{(2\pi)^3} \tag{5.25}$$

to the cross section. Consider

$$\begin{aligned} \int_{-\infty}^{\infty} d^4p\,\delta(p^2 - m_0^2)\Theta(p_0) &= \int_0^{\infty} dp_0\,\delta(p_0^2 - \mathbf{p}^2 - m_0^2)\,d^3p \\ &= \int_0^{\infty} dp_0\,\delta(p_0^2 - E^2)\,d^3p \\ &= \int_0^{\infty} dp_0\,\delta[(p_0 - E)(p_0 + E)]\,d^3p \\ &= \int_0^{\infty} dp_0\,\delta[2E(p_0 - E)]\,d^3p \\ &= \frac{d^3p}{2E} \end{aligned}$$

where

$$\Theta(p_0) = \begin{cases} 1 & \text{for } p_0 > 0 \\ 0 & \text{for } p_0 < 0 \end{cases}$$

This step function is obviously Lorentz invariant since Lorentz transformations always transform timelike four vectors into timelike four vectors. Thus we have established that $d^3p/2E$ is a Lorentz-invariant factor.

Now we consider the factor

$$\frac{m_0}{E_i}\frac{M_0}{E_i^P}\frac{1}{|\mathbf{J}_{inc}|V}.$$

$$|\mathbf{J}_{inc}| = \frac{1}{V}|\mathbf{v}_i - \mathbf{V}_i| \tag{5.26}$$

$$\mathbf{v}_i = \frac{\mathbf{p}_i}{E_i}, \quad \mathbf{V}_i = \frac{\mathbf{P}_i}{E_i^P}$$

This gives

$$\begin{aligned} \frac{m_0}{E_i}\frac{M_0}{E_i^P}\frac{1}{|\mathbf{J}_{inc}|V} &= \frac{m_0 M_0}{E_i E_i^P |\mathbf{v}_i - \mathbf{V}_i|} \\ &= \frac{m_0 M_0}{E_i E_i^P \sqrt{\mathbf{v}_i^2 + \mathbf{V}_i - 2\mathbf{v}_i \cdot \mathbf{V}_i}} \\ &= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + \mathbf{P}_i^2 E_i^2 - 2\mathbf{p}_i \cdot \mathbf{P}_i E_i E_i^P}} \end{aligned} \tag{5.27}$$

We prove that for collinear collisions that this is equivalent to the Lorentz invariant scalar

$$\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}},$$

because

$$
\begin{aligned}
\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} &= \frac{m_0 M_0}{\sqrt{(E_i E_i^P - \mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{E_i^2 E_i^{P2} - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{(m_0^2 + \mathbf{p}_i^2)(M_0^2 + \mathbf{P}_i^2) - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2 - m_0^2 M_0^2}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + m_0^2 \mathbf{P}_i^2 - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + (\mathbf{p}_i \cdot \mathbf{P}_i)^2}} \qquad (5.28)
\end{aligned}
$$

As the velocity vectors are collinear we have that $(\mathbf{p}_i \cdot \mathbf{P}_i)^2 = \mathbf{p}_i^2 \mathbf{P}_i^2$.

$$
\begin{aligned}
\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} &= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + m_0^2 \mathbf{P}_i^2 - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i + \mathbf{p}_i^2 \mathbf{P}_i^2}} \\
&= \frac{m_0 M_0}{\sqrt{\mathbf{p}_i^2 E_i^{P2} + \mathbf{P}_i^2 E_i^2 - 2E_i E_i^P \ \mathbf{p}_i \cdot \mathbf{P}_i}} \qquad (5.29)
\end{aligned}
$$

We can use this Lorentz-invariant flux factor to write the cross section in a invariant form

$$
d\sigma = \frac{m_0 M_0}{\sqrt{(p_i \cdot P_i)^2 - m_0^2 M_0^2}} \ |M_{fi}|^2 \ (2\pi)^4 \delta^4(P_f - P_i + p_f - p_i) \ \frac{m_0 d^3 p_f}{(2\pi)^3 E_f} \ \frac{M_0 d^3 p_f}{(2\pi)^3 E_f^P}. \qquad (5.30)
$$

5.7 Averaging over Spin

The squared invariant matrix element averaged over initial and final spin is

$$
\overline{|M_{fi}|^2} = \frac{1}{4} \sum_{S_f, S_i, s_f, s_i} \left| \overline{u}(p_f, s_f) \gamma^\mu u(p_i, s_i) \ \frac{e e_p (4\pi)}{q^2 + i\epsilon} \ \overline{u}(P_f, S_f) \gamma_\mu u(P_i, S_i) \right|^2 \qquad (5.31)
$$

$$
\begin{aligned}
&\sum_{S_f,S_i,s_f,s_i} \left| [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(P_f,S_f)\gamma_\mu u(P_i,S_i)] \right|^2 \\
= &\sum_{S_f,S_i,s_f,s_i} [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(P_f,S_f)\gamma_\mu u(P_i,S_i)] \\
&[\overline{u}(p_f,s_f)\gamma^\nu u(p_i,s_i)]^*\ [\overline{u}(P_f,S_f)\gamma_\nu u(P_i,S_i)]^* \\
= &\sum_{s_f,s_i} [\overline{u}(p_f,s_f)\gamma^\mu u(p_i,s_i)]\ [\overline{u}(p_f,s_f)\gamma^\nu u(p_i,s_i)]^* \\
&\sum_{S_f,S_i} [\overline{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)]\ [\overline{u}(P_f,S_f)\gamma^\mu u(P_i,S_i)]^*
\end{aligned} \tag{5.32}
$$

At this point the reader should go throught section D. The answer according to (D.5) is

$$
Tr\left[\frac{p_{f\alpha}\gamma^\alpha + m_0}{2m_0}\ \gamma^\mu\ \frac{p_{i\beta}\gamma^\beta + m_0}{2m_0}\ \gamma^\nu\right]\ Tr\left[\frac{P_{f\gamma}\gamma^\gamma + M_0}{2M_0}\ \gamma_\mu\ \frac{P_{i\delta}\gamma^\delta + M_0}{2M_0}\ \gamma_\nu\right] \tag{5.33}
$$

$$
\overline{|M_{fi}|^2} = \frac{e^2 e_p^2 (4\pi)^2}{q^4} L^{\mu\nu} H_{\mu\nu} \tag{5.34}
$$

where we have introduced the lepton tensor $L^{\mu\nu}$ and the hadron tensor $H_{\mu\nu}$, defined as

$$
L^{\mu\nu} = Tr\left[\frac{p_{f\alpha}\gamma^\alpha + m_0}{2m_0}\ \gamma^\mu\ \frac{p_{i\beta}\gamma^\beta + m_0}{2m_0}\ \gamma^\nu\right] \tag{5.35}
$$

and

$$
H_{\mu\nu} = Tr\left[\frac{P_{f\gamma}\gamma^\gamma + M_0}{2M_0}\ \gamma_\mu\ \frac{P_{i\delta}\gamma^\delta + M_0}{2M_0}\ \gamma_\nu\right] \tag{5.36}
$$

Using methods already introduced in the previous section on Coulomb scattering, we can easily evaluate the trace in the lepton tensor $L^{\mu\nu}$ to obtain:

$$
L^{\mu\nu} = \frac{1}{2}\frac{1}{m_0^2}\left[p_f^\mu p_i^\nu + p_i^\mu p_f^\nu - \eta^{\mu\nu}(p_f \cdot p_i - m_0^2)\right] \tag{5.37}
$$

The Hadron trace has the same structure, we just replace small letters by capitals and lower the spacetime indices.

5.8 Differential Cross Section in Rest Frame of Proton

The calculation of $\overline{|M_{fi}|^2}$ which we leave to the reader results in

$$\begin{aligned}\overline{|M_{fi}|^2} &= \frac{e^2 e_p^2 (4\pi)^2}{4m_0^2 M_0^2 (q^2)^2}\Big[(p_i \cdot P_i)(p_f \cdot P_f) + (p_i \cdot P_f)(p_f \cdot P_i) \\ &\quad -(p_i \cdot p_f)M_0^2 - (P_i \cdot P_f)m_0^2 + 2m_0^2 M_0^2\Big] . \end{aligned} \tag{5.38}$$

Let us work in the rest frame of the proton. We define

$$\begin{aligned} p_f &= (E', \mathbf{p}') =: p' \\ p_f &= (E, \mathbf{p}) =: p \\ P_i &= (M_0, 0) \end{aligned} \tag{5.39}$$

We calculate the differential cross section for electron scattering into a solid angle $d\Omega'$ centered around the scattering angle θ. Thus we will integrate the differential cross section over all momentum variable except for the direction of $\mathbf{p_f}$. First we will want to write down the spin averaged differential cross section in the proton rest system. The invariant flux factor reduces to

$$\frac{m_0 M_0}{\sqrt{(p_i \cdot P_i) - m_0^2 M_0^2}} = \frac{m_0 M_0}{\sqrt{E^2 M_0^2 - m_0^2 M_0^2}} = \frac{m_0}{|\mathbf{p}|} \tag{5.40}$$

We will use

$$\frac{M_0 d^3 P_f}{(2\pi)^3 E_f^p} = \frac{2M_0}{(2\pi)^3}\int_{-\infty}^{\infty} d^4 P_f \; \delta(P_f^2 - M_0^2)\Theta(P_f^0).$$

The spin averaged differential cross section $d\overline{\sigma}$ is then

$$\begin{aligned} d\overline{\sigma} &= \frac{m_0}{|\mathbf{p}|}\,\overline{|M_{fi}|^2}(2\pi)^4\delta^4(P_f + p' - P_i - p) \\ &\quad \times \frac{m_0}{(2\pi)^3}|\mathbf{p}'|dE' d\Omega' \times \frac{2M_0}{(2\pi)^3}\int_{-\infty}^{\infty} d^4 P_f \; \delta(P_f^2 - M_0^2)\Theta(P_f^0) \end{aligned} \tag{5.41}$$

Now integrating over dE' and $d^3 P_f$ we obtain

$$
\begin{aligned}
\frac{d\overline{\sigma}}{d\Omega'} &= \int dE'|\mathbf{p}'|E' \, (d\overline{\sigma}) \\
&= \frac{m_0^2 M_0}{|\mathbf{p}|2\pi^2} \int dE'|\mathbf{p}'| \, \overline{|M_{fi}|^2} \delta\left((p' - P_i - p)^2 - M_0^2\right) \Theta(P_i^0 + E - E') \\
&= \frac{m_0^2 M_0}{|\mathbf{p}|2\pi^2} \int_{m_0}^{M_0+E} dE'|\mathbf{p}'| \, \overline{|M_{fi}|^2} \delta(2m_0^2 - 2(E' - E) - 2E'E2|\mathbf{p}||\mathbf{p}'|\cos\theta)
\end{aligned}
\tag{5.42}
$$

where the upper limit in the integral comes from the step function and the lower from the fact that E' cannot be less than m_0. Furthermore, the argument of the delta function has been expressed in terms of the kinematical variables of the laboratory frame. The remaining integral over E' can be performed using

$$
\delta\left(f(x)\right) = \sum_k \frac{\delta(x - x_k)}{|\frac{df}{dx}|_{x_k}}
$$

x_i being the roots of $f(x)$ inside the range of integration. We get

$$
\frac{d\overline{\sigma}}{d\Omega'} = \frac{m_0^2 M_0}{4\pi^2} \frac{|\mathbf{p}'|}{|\mathbf{p}|} \frac{\overline{|M_{fi}|^2}}{M_0 + E - |\mathbf{p}|(E'/|\mathbf{p}'|)\cos\theta} \tag{5.43}
$$

where we have used $d|\mathbf{p}'|/dE' = E'/|\mathbf{p}'|$ and we have for E'

$$
E'(M_0 + E) - |\mathbf{p}||\mathbf{p}'|\cos\theta = EM_0 + m_0^2 \tag{5.44}
$$

For given scattering angle θ the final energy E' of the electrons can be determined as a function of E and θ. The resulting E' and the corresponding $|\mathbf{p}'| = E'^2 - m_0^2$ have to be inserted into (5.43).

Chapter 6

Scattering of Identical Fermions

We can take over many aspects of electron-proton scattering. But now because the two particles are of the same type there is no way to tell which of the two emerging electrons was the "incident" and which was the "target" particle. This is taken into account by adding the amplitudes for both processes but incuding a change of sign since we are exchanging two identical fermions. The resulting total amplitude is then

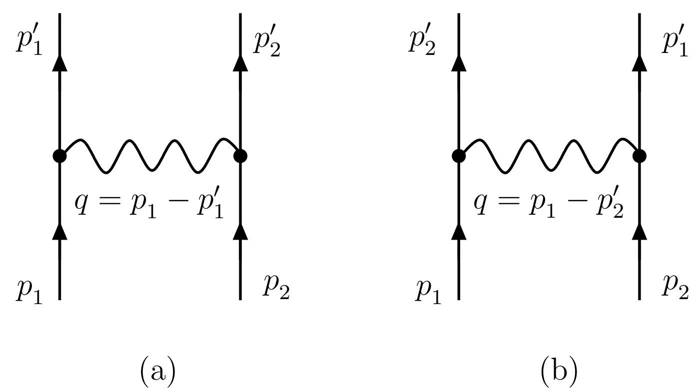


Figure 6.1:

$$
\begin{aligned}
S_{fi} \quad = \quad & -\frac{1}{2}\sqrt{\frac{m_0^2}{E_1 E_2}}\sqrt{\frac{m_0^2}{E_1' E_2'}}(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2') \\
& \times \left\{ + \, [\overline{u}(p_1', s_1')(-ie\gamma_\mu)u(p_1, s_1)]\frac{i4\pi}{(p_1 - p_1')^2 + i\epsilon}[\overline{u}(p_2', s_2')(-ie\gamma_\mu)u(p_2, s_2)] \right. \\
& \left. - \, [\overline{u}(p_2', s_2')(-ie\gamma_\mu)u(p_1, s_1)]\frac{i4\pi}{(p_1 - p_2')^2 + i\epsilon}[\overline{u}(p_1', s_1')(-ie\gamma_\mu)u(p_2, s_2)] \right\}
\end{aligned} \tag{6.1}
$$

6.1 Averaging over Spin

The squared invariant matrix element averaged over initial and final spin is

$$
\begin{aligned}
&\overline{|M_{fi}|^2} \\
= \;& e^4(4\pi)^2\frac{1}{4}\sum_{s_1',s_1}\sum_{s_2',s_2}\Bigg|\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\frac{1}{(p_1-p_1')^2}\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2) \\
&- \overline{u}(p_2',s_2')\gamma_\mu u(p_1,s_1)\frac{1}{(p_1-p_2')^2}\overline{u}(p_1',s_1')\gamma^\mu u(p_2,s_2)\Bigg|^2
\end{aligned}
\tag{6.2}
$$

We gain familiarity with calculating spin averaging, the reader will be left to put together the results to obtain the final answer. Consider the mod-squared terms in the square $|\cdots|^2$. Take the first such term:

$$
\begin{aligned}
&\sum_{s_1',s_1}\sum_{s_2',s_2}\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right) \\
&\quad\times\left(\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right)^*\,\left(\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right)^* \\
= &\left[\sum_{s_1',s_1}\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_1',s_1')\gamma_\nu u(p_1,s_1)\right)^*\right] \\
&\quad\times\left[\sum_{s_2',s_2}\left(\overline{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\right)\,\left(\overline{u}(p_2',s_2')\gamma^\nu u(p_2,s_2)\right)^*\right]
\end{aligned}
\tag{6.3}
$$

It is sufficient to consider just the first sum over s_1', s_1, since the second term s_2', s_2 has the same structure. The reader should see section D on how to turn this into the following trace

$$
Tr\left[\gamma_\mu\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu\frac{p_{1\beta}'\gamma^\beta+m_0}{2m_0}\right].
\tag{6.4}
$$

Now we consider the more complicated mixed terms in the square $|\cdots|^2$. The first of these is

$$
\begin{aligned}
&\sum_{s_1',s_1}\sum_{s_2',s_2}\left[\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_2',s_2')\gamma^\mu u(p_2,s_2)\right)\right] \\
&\quad\times\left[\left(\overline{u}(p_2',s_2')\gamma_\nu u(p_1,s_1)\right)\,\left(\overline{u}(p_1',s_1')\gamma^\nu u(p_2,s_2)\right)\right]^* \\
&\sum_{s_1',s_1}\sum_{s_2',s_2}\left(\overline{u}(p_1',s_1')\gamma_\mu u(p_1,s_1)\right)\,\left(\overline{u}(p_1,s_1)\gamma_\nu u(p_2',s_2')\right) \\
&\quad\times\left(\overline{u}(p_2',s_2')\gamma_\mu u(p_2,s_2)\right)\,\left(\overline{u}(p_2,s_2)\gamma_\nu u(p_1',s_1')\right) \\
= \;&\sum_{s_1'}\sum_{s_2'}\left(\overline{u}(p_1',s_1')\gamma_\mu\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu u(p_2',s_2')\right) \\
&\quad\times\left(\overline{u}(p_2',s_2')\gamma_\mu\frac{p_{2\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu u(p_1',s_1')\right)
\end{aligned} \tag{6.5}
$$

where we have used the identity

$$
\sum_{s_1} u_\alpha(p_1,s_1)\overline{u}_\beta(p_1,s_1) = \left(\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\right)_{\alpha\beta}
$$

We use this identity another two times to obtain the final result

$$
Tr\left[\gamma_\mu\frac{p_{1\alpha}\gamma^\alpha+m_0}{2m_0}\gamma_\nu\frac{p_{2\beta}'\gamma^\beta+m_0}{2m_0}\gamma^\mu\frac{p_{2\gamma}\gamma^\gamma+m_0}{2m_0}\gamma^\nu\frac{p_{1\delta}'\gamma^\delta+m_0}{2m_0}\right]. \tag{6.6}
$$

In section A.1 we provide theorems on the trace of $\gamma-$matrices sufficient for the reader to evaluate the above expresions.

Chapter 7

Electron-Positron Scattering

7.1 Scattering of an Positron off a Coulomb Potential

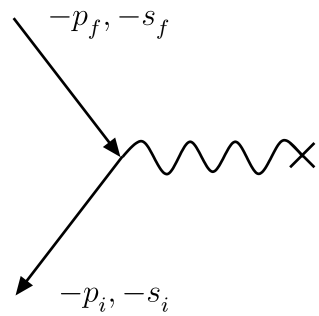


Figure 7.1: The incoming positron with momentum p_i and spin s_i is described by an outgoing electron with negative energy, with momentum $-p_i$ and spin $-s_i$. Similarly for the outgoing positron.

$$S_{fi} = -iZe^2 \frac{1}{V}\sqrt{\frac{m_0^2}{E_f E_i}}\overline{v}(p_i, s_i)\gamma^0 v(p_f, s_f) \int d^4x\, e^{i(p_f - p_i)\cdot x}\frac{1}{|x|}. \tag{7.1}$$

7.2 Electron-Positron Scattering Amplitude

Make the replacements

an incoming electron spinor $u(p_i, s_i) \rightarrow$ an outgoing positron spinor $\overline{v}(p_f, s_f)$

$$
\begin{aligned}
S_{fi}(dir.) \;\; = \;\; & -\frac{1}{2}\sqrt{\frac{m_0^2}{E_1\overline{E}_2'}}\sqrt{\frac{m_0^2}{E_1'\overline{E}_2}}(2\pi)^4\delta^4(p_1+\overline{p}_2'-p_1'-\overline{p}_2) \\
& \times [\overline{u}(p_1',s_1')(-ie\gamma_\mu)u(p_1,s_1)]\frac{i4\pi}{(p_1-p_1')^2+i\epsilon}[\overline{v}(\overline{p}_2,\overline{s}_2)(-i(-e)\gamma_\mu)v(p_2',s_2')]
\end{aligned} \tag{7.2}
$$

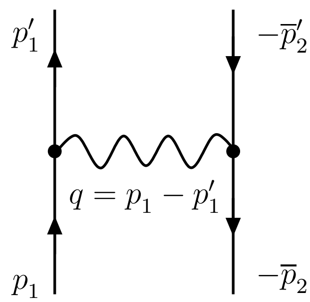


Figure 7.2:

The exchange amplitude

$$
\begin{aligned}
S_{fi}(exch.) \;\; = \;\; & \frac{1}{2}\sqrt{\frac{m_0^2}{E_1E_2}}\sqrt{\frac{m_0^2}{E_1'E_2'}}(2\pi)^4\delta^4(p_1+\overline{p}_2'-p_1'-\overline{p}_2) \\
& \times [\overline{v}(\overline{p}_2,\overline{s}_2)(-i(-e)\gamma_\mu)u(p_1,s_1)]\frac{i4\pi}{(p_1-p_1')^2+i\epsilon}[\overline{u}(p_1',s_1')(-ie\gamma_\mu)u(\overline{p}_2',\overline{s}_2')]
\end{aligned} \tag{7.3}
$$

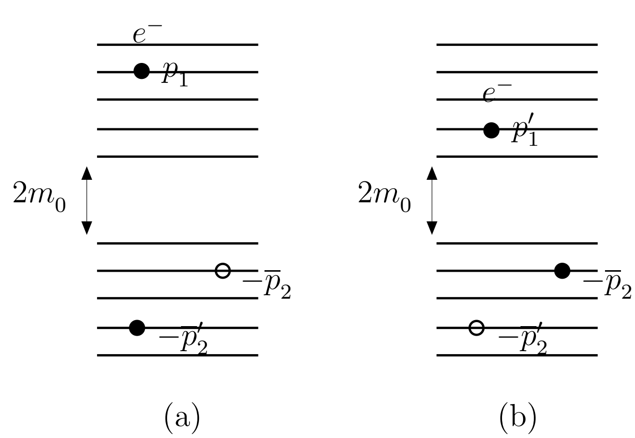


Figure 7.3: (a) The intial state. (b) The final state.

The exclusion principle requires that antisymmetric combinations of amplitudes be chosen for processes which differ only by an exchange of particles. In the final state, fig (**??**), has to be antisymmetric with respect to the exchange $p_1' \leftrightarrow -\overline{p}_2$

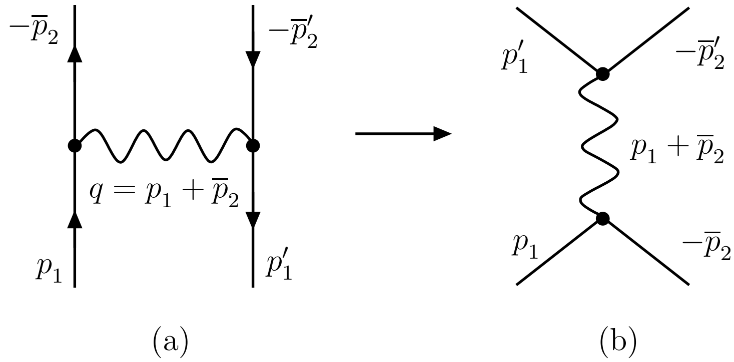


Figure 7.4: (a) . (b) The exchange graph is usually written this way.

7.3 Remarks on the Form of the $S-$Matrix element

7.4 Crossing Symmetry

The squared invariant matrix element for electron-positron scattering can be obtained from the squared invariant matrix element for electron-electron scattering by making the following substitutions of four-momenta

$$\begin{aligned} p_1 &\rightarrow p_1 \\ p_1' &\rightarrow p_1' \\ p_2 &\rightarrow -\overline{p}_2' \\ p_2' &\rightarrow -\overline{p}_2. \end{aligned} \tag{7.4}$$

Chapter 8

Scattering of Polarised Dirac Particles

s^μ is a Lorentz vector which is properly defined in the rest system of the particle where it reduces to a spacial unit vector

$$(s^\mu)_{RS} = (0, \mathbf{s}'). \tag{8.1}$$

We wish to obtain the components of s^μ in a frame in which the particle moves with momentum $\mathbf{p}$. What is the Lorentz transformation formula os an aritrary four-vecto a^μ for when $\mathbf{v}$ is not parallel to the x-axis? Consider

$$a^{0'} = \gamma\left(a^0 - \mathbf{v}\cdot\mathbf{a}\right), \qquad \mathbf{a}' = \mathbf{a} + \left(\frac{\mathbf{v}\cdot\mathbf{a}}{v^2}(\gamma-1) - \gamma a^0\right)\mathbf{v} \tag{8.2}$$

where

$$\gamma = \frac{1}{\sqrt{1-\mathbf{v}^2}}.$$

Specialising to $\mathbf{v} = (v, 0, 0)$ we find

$$a^{0'} = \gamma\left(a^0 - v a_x\right), \tag{8.3}$$

$$(a'_x, a'_y, a'_z) = (a_x, a_y, a_z) - \left(\frac{\mathbf{v}\cdot a}{v^2}(1-\gamma) - \gamma a_t\right)(v, 0, 0) \tag{8.4}$$

which reads separately

$$a'_t = \gamma(a_t - va_x), \quad a'_x = \gamma(a_x - va_t), \quad a'_y = a_y, \quad a'_z = a_z \tag{8.5}$$

We want the inverse of this transformation which is easily obtained by making the replacement $\mathbf{v} \to -\mathbf{v}$. We have that $s^{0'} = 0$. We obtain for s^μ,

$$s^\mu = \left[\gamma\mathbf{v}\cdot\mathbf{s}' \; , \; \mathbf{s}' + \frac{\mathbf{v}\cdot\mathbf{s}'}{v^2}(\gamma - 1)\mathbf{v}\right] \tag{8.6}$$

We use the following:

$$\mathbf{v} = \frac{\mathbf{p}}{E}, \tag{8.7}$$

$$E/\gamma = m_0, \tag{8.8}$$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \mathbf{p}^2/E^2}} \\ &= \frac{E}{\sqrt{E^2 - \mathbf{p}^2}} = \frac{E}{m_0}, \end{aligned} \tag{8.9}$$

$$\begin{aligned} \frac{\gamma - 1}{E^2 - \mathbf{p}^2} &= \frac{E/m_0 - 1}{(E + m_0)(E - m_0)} \\ &= \frac{1}{m_0(E + m_0)}, \end{aligned} \tag{8.10}$$

so that () finally becomes

$$s^\mu = \left[\frac{\mathbf{p}\cdot\mathbf{s}'}{m_0} \; , \; \mathbf{s}' + \frac{\mathbf{p}\cdot\mathbf{s}'}{m_0(E + m_0)}\mathbf{p}\right] \tag{8.11}$$

Because of the Lorentz invariance of the four-diemnsional scalar product it follows

$$s_\mu s^\mu = (s_\mu)_{RS}(s^\mu)_{RS} = -\mathbf{s}\cdot\mathbf{s} = -1 \tag{8.12}$$

and

$$p^\mu s_\mu = (p^\mu)_{RS}(s_\mu)_{RS} = (m_0, 0, 0, 0)\begin{pmatrix} 0 \\ -s_x \\ -s_y \\ -s_z \end{pmatrix} = 0. \tag{8.13}$$

So we have the normalisation and orthogonality relations

$$s^2 = -1, \quad p \cdot s = 0. \tag{8.14}$$

We will specialise to helicity states, namely states where the spin points in the direction (or opposite direction) of the momentum:

$$s'_\lambda = \lambda \frac{\mathbf{p}}{|\mathbf{p}|}, \quad \text{where} \quad \lambda = \pm 1. \tag{8.15}$$

Substituting this into (8.11) leads to the spin four-vector

$$s^\mu = \lambda \left(\frac{|\mathbf{p}|}{m_0}\, , \, \frac{\mathbf{p}}{|\mathbf{p}|} + \frac{\mathbf{p}^2}{m_0(E+m_0)} \frac{\mathbf{p}}{|\mathbf{p}|} \right) = \lambda \left(\frac{|\mathbf{p}|}{m_0}\, , \, \frac{E}{m_0} \frac{\mathbf{p}}{|\mathbf{p}|} \right) \tag{8.16}$$

After this preliminar work we now look the cross section for Coulomb scattering.

8.1 Polarised Electron Scattering of a Coulomb Potential

$$\frac{d\sigma}{d\Omega}(s_i, s_f) = \frac{4Z^4\alpha^2 m_0^2}{|q|^4} |\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)|^2 \tag{8.17}$$

We introduce auxialiary summations over the spin orientations s_i and s_f using the spin projection operator $\mathbf{\Sigma}(s)$ which suppresses the "wrong" spin state $u(p, -s)$.

$$\begin{aligned} \frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4} \left(\overline{u}(p_f, s_f)\gamma^0 u(p_i, s_i)\right) \left(u^\dagger(p_i, s_i)\gamma_0^\dagger \gamma_0^\dagger u(p_f, s_f)\right) \\ &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4} \sum_{s'_i, s'_f} \left(\overline{u}(p_f, s'_f)\gamma^0 \hat{\Sigma}(s_i) u(p_i, s'_i)\right) \left(\overline{u}(p_i, s'_i)\gamma^0 \hat{\Sigma}(s_f) u(p_f, s'_f)\right) \end{aligned} \tag{8.18}$$

The same calculation as in (4.29) but with the replacement $\gamma^0_{\alpha\beta} \to \sum_{\delta=1}^4 \gamma^0_{\alpha\delta}(\hat{\Sigma})_{\delta\beta}$

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(s_i, s_f) &= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0 \hat{\Sigma}(s_i)\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\hat{\Sigma}(s_f)\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \\
&= \frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0 \frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\frac{1+\gamma_5 s_f^\delta\gamma_\delta}{2}\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right]
\end{aligned}
\tag{8.19}
$$

8.2 When the Incoming Beam is Unpolarised

Before examinig the above we warm up by looking at when the scattering process in which the incoming beam is unpolarised, and ask if polarisation of the scaterred particle takes place. The above cross section is then replaced by

$$
\frac{d\sigma}{d\Omega}(s_i, s_f) = \frac{1}{2}\frac{4Z^4\alpha^2 m_0^2}{|q|^4}\, Tr\left[\gamma_0 \frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu + m_0}{2m_0}\gamma_0\frac{(p_f)_\nu\gamma^\nu + m_0}{2m_0}\right] \tag{8.20}
$$

The factor 1/2 comes from averaging over the initial spins.

Expanding (8.20) we find the traces

$$
Tr\gamma_0\gamma_5\gamma_\sigma\gamma_0, \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_\mu\gamma_0\gamma_\nu, \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_\mu\gamma_0, \quad \text{and} \quad Tr\gamma_0\gamma_5\gamma_\sigma\gamma_0\gamma_\nu.
$$

It is easily seen this reduces to evaluating

$$
Tr\gamma_5\gamma_\sigma, \quad Tr\gamma_5\gamma_\sigma\gamma_\mu\gamma_\nu, \quad \text{and} \quad Tr\gamma_5\gamma_\sigma\gamma_\mu. \tag{8.21}
$$

because of the cyclic property of the trace, $(\gamma_0)^2 = \mathbf{1}$, and $\gamma_0\gamma_\nu = -\gamma_\nu\gamma_0$ (for $\mu \neq 0$). The first two of (8.21) vanish as the trace of the product of γ_5 and an odd number of γ matrices is zero:

$$
Tr\gamma_5\gamma_\mu \dots \gamma_\nu = (-1)^n Tr\gamma_\mu \dots \gamma_\nu\gamma_5 = (-1)^n Tr\gamma_5\gamma_\mu \dots \gamma_\nu
$$

where we first used $\gamma_5\gamma_\mu = \gamma_\mu\gamma_5$ and then the cyclic property of the trace. We consider the last trace in (8.21). Say first that $\mu = \nu$ then

$$
Tr\gamma_5\gamma_\mu\gamma_\nu = Tr\gamma_5(\gamma_\mu)^2 = \eta^{\mu\mu}Tr\gamma_5. \tag{8.22}
$$

As $\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0$ and thus in particular $\gamma^0\gamma^5 = -\gamma^5\gamma^0$ we get

$$
\begin{aligned}
Tr\gamma_5 &= Tr\gamma_5(\gamma^0)^2 \\
&= Tr\gamma^0\gamma_5\gamma^0 \\
&= -Tr\gamma_5(\gamma^0)^2 \\
&= -Tr\gamma_5 = 0.
\end{aligned}
\tag{8.23}
$$

where in the second line we have used the cyclic property of the trace and then in the next line swapped γ^0 and γ_5. Now if $\mu \neq \nu$ we choose λ that differs from μ and ν and use $\gamma_\lambda\gamma_5\gamma_\mu\gamma_\nu = (-1)^3\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda$

$$
\begin{aligned}
Tr\gamma_5\gamma_\mu\gamma_\nu &= Tr\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1}\gamma_\lambda \\
&= Tr\gamma_\lambda\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda^{-1} \\
&= (-1)^3 Tr\gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\lambda^{-1} \\
&= -Tr\gamma_5\gamma_\mu\gamma_\nu = 0.
\end{aligned}
\tag{8.24}
$$

Thus the cross section is independent of the final spin and agrees with half the unpolarised Mott scattering cross section

$$
\frac{d\sigma}{d\Omega}(s_f) = \frac{1}{2}\frac{d\sigma_{Mott}}{d\Omega} \tag{8.25}
$$

Thus at first order in perturbatin theory Coulomb scattering of electrons does not lead to polarisation of the incoming beam.

8.3 Polarised Scattering

We assume that the spin of the incoming electron is parallel to its direction of motion, i.e. it has well defined helicity $\lambda_i = +1$

$$
\begin{aligned}
s_{i\lambda_i} &= \lambda_i\left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|}\right) \equiv \lambda_i s_i \\
s_{f\lambda_f} &= \lambda_i\left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|}\right) \equiv \lambda_f s_f
\end{aligned}
\tag{8.26}
$$

Dropping terms we know to zanish from the previous example, the polarised scattering cross section becomes

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(s_i,s_f) &= \frac{4Z^4\alpha^2m_0^2}{|q|^4}\,Tr\left[\gamma_0\frac{1+\gamma_5 s_i^\sigma\gamma_\sigma}{2}\frac{(p_i)_\mu\gamma^\mu+m_0}{2m_0}\gamma_0\frac{1+\gamma_5 s_f^\mu\gamma_\mu}{2}\frac{(p_f)_\nu\gamma^\nu+m_0}{2m_0}\right]\\
&= \frac{4Z^4\alpha^2m_0^2}{|q|^4}\frac{1}{4}\frac{1}{(2m_0)^2}\Big(Tr[\gamma_0((p_i)_\mu\gamma^\mu+m_0)\gamma_0((p_f)_\nu\gamma^\nu+m_0)]\\
&\quad +\lambda_i\lambda_f\,Tr[\gamma_0\gamma_5 s_i^\sigma\gamma_\sigma((p_i)_\mu\gamma^\mu+m_0)\gamma_0\gamma_5 s_f^\delta\gamma_\delta((p_f)_\nu\gamma^\nu+m_0)]\Big)
\end{aligned}
\tag{8.27}
$$

Here we define the degree of polarisation P of the scattered particles by as the difference between counting rates for the positive and negative helicities, normalised by the total counting rate:

$$
P=\frac{d\sigma(\lambda_f=+1)-d\sigma(\lambda_f=-1)}{d\sigma(\lambda_f=+1)+d\sigma(\lambda_f=-1)} \tag{8.28}
$$

If the initial state is fully polarised, e.g. $\lambda_i=+1$, the final degree of polarisation becomes, using ()

$$
P=\frac{Tr[\gamma_0\gamma_5 s_i^\sigma\gamma_\sigma((p_i)_\mu\gamma^\mu+m_0)\gamma_0\gamma_5 s_f^\mu\gamma_\mu((p_f)_\nu\gamma^\nu+m_0)]}{Tr[\gamma_0((p_i)_\mu\gamma^\mu+m_0)\gamma_0((p_f)_\nu\gamma^\nu+m_0)]} \tag{8.29}
$$

The evaluation of the trace in the denominator is done along the same lines as early calculations. Expand the numerator, using that the trace of an odd number of γ matrices vanishes,

$$
\begin{aligned}
& Tr[\gamma_0\gamma_5 s_i^\sigma\gamma_\sigma((p_i)^\mu\gamma^\mu+m_0)\gamma_0\gamma_5 s_f^\mu\gamma_\mu((p_f)^\nu\gamma^\nu+m_0)\\
=\;& s_i^\sigma p_i^\mu s_f^\delta p_f^\nu\,Tr[\gamma_0\gamma_\sigma\gamma_\mu\gamma_0\gamma_\delta\gamma_\nu]+m_0^2 s_i^\sigma s_f^\delta\,Tr[\gamma_0\gamma_\sigma\gamma_0\gamma_\delta]
\end{aligned}
\tag{8.30}
$$

To evaluate the first trace we generalise the result of () to arbitrary even number of γ matrices. We know

$$
Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n}=2\eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n}-Tr\gamma^{\mu_2}\gamma^{\mu_1}\gamma^{\mu_3}\ldots\gamma^{\mu_n}
$$

Repeating this procedure we get

$$
\begin{aligned}
Tr\gamma^{\mu_1}\ldots\gamma^{\mu_n} &= 2\eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\ldots\gamma^{\mu_n}-\\
&\quad +2\eta^{\mu_1\mu_n}Tr\gamma^{\mu_2}\ldots\gamma^{\mu_n}\\
&\quad -Tr\gamma^{\mu_2}\ldots\gamma^{\mu_n}\gamma^{\mu_1}
\end{aligned}
$$

Using the cyclic property of traces we get

$$Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} = \eta^{\mu_1\mu_2}\, Tr\gamma^{\mu_3}\dots\gamma^{\mu_n} - \dots + \eta^{\mu_1\mu_n}\, Tr\gamma^{\mu_2}\dots\gamma^{\mu_n} \tag{8.31}$$

To do the calculation we need the following scalar products:

It satifies the orthogonality relations. Firstly

$$\begin{aligned} p_i \cdot s_i &= (E, \mathbf{p}_i) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|}\right) \\ &= \frac{E}{m_0}(|\mathbf{p}| - \frac{\mathbf{p}_i \cdot \mathbf{p}_i}{|\mathbf{p}|}) = 0. \end{aligned} \tag{8.32}$$

Similarly we have

$$p_f \cdot s_f = 0. \tag{8.33}$$

$$\begin{aligned} p_i \cdot s_f &= (E, \mathbf{p}_i) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|}\right) \\ &= \frac{E}{m_0}(|\mathbf{p}| - \frac{\mathbf{p}_i \cdot \mathbf{p}_f}{|\mathbf{p}|}) \\ &= \frac{E|\mathbf{p}|}{m_0}(1 - \cos\theta) \end{aligned} \tag{8.34}$$

$$p_f \cdot s_i = \frac{E|\mathbf{p}|}{m_0}(1 - \cos\theta) \tag{8.35}$$

Similarly we have

$$\begin{aligned} s_i \cdot s_f &= \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_i}{|\mathbf{p}|}\right) \cdot \left(\frac{|\mathbf{p}|}{m_0}, \frac{E}{m_0}\frac{\mathbf{p}_f}{|\mathbf{p}|}\right) \\ &= \frac{1}{m_0^2}(\mathbf{p}^2 - \frac{E^2}{\mathbf{p}^2}\mathbf{p}_i \cdot \mathbf{p}_f) \\ &= \frac{1}{m_0^2}(\mathbf{p}^2 - E^2\cos\theta) \end{aligned} \tag{8.36}$$

$$p_i \cdot p_f = E^2 - \mathbf{p}^2\cos\theta \tag{8.37}$$

We leave the details of the calculation to the reader. The result leads to

$$P = 1 - \frac{2 \sin \frac{\theta}{2}}{\left(\frac{E}{m_0}\right)^2 \cos \frac{\theta}{2} + \sin \frac{\theta}{2}} \tag{8.38}$$

In the nonrelativistic limit $E \to m_0$ this reduces to

$$P \simeq 1 - 2 \sin \frac{\theta}{2} = \cos \theta. \tag{8.39}$$

Chapter 9

Bremsstrahlung

When electrons scatter at protons or in the field of a nucleus, they can emit real photons.

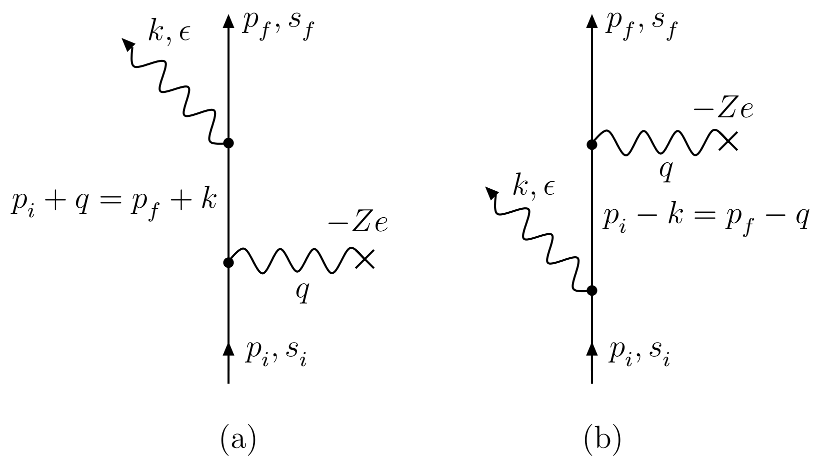


Figure 9.1: (a) . (b) .

Bremsstrahlung is a second order process

$$S_{fi}^{(2)} = -ie^2 \int d^4y d^4x \overline{\psi}_f(x) A_\mu(x) \gamma^\mu S_F(x-y) A_\nu(y) \gamma^\nu \psi_i(y) \tag{9.1}$$

the outgoing photon by

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\, \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{9.2}$$

the incoming electron

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot x} \tag{9.3}$$

the outgoing electron

$$\psi_f(x) = \sqrt{\frac{m_0}{E_f V}} u(p_f, s_f) e^{ip_f \cdot x}. \tag{9.4}$$

and the Coulomb potetial is

$$A_0^{coul}(x) = -\frac{Ze}{|\mathbf{x}|} \tag{9.5}$$

$$\begin{aligned} S_{fi} \;\; &= \;\; e^2 \int d^4y d^4x \overline{\psi}_f(x) \Big[(-iA_\mu(x,k)\gamma^\mu) iS_F(x-y)(-i\gamma^0) A_0^{coul}(y) \\ &\quad + (-i\gamma^0) A_0^{coul}(x) iS_F(x-y)(-iA_\mu(y,k)\gamma^\mu) \Big] \psi_i(x) \end{aligned} \tag{9.6}$$

Again we transform to momentum space. The Fourier transformation of the Coulomb potential

$$-\frac{Ze}{|\mathbf{x}|} = -Ze4\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} e^{-iq\cdot x} \tag{9.7}$$

Substituting all of the above into (9.6)

$$\begin{aligned} S_{fi} \;\; &= \;\; e^2 \int d^4y d^4x \left(\sqrt{\frac{m_0}{E_f V}} \overline{u}(p_f, s_f) e^{ip_f \cdot x} \right) \left[-i \left(\sqrt{\frac{4\pi}{2\omega V}}\; \epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \right) \gamma^\mu \times \right. \\ &\quad \times \left(\int \frac{d^4p}{(2\pi)^4} \frac{ie^{-ip\cdot(x-y)}}{p_\alpha \gamma^\alpha - m_0 + i\epsilon} \right) \times (-i\gamma^0) \left. \left(-Ze4\pi \int \frac{d^3q}{(2\pi)^3} \frac{1}{|\mathbf{q}|^2} e^{-iq\cdot y} \right) \right] \\ &\quad \times \left(\sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i \cdot y} \right) + exch. \end{aligned} \tag{9.8}$$

rearanging

$$
\begin{aligned}
S_{fi} &= -\frac{Ze^3 4\pi}{V^{3/2}}\sqrt{\frac{4\pi}{2\omega}}\sqrt{\frac{m_0^2}{E_f E_i}}\int d^4y d^4x \frac{d^3q}{(2\pi)^3}\frac{d^4p}{(2\pi)^4} \\
&\times \overline{u}(p_f,s_f)e^{ip_f\cdot x}\left[-i\epsilon_\mu\gamma^\mu(e^{-ik\cdot x}+e^{ik\cdot x})\ \frac{ie^{-ip\cdot(x-y)}}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}\ (-i\gamma^0)\frac{e^{-iq\cdot y}}{|\mathbf{q}|^2}\right. \\
&\left.(-i\gamma^0)\frac{e^{-iq\cdot y}}{|\mathbf{q}|^2}\ \frac{ie^{-ip\cdot(x-y)}}{p_\alpha\gamma^\alpha - m_0 + i\epsilon}\ (-i\epsilon_\mu\gamma^\mu)(e^{-ik\cdot x}+e^{ik\cdot x})\right] u(p_i,s_i)e^{-ip_i\cdot y}
\end{aligned}
\tag{9.9}
$$

Peforming the integrations

$$
\begin{aligned}
& \int d^4x(e^{ix\cdot(p_f-k-p)}+e^{ix\cdot(p_f+k-p)})\int d^4y e^{iy\cdot(-q+p-p_i)} \\
&= [(2\pi)^4\delta^4(p_f-k-p)+(2\pi)^4\delta^4(p_f+k-p)](2\pi)^4\delta^4(q+p-p_i)
\end{aligned}
$$

$$
\begin{aligned}
& \int d^4y(e^{iy\cdot(p-k-p_i)}+e^{iy\cdot(p+k-p_i)})\int d^4x e^{ix\cdot(p_f+q-p)} \\
&= [(2\pi)^4\delta^4(p-k-p_i)+(2\pi)^4\delta^4(p+k-p_i)](2\pi)^4\delta^4(-q+p-p_i)
\end{aligned}
$$

The $S-$matrix becomes

$$
\begin{aligned}
S_{fi} &= -\frac{Ze^3 4\pi}{V^{3/2}}\sqrt{\frac{4\pi}{2\omega}}\sqrt{\frac{m_0^2}{E_f E_i}}\int d^4y d^4x \frac{d^3q}{(2\pi)^3}\frac{d^4p}{(2\pi)^4} \\
&\times\Big\{[(2\pi)^4\delta^4(p_f-k-p)+(2\pi)^4\delta^4(p_f+k-p)](2\pi)^4\delta^4(q+p-p_i) \\
&\times\overline{u}(p_f,s_f)(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_\alpha\gamma^\alpha-m_0+i\epsilon}(-i\gamma^0)\frac{1}{|\mathbf{q}|^2}u(p_i,s_i) \\
&+[(2\pi)^4\delta^4(p-k-p_i)+(2\pi)^4\delta^4(p+k-p_i)](2\pi)^4\delta^4(-q+p-p_i) \\
&\times\overline{u}(p_f,s_f)(-i\gamma^0)\frac{1}{|\mathbf{q}|^2}\frac{i}{p_\alpha\gamma^\alpha-m_0+i\epsilon}(-i\epsilon_\mu\gamma^\mu)u(p_i,s_i)\Big\}
\end{aligned}
\tag{9.10}
$$

In the following we will need the formula

$$
\int dx\delta(x-a)\delta(x-b)=\delta(a-b).
$$

Consider the momentum integrals coming from the direct graph, we find

$$\begin{aligned}
&\int \frac{d^3q}{(2\pi)^3}\int \frac{d^4q}{(2\pi)^4}(2\pi)^4\delta^4(p_f \pm k - p)(2\pi)^4\delta^4(p-q-p_i)f(p,|\mathbf{q}|) \\
=&\int \frac{d^3q}{(2\pi)^3}(2\pi)^4\delta^4(p_f \pm k - q - p_i)f(p,|\mathbf{q}|) \\
=&\ 2\pi\delta(E_f - E_i \pm \omega)f(p,|\mathbf{q}|) \qquad (9.11)
\end{aligned}$$

where $q = p_f \pm k - p_i$ and $p = p_f \pm k$. There is something similary for the exchange graph.

Since we want to describe photon emmission the electron loses energy, $E_f < E_i$, which correspnds to $E_f = E_i - \omega$ - this is the arrangement measured experimentally. The $S-$matrix we require to describe emmision of a photon is then

$$\begin{aligned}
S_{fi} &= -Ze^3 2\pi\delta(E_f + \omega - E_i)\sqrt{\frac{4\pi}{2\omega V}}\sqrt{\frac{m_0^2}{E_f E_i V^2}}\frac{4\pi}{|\mathbf{q}|^2} \\
&\times\ \overline{u}(p_f, s_f)\left[(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\gamma_0)\right. \\
&\qquad \left. +(-i\gamma_0)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\epsilon_\mu\gamma^\mu)\right] u(p_i, s_i) \qquad (9.12)
\end{aligned}$$

Using the relations $p_i^2 = p_f^2 = m_0^2$, $k^2 = 0$, we have

$$\begin{aligned}
\frac{1}{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu - m_0 + i\epsilon} &= \frac{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu + m_0}{(p_f + k)^2 - m_0^2 + i\epsilon} \\
&= \frac{p_{f\mu}\gamma^\mu + k_\mu\gamma^\mu + m_0}{2p_f \cdot k + i\epsilon}. \qquad (9.13)
\end{aligned}$$

9.1 Remarks on the Form of the $S-$Matrix element

• At the free vertex, where a free photon with polarisation vector ϵ_μ is emitted, a factor

i) $(-i\epsilon_\mu\gamma^\mu)$ occurs

ii) and the normalisation factor of the photon $\sqrt{4\pi/2\omega V}$ enters.

9.2 Bremsstrahlung Cross Section

We can simplify the notation by writing

$$S_{fi} = iZe^3 2\pi\delta(E_f + \omega - E_i)\sqrt{\frac{4\pi}{2\omega V}}\sqrt{\frac{m_0^2}{E_f E_i V^2}}\frac{4\pi}{|\mathbf{q}|^2}\,\epsilon^\mu M_\mu(k), \tag{9.14}$$

where

$$M_\mu(k) = \overline{u}(p_f, s_f)\left[\gamma_\mu \frac{p_{f\alpha}\gamma^\alpha + k_\alpha\gamma^\alpha + m_0}{2p_f \cdot k + i\epsilon}\gamma_0 + \gamma_0 \frac{p_{i\alpha}\gamma^\alpha - k_\alpha\gamma^\alpha + m_0}{-2p_i \cdot k + i\epsilon}\gamma_\mu\right] u(p_i, s_i) \tag{9.15}$$

The bremsstrahlung cross section is given by

$$\begin{aligned} d\sigma &= \frac{1}{\frac{|\mathbf{v_i}|}{V}T}\,|S_{fi}|^2\,\frac{V d^3k}{(2\pi)^3}\frac{V d^3 p_f}{(2\pi)^3} \\ &= \frac{Z^2 e^6}{|\mathbf{v_i}|}\frac{4\pi}{2\omega}\frac{m_0^2}{E_f E_i}\,\frac{(4\pi)^2}{|\mathbf{q}|^4}\,|\epsilon^\mu M_\mu(k)|^2\,2\pi\delta(E_f + \omega - E_i)\frac{d^3k}{(2\pi)^3}\frac{d^3p_f}{(2\pi)^3} \end{aligned} \tag{9.16}$$

9.3 Sum Over Polarisations of Photon

We know that gauge invariance implies the condition of conservered current

$$\frac{\partial J_\mu(x)}{\partial x_\mu} = 0, \tag{9.17}$$

where $J_\mu(x) = \overline{\psi}(x)\gamma_\mu\psi(x)$. In momentum space the conservation condition reads

$$k^\mu J_\mu(k) = 0 \tag{9.18}$$

Now the matrix element $M_\mu(k)$ given in () is a quantum mchanical transition current for Bremsstrahlung in lowest order perurbation theory and also satifies

$$k^\mu M_\mu(k) = 0 \tag{9.19}$$

This condition is easily verified using $k_\mu\gamma^\mu p_\nu\gamma^\nu = -p_\mu\gamma^\mu k_\nu\gamma^\nu + 2p\cdot k$ and the dirac equation:

$$\overline{u}(p_f,s_f)(p_{f\mu}\gamma^\mu - m_0) = 0, \quad (p_{i\mu}\gamma^\mu - m_0)u(p_i,s_i) = 0$$

$$\begin{aligned} k^\mu M_\mu(k) &= \overline{u}(p_f,s_f)\left[k_\mu\gamma^\mu \, \frac{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu + m_0}{2p_f\cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{p_{i\mu}\gamma^\mu - k_\mu\gamma^\mu + m_0}{-2p_i\cdot k + i\epsilon}\, k_\nu\gamma^\nu\right]u(p_i,s_i) \\ &= \overline{u}(p_f,s_f)\left[\frac{-(p_{f\nu}\gamma^\nu - m_0)k_\mu\gamma^\mu + 2p_f\cdot k + k^2}{2p_f\cdot k + i\epsilon}\gamma_0\right. \\ &\quad \left. +\gamma_0\frac{-k_\nu\gamma^\nu(p_{i\mu}\gamma^\mu - m_0) + 2p_i\cdot k - k^2}{-2p_i\cdot k + i\epsilon}\right]u(p_i,s_i) \\ &= \overline{u}(p_f,s_f)\left[\frac{2p_f\cdot k}{2p_f\cdot k + i\epsilon}\gamma_0 + \gamma_0\frac{2p_i\cdot k}{-2p_i\cdot k + i\epsilon}\right]u(p_i,s_i) \\ &= 0. \end{aligned} \tag{9.20}$$

We now perform the summation over the photon polarisation vectors $\epsilon_\mu(\mathbf{k},\lambda)$ with $\lambda = 1,2$. The quantity of interest is

$$\overline{|\epsilon\cdot M_\mu(k)|^2} = \sum_{\lambda=1,2}|\epsilon_\mu(\mathbf{k},\lambda)M^\mu(k)|^2 = \sum_{\lambda=1,2}\epsilon_\mu(\mathbf{k},\lambda)\epsilon^*_\nu(\mathbf{k},\lambda)M^\mu(k)M^{*\nu}(k) \tag{9.21}$$

We work in the radiation gauge and choose a particular coordinate which simplifies the calculation. Consider the coordinate system such that the momentum vector $\mathbf{k}$ points in the z-direction

$$k^\mu = \omega(1,0,0,1) \tag{9.22}$$

We choose the two transverse polarisation vectors

$$\begin{aligned} \epsilon(\mathbf{k},1) &= (0,1,0,0), \\ \epsilon(\mathbf{k},2) &= (0,0,1,0). \end{aligned} \tag{9.23}$$

Now we use the condition of current conservation

$$k^\mu M^\mu = \omega(M^0 - M^3) = 0, \tag{9.24}$$

which implies $M^0 = M^3$. Therefore we can write

$$\overline{|\epsilon \cdot M|^2} = M^1 M^{*1} + M^2 M^{*2} + M^3 M^{*3} - M^0 M^{*0} = -M_\mu M^{*\mu} \tag{9.25}$$

Obviously this is Lorentz covariant. In general we have

$$\sum_{\lambda=1,2} \epsilon_\mu(\mathbf{k}, 1)\epsilon_\nu(\mathbf{k}, 2) = -\eta_{\mu\nu} + \text{gauge terms.} \tag{9.26}$$

The additional terms are proportional to k_μ or k_ν and thus do not contribute to any observable quantitiy since the sum is multiplied by conserved currents which satisfy $k \cdot J$.

9.4 The Infrared Catastrophe

A photon may be emmited which is too soft to be detected because of the energy resolution ΔE of the apparatus. Consequently, the experimental cross section is the sum the cross section for bremsstrahlung of energy less than ΔE and second order (raidiative corrected) elastic cross section, i.e.,

$$\left(\frac{d\sigma}{d\Omega'}\right)_{Exp} = \left(\frac{d\sigma}{d\Omega'}\right)_B + \left(\frac{d\sigma}{d\Omega'}\right)_{El}. \tag{9.27}$$

Here $(d\sigma/\Omega')_B$ is the soft bremsstrahlung cross section integrated over the range of photon energy $0 \ll \omega \ll \Delta E$ and $(d\sigma/\Omega')_{El}$ is the cross section for raidiative corrected elastic electron off the Coulomb potential.

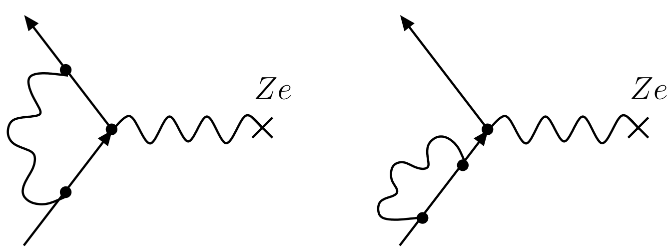


Figure 9.2: The two types of lowest order radiative corrections to elastic scattering of an electron of a Coulomb potential.

Chapter 10

Compton Scattering

We describe the incoming photon as a plane wave:

$$A_\mu(x,k) = \sqrt{\frac{4\pi}{2\omega V}}\,\epsilon_\mu(k,\lambda)(e^{-ik\cdot x} + e^{ik\cdot x}) \tag{10.1}$$

and the outgoing (scattered) photon by

$$A'_\mu(x',k') = \sqrt{\frac{4\pi}{2\omega' V}}\,\epsilon_\mu(k,\lambda)(e^{-ik'\cdot x'} + e^{ik'\cdot x'}) \tag{10.2}$$

the incoming electron by

$$\psi_i(x) = \sqrt{\frac{m_0}{E_i V}} u(p_i, s_i) e^{-ip_i\cdot x} \tag{10.3}$$

the outgoing electron by

$$\psi_f(x) = \sqrt{\frac{m_0}{E_f V}} u(p_f, s_f) e^{ip_f\cdot x}. \tag{10.4}$$

The $S-$matrix is

$$\begin{aligned} S_{fi} &= e^2 \int d^4x d^4y\, \overline{\psi}_f(x) \Big[(-iA_\mu(y,k')\gamma^\mu)\; iS_F(x-y)(-iA_\nu(y,k)\gamma^\nu) \\ &\quad + (-iA_\mu(y,k)\gamma^\mu)\; iS_F(x-y)(-iA_\nu(y,k')\gamma^\nu)\Big]\, \psi_i(y). \end{aligned} \tag{10.5}$$

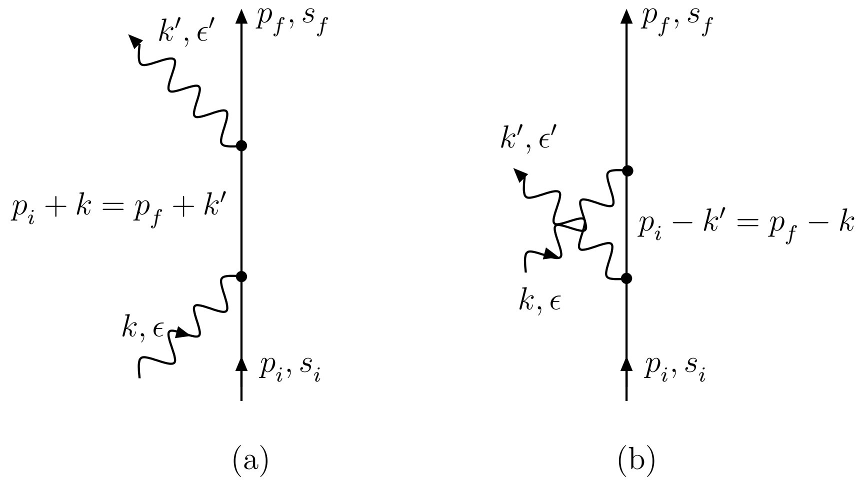


Figure 10.1: The direct and exchange diagrams describing Compton scattering.

From previous experience we know to write this in momentum space to be

$$\begin{aligned} S_{fi} &= \frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_i E_f}}\sqrt{\frac{(4\pi)^2}{2\omega 2\omega'}}\,(2\pi)^4\delta^4(p_f + k' - p_i - k) \\ &\times\ \overline{u}(p_f, s_f)\left[(-i\epsilon'_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu + k_\nu\gamma^\nu - m_0}(-i\epsilon_\sigma\gamma^\sigma)\right. \\ &\qquad \left. +(-i\epsilon_\mu\gamma^\mu)\frac{i}{p_{f\nu}\gamma^\nu - k'_\nu\gamma^\nu - m_0}(-i\epsilon'_\sigma\gamma^\sigma)\right] u(p_i, s_i) \end{aligned} \tag{10.6}$$

In going from (10.5) to (10.6) we integrated over plane waves which gave delta functions which impose energy-momentum conservation at the vertices. It results in four different processes, two of which are not allowed kinematically: the emission or absorption of two photons by a free electron. A third process is not compatible with the kinematic conditions fixed by the experiment. The process describing Compton scattering corresponds to the followings constraints on four momentum:

$$+k + p_i = +k' + p_f \tag{10.7}$$

The situation is similarly to what we encoutered in Bremsstrahlung in that not every term is physically relevant for the process considered. The term in the Compton scattering amplitude stems from the part $\exp(-ik \cdot x)$ of the photon field in (10.1) describing the absorption of a photon with four-momentum k^μ by the electron and from the part $\exp(-ik' \cdot x')$ of the photon field in (10.2) describing a photon emitted by the electron with four-momentum k'^μ.

10.1 Compton Scattering Cross Section

We split the S-matrix into two parts:

$$S_{fi} = -i\frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_i E_f}}\sqrt{\frac{(4\pi)^2}{2\omega 2\omega'}}(2\pi)^4\delta^4(p_f + k' - p_i - k)\ \epsilon^\mu(\mathbf{k}', \lambda')\epsilon^\nu(\mathbf{k}, \lambda)\ M_{\mu\nu} \tag{10.8}$$

Here $M_{\mu\nu}$ is the Compton tensor

$$M_{\mu\nu} = \overline{u}(p_f, s_f)\left[\gamma_\mu \frac{p_{i\alpha}\gamma^\alpha + k_\alpha\gamma^\alpha + m_0}{2p_i\cdot k + i\epsilon}\gamma_\nu + \gamma_\nu \frac{p_{i\alpha}\gamma^\alpha - k'_\alpha\gamma^\alpha + m_0}{-2p_i\cdot k' + i\epsilon}\gamma_\mu\right] u(p_i, s_i) \tag{10.9}$$

we have

$$k'^\mu M_{\mu\nu} = k^\nu M_{\mu\nu} = 0. \tag{10.10}$$

The proof is analogous to the bremsstrahlung case.

The cross section starts as

$$d\sigma = \int \frac{|S_{fi}|^2}{T|\mathbf{v}_{rel}|/V}\,\frac{V d^3 p_f}{(2\pi)^3}\frac{V d^3 k'}{(2\pi)^3} \tag{10.11}$$

with

$$\frac{|S_{fi}|^2}{T} = \frac{|S_{fi}|^2}{VT/V}$$

being the transition rate per unit volume and normalised to one electron per volume. $|\mathbf{v_{rel}}|/V$ is the incoming photon flux.

$$d\sigma \quad = \quad \frac{e^4}{V^4}\frac{m_0^2}{E_i E_f} \tag{10.12}$$

We calculate the cross section in the laboratory frame.

$$p_i = (m_0, 0)$$

Also

$$|\mathbf{v}_{rel}| = |\mathbf{c} - \mathbf{v}_{\mathbf{e}}| = |\mathbf{c}| = c$$

We use the covariant expression for the density of final states:

$$\frac{d^3p}{2E} = \int_{-\infty}^{\infty} d^4p\ \delta(p^2 - m_0^2)\Theta(p_0) \tag{10.13}$$

Averaging over initial and final electron spins and polarisations

$$\frac{1}{4}\sum_{pol}\sum_{spin} |\epsilon^\mu(\mathbf{k}', \lambda')\epsilon^\nu(\mathbf{k}, \lambda)\ M_{\mu\nu}|^2 = \frac{1}{4}\sum_{spin} M^{\mu\nu} M^*_{\mu\nu} \tag{10.14}$$

(on account of (10.10) and (9.26)).

the unpolarised differential cross section for Compton scattering is then

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m}\left(\frac{\omega'}{\omega}\right)^2 \left\{\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\theta\right\}. \tag{10.15}$$

10.2 Annihilation of Particle and Antiparticle

$$\begin{aligned} S_{fi} &= e^2 \int d^4x d^4y\, \overline{\psi}_+(x) \Big[(-iA_\mu(y,k')\gamma^\mu)\ iS_F(x-y)(-iA_\nu(y,k)\gamma^\nu) \\ &\quad + (-iA_\mu(y,k)\gamma^\mu)\ iS_F(x-y)(-iA_\nu(y,k')\gamma^\nu)\Big]\, \psi_-(y). \end{aligned} \tag{10.16}$$

To fit the experimental situation both photon outgoing plane waves should be used. We are lead to the following expression in momentum space:

$$\begin{aligned} S_{fi} &= \frac{e^2}{V^2}\sqrt{\frac{m_0^2}{E_+E_-}}\sqrt{\frac{(4\pi)^2}{\omega_1\omega_2}}(2\pi)^4\delta^4(k_1 + k_2 - p_+ - p_-) \\ &\times\ \overline{v}(p_+, s_+)\left[(-i\epsilon_{2\mu}\gamma^\mu)\frac{i}{p_{-\alpha}\gamma^\alpha - k_{1\alpha}\gamma^\alpha - m_0}(-i\epsilon_{1\nu}\gamma^\nu)\right. \\ &\quad \left. +(-i\epsilon_{1\mu}\gamma^\mu)\frac{i}{p_{-\alpha}\gamma^\alpha - k_{2\alpha}\gamma^\alpha - m_0}(-i\epsilon_{2\nu}\gamma^\nu)\right] u(p_-, s_-) \end{aligned} \tag{10.17}$$

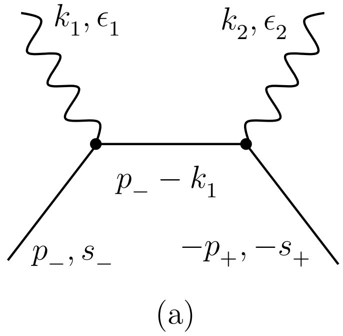


Figure 10.2: Direct and exchange graph of pair annihilation into two photons.

10.3 Second Order Electron-Proton Scattering

Before we state Feynman's rules for QED we wish to examine how to formulate the amplitude for a second order scattering problem, namely the secon order S-matrix for electron-proton scattering.

The amplitude for second order electron proton scattering

$$S^{(2)}_{fi} = -ie^2 \int d^4x d^4y \overline{\psi}_f(x) A_\mu \gamma^\mu S_F(x-y) A_\nu \gamma^\mu \psi_i(y) \tag{10.18}$$

The second order electron current is given by

$$J^{(2)}_{\mu\nu}(x,y) = ie^2 \overline{\psi}_f(x)\ \gamma_\mu S_F(x-y)\gamma_\nu\ \psi_i(y) \tag{10.19}$$

We generalise the relation

$$A^\mu(x) = \int d^4y\ D_F(x-y)\ J^\mu(y).$$

by conjecturing

$$A_\mu(x)A_\nu(y) = \int d^4X d^4Y\ D_F(x-X)D_F(y-Y)J^{p(2)}_{\mu\nu}(X,Y). \tag{10.20}$$

By symmetry the proton current $J^{p(2)}_{\mu\nu}(X,Y)$ should be

$$J^{p(2)}_{\mu\nu}(X,Y) = ie^2\ \overline{\psi}^p_f(X)\ \gamma_\mu S_F(X-Y)\gamma_\nu\ \psi^p_i(Y) \tag{10.21}$$

Substituting () and (10.20) into

$$
\begin{aligned}
S^{(2)}_{fi}(dir.) &= e^2 e_p^2 \int d^4x d^4y d^4X d^4Y \\
&\quad \times [\overline{\psi}_f(x)\gamma^\mu S_F(x-y)\gamma^\nu \, \psi_i(y)] \\
&\quad \times D_F(x-X) D_F(y-Y) \\
&\quad \times [\overline{\psi}^p_f(X)\gamma_\mu S_F(X-Y)\gamma_\nu \, \psi^p_i(Y)] \qquad (10.22)
\end{aligned}
$$

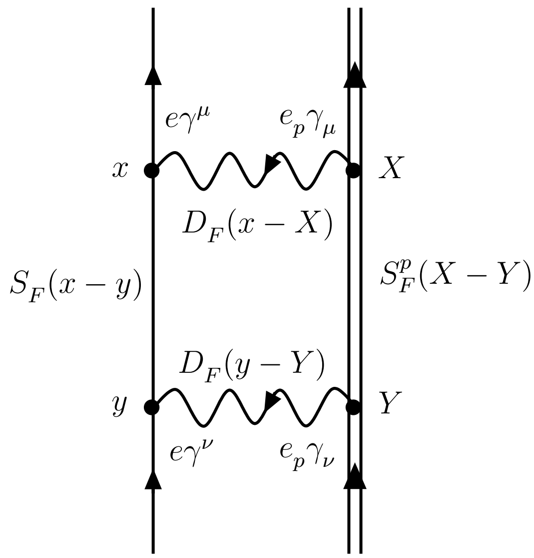


Figure 10.3:

Since the two photons emitted by the proton current are indistiguishable the electron at x does not know whether the photon absorbed there has been emitted at X or Y. According to quantum mechanics we must coherently add the contribution coming from the corresponding exchange graph in (10.3).

$$
\begin{aligned}
S^{(2)}_{fi}(exch.) &= e^2 e_p^2 \int d^4x d^4y d^4X d^4Y \\
&\quad \times [\overline{\psi}_f(x)\gamma^\mu S_F(x-y)\gamma^\nu \, \psi_i(y)] \\
&\quad \times D_F(x-Y) D_F(y-X) \\
&\quad \times [\overline{\psi}^p_f(X)\gamma_\nu S_F(X-Y)\gamma_\mu \, \psi^p_i(Y)] \qquad (10.23)
\end{aligned}
$$

Notice how the indicies μ and ν in the ‘proton current’ term have been exchanged with respect to the direct term.

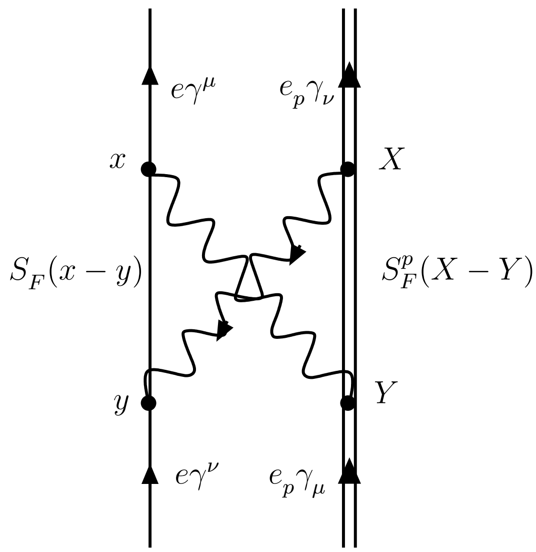


Figure 10.4:

10.4 Feynman Diagram in Momentum Space

As always, the external particles (incoming and outgoing electron and proton) are described by plane waves. The direct term becomes

$$
\begin{aligned}
S_{fi}^{(2)}(dir.) &= \frac{(4\pi)^2 e^4}{V^2} \int d^4x d^4y d^4X d^4Y \sqrt{\frac{m_0^2}{E_f E_i}} \sqrt{\frac{M_0^2}{E_f^p E_i^p}} \\
& \int \frac{d^4q_1}{(2\pi)^4} \frac{d^4q_2}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4P}{(2\pi)^4} \qquad (10.24)
\end{aligned}
$$

It is easy to perform the integration over spacetime coordinates. It results in the product δ^4-functions:

$$
\begin{aligned}
&(2\pi)^4\delta^4(q_1 + p - p_f)(2\pi)^4\delta^4(q_2 - p + p_i) \\
&\times(2\pi)^4\delta^4(-q_2 - P + P_i)(2\pi)^4\delta^4(-q_1 + P - P_f). \qquad (10.25)
\end{aligned}
$$

Each δ^4-function expresses the energy-momentum conservation at one of the four vertices. Now we can integrate over

$$
\begin{aligned}
S_{fi}^{(2)}(dir.) \;\; = \;\; & \frac{(4\pi)^2 e^4}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E_f^p E_i^p}}\,(2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \\
& \times \int \frac{d^4 q_1}{(2\pi)^4}\frac{1}{q_1^2 + i\epsilon}\frac{1}{(q-q_1)^2 + i\epsilon} \\
& \times \left[\overline{u}(p_f, s_f)\gamma^\mu \frac{1}{p_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - m_0 + i\epsilon}\gamma^\nu u(p_i, s_i)\right] \\
& \times \left[\overline{u}(P_f, S_f)\gamma_\mu \frac{1}{P_{f\alpha}\gamma^\alpha + q_{1\alpha}\gamma^\alpha - M_0 + i\epsilon}\gamma_\nu u(P_i, S_i)\right] \qquad (10.26)
\end{aligned}
$$

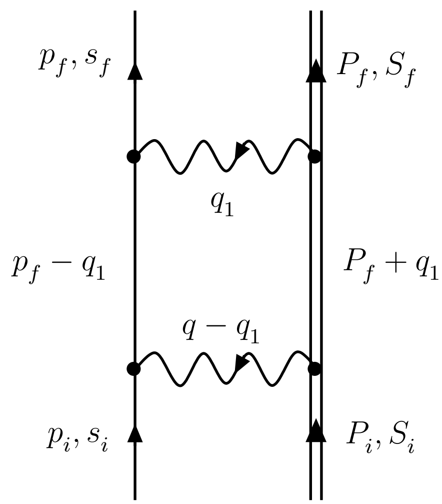


Figure 10.5:

$$
\begin{aligned}
S_{fi}^{(2)}(exch.) \;\; = \;\; & \frac{(4\pi)^2 e^4}{V^2}\sqrt{\frac{m_0^2}{E_f E_i}}\sqrt{\frac{M_0^2}{E_f^p E_i^p}}\,(2\pi)^4\delta^4(P_f + p_f - P_i - p_i) \\
& \times \int \frac{d^4 q_1}{(2\pi)^4}\frac{1}{q_1^2 + i\epsilon}\frac{1}{(q-q_1)^2 + i\epsilon} \\
& \times \left[\overline{u}(p_f, s_f)\gamma^\mu \frac{1}{p_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - m_0 + i\epsilon}\gamma^\nu u(p_i, s_i)\right] \\
& \times \left[\overline{u}(P_f, S_f)\gamma_\nu \frac{1}{P_{f\alpha}\gamma^\alpha - q_{1\alpha}\gamma^\alpha - M_0 + i\epsilon}\gamma_\mu u(P_i, S_i)\right] \qquad (10.27)
\end{aligned}
$$

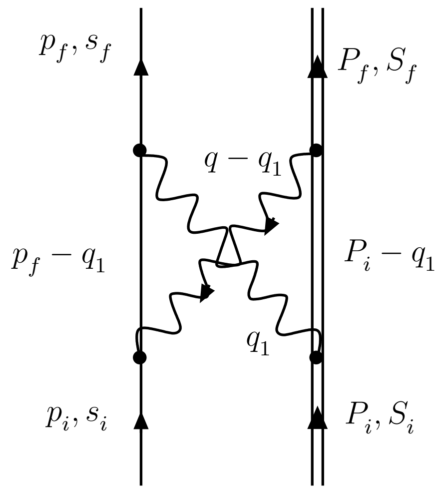


Figure 10.6:

10.5 Remarks on form of scattering Matrix

- each vertex contributes a factor of the form $-ie\gamma_\mu ...$
- each external particle yields a facttor $\sqrt{m_0/E}$.

Chapter 11

Feynmann Rules of QED

There origins should be clear to the reader.

11.1 Scattering Amplitudes

We consider a scattering process in which two particles, they may be electrons, positrons or photons, with four-momenta

$$p_i = (E_i, \mathbf{p}_i), \quad i = 1, 2,$$

collide and produce N final particles with momenta

$$p_f = (E_f, \mathbf{p}_f), \quad f = 1, \ldots, N.$$

Individual energy-momentum conservation at each verrtex leads to conservation of total energy-momentum, represented by the delta function

$$\delta^4 \left(p_1 + p_2 - \sum_{i=1}^{n} p_i' \right).$$

The scattering matrix element S_{fi} is given by

$$S_{fi} = i(2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_{i=1}^{n} p_i' \right) \; M_{fi} \; \prod_{i=1}^{2} \sqrt{\frac{N_i}{2E_i V}} \prod_{i=1}^{n} \sqrt{\frac{N_i'}{2E_i' V}} \tag{11.1}$$

The normalisation factors N_i:

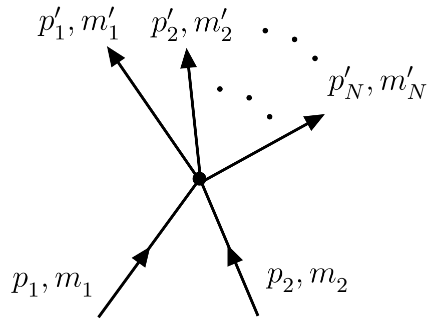


Figure 11.1: We are considering reactions in which there are two particles in the initial state and n particles in the final state.

$$N_i = \begin{cases} 4\pi & photons \\ 2m_0 & spin - \frac{1}{2} particles \end{cases} \tag{11.2}$$

After drawing any Feynman diagram in momentummm space we see clearly how to translate the various lines in the graph directly into mathematical expressions.

The Feynman rules concern the calculation of the reduced scattering matrix element M_{fi}. A Feynman graph describing a scattering process consists of three parts:

(1) the external lines representing the wave functions of incomingand outgoing particles,

(2) the internal lines described by propagators, and

(3) the vertices representing the interactions between the particles.

With each external line one associates the following factors:

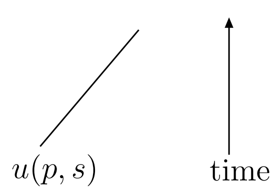


Figure 11.2: An electron entering an interaction.

electron:

$$iS_F = \frac{i(p_\mu \gamma^\mu + m_0)}{p^2 - m_0^2 + i\epsilon} \tag{11.3}$$

photon:

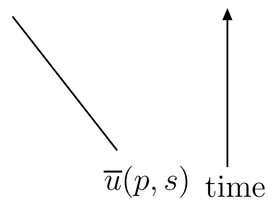


Figure 11.3: An electron leaving an interaction.

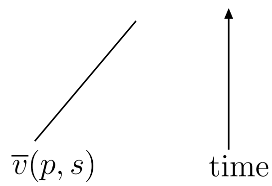


Figure 11.4: .

$$D_F^{\mu\nu}(k) = \frac{-i4\pi\ \eta^{\mu\nu}}{k^2 + i\epsilon} \tag{11.4}$$

Each vertex is associated with a factor

$$-ie\gamma_\mu. \tag{11.5}$$

a) a factor of -1 for each incoming positron (outgoing electtron with negative energy)

b) a factror of -1 in the case that two graphs which differ only by the exchange of two fermion lines.

c) a factor of -1 for each closed fermion loop.

For each internal loop, integrate over

$$\int \frac{d^4q}{(2\pi)^4}$$

11.2 Differential Cross Section

To obtain the transition rate to a group of final states with momenta in the intervals $f = 1, \ldots, N$, we multiply by the number of these states which is

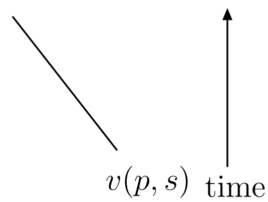


Figure 11.5: .

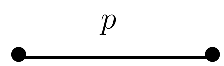


Figure 11.6: Electron propagator.

$$\prod_f \frac{V d^3\mathbf{p}'_f}{(2\pi)^3} \tag{11.6}$$

$$d\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} N_1 N_2 (2\pi)^4 \delta^4(p_1 + p_2 - \sum_{i=1}^{N} p'_i) S |M_{fi}|^2 \prod_f \frac{d^3\mathbf{p}'_f}{2E'_f (2\pi)^3} \tag{11.7}$$

The degeneracy factor S exists when the final state contains identical particles. Its taken into account by

$$S = \prod_k \frac{1}{g_k!}, \tag{11.8}$$

where g_k particles of the kind k in the final state.

11.2.1 External static electromagnetic fields

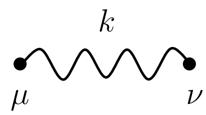


Figure 11.7: Photon propagator.

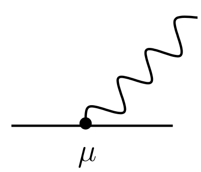


Figure 11.8: Vertex

Part V

Divergences, Renormalisation and Radiative Corrections

Chapter 12

Renormalisation

12.1 Radiative Corrections and Divergent Feynmann Diagrams

12.1.1 Radiative Corrections

The leading terms in the perturbative series can be represented by Feynman diagrams without loops, the so-called tree diagrams.

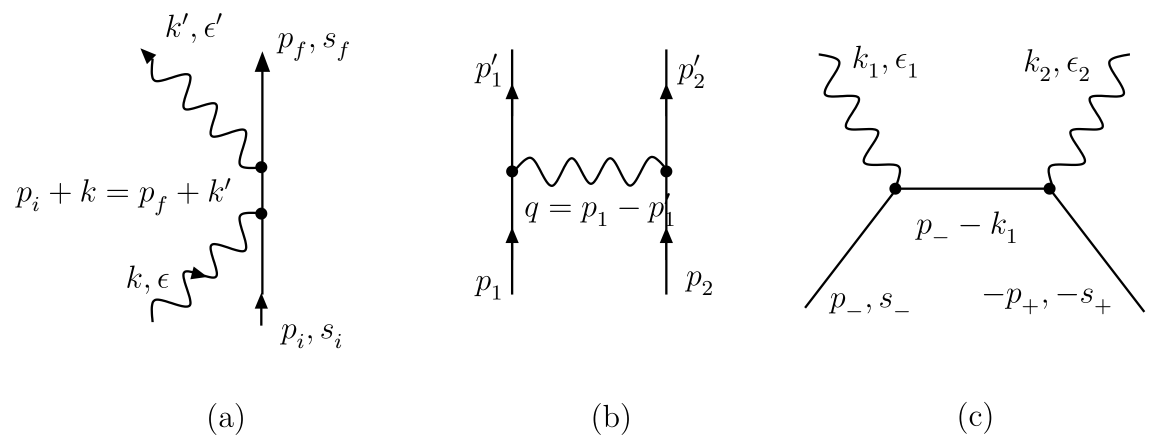


Figure 12.1: Examples of tree diagrams in QED. (a) Comptom scattering. (b) Electron-electron scattering. (c) Particle-anti-particle annihilation.

12.2 Divergence of Integrals

The integral corresponding to the diagram showm in 12.2 (a) has the form

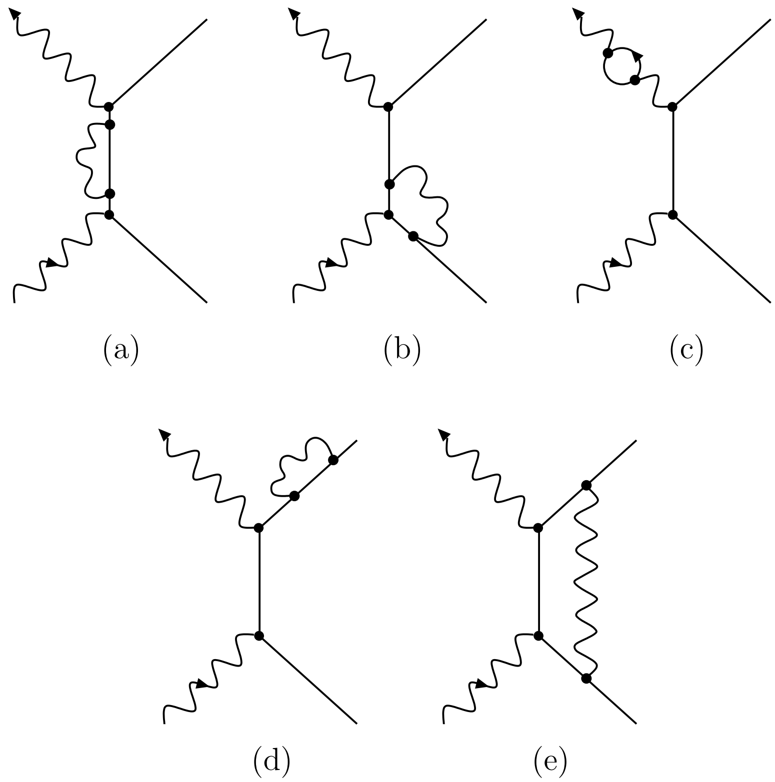


Figure 12.2: The one-loop diagrams for the process of Comptom scattering.

$$\int d^d q \frac{\gamma_\mu(p-q+m)\gamma_\nu}{[(q^2+i\epsilon)][(p-q)^2-m^2+i\epsilon]} \tag{12.1}$$

12.3 Feynmann Parameterisation

We introduce the $\Gamma-$function:

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du \tag{12.2}$$

Integrating by parts

$$\begin{aligned} \Gamma(z) &= \int_0^\infty e^{-u} u^{z-1} du \\ &= [-e^{-u} u^{z-1}]_0^\infty + \int_0^\infty e^{-u}(z-1) u^{z-2} du \\ &= (z-1)\Gamma(z-1). \end{aligned} \tag{12.3}$$

If z is equal to an integer n then,

$$\begin{aligned}\Gamma(n) &= \int_0^\infty e^{-u} u^{n-1} du \\ &= (n-1)(n-2)\ldots 1 \int_0^\infty e^{-u} du \\ &= (n-1)! \end{aligned} \tag{12.4}$$

So the $\Gamma-$functionis also the factorial function. The following forumal can be

The product of several brackets in the denominator into a single bracket can be done with the use of the Feynman parameterisation.

$$\frac{1}{a^m b^n} = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} \int_0^1 du \frac{u^{m-1}(1-u)^{n-1}}{[au + b(1-u)]^{m+n}} \tag{12.5}$$

Derivation.

$$\frac{1}{ab} = \frac{1}{b-a}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{1}{b-a}\int_a^b \frac{dt}{t^2} \tag{12.6}$$

Defining $t = b + (a-b)u$,

$$\frac{1}{ab} = \int_0^1 \frac{du}{[b + (a-b)u]^2} = \int_0^1 \frac{du}{[au + b(1-u)]^2} \tag{12.7}$$

This can be rewritten as

$$\frac{1}{ab} = \int_0^1 du_1 du_2 \delta(u_1 + u_2 - 1) \frac{1}{[a_1 u_1 + a_2 u_2]^2} \tag{12.8}$$

Differentiation with respect to b gives

$$\begin{aligned}\frac{1}{ab^n} &= \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{db^{n-1}}\frac{1}{ab} \\ &= \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{db^{n-1}}\int_0^1 \frac{du}{[au + b(1-u)]^2} \\ &= n\int_0^1 \frac{du(1-u)^{n-1}}{[au + b(1-u)]^{n+1}} \end{aligned} \tag{12.9}$$

Differentiation the previous result with respect to a gives

$$\begin{aligned}
\frac{1}{a^m b^n} &= \frac{(-1)^{n-1}}{(m-1)!}\frac{d^{m-1}}{da^{m-1}}\frac{1}{ab^n} \\
&= n\frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{da^{m-1}}\int_0^1 \frac{du}{[au+b(1-u)]^{n+1}} \\
&= \frac{n\times(n+1)\cdots(n+m-1)}{(m-1)!}\int_0^1 \frac{du u^{m-1}(1-u)^{n-1}}{[au+b(1-u)]^{m+n}} \\
&= \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}\int_0^1 \frac{du u^{m-1}(1-u)^{n-1}}{[au+b(1-u)]^{m+n}} \qquad (12.10)
\end{aligned}$$

Equation (12.8) generalises to

$$\frac{1}{a_1 a_2 \ldots a_n} = \Gamma(n)\int_0^1 du_1 \cdots \int_0^1 du_n \frac{\delta(u_1+\cdots+u_n-1)}{[u_1 a_1 + u_2 a_2 + \cdots + u_n a_n]^n} \qquad (12.11)$$

We can prove this by induction, see appendix E. By repeated differentiation of (12.11), we obtain the more general result

$$\frac{1}{a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}} = \frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\ldots\Gamma(m_n)}\int_0^1 du_1 \cdots \int_0^1 du_n \delta(\sum_{i=1}^n u_i - 1)\frac{u_1^{m_1-1}\ldots u_n^{m_n-1}}{[\sum_{i=1}^n u_i a_i]^{\sum m_i}} \qquad (12.12)$$

This formula is true even when the m_i's are not integers.

12.3.1 Alternative forumla

We can generalise the original form

$$\frac{1}{a_1 a_2} = \int_0^1 \frac{du_1}{[a_1 + (a_2 - a_1)u_1]^2} \qquad (12.13)$$

generalises to three factors,

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{[a_1 + (a_2 - a_1)u_1 + (a_3 - a_2)u_2]^3}. \qquad (12.14)$$

We prove this by integrating with respect to u_2,

$$
\begin{aligned}
& 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{[a_1+(a_2-a_1)u_1+(a_3-a_2)u_2]^3} \\
= & \; 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{(a_3-a_2)^3\left[\frac{a_1}{a_3-a_2}+\frac{(a_2-a_1)}{a_3-a_2}u_1+u_2\right]^3} \\
= & \; 2\int_0^1 du_1 \frac{1}{(a_3-a_2)^3}\left[-\frac{1}{2}\frac{1}{\left[\frac{a_1}{a_3-a_2}+\frac{(a_2-a_1)}{a_3-a_2}u_1+u_2\right]^2}\right]_0^{u_1} \\
= & \; \frac{1}{a_3-a_2}\int_0^1 du_1 \left(\frac{1}{[a_1+(a_2-a_1)u_1]^2}-\frac{1}{[a_1+(a_2-a_1)u_1+(a_3-a_2)u_1]^2}\right) \\
= & \; \frac{1}{a_3-a_2}\left(\frac{1}{a_1a_2}-\frac{1}{a_1a_3}\right) \\
= & \; \frac{1}{a_1a_2a_3}
\end{aligned}
\tag{12.15}
$$

where we used (12.13). This generalises to an arbitrary number of factors

$$
\begin{aligned}
\frac{1}{a_1a_2\ldots a_n} = & \; \Gamma(n)\int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \\
& \times \frac{1}{[a_1+(a_2-a_1)u_1+\cdots+(a_n-a_{n-1})u_{n-1}]^n}
\end{aligned}
\tag{12.16}
$$

We prove this by induction, see appendix E.0.1. By repeated differentiation of (12.16), we obtain the more general result

$$
\begin{aligned}
\frac{1}{a_1^{m_1}a_2^{m_2}\ldots a_n^{m_n}} = & \; \frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\ldots\Gamma(m_n)}\int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \\
& \times \frac{(1-u_1)^{m_1-1}\cdots(u_{n-2}-u_{n-1})^{m_{n-2}-1}u_{n-1}^{m_{n-1}-1}}{[a_1(1-u_1)+\cdots+a_{n-1}(u_{n-2}-u_{n-1})+a_nu_{n-1}]^{\sum_i m_i}}
\end{aligned}
\tag{12.17}
$$

Variation of the forumla

We include a variation of equations (12.16) and (12.17) found in some text books. We rewrite (12.16) as

$$
\begin{aligned}
\frac{1}{a_1 a_2 \ldots a_n} &= \Gamma(n) \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \\
&\qquad \times \frac{1}{[a_n + (a_{n-1} - a_n)u_1 + \cdots + (a_1 - a_2)u_{n-1}]^n} \\
&= \Gamma(n) \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \\
&\qquad \times \frac{1}{[a_1 u_{n-1} + a_2(u_{n-2} - u_{n-1}) + \cdots + a_n(1 - u_1)]^n}
\end{aligned} \tag{12.18}
$$

Repeated differentiation gives

$$
\begin{aligned}
\frac{1}{a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}} &= \frac{\Gamma(m_1 + \cdots + m_m)}{\Gamma(m_1) \cdots \Gamma(m_n)} \int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \\
&\qquad \times \frac{u_{n-1}^{m_1 - 1} (u_{n-2} - u_{n-1})^{m_2 - 1} \cdots (1 - u_1)^{m_n - 1}}{[a_1 u_{n-1} + a_2(u_{n-2} - u_{n-1}) + \cdots + a_n(1 - u_1)]^{\sum_i m_i}}
\end{aligned} \tag{12.19}
$$

12.4 Divergent Feynmann Diagrams

12.5 Dimensional Regularisation

12.5.1 The d-Dimensional Integral

Consider integrals of the form

$$
\int d^d q f(q^2). \tag{12.20}
$$

As the integrand is “sperically symmetric”, this integral becomes

$$
\int_0^\infty \int_{S^{d-1}(r)} f(r^2) r^{d-1} d\Omega dr = \left(\int_0^\infty f(r^2) r^{d-1} dr \right) \left(\int_{S^{d-1}(1)} d\Omega \right). \tag{12.21}
$$

12.5.2 The Volume of a $(d-1)$-sphere via Gaussian Integration

We can easily derive the volume of a $(d-1)-$sphere by writing the integral

$$I = \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}\sum_{i=0}^{d} x_i^2\right) dV. \tag{12.22}$$

two different ways. First of all, we use that the integral decomposes as

$$I = \prod_{i=1}^{d} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}x_i^2\right) dx_i \tag{12.23}$$

and using

$$\int_0^\infty \exp\left(-\frac{1}{2}x^2\right) dx = (2\pi)^{1/2}. \tag{12.24}$$

we find

$$I = (2\pi)^{d/2} \tag{12.25}$$

Now we write the integral in spherical polar coordinates and use that the function

$$\exp\left(-\frac{1}{2}\sum_{i=0}^{d} x_i^2\right) \tag{12.26}$$

is rotationally invariant:

$$I = \int_0^\infty \int_{S^{d-1}(r)} \exp\left(-\frac{1}{2}r^2\right) r^{d-1} d\Omega dr = \left(\int_0^\infty \exp\left(-\frac{1}{2}r^2\right) r^{d-1} dr\right)\left(\int_{S^{d-1}(1)} d\Omega\right). \tag{12.27}$$

Notice that on the LHS that the intgration is over a $(d-1)-$spheres of radius r, but on the RHS the integration is over a three-sphere of radius $r=1$. The integral

$$\int_{S^{d-1}(1)} d\Omega \tag{12.28}$$

is the volume of a $(d-1)-$sphere of unit radius.

We perform the integration over r,

$$\begin{aligned}\int_0^\infty \exp\left(-\frac{1}{2}r^2\right) r^{d-1}dr &= \int_0^\infty \exp(-u)(\sqrt{2}u^{1/2})^{d-2}du \\ &= 2^{(d-2)/2}\int_0^\infty \exp(-u)u^{d/2-1}du \\ &= 2^{(d-2)/2}\Gamma(d/2) \qquad (12.29)\end{aligned}$$

where we have used

$$\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du \qquad (12.30)$$

So that

$$\int_{S^{d-1}(1)} d\Omega = \frac{I}{2^{(d-2)/2}\Gamma(d/2)} = \frac{(2\pi)^{d/2}}{2^{(d-2)/2}\Gamma(d/2)}. \qquad (12.31)$$

Finally

$$\int d\Omega_{d-1} = \frac{2(\pi)^{d/2}}{\Gamma(d/2)} \qquad (12.32)$$

□

12.5.3 The Γ and Beta Functions

$$\Gamma(z) = \int_0^\infty e^{-u}u^{z-1}du \qquad (12.33)$$

We define the *beta* function by

$$B(p,q) = \int_0^1 u^{p-1}(1-u)^{q-1}du \qquad (12.34)$$

We set $u = t/(1+t)$ so $du = dt/(1+t)^2$ and $t = u/(1-u)$. Then

$$
\begin{aligned}
B(p,q) &= \int_0^\infty \left(\frac{t}{1+t}\right)^{p-1}\left(\frac{1}{1+t}\right)^{q-1}\frac{dt}{(1+t)^2} \\
&= \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}dt
\end{aligned}
\tag{12.35}
$$

It is possible to express $B(p,q)$ in terms of gamma functions

$$
\begin{aligned}
\Gamma(p)\Gamma(q) &= \int_0^\infty e^{-u}u^{p-1}du\int_0^\infty e^{-t}t^{q-1}dt \\
&= \int_0^\infty e^{-u}u^{p-1}\left(u^q\int_0^\infty e^{-uv}v^{q-1}dv\right)du \\
&= \int_0^\infty v^{q-1}\left(\int_0^\infty u^{p+q-1}e^{-u(1+v)}du\right)dv \\
&= \int_0^\infty v^{q-1}\left(\int_0^\infty \frac{1}{(1+v)^{p+q}}\tau^{p+q-1}e^{-\tau}d\tau\right)dv \\
&= \int_0^\infty \frac{v^{q-1}}{(1+v)^{p+q}}dv\int_0^\infty e^{-\tau}\tau^{p+q-1}d\tau \\
&= B(p,q)\Gamma(p+q).
\end{aligned}
\tag{12.36}
$$

So that

$$
B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{12.37}
$$

Comparing (12.35) and (12.37) gives

$$
\int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}. \tag{12.38}
$$

12.5.4 Expressing Integrals in terms of Γ Functions

$$
I = \int d^dq\frac{1}{(q^2+2q\cdot p-m^2)^\alpha}. \tag{12.39}
$$

The integral can be written

$$
\int d^dq\frac{1}{(q^2+2q\cdot p-m^2)^\alpha} = \int d^dq\frac{1}{((q+p)^2-p^2-m^2)^\alpha}. \tag{12.40}
$$

Making the substitution $q'_\mu = q_\mu + p_\mu$ so that

$$I = \int d^d q \frac{1}{(q^2 - p^2 - m^2)^\alpha}. \tag{12.41}$$

Then

$$I = \int d\Omega_{d-1} \int dr \frac{r^{d-1}}{(r^2 - p^2 - m^2)^\alpha}. \tag{12.42}$$

In general

$$\begin{aligned} \int_0^\infty dr \frac{r^\beta}{(r^2 + M^2)^\alpha} &= \frac{1}{M^{2\alpha-\beta-1}} \int_0^\infty \frac{dr^2}{2} \frac{(r^2)^{(\beta-1)/2}}{(r^2+1)^\alpha} \\ &= \frac{1}{M^{2\alpha-\beta-1}} \frac{1}{2} \int_0^\infty dt \frac{t^{(\beta-1)/2}}{(1+t)^\alpha} \end{aligned} \tag{12.43}$$

We have $p = (\beta+1)/2$ and $p + q = \alpha$, so that

$$\int_0^\infty dr \frac{r^\beta}{(r^2 + M^2)^\alpha} = \frac{\Gamma\left(\frac{1}{2}(1+\beta)\right)\Gamma\left(\alpha - \frac{1}{2}(1+\beta)\right)}{2\Gamma(\alpha)(M^2)^{\alpha-(\beta+1)/2}} \tag{12.44}$$

The final result is

$$I = \int d^d q \frac{1}{(q^2 + 2q\cdot p - m^2)^\alpha} = \frac{i\pi^{\frac{d}{2}}\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)(-p^2 - m^2)^{\alpha - d/2}}. \tag{12.45}$$

Now

$$\begin{aligned} \int d^d q \frac{q_\mu}{(q^2 + 2q\cdot p - m^2)^\alpha} &= -\frac{1}{2(\alpha-1)} \frac{\partial}{\partial p_\mu} \int d^d q \frac{1}{(q^2+2q\cdot p - m^2)^{\alpha-1}} \\ &= \frac{i\pi^{\frac{d}{2}}\left(\alpha - 1 - \frac{d}{2}\right)\Gamma\left(\alpha - 1 - \frac{d}{2}\right)}{(\alpha-1)\Gamma(\alpha-1)(-p^2-m^2)^{\alpha-d/2}}(-p_\mu) \\ &= \frac{i\pi^{\frac{d}{2}}\Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)(-p^2-m^2)^{\alpha-d/2}}(-p_\mu). \end{aligned} \tag{12.46}$$

Next

$$\begin{aligned}
\int d^d q \frac{q_\mu q_\nu}{(q^2+2q\cdot p-m^2)^\alpha} &= -\frac{1}{2(\alpha-1)}\frac{\partial}{\partial p_\nu}\int d^d q \frac{q_\mu}{(q^2+2q\cdot p-m^2)^{\alpha-1}} \\
&= \frac{i\pi^{\frac{d}{2}}\Gamma\left(\alpha-1-\frac{d}{2}\right)}{\Gamma(\alpha)} \times -\frac{1}{2}\frac{\partial}{\partial p_\nu}\frac{(-p_\mu)}{(-p^2-m^2)^{\alpha-1-d/2}} \\
&= \frac{i\pi^{\frac{d}{2}}}{(-p^2-m^2)^{\alpha-d/2}}\frac{1}{\Gamma(\alpha)}\left[\Gamma\left(\alpha-\frac{d}{2}\right)p_\mu p_\nu\right. \\
&\quad \left. + \Gamma\left(\alpha-1-\frac{d}{2}\right)\frac{1}{2}\delta_{\mu\nu}(-p^2-m^2)\right].
\end{aligned} \tag{12.47}$$

Contracting the previous result with $g^{\mu\nu}$ yields

$$\begin{aligned}
\int d^d q \frac{q^2}{(q^2+2q\cdot p-m^2)^\alpha} &= \frac{i\pi^{\frac{d}{2}}}{(-p^2-m^2)^{\alpha-d/2}}\frac{1}{\Gamma(\alpha)}\left[\Gamma\left(\alpha-\frac{d}{2}\right)p^2\right. \\
&\quad \left. + \Gamma\left(\alpha-1-\frac{d}{2}\right)\frac{d}{2}(-p^2-m^2)\right].
\end{aligned} \tag{12.48}$$

Next

$$\begin{aligned}
\int d^d q \frac{q_\mu q_\nu q_\lambda}{(q^2+2q\cdot p-m^2)^\alpha} &= -\frac{1}{2(\alpha-1)}\frac{\partial}{\partial p_\lambda}\int d^d q \frac{q_\mu q_\nu}{(q^2+2q\cdot p-m^2)^{\alpha-1}} \\
&= -\frac{i\pi^{\frac{d}{2}}}{\Gamma(\alpha)}\frac{1}{2}\frac{\partial}{\partial p_\lambda}\left[\Gamma\left(\alpha-1-\frac{d}{2}\right)\frac{p_\mu p_\nu}{(-p^2-m^2)^{\alpha-1-d/2}}\right. \\
&\quad \left. + \Gamma\left(\alpha-2-\frac{d}{2}\right)\frac{1}{2}\delta_{\mu\nu}\frac{1}{(-p^2-m^2)^{\alpha-2-d/2}}\right] \\
&= \frac{i\pi^{\frac{d}{2}}}{(-p^2-m^2)^{\alpha-d/2}}\frac{1}{\Gamma(\alpha)}\left[-\Gamma\left(\alpha-\frac{d}{2}\right)p_\mu p_\nu p_\lambda\right. \\
&\quad \left. - \Gamma\left(\alpha-1-\frac{d}{2}\right)(-p^2-m^2)\frac{1}{2}(\delta_{\mu\nu}p_\lambda+\delta_{\mu\lambda}p_\nu+\delta_{\nu\lambda}p_\mu)\right].
\end{aligned} \tag{12.49}$$

12.6 Renormalisation

12.7 Ward-Takahashi Identities - differentiation

Observe that

$$-\frac{\partial}{\partial p^\mu}\frac{1}{\not p - m + i\epsilon} = \frac{1}{\not p - m + i\epsilon}\gamma_\mu\frac{1}{\not p - m + i\epsilon} \tag{12.50}$$

which follows from the matrix equation

$$\frac{1}{X+Y} = \frac{1}{X} - \frac{1}{X}Y\frac{1}{X} + \ldots \tag{12.51}$$

viz

$$-\left(\frac{1}{\not p + \delta p^\mu \gamma_\mu - m + i\epsilon} - \frac{1}{\not p - m + i\epsilon}\right) = \frac{1}{\not p - m + i\epsilon}\delta p^\mu \gamma_\mu \frac{1}{\not p - m + i\epsilon} + \mathcal{O}\left((\delta p^\mu)^2\right) \tag{12.52}$$

where no summation is implied over μ.

12.8 Ward-Takahashi Identities

From

$$-iek_\mu\gamma^\mu = -ie\left[(\not p_i + \not k - m) - (\not p_i - m)\right] \tag{12.53}$$

we easily have

$$\frac{i}{\not p_i + \not k - m}\left(-ie\not k\right)\frac{i}{\not p_i - m} = e\left(\frac{i}{\not p_i - m} - \frac{i}{\not p_i + \not k - m}\right) \tag{12.54}$$

12.8.1 Ward-Takahashi Identity - Two external points

Ward-Takahashi Identity - one q_1

The diagram with the photon inserted in the first position

$$\begin{aligned}\left(\frac{i}{\not p' + \not k - m}\right)\gamma^\lambda\left(\frac{i}{\not p + \not k - m}\right)\left(-iek_\mu\gamma^\mu\right)\left(\frac{i}{\not p - m}\right)& \\ = e\left(\frac{i}{\not p' + \not k - m}\right)\gamma^\lambda\left(\frac{i}{\not p - m} - \frac{i}{\not p + \not k - m}\right)&\end{aligned} \tag{12.55}$$

The diagram with the photon inserted in the second position

$$\left(\frac{i}{\not p' + \not k - m}\right)\left(-iek_\mu \gamma^\mu\right)\left(\frac{i}{\not p' - m}\right)\gamma^\lambda\left(\frac{i}{\not p - m}\right)$$
$$= e\left(\frac{i}{\not p' - m} - \frac{i}{\not p' + \not k - m}\right)\gamma^\lambda\left(\frac{i}{\not p - m}\right) \tag{12.56}$$

Adding together (12.55) and (12.56) the RHS gives

$$\begin{aligned} &= e\left(\left(\frac{i}{\not p' - m}\right)\gamma^\lambda\left(\frac{i}{\not p - m}\right) - \left(\frac{i}{\not p' + \not k - m}\right)\gamma^\lambda\left(\frac{i}{\not p + \not k - m}\right)\right) \\ &= e\left(\left(\frac{i}{\not q - \not k - m}\right)\gamma^\lambda\left(\frac{i}{\not p - m}\right) - \left(\frac{i}{\not q - m}\right)\gamma^\lambda\left(\frac{i}{\not p + \not k - m}\right)\right) \end{aligned} \tag{12.57}$$

where we have renamed $p' + k = q$.

Ward-Takahashi Identity - two q_1, q_2

The diagram with the photon inserted in the first position

$$\left(\frac{i}{\not p' + \not k - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not p_1 + \not k - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not p + \not k - m}\right)\left(-iek_\mu \gamma^\mu\right)\left(\frac{i}{\not p - m}\right)$$
$$= e\left(\frac{i}{\not p' + \not k - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not p_1 + \not k - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not p - m} - \frac{i}{\not p + \not k - m}\right) \tag{12.58}$$

The diagram with the photon inserted in the second position

$$\left(\frac{i}{\not p' + \not k - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not p_1 + \not k - m}\right)\left(-iek_\mu \gamma^\mu\right)\left(\frac{i}{\not p_1 - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not p - m}\right)$$
$$= e\left(\frac{i}{\not p' + \not k - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not p_1 - m} - \frac{i}{\not p_1 + \not k - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not p - m}\right) \tag{12.59}$$

The diagram with the photon inserted in the third position

$$
\begin{aligned}
&\left(\frac{i}{\not{p}' + \not{k} - m}\right)\left(-iek_\mu \gamma^\mu\right)\left(\frac{i}{\not{p}' - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not{p}_1 - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not{p} - m}\right) \\
&\qquad = e\left(\frac{i}{\not{p}' - m} - \frac{i}{\not{p}' + \not{k} - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not{p}_1 - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not{p} - m}\right)
\end{aligned} \tag{12.60}
$$

Adding together (12.58), (12.59) and (12.60) the RHS gives

$$
\begin{aligned}
= \; & e\left(\left(\frac{i}{\not{p}' - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not{p}_1 - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not{p} - m}\right)\right. \\
& \quad \left. - \left(\frac{i}{\not{p}' + \not{k} - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not{p}_1 + \not{k} - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not{p} + \not{k} - m}\right)\right) \\
= \; & e\left(\left(\frac{i}{\not{q} - \not{k} - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not{p}_1 - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not{p} - m}\right)\right. \\
& \quad \left. - \left(\frac{i}{\not{q} - m}\right)\gamma^{\lambda_2}\left(\frac{i}{\not{p}_1 + \not{k} - m}\right)\gamma^{\lambda_1}\left(\frac{i}{\not{p} + \not{k} - m}\right)\right)
\end{aligned} \tag{12.61}
$$

where we have renamed $p' + k = q$.

Ward-Takahashi Identity - general case

12.8.2 Ward-Takahashi Identity - Closed electron loop

Chapter 13

Vacuum Polarisation

13.1 Modified Photon Propagator

The influence of the creation of a virtual electron-positron pair on the propagagtor of a photon. We investigate how original photon propagator

$$iD_{F\mu\nu}(q) = \frac{-4\pi i}{q^2 + i\epsilon} g_{\mu\nu} \tag{13.1}$$

is modified by the correction of the order e^2

in equation form:

$$iD'_{F\mu\nu}(q) = iD_{F\mu\nu}(q) = iD_{F\mu\lambda}(q)\frac{i\Pi^{\lambda\sigma}(q)}{4\pi}iD_{F\sigma\nu}(q) \tag{13.2}$$

where the polarisation tensor is given by

$$\frac{i\Pi^{\lambda\sigma}(q)}{4\pi} = -e^2 \int \frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[\gamma_\lambda \frac{1}{\not{k} - m + i\epsilon}\gamma_\sigma \frac{1}{\not{k} - \not{q} - m + i\epsilon}\right] \tag{13.3}$$

13.1.1 Calculation of Π

$$
\begin{aligned}
\frac{i\Pi^{\mu\nu}(q)}{4\pi} &= -e^2 \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\left[\gamma^\mu \frac{\not{k} - m}{k^2 - m^2 + i\epsilon} \gamma^\nu \frac{\not{k} - \not{q} - m}{(k-q)^2 - m^2 + i\epsilon}\right] \\
&= -e^2 \int \frac{d^dk}{(2\pi)^d} \mathrm{Tr}\left[\frac{(k_\lambda \gamma^\mu \gamma^\lambda - m\gamma^\mu)((k-q)_\sigma \gamma^\nu \gamma^\sigma - m\gamma^\nu)}{[k^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}\right] \\
&= -e^2 \int \frac{d^dk}{(2\pi)^d} \left[\frac{k_\lambda (k-q)_\sigma \mathrm{Tr}(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma) + m^2 \mathrm{Tr}(\gamma^\mu \gamma^\nu)}{[k^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}\right] \qquad (13.4)
\end{aligned}
$$

where we have used that the trace of an odd number of gamma matrtices is zero. We use the results of appendix A.1

$$
\mathrm{Tr}(\gamma^\mu \gamma^\lambda \gamma^\nu \gamma^\sigma) = 4(g^{\mu\lambda} g^{\nu\sigma} - g^{\mu\nu} g^{\lambda\sigma} + g^{\mu\sigma} g^{\lambda\nu}), \qquad \mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}.
$$

in the evaluation of the numerator. Going to Euclidean space

$$
\frac{i\Pi^{\mu\nu}(q)}{4\pi} = -4ie^2 \int \frac{d_E^dk}{(2\pi)^d} \left[\frac{k^\mu (k-q)^\nu + k^\nu (k-q)^\mu - g^{\mu\nu}[k \cdot (k-q) - m^2]}{[k^2 - m^2 + i\epsilon][(k-q)^2 - m^2 + i\epsilon]}\right] \qquad (13.5)
$$

We now introduce the Feynman parameterisation

$$
\frac{1}{ab} = \int_0^1 d\beta \frac{1}{[a + (b-a)\beta]^2}
$$

to combine the denominator factors in (13.5)

$$
\begin{aligned}
\frac{1}{[k^2 - m^2][(k-q)^2 - m^2]} &= \int_0^1 d\beta \frac{1}{[(k^2 - m^2) + ((k-q)^2 - m^2 - (k^2 - m^2))\beta]} \\
&= \int_0^1 d\beta \frac{1}{[k^2 - k \cdot q\beta + q^2\beta - m^2]^2} \qquad (13.6)
\end{aligned}
$$

So that

$$
\begin{aligned}
\frac{i\Pi^{\mu\nu}(q)}{4\pi} &= -4ie^2 \int \frac{d_E^dk}{(2\pi)^d} \int_0^1 d\beta \left[\frac{k^\mu (k-q)^\nu + k^\nu (k-q)^\mu - g^{\mu\nu}[k \cdot (k-q) - m^2]}{[k^2 - 2k \cdot q\beta + q^2\beta - m^2]^2}\right] \\
&= -4ie^2 \int_0^1 d\beta \int \frac{d_E^d\ell}{(2\pi)^d} \frac{(2\ell^\mu \ell^\nu - 2\beta(1-\beta)q^\mu q^\nu - g^{\mu\nu}[\ell^2 - \beta(1-\beta)q^2 - m^2]}{[\ell^2 + \beta(1-\beta)q^2 - m^2]^2}
\end{aligned}
$$
(13.7)

where $\ell = k - \beta q$ and eliminated terms linear in ℓ^μ in the numerator. Now make the replacement

$$\ell^\mu \ell^\nu \mapsto \frac{1}{d}\ell^2 g^{\mu\nu} \tag{13.8}$$

in the above integral,

$$\begin{aligned}
\frac{i\Pi^{\mu\nu}(q)}{4\pi} &= -4ie^2 \int_0^1 d\beta \int \frac{d_E^d \ell}{(2\pi)^d} \frac{\left(\frac{2}{d}-1\right)\ell^2 g^{\mu\nu} - 2\beta(1-\beta)q^\mu q^\nu + g^{\mu\nu}[\beta(1-\beta)q^2 + m^2]}{[\ell^2 + q^2\beta(1-\beta) - m^2]^2} \\
&= -4ie^2 \int_0^1 d\beta \int \frac{d_E^d \ell}{(2\pi)^d} \frac{\left(\frac{2}{d}-1\right)\ell^2 g^{\mu\nu} - 2\beta(1-\beta)q^\mu q^\nu + g^{\mu\nu}[\beta(1-\beta)q^2 + m^2]}{[\ell^2 + \Delta]^2}
\end{aligned} \tag{13.9}$$

where $\Delta = \beta(1-\beta)q^2 - m^2$. Now

$$\begin{aligned}
\int \frac{d_E^d \ell}{(2\pi)^d} \frac{1}{[\ell^2 + \Delta]^2} &= \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \\
\int \frac{d_E^d \ell}{(2\pi)^d} \frac{\ell^2}{[\ell^2 + \Delta]^2} &= \frac{1}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma\left(2 - \frac{d}{2} - 1\right)}{\Gamma(2)} \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}-1}
\end{aligned} \tag{13.10}$$

Let us apply these dimensional regularisation formulae to the integral ().

$$\begin{aligned}
\int \frac{d_E^d \ell}{(2\pi)^d} \frac{\left(\frac{2}{d}-1\right) g^{\mu\nu}\ell^2}{[\ell^2 + \Delta]^2} &= \frac{1}{(4\pi)^{d/2}} \left(1 - \frac{d}{2}\right) \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{1-\frac{d}{2}} \\
&= \frac{1}{(4\pi)^{d/2}} \Gamma\left(2 - \frac{d}{2}\right) \left(\frac{1}{\Delta}\right)^{2-\frac{d}{2}} \cdot (g^{\mu\nu}\Delta)
\end{aligned} \tag{13.11}$$

Dimensional regularisation preserves the Ward identity.

$$\begin{aligned}
\frac{i\Pi^{\mu\nu}(q)}{4\pi} &= -4ie^2 \int_0^1 d\beta \frac{1}{(4\pi)^{d/2}} \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \\
&\quad \times \left[g^{\mu\nu}(\beta(1-\beta)q^2 - m^2) + g^{\mu\nu}(\beta(1-\beta)q^2 + m^2) - 2\beta(1-\beta)q^\mu q^\nu\right] \\
&= (q^2 g^{\mu\nu} - q^\mu q^\nu) \times i\frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 d\beta \beta(1-\beta) \frac{\Gamma\left(2 - \frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \\
&= (q^2 g^{\mu\nu} - q^\mu q^\nu) \cdot i\Pi(q^2).
\end{aligned} \tag{13.12}$$

where

$$
\begin{aligned}
\Pi(q^2) \quad &= \quad \frac{-8e^2}{(4\pi)^{d/2}} \int_0^1 d\beta\beta(1-\beta)\frac{\Gamma\left(2-\frac{d}{2}\right)}{\Delta^{2-\frac{d}{2}}} \\
&\to_{d\to 4} \quad \frac{-8e^2}{(4\pi)^2} \int_0^1 d\beta\beta(1-\beta)(4\pi)^{\epsilon/2}\frac{\Gamma(\epsilon/2)}{\Delta^{\epsilon/2}} \\
&\to_{d\to 4} \quad -\frac{e^2}{2\pi^2} \int_0^1 d\beta\beta(1-\beta)e^{(\epsilon/2)\ln(4\pi)}e^{-(\epsilon/2)\ln\Delta}\left(\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right) \\
&\to_{d\to 4} \quad -\frac{2\alpha}{\pi} \int_0^1 d\beta\beta(1-\beta)\left(\frac{2}{\epsilon}-\ln\Delta-\gamma+\ln(4\pi)+\mathcal{O}(\epsilon)\right). \qquad (13.13)
\end{aligned}
$$

13.2 Uehling Potential

The potential of a stationary external charge is

$$
j_\mu(x) = Ze\delta^3(\mathbf{x})\,\delta_{\mu 0}. \qquad (13.14)
$$

With the use of the modified photon propagator D'_F we can take into account the correction to the potential due to vacuum polarisation.

In the following we will need

$$
\left(\nabla^2-\mu^2\right)\frac{e^{-\mu r}}{r} = -4\pi\delta^3(\mathbf{x}). \qquad (13.15)
$$

which can be derived from the well known identity

$$
\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\mathbf{x}). \qquad (13.16)
$$

The modified 4-potential generally in the momentum space is given by

$$
\begin{aligned}
A'_\mu(x) \quad &= \quad \int d^4y\; D'_{F\mu\nu}(x-y)\; j_\nu(y) \\
&= \quad \int \frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}D'_{F\mu\nu}(q)j_\nu(q). \qquad (13.17)
\end{aligned}
$$

In momentum space the modified photon propagator

$$
\begin{aligned}
D'_{F\mu\nu}(q) &= D_{F\mu\nu}(q) - D_{F\mu\lambda}(q)\frac{\Pi^{\lambda\sigma}(q)}{4\pi}D_{F\sigma\nu}(q) \\
&= \frac{-4\pi g_{\mu\nu}}{q^2} - \frac{-4\pi g_{\mu\lambda}}{q^2}\frac{q^2 g^{\lambda\sigma} - q^\lambda q^\sigma}{4\pi}\Pi^R(q^2)\frac{-4\pi g_{\sigma\nu}}{q^2} \\
&= \frac{-4\pi g_{\mu\nu}}{q^2}\left(1+\Pi^R(q^2)\right) + \frac{4\pi q_\mu q_\nu}{q^4}\Pi^R(q^2) \qquad (13.18)
\end{aligned}
$$

Thus the modified potential (13.17) is

$$
\begin{aligned}
A'_\mu(x) &= \int \frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\left(1+\Pi^R(q^2)\right)D_{F\mu\nu}(q)j^\nu(q) \\
&= \int \frac{d^4q}{(2\pi)^4}e^{-iq\cdot x}\left(1+\Pi^R(q^2)\right)A_\mu(q). \qquad (13.19)
\end{aligned}
$$

In the case we are considering the current source is stationary, $j^\nu(x) = j^\nu(\mathbf{x})$, and the q_0-dependence drops out according to

$$
\begin{aligned}
j^\nu(q) &= \int d^4y j^\nu(y) \\
&= \int dy_0 e^{iq_0y_0}\int d^3y e^{-i\mathbf{q}\cdot\mathbf{y}}j^\nu(\mathbf{y}) \\
&= 2\pi\delta(q_0)j^\nu(q). \qquad (13.20)
\end{aligned}
$$

Thus (13.19) reduces to

$$
\begin{aligned}
A'_\mu(\mathbf{x}) &= \int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\left(1+\Pi^R(-\mathbf{q}^2)\right)D_{F\mu\nu}(0,\mathbf{q})j^\nu(\mathbf{q}) \\
&= \int \frac{d^3q}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\left(1+\Pi^R(-\mathbf{q}^2)\right)A_\mu(\mathbf{q}). \qquad (13.21)
\end{aligned}
$$

For the electrostatic point charge (13.14) we have

$$
j^\nu(\mathbf{q}) = \int d^3y e^{-i\mathbf{q}\cdot\mathbf{y}}\left(-Ze\delta_{\nu0}\delta^3(\mathbf{y})\right) = -Ze\ \delta_{\nu0}, \qquad (13.22)
$$

so that

$$A'_\mu(\mathbf{x}) = -Ze \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \left(1 + \Pi^R(-\mathbf{q}^2)\right) D_{F\mu 0}(0, \mathbf{q}). \tag{13.23}$$

We now insert the explicit formula for the polarisation function

$$A'_\mu(\mathbf{x}) = -Ze \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{4\pi}{\mathbf{q}^2} \left[1 + \frac{2\alpha}{\pi} \int_0^1 d\beta\, \beta(1-\beta) \ln\left(1 + \frac{\mathbf{q}^2}{m^2}\beta(1-\beta)\right)\right] \tag{13.24}$$

Here we have assumed that charge renormalisation has already been performed. From the identity (13.16) we have

$$\int \frac{d^3q}{(2\pi)^3} \frac{\exp(i\mathbf{q}\cdot\mathbf{x})}{\mathbf{q}^2} = \frac{1}{4\pi r} \tag{13.25}$$

from which we see that the Fourier transform of the first term yields the usual Coulomb potential. Next we treat the correction term,

$$\begin{aligned}
\Pi^R(-\mathbf{q}^2) &= \frac{2\alpha}{\pi} \int_0^1 d\beta\beta(1-\beta) \ln\left(1 + \frac{\mathbf{q}^2}{m^2}\beta(1-\beta)\right) \\
&= \frac{2\alpha}{\pi} \left[\left(\frac{\beta^2}{2} - \frac{\beta^3}{3}\right) \ln\left(1 + \frac{\mathbf{q}^2}{m^2}\beta(1-\beta)\right)\right]_0^1 \\
&\quad -2\frac{\alpha}{\pi}\frac{\mathbf{q}^2}{m^2} \int_0^1 d\beta \left(\frac{\beta^2}{2} - \frac{\beta^3}{3}\right) \frac{1-2\beta}{1 + \frac{\mathbf{q}^2}{m^2}\beta(1-\beta)} \\
&= 2\frac{\alpha}{\pi}\frac{\mathbf{q}^2}{m^2} \int_0^1 d\beta \left(\frac{\beta^2}{2} - \frac{\beta^3}{3}\right) \frac{2\beta-1}{1 + \frac{\mathbf{q}^2}{m^2}\beta(1-\beta)}
\end{aligned}$$

We now introduce the new variable $v = 2\beta - 1$ (or $\beta = (v+1)/2$). Then $v^2 - 1 = 4\beta(\beta - 1)$ and

$$\begin{aligned}
\frac{\beta^2}{2} - \frac{\beta^3}{3} &= \frac{(v+1)^2}{4}\left(\frac{1}{2} - \frac{1}{3}\frac{v+1}{2}\right) \\
&= \frac{1}{8}(v^2 + 2v + 1)(\frac{2}{3} - \frac{1}{3}v) \\
&= \frac{1}{8}v(-\frac{1}{3}v^2 + \frac{4}{3} - \frac{1}{3}) + \text{even in } v \\
&= \frac{1}{8}v(1 - \frac{1}{3}v^2) + \text{even in } v.
\end{aligned} \tag{13.26}$$

so that

$$
\begin{aligned}
\Pi^R(-\mathbf{q}^2) &= \frac{1}{2}\frac{\alpha}{\pi}\frac{\mathbf{q}^2}{4m^2}\int_{-1}^{1} dv \frac{v^2(1-\frac{1}{3}v^2)}{1+\frac{\mathbf{q}^2}{4m^2}(1-v^2)} \\
&= \frac{\alpha}{\pi}\frac{\mathbf{q}^2}{4m^2}\int_{0}^{1} dv \frac{v^2(1-\frac{1}{3}v^2)}{1+\frac{\mathbf{q}^2}{4m^2}(1-v^2)} \qquad (13.27)
\end{aligned}
$$

The identity (13.15) results in the formula

$$
\int \frac{d^3q}{(2\pi)^3}\frac{\exp(i\mathbf{q}\cdot\mathbf{x})}{\mathbf{q}^2+a^2} = \frac{1}{4\pi}\frac{\exp(-ar)}{r} \qquad (13.28)
$$

which easily follows from the residue integration.

The potential reads

$$
\begin{aligned}
A_0'(\mathbf{x}) &= -Ze\left[\frac{1}{r} + \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{4\pi}{\mathbf{q}^2} \times \frac{\alpha}{\pi}\frac{\mathbf{q}^2}{4m^2}\int_0^1 dv \frac{v^2(1-\frac{1}{3}v^2)}{1+\frac{\mathbf{q}^2}{4m^2}(1-v^2)}\right] \\
&= -Ze\left[\frac{1}{r} + \frac{\alpha}{m^2}\int_0^1 dv \frac{v^2(1-\frac{1}{3}v^2)}{\frac{1}{4m^2}(1-v^2)} \int \frac{d^3q}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} \frac{1}{\frac{4m^2}{1-v^2}+\mathbf{q}^2}\right] \\
&= -Ze\left[\frac{1}{r} + \frac{\alpha}{\pi}\int_0^1 dv \frac{v^2(1-\frac{1}{3}v^2)}{(1-v^2)} \exp\left(-\frac{2m}{\sqrt{1-v^2}}r\right)\right]. \qquad (13.29)
\end{aligned}
$$

A further transformation $\zeta = (1-v^2)^{-1/2}$ or $v^2 = 1 - 1/\zeta^2$ with $vdv = \zeta^{-3}d\zeta$,

$$
\begin{aligned}
A_0(r) &= -Ze\left[\frac{1}{r} + \frac{\alpha}{\pi}\frac{1}{3}\int_1^\infty \frac{d\zeta}{\zeta^3} v(3-v^2)\zeta^2 e^{-2m\zeta r}\right] \\
&= -Ze\left[\frac{1}{r} + \frac{2\alpha}{3\pi}\int_1^\infty d\zeta \left(1+\frac{1}{2\zeta^2}\right)\frac{\sqrt{\zeta^2-1}}{\zeta^2} e^{-2m\zeta r}\right] \qquad (13.30)
\end{aligned}
$$

This is the often quoted integral representation of the Uehling potential.

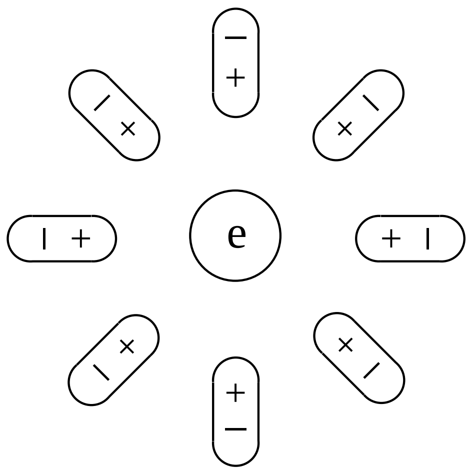


Figure 13.1: Virtual e^+e^- pairs acting as dipoles to screen the bare charge.

Chapter 14

Electron Self-Energy Graph

The electron propagator will be modified by the process where by a virtual photon is emitted and reabsorbed. We want to calculate how the undisturbed electron prpagator

$$iS_F(p) = \frac{i}{\not{p} - m + i\epsilon} \tag{14.1}$$

is modified by the Feynman graph shown in fig ?. With corresponding equation

$$iS'_F = iS_F(p) + iS_F(p)\left(-i\Sigma(p)\right) iS_F(p) \tag{14.2}$$

where the self-energy function is given by

$$-i\Sigma(p) = (-ie)^2 \int d^d q (2\pi)^4 \frac{-4\pi i}{q^2 + i\epsilon} \gamma^\mu \frac{i}{\not{p} - \not{q} - m + i\epsilon} \gamma_\mu \tag{14.3}$$

Chapter 15

The Vertex Function

We examine the change in the vertex due to a virtual photon.

With corresponding equation

$$-ie\Lambda_{\mu}(p', p) = -ie\gamma_{\mu} - ie\Sigma_{\mu}(p', p) \tag{15.1}$$

where the vertex function is given by

$$\Sigma_{\mu}(p', p) = -4\pi i e^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 - \mu^2 + i\epsilon} \left(\gamma^{\nu} \frac{1}{\not{p}' - \not{q} - m + i\epsilon} \right) \gamma_{\mu} \left(\gamma^{\nu} \frac{1}{\not{p} - \not{q} - m + i\epsilon} \right) \gamma_{\nu} \tag{15.2}$$

Chapter 16

Lamb Shift

According to the Dirac theory of the hydrogen atom, the $2_{s_{1/2}}$ and $2_{p_{1/2}}$ levels are degenerate.

Chapter 17

Anomalous Magnetic Moment of the Electron

Part VI

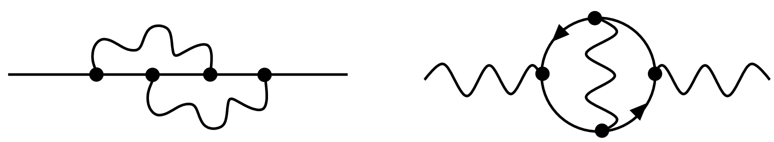
Renormalisability

Chapter 18

Renormalisability of QED - Dyson/Ward

18.1 Overlapping Divergences

The proof of renormalisation is straightforward for diagrams which do not have overlapping divergences. These overlapping divergences occur with self-energy diagrams, examples of which are given in fig 18.1.



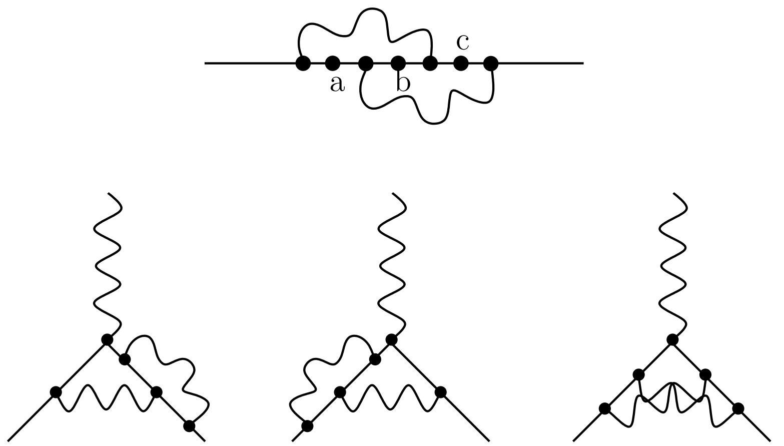
(a) (b)

Figure 18.1: QED diagrams with overlapping divergences.

18.2 Renormalisability to all Orders - Dyson/Ward

Because of the Ward-Takahashi identities we dont have to consider the self-energy diagrams explicitly. The Ward-Takahashi identity for the electron self-energy relates it directly to the electron vertex

$$\Lambda^{\mu}(p,p) = -\frac{\partial}{\partial p_{\mu}}\Sigma(p) \tag{18.1}$$



(a) (b) (c)

Figure 18.2: Taking derivative with respect to p_μ of the overlapping divergence.

Since the vertex contains no overlapping divergences, the proof of renormalisability is straighforward. Once the vertex is renormalised, the self-energy can be obtained from it.

There are unresolved concerns about overlapping divergences that emerge at the 14th order diagrams.

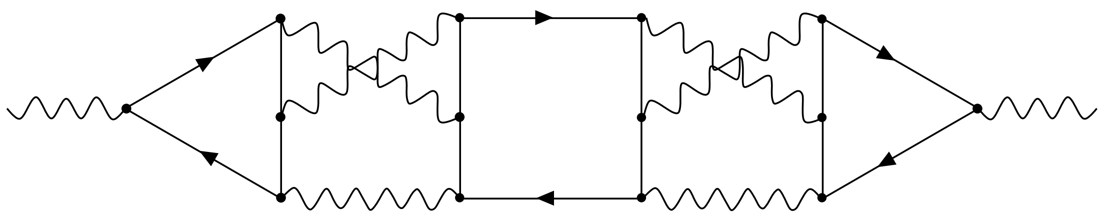


Figure 18.3: The graph apparently invalidates the original Dyson/Ward renormalisation proof.

Chapter 19

BPHZ Renormalisation

BPHZ theorem: All divergences can be removed by counterterms corresponding to superficially divergent 1PI amplitudes.

Part VII

Appendices

Appendix A

Appendix

A.1 Traces of Products of $\gamma-$Matrices

As we have already seen, when calculating Feynman diagrams and the resulting cross sections, one has to evaluate traces of certain combinations of $\gamma-$matrices. Here we collect useful theorems here, some of which have already been proved in the main text..

We will use the Dirac/Feynman slash notation: $a_\mu \gamma^\mu \equiv \not{a}$.

Theorem **1.** The trace of an odd number of $\gamma-$matrices vanishes. So $\mathrm{Tr}\not{a}_1 \cdots \not{a}_n = 0$ for odd number of terms.

Theorem **2.** We have $a_\mu b_\nu \, Tr\gamma^\mu\gamma^\nu = 4a \cdot b$.

Theorem **3.** We have

$$\begin{aligned} Tr\gamma^{\mu_1} \cdots \gamma^{\mu_n} &= \eta^{\mu_1\mu_2} Tr\gamma^{\mu_3} \cdots \gamma^{\mu_n} \\ &- \eta^{\mu_1\mu_3} Tr\gamma^{\mu_2}\gamma^{\mu_4} \ldots \gamma^{\mu_n} + \cdots \\ &+ \eta^{\mu_1\mu_n} Tr\gamma^{\mu_2} \cdots \gamma^{\mu_{n-1}}. \end{aligned} \tag{A.1}$$

A special case is

$$Tr\not{a}_1\not{a}_2\not{a}_3\not{a}_4 = 4(a_1 \cdot a_2 a_3 \cdot a_4 - a_1 \cdot a_3 a_2 \cdot a_4 + a_1 \cdot a_4 a_2 \cdot a_3). \tag{A.2}$$

Theorem **4.** We have $Tr\gamma^5 = 0$.

Theorem **5.** We have $Tr\gamma^5\gamma^\mu\gamma^\nu = 0$.

Theorem **6.** We have $a_\mu b_\nu c_\sigma d_\gamma Tr\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\gamma = -4i\epsilon^{\mu\nu\sigma\gamma}a_\mu b_\nu c_\sigma d_\gamma$. Where $\epsilon^{\mu\nu\sigma\gamma}$ is the completely antisymmetric tensor: $\epsilon^{\mu\nu\sigma\gamma} = +1$ if (μ,ν,σ,γ) is an even permutation of $(0,1,2,3)$, $\epsilon^{\mu\nu\sigma\gamma} = -1$ if (μ,ν,σ,γ) is an odd permutation of $(0,1,2,3)$, and $\epsilon^{\mu\nu\sigma\gamma} = 0$ if any two indices are identical.

Theorem **7.** We have $Tr\gamma_{\mu_1}\gamma_{\mu_2}\dots\gamma_{\mu_{2n}} = Tr\gamma_{\mu_{2n}}\dots\gamma_{\mu_1}$.

Theorem **8.** We have

i) $\gamma_\mu\gamma^\mu = 4\mathbb{1}$

ii) $\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu$

iii) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu = 4\eta^{\nu\sigma}\mathbb{1}$

iv) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\mu = -2\gamma^\gamma\gamma^\sigma\gamma^\nu$

v) $\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\delta\gamma^\mu = 2\gamma^\delta\gamma^\nu\gamma^\sigma\gamma^\gamma + 2\gamma^\gamma\gamma^\sigma\gamma^\nu\gamma^\delta$

Proof.

1. See (4.31).

2. See (4.33).

3. We use $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu + 2\eta^{\mu\nu}\mathbb{1}$ to shift γ^{μ_1} to the right-hand side of γ^{μ_2}, giving

$$Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} = 2\eta^{\mu_1\mu_2}\,Tr\gamma^{\mu_3}\cdots\gamma^{\mu_n} - Tr\gamma^{\mu_2}\gamma^{\mu_1}\dots\gamma^{\mu_n}$$

Doing it again we obtain

$$\begin{aligned} Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} &= 2\eta^{\mu_1\mu_2}\,Tr\gamma^{\mu_3}\cdots\gamma^{\mu_n} - Tr\gamma^{\mu_2}(\gamma^{\mu_1}\gamma^{\mu_3})\dots\gamma^{\mu_n} \\ &= 2\eta^{\mu_1\mu_2}\,Tr\gamma^{\mu_3}\cdots\gamma^{\mu_n} - 2\eta^{\mu_1\mu_3}\,Tr\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_n} + Tr\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_1}\dots\gamma^{\mu_n} \end{aligned}$$

Repeating this (and remembering that n must be even by part **1**)),

$$\begin{aligned} Tr\gamma^{\mu_1}\dots\gamma^{\mu_n} &= 2\eta^{\mu_1\mu_2}\,Tr\gamma^{\mu_3}\cdots\gamma^{\mu_n} - 2\eta^{\mu_1\mu_3}\,Tr\gamma^{\mu_2}\gamma^{\mu_4}\cdots\gamma^{\mu_n} + \\ &+ 2\eta^{\mu_1\mu_4}\,Tr\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_5}\cdots\gamma^{\mu_n} - \cdots \\ &+ 2\eta^{\mu_1\mu_n}Tr\gamma^{\mu_2}\cdots\gamma^{\mu_{n-1}} - Tr\gamma^{\mu_2}\cdots\gamma^{\mu_n}\gamma^{\mu_1} \end{aligned}$$

Lastly we use the invariance of the trace under cyclic permutation to move γ^{μ_1} back to the left-hand side of the trace.

The special case was computed in (4.33)-(4.34). We use the above theorem 3 to prove it again,

$$\begin{aligned} Tr\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\gamma^{\mu_4} &= \eta^{\mu_1\mu_2}Tr\gamma^{\mu_3}\gamma^{\mu_4} - \eta^{\mu_1\mu_3}Tr\gamma^{\mu_2}\gamma^{\mu_4} + \eta^{\mu_1\mu_4}Tr\gamma^{\mu_2}\gamma^{\mu_3} \\ &= 4(\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} - \eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}) \end{aligned}$$

where we have used theorem 2. Contracting with $a_{\mu_1}a_{\mu_2}a_{\mu_3}a_{\mu_4}$ we obtain (A.2).

4. See (8.23).

5. See (8.22)-(8.24)

6. We evaluate

$$Tr\gamma^5 \not{a}\not{b}\not{c}\not{d} = a_\mu b_\nu c_\sigma d_\gamma Tr\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\gamma$$

where summation over all repeated indices is implied. Most terms do not contribute; if any two of the indices μ, ν, σ, γ take indentical values, the trace will vanish. Take the first and third indices to be equal,

$$\begin{aligned} Tr\gamma^5\gamma^\mu\gamma^\nu\gamma^\mu\gamma^\gamma &= Tr\gamma^5\gamma^\mu(-\gamma^\mu\gamma^\nu + 2\eta^{\mu\nu}\mathbf{1})\gamma^\gamma \\ &= Tr\gamma^5(-\gamma^\mu\gamma^\mu\gamma^\nu + 2\gamma^\mu\eta^{\mu\nu}\mathbf{1})\gamma^\gamma \\ &= -\eta^{\mu\mu}Tr\gamma^5\gamma^\nu\gamma^\gamma + 2\eta^{\mu\nu}Tr\gamma^5\gamma^\mu\gamma^\gamma \\ &= 0 \end{aligned}$$

by theorem 5.

Thus only indices $(\mu, \nu, \sigma, \gamma)$ that are a permutation of $(0, 1, 2, 3)$ contribute. Let us evaluate the trace

$$\begin{aligned} Tr\gamma^5\gamma^0\gamma^1\gamma^2\gamma^3 &= Tr\gamma^5(-i\gamma^5) \\ &= -iTr\mathbb{1} \\ &= -4i \\ &= -4i\epsilon^{0123}. \end{aligned}$$

Since the four γ^μ matrices are mutually anticommuting, an odd permutation of the indices $(0, 1, 2, 3)$ introduces an additional minus sign.

□

7. We make use of the matrix $\hat{C}$, involved in charge conjugation. $\hat{C}$ has the property $\hat{C}\gamma_\mu \hat{C}^{-1} = -\gamma_\mu^T$. It follows that

$$
\begin{aligned}
Tr\gamma_{\mu_1}\gamma_{\mu_2}\dots\gamma_{\mu_{2n}} &= Tr(\hat{C}\gamma_{\mu_1}\hat{C}^{-1})(\hat{C}\gamma_{\mu_2}\hat{C}^{-1})\dots(\hat{C}\gamma_{\mu_{2n}}\hat{C}^{-1}) \\
&= (-1)^{2n}\, Tr\gamma_{\mu_1}^T\gamma_{\mu_2}^T\dots\gamma_{\mu_{2n}}^T \\
&= Tr[\gamma_{\mu_{2n}}\dots\gamma_{\mu_1}]^T \\
&= Tr\gamma_{\mu_{2n}}\dots\gamma_{\mu_1}.
\end{aligned}
$$

8.

i) $\gamma_\mu\gamma^\mu = \frac{1}{2}(\gamma_\mu\gamma^\mu + \gamma^\mu\gamma_\mu) = \frac{1}{2}2g^\mu{}_\mu \mathbb{1} = 4\mathbb{1}$.

ii) In the following we use i)

$$
\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\mu &= \gamma_\mu(2\eta^{\mu\nu} - \gamma^\mu\gamma^\nu) \\
&= 2\gamma^\nu - 4\gamma^\nu \\
&= -2\gamma^\nu.
\end{aligned}
$$

iii) In the following we use ii)

$$
\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu &= \gamma_\mu\gamma^\nu(2\eta^{\mu\sigma} - \gamma^\mu\gamma^\sigma) \\
&= 2\gamma^\sigma\gamma^\nu - \gamma_\mu\gamma^\nu\gamma^\mu\gamma^\sigma \\
&= 2\gamma^\sigma\gamma^\nu + 2\gamma^\nu\gamma^\sigma \\
&= 2(2\eta^{\nu\sigma} - \gamma^\nu\gamma^\sigma) + 2\gamma^\nu\gamma^\sigma \\
&= 4\eta^{\nu\sigma}
\end{aligned}
$$

iv) In the following we use iii)

$$
\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\mu &= \gamma_\mu\gamma^\nu\gamma^\sigma(2\eta^{\mu\gamma} - \gamma^\mu\gamma^\gamma) \\
&= 2\gamma^\gamma\gamma^\nu\gamma^\sigma - (\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu)\gamma^\gamma \\
&= 2\gamma^\gamma\gamma^\nu\gamma^\sigma - 4\eta^{\nu\sigma}\gamma^\gamma \\
&= 2\gamma^\gamma(2\eta^{\nu\sigma} - \gamma^\sigma\gamma^\nu) - 4\eta^{\nu\sigma}\gamma^\gamma \\
&= -2\gamma^\gamma\gamma^\sigma\gamma^\nu.
\end{aligned}
$$

v)

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\delta \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma (2\eta^{\mu\delta} - \gamma^\mu \gamma^\delta) \\
&= 2\gamma^\delta \gamma^\nu \gamma^\sigma \gamma^\gamma - (\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\mu)\gamma^\delta \\
&= 2\gamma^\delta \gamma^\nu \gamma^\sigma \gamma^\gamma + 2\gamma^\gamma \gamma^\sigma \gamma^\nu \gamma^\delta.
\end{aligned}
$$

□

A.2 Complete Set of 4×4 Matrices

Any 4×4 matrix can be written as

$$
\sum_{A=1}^{16} a_A \hat{\Gamma}_A \tag{A.3}
$$

where

$$
\begin{aligned}
\hat{\Gamma}_A \quad = \quad & \mathbb{1}, \\
& \gamma_0,\ i\gamma_1,\ i\gamma_2,\ i\gamma_3, \\
& i\gamma_2\gamma_3,\ i\gamma_3\gamma_1,\ i\gamma_1\gamma_2,\ \gamma_1\gamma_0,\ \gamma_2\gamma_0,\ \gamma_3\gamma_0, \\
& \gamma_1\gamma_2\gamma_3,\ i\gamma_1\gamma_2\gamma_0,\ i\gamma_3\gamma_1\gamma_0,\ i\gamma_2\gamma_3\gamma_0 \\
& i\gamma_1\gamma_2\gamma_3\gamma_0
\end{aligned} \tag{A.4}
$$

Proof:

Note

$$
\hat{\Gamma}_A^2 = \mathbb{1} \quad (A = 1, \ldots, 16) \tag{A.5}
$$

For all $\hat{\Gamma}_A$ but **I** there exists a $\hat{\Gamma}_B$ with

$$
\hat{\Gamma}_B \hat{\Gamma}_A \hat{\Gamma}_B = -\hat{\Gamma}_A \tag{A.6}
$$

The trace of all $\hat{\Gamma}_A$ $(A = 2, \ldots, 16)$ are zero,

$$
Tr(\hat{\Gamma}_A) = -Tr(\hat{\Gamma}_B \hat{\Gamma}_A \hat{\Gamma}_B) = -Tr(\hat{\Gamma}_B^2 \hat{\Gamma}_A) = -Tr(\hat{\Gamma}_A).
$$

A.3 Linear independence

Say

$$\sum_{A=1}^{16} a_A \hat{\Gamma}_A = 0.$$

Multiply this sum from the right by $\hat{\Gamma}_B$,

$$a_B \mathbb{1} + \sum_{A \neq B} a_A \hat{\Gamma}_A \hat{\Gamma}_B = 0. \tag{A.7}$$

Then take the trace

$$4a_B + \sum_{A \neq B} a_A Tr(\hat{\Gamma}_A \hat{\Gamma}_B) = 0. \tag{A.8}$$

Now from (A.4) we see that $\hat{\Gamma}_A \hat{\Gamma}_B = Const. \hat{\Gamma}_C$. In the case where $A \neq B$, $\hat{\Gamma}_C \neq \mathbb{1}$. This implies in (A.8) that $a_B = 0$.

A.4 Expansion of 4×4

Each 4×4 matrix can be expanded as

$$\hat{X} = \sum_{A=1}^{16} x_A \hat{\Gamma}_A. \tag{A.9}$$

This is evident since 4×4 matrices represents a 16-dimension space and the $\hat{\Gamma}_A$ are linearly independent. The coefficents are then given by

$$x_B = \frac{1}{4} Tr(\hat{\Gamma}_A \hat{X}).$$

$\square$

A.5 Unitary Equivalence of Representations of the Dirac Algebra

All representations of the Dirac algebra $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}$ which satify $\gamma^{0\dagger} = \gamma^0$, $\gamma^{i\dagger} = -\gamma^i$ are unitary equivalent.

Proof: The proof is split into five parts.

i) First we prove that each 4×4 matrix which commutes with all $\hat{\Gamma}_A$ is a multiple of $\mathbb{1}$. Consider such a the matrix which write

$$\hat{X} = x_B\hat{\Gamma}_B + \sum_{A\neq B} x_A\hat{\Gamma}_A \tag{A.10}$$

where we have picked out a particular matrix which is not $\mathbb{1}$. We choose $\hat{\Gamma}_C$ such that

$$\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C = -\hat{\Gamma}_B. \tag{A.11}$$

Since $\hat{X}$ comutes wth all $\hat{\Gamma}_A$,

$$\hat{X} = \hat{\Gamma}_C\hat{X}\hat{\Gamma}_C$$

we have

$$\begin{aligned} x_B\hat{\Gamma}_B + \sum_{A\neq B} x_A\hat{\Gamma}_A &= x_B\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C + \sum_{A\neq B} x_A\hat{\Gamma}_C\hat{\Gamma}_A\hat{\Gamma}_C \\ &= x_B\hat{\Gamma}_C\hat{\Gamma}_B\hat{\Gamma}_C + \sum_{A\neq B} x_A(\pm\hat{\Gamma}_A\hat{\Gamma}_C)\hat{\Gamma}_C \\ &= -x_B\hat{\Gamma}_B + \sum_{A\neq B}(\pm)x_A\hat{\Gamma}_A \end{aligned} \tag{A.12}$$

where we have used $\hat{\Gamma}_C\hat{\Gamma}_A = (\pm)\hat{\Gamma}_A\hat{\Gamma}_C$ (this established by inspection (A.4)). Next multiply by $\hat{\Gamma}_B$ and take the trace

$$4x_B + \sum_{A\neq B} x_A Tr(\hat{\Gamma}_A\hat{\Gamma}_B) = -4x_B + \sum_{A\neq B}(\pm)x_A Tr(\hat{\Gamma}_A\hat{\Gamma}_B) \tag{A.13}$$

implying

$$x_B = -x_B = 0.$$

So we conclude that

$$\hat{X} = x_1 \mathbb{1}. \tag{A.14}$$

This result is actually just a special case of Schur's lemma which states that every matrix which commutes with every element of and irreducible representation must be a muliply of the identity matrix; we know 4×4 matrix reprersentations of the Dirac algebra is irreducible as there are no lower dimensional representations.

ii) Let γ_μ and γ'_μ be two representations of the Dirac algebra and $\hat{\Gamma}_A, \hat{\Gamma}'_A$ are respectively their basis. We wish to prove that

$$\hat{\Gamma}'_A \hat{S} = \hat{S} \hat{\Gamma}_A \tag{A.15}$$

where

$$\hat{S} = \sum_{B=1}^{16} \hat{\Gamma}'_B \hat{F} \hat{\Gamma}_B \tag{A.16}$$

and $\hat{F}$ is an aritrary 4×4 matrix. To this end, consider the matrix

$$\hat{\Gamma}'_A \hat{S} \hat{\Gamma}_A = \sum_{B=1}^{16} \hat{\Gamma}'_A \hat{\Gamma}'_B \hat{F} \hat{\Gamma}_B \hat{\Gamma}_A. \tag{A.17}$$

By inspection, from (A.4) we have $\hat{\Gamma}_B \hat{\Gamma}_A = \alpha_C \hat{\Gamma}_C$ where $\alpha_C \in \{\pm 1, \pm i\}$.

$$\hat{\Gamma}'_B \hat{\Gamma}'_A = \alpha_C \hat{\Gamma}'_C. \tag{A.18}$$

Multiply from the right by $\hat{\Gamma}'_A$

$$\hat{\Gamma}'_B = \alpha_C \hat{\Gamma}'_C \hat{\Gamma}'_A$$

then the right by $\hat{\Gamma}'_C$

$$\hat{\Gamma}'_C \hat{\Gamma}'_B = \alpha_C \hat{\Gamma}'_A$$

and from the left by $\hat{\Gamma}'_B$ gives

$$\hat{\Gamma}'_A \hat{\Gamma}'_B = \frac{1}{\alpha_C} \hat{\Gamma}'_C. \tag{A.19}$$

Substituting this into (A.17) then gives

$$\hat{\Gamma}'_A \hat{S} \hat{\Gamma}_A = \sum_{C=1}^{16} \left(\frac{1}{\alpha_C} \hat{\Gamma}'_C \right) \hat{F} \, (\alpha_C \hat{\Gamma}'_C) = \hat{S} \tag{A.20}$$

proving (A.15).

Suppose we could choose $\hat{F}$ (recall $\hat{F}$ is completely arbitrary) so that $\hat{S}$ is non-singular, we would then have in particular

$$\hat{\Gamma}'_2 = \hat{S} \hat{\Gamma}_2 \hat{S}^{-1}, \quad \hat{\Gamma}'_3 = \hat{S} \hat{\Gamma}_3 \hat{S}^{-1}, \quad \hat{\Gamma}'_4 = \hat{S} \hat{\Gamma}_4 \hat{S}^{-1}, \quad \hat{\Gamma}'_5 = \hat{S} \hat{\Gamma}_5 \hat{S}^{-1}$$

equivalently

$$\gamma'_\mu = \hat{S} \gamma_\mu \hat{S}^{-1}.$$

The next two steps are to prove we can choose $\hat{F}$ so the above conditions for $\hat{S}$ are fulfilled. In the final step we show that the additional conditions $\gamma_0 = \gamma_0^\dagger$, $\gamma_i = -\gamma_i^\dagger$, $\gamma'_0 = \gamma_0'^\dagger$, $\gamma'_i = -\gamma_i'^\dagger$ imply that $\hat{S}$ can be choosen to be unitary, proving the entire result.

iii) The matrix $\hat{F}$ can be choosen so that $\hat{S}$ does not vanish. We prove this by contraction. Say $\hat{S} = 0$ held for all choices of $\hat{F}$, then

$$0 = (\hat{S})_{\mu\rho} = \sum_{B=1}^{16} \sum_{\alpha,\beta=1}^{4} (\hat{\Gamma}'_B)_{\mu\alpha} (\hat{F})_{\alpha\beta} (\hat{\Gamma}_B)_{\beta\rho}. \tag{A.21}$$

for all μ and ρ. Now let us choose $\hat{F}$ such that a single element has the value 1 with all other elements being zero. Say it is the element $(\hat{F})_{\nu\sigma}$ that is equal to 1, then (A.21) reads

$$\sum_{B=1}^{16} (\hat{\Gamma}'_B)_{\mu\nu} (\hat{\Gamma}_B)_{\sigma\rho} = 0 \tag{A.22}$$

This equation can be written for all possible choices of ν and σ, so we can infer

$$\sum_{B=1}^{16}(\hat{\Gamma}'_B)_{\mu\nu}\,\hat{\Gamma}_B = 0 \tag{A.23}$$

holds for all μ and ν. Since $\hat{\Gamma}_B^2 = \mathbb{1}$ this would imply

$$\sum_{B=1}^{16}(\hat{\Gamma}'_B)_{\mu\nu} = 0 \quad \text{for all} \quad \mu, \nu. \tag{A.24}$$

As $(\hat{\Gamma}'_B)_{\mu\nu}$ cannot be equal to zero simultaneously, we have a contradiction to the linear independence of the $\hat{\Gamma}_B$.

iv) Now we prove that $\hat{S}$ is not singular with appropriate choice of $\hat{F}$. To this end construct

$$\hat{T} = \sum_{B=1}^{16}\hat{\Gamma}_B\hat{G}\hat{\Gamma}'_B \tag{A.25}$$

where $\hat{G}$ is arbitrary. Obviously we have

$$\hat{\Gamma}_A\hat{T} = \hat{T}\hat{\Gamma}'_A \tag{A.26}$$

(same argument as in iii) which together with (A.15) implies

$$(\hat{\Gamma}_A\hat{T})\hat{S} = (\hat{T}\hat{\Gamma}'_A)\hat{S} = \hat{T}(\hat{\Gamma}'_A\hat{S}) = \hat{T}(\hat{S}\hat{\Gamma}_A)$$

i.e.

$$\hat{\Gamma}_A(\hat{T}\hat{S}) = (\hat{T}\hat{S})\hat{\Gamma}_A, \tag{A.27}$$

accordingly $\hat{T}\hat{S}$ must be a multiple of the identity

$$\hat{T}\hat{S} = k\,\mathbb{1}. \tag{A.28}$$

Obviously we can choose $\hat{G}$ so that $\hat{T} \neq 0$ (same argument as iii). With the same kind of choice of $\hat{F}$ as in part iv) we will now show that we must have $k \neq 0$, and hence that $\hat{S}$ is not singular! We prove it by contradiction

$$\sum_{B=1}^{16} \hat{T}\hat{\Gamma}'_B \hat{F}\hat{\Gamma}_B = 0 \tag{A.29}$$

with the choice of $(\hat{F})_{\nu\rho} = 1$ with all other terms zero,

$$\sum_{B=1}^{16} (\hat{T}\hat{\Gamma}'_B)_{\mu\nu}(\hat{\Gamma}_B)_{\rho\sigma} = 0 \tag{A.30}$$

or

$$\sum_{B=1}^{16} (\hat{T}\hat{\Gamma}'_B)_{\mu\nu}\hat{\Gamma}_B = 0. \tag{A.31}$$

From $\hat{\Gamma}'^2_A = \mathbb{1}$ and the fact that $(\hat{T}\hat{\Gamma}'_B)_{\mu\nu}$ cannot all be simulateously zero as $\hat{\Gamma}'_1 = \mathbb{1}$ and $\hat{T} \neq 0$. This is in contradiction to the linear independence of the $\hat{\Gamma}_B$.

v) We now show that in the case of

$$\gamma_0 = \gamma_0^\dagger, \quad \gamma_i = -\gamma_i^\dagger, \quad \gamma'_0 = \gamma_0'^\dagger, \quad \gamma'_i = -\gamma_i'^\dagger,$$

equivalently

$$\gamma_\mu^\dagger = \eta_{\mu\mu}\gamma_\mu, \quad \gamma_\mu'^\dagger = \eta_{\mu\mu}\gamma'_\mu, \tag{A.32}$$

then $\hat{S}$ can be choosen as a unitary operator. To see this put $\hat{V} \equiv (det\ \hat{S})^{-1}\hat{S}$ then

$$\gamma'_\mu = \hat{V}\gamma_\mu\hat{V}^{-1}, \quad det\ \hat{V} = 1. \tag{A.33}$$

Let us see if there exist another choice for $\hat{V}$. We must have $det\ \hat{V}_1 = det\ \hat{V}_2 = 1$ and

$$\gamma'_\mu = \hat{V}_1\gamma_\mu\hat{V}_1^{-1} = \hat{V}_2\gamma_\mu\hat{V}_2^{-1}. \tag{A.34}$$

Eq (A.34) implies

$$\hat{V}_1\hat{\Gamma}_A\hat{V}_1^{-1} = \hat{V}_2\hat{\Gamma}_A\hat{V}_2^{-1}, \tag{A.35}$$

for example

$$\begin{aligned}
\hat{V}_1(i\gamma_1\gamma_2)\hat{V}_1^{-1} &= i(\hat{V}_1\gamma_1\hat{V}_1^{-1})(\hat{V}_1\gamma_2\hat{V}_1^{-1}) \\
&= i(\hat{V}_2\gamma_1\hat{V}_2^{-1})(\hat{V}_2\gamma_2\hat{V}_2^{-1}) \\
&= \hat{V}_2(i\gamma_1\gamma_2)\hat{V}_2^{-1}.
\end{aligned} \tag{A.36}$$

Eq (A.35) rearranged becomes

$$(\hat{V}_2^{-1}\hat{V}_1)\hat{\Gamma}_A = \hat{\Gamma}_A(\hat{V}_2^{-1}\hat{V}_1). \tag{A.37}$$

By result i) (Schur's lemma)

$$\hat{V}_2^{-1}\hat{V}_1 = k'\mathbb{1}$$

hence

$$\hat{V}_1 = k'\hat{V}_2. \tag{A.38}$$

As $det\ \hat{V}_2 = det\ \hat{V}_1 = k'^4 det\ \hat{V}_2$, we must have $k' \in \{\pm 1, \pm i\}$. Now take the Hermitian conjugate of (A.33),

$$\gamma_\mu{'}\dagger = (\hat{V}^{-1})^\dagger \gamma_\mu^\dagger \hat{V}^\dagger \tag{A.39}$$

then by means of (A.32),

$$\gamma'_\mu = (\hat{V}^\dagger)^{-1}\gamma_\mu \hat{V}^\dagger \tag{A.40}$$

We see that $(\hat{V}^\dagger)^{-1}$ fulfills (A.33) as does $\hat{V}$. From (A.38) ($k' \in \{\pm 1, \pm i\}$) it follows

$$\begin{aligned}
(\hat{V}^\dagger)^{-1} &= k'\hat{V}, \quad \hat{V}^\dagger = k'^{-1}\hat{V}^{-1} \\
\hat{V}^\dagger\hat{V} &= k'^{-1}\mathbb{1}.
\end{aligned} \tag{A.41}$$

Since

$$(\hat{V}^\dagger\hat{V})_{ii} = \sum_j (\hat{V}^\dagger)_{ij}(\hat{V})_{ji} = \sum_j |V_{ji}|^2 = k'^{-1} \tag{A.42}$$

Hence $k^{'-1}$ must be real and positive, i.e. $k^{'-1} = 1$. Hence,

$$\hat{V}^{\dagger}\hat{V} = \mathbb{1}. \tag{A.43}$$

□

A.6 Coefficients of Infintesimal Lorentz Transformation

We prove that the

$$\hat{\sigma}_{\alpha\beta} = \frac{i}{2}(\gamma_\alpha\gamma_\beta - \gamma_\alpha\gamma_\beta)$$

fulfill ()

Proof: Insert the above expression in the RHS of ()

$$\begin{aligned}
[\gamma^\nu, \hat{\sigma}_{\alpha\beta}] &= \frac{i}{2}\Big[\gamma^\nu, [\gamma_\alpha, \gamma_\beta]\Big] \\
&= \frac{i}{2}\Big(\Big[\gamma^\nu, \gamma_\alpha\gamma_\beta\Big] - \Big[\gamma^\nu, \gamma_\beta\gamma_\alpha\Big]\Big) \\
&= \frac{i}{2}\Big(2\Big[\gamma^\nu, \gamma_\alpha\gamma_\beta\Big] - 2\Big[\gamma^\nu, \eta_{\alpha\beta}\Big]\Big) \\
&= i[\gamma^\nu, \gamma_\alpha\gamma_\beta]
\end{aligned} \tag{A.44}$$

where we used $\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\eta_{\alpha\beta}$. Furthermore we have

$$\begin{aligned}
i[\gamma^\nu, \gamma_\alpha\gamma_\beta] &= i(\gamma^\nu\gamma_\alpha\gamma_\beta - \gamma_\alpha\gamma_\beta\gamma^\nu) \\
&= i(\gamma^\nu\gamma_\alpha\gamma_\beta - 2\eta^\nu{}_\beta\gamma_\alpha + \gamma_\alpha\gamma^\nu\gamma_\beta) \\
&= i(\gamma^\nu\gamma_\alpha\gamma_\beta - 2\eta^\nu{}_\beta\gamma_\alpha + 2\eta^\nu{}_\alpha\gamma_\beta - \gamma^\nu\gamma_\alpha\gamma_\beta) \\
&= 2i(\eta^\nu{}_\alpha\gamma_\beta - \eta^\nu{}_\beta\gamma_\alpha).
\end{aligned} \tag{A.45}$$

□

A.7 Proof of Relation $\hat{S}^{-1} = \gamma_0 \hat{S}^\dagger \gamma_0$

We show that for

$$\hat{S} = \exp\left(-\frac{i}{4}\omega\hat{\sigma}_{\mu\nu}(\hat{I}_{\mathbf{n}})^{\mu\nu}\right)$$

the inverse operator is given by

$$\hat{S}^{-1} = \gamma_0 \hat{S}^\dagger \gamma_0 \tag{A.46}$$

Proof:

(i) Rotations:

For spacial rotations we can write:

$$\hat{S} = \exp\left(-\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right). \tag{A.47}$$

The $\hat{\sigma}_{ij}$ are Hermitian because

$$\begin{aligned}
\hat{\sigma}_{ij}^\dagger &= -\frac{i}{2}\left\{(\gamma_i\gamma_j)^\dagger - (\gamma_j\gamma_i)^\dagger\right\} \\
&= -\frac{i}{2}\left\{\gamma_j\gamma_i - \gamma_i\gamma_j\right\} \\
&= \hat{\sigma}_{ij}.
\end{aligned} \tag{A.48}$$

This implies

$$\hat{S}^\dagger = \exp\left(\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}^\dagger\right) = \exp\left(\frac{i}{4}\omega^{ij}\hat{\sigma}_{ij}\right). \tag{A.49}$$

Obviously, γ_0 commutes with $\hat{\sigma}_{ij}$ and thus with $\hat{S}^\dagger$. Hence we have

$$\gamma_0 \hat{S}^\dagger \gamma_0 = \hat{S}^\dagger = \hat{S}^{-1}. \tag{A.50}$$

(ii) Lorentz boosts:

Given that a general Loreentz transformation can be decomposed into first a rotation, a Lorentz boost along the x-direction and then undoing the rotation, and given that $\gamma^i\gamma^0 + \gamma^0\gamma^i = 0$, it suffices to prove the result for a simple boost along the $x-$direction. For this transformation we have

$$\hat{S} = \exp\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right)$$

$\hat{\sigma}_{01}$ is antihermitian because

$$\begin{aligned}
\hat{\sigma}_{01}^\dagger &= -\frac{i}{2}\left\{(\gamma_0\gamma_1)^\dagger - (\gamma_1\gamma_0)^\dagger\right\} \\
&= \frac{i}{2}\left\{\gamma_1\gamma_0 - \gamma_0\gamma_1\right\} \\
&= -\hat{\sigma}_{01}.
\end{aligned} \tag{A.51}$$

Therefore

$$\hat{S}^\dagger = \exp\left(\frac{i}{2}\omega\hat{\sigma}_{01}^\dagger\right) = \exp\left(-\frac{i}{2}\omega\hat{\sigma}_{01}\right) = \hat{S}. \tag{A.52}$$

From

$$\begin{aligned}
\gamma_0\hat{\sigma}_{01} &= \frac{i}{2}\left\{\gamma_0\gamma_0\gamma_1 - \gamma_0\gamma_1\gamma_0\right\} \\
&= \frac{i}{2}\left\{\gamma_1\gamma_0\gamma_0 - \gamma_0\gamma_1\gamma_0\right\} \\
&= \hat{\sigma}_{10}\gamma_0 = -\hat{\sigma}_{01}\gamma_0.
\end{aligned} \tag{A.53}$$

we get

$$
\begin{aligned}
\gamma_0 \hat{S}^\dagger \gamma_0 &= \gamma_0 \left[\sum_{n=0}^{\infty} \left(-\frac{i}{2} \omega \hat{\sigma}_{01} \right)^n \right] \gamma_0 \\
&= \sum_{n=0}^{\infty} \gamma_0 \left(-\frac{i}{2} \omega \hat{\sigma}_{01} \right)^n \gamma_0 \\
&= \sum_{n=0}^{\infty} \gamma_0 \left(-\frac{i}{2} \omega \hat{\sigma}_{01} \right) \gamma_0 \gamma_0 \left(-\frac{i}{2} \omega \hat{\sigma}_{01} \right) \gamma_0 \dots \gamma_0 \left(-\frac{i}{2} \omega \hat{\sigma}_{01} \right) \gamma_0 \\
&= \sum_{n=0}^{\infty} \left(+\frac{i}{2} \omega \hat{\sigma}_{01} \right)^n \\
&= \exp \left(\frac{i}{2} \omega \hat{\sigma}_{01} \right) = \hat{S}^{-1}. \qquad \text{(A.54)}
\end{aligned}
$$

□

A.8 Proof of the Completenes Relation for spinors

A.8.1 Proof of completenes relation: $\omega^{r\dagger}(\epsilon_r \mathbf{p}) \omega^{r'}(\epsilon_{r'} \mathbf{p}) = \delta_{rr'}(E/m_0)$

Proof:

We calculate some examples

$r = 1, r' = 1$:

$$
\begin{aligned}
& \frac{E+m_0}{2m_0} \left(1, 0, \frac{p_z}{E+m_0}, \frac{p_-}{E+m_0} \right) \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m_0} \\ \frac{p_-}{E+m_0} \end{pmatrix} \\
&= \frac{E+m_0}{2m_0} \left\{ 1 + \frac{\mathbf{p}^2}{(E+m_0)^2} \right\} \\
&= \frac{E+m_0}{2m_0} \left\{ \frac{(E+m_0)^2 + \mathbf{p}^2}{(E+m_0)^2} \right\} \\
&= \left\{ \frac{2E + 2m_0 E}{2m_0(E+m_0)} \right\} \\
&= \frac{E}{m_0} \delta_{11} \qquad \text{(A.55)}
\end{aligned}
$$

$r = 2, r' = 4$:

$$\frac{E+m_0}{2m_0}\left(0, 1, \frac{p_+}{E+m_0}, -\frac{p_z}{E+m_0}\right)\left(\begin{array}{c} -\frac{p_-}{E+m_0} \\ \frac{p_z}{E+m_0} \\ 0 \\ 1 \end{array}\right)$$

$$= \frac{E+m_0}{2m_0}\left\{\frac{p_z}{E+m_0} - \frac{p_z}{E+m_0}\right\} = 0. \tag{A.56}$$

$r = 4, r' = 4$:

$$\frac{E+m_0}{2m_0}\left(1, 0, \frac{p_+}{E+m_0}, -\frac{p_z}{E+m_0}\right)\left(\begin{array}{c} -\frac{p_-}{E+m_0} \\ \frac{p_z}{E+m_0} \\ 0 \\ 1 \end{array}\right)$$

$$= \frac{E}{m_0}\delta_{44} \tag{A.57}$$

The other combinations can be calculated similarly.

$\square$

Appendix B

Dirac's Delta Function

B.1 Definition

The Dirac delta-function satisfies

$$f(x) = \int_{-L}^{L} f(x')\delta(x - x')dx'. \tag{B.1}$$

In particular, if $f(x) = 1$ over the entire interval $[-L, L]$, we get 1. Hence

$$\int_{-L}^{L} \delta(x - x')dx' = 1, \quad \text{if } -L \leq x \leq L. \tag{B.2}$$

The simplest object is a rectangular spike of width η and height η^{-1} centered at x:

$$D_\epsilon(x - x') = \begin{cases} \eta^{-1}, & |x - x'| \leq \frac{1}{2}\eta \\ 0, & \text{otherwise,} \end{cases} \tag{B.3}$$

As the area under $D_\epsilon(x - x')$ is 1, equation (B.2) is satified.

Properties are

$$\begin{aligned} \delta(-x) &= \delta(x) && \text{(B.4)} \\ \delta(ax) &= \delta(x)/|a| && \text{(B.5)} \\ \delta(x^2 - b^2) &= [\delta(x - b) + \delta(x + b)]/|b| && \text{(B.6)} \\ \delta(g(x)) &= \frac{f(x_i)}{|\frac{dg}{dx}(x_i)|} && \text{(B.7)} \end{aligned}$$

where in the x_i are the roots of $g(x)$.

Proof:

Property (B.5) comes from a change of variables $ax = y$. First take $a > 0$, then

$$\int_{-L}^{L} f(x)\delta(ax)dx = \int_{-aL}^{aL} f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} = \frac{f(0)}{a}. \tag{B.8}$$

This is just what is obtained if $\delta(ax) = \delta(x)/a$. If $a < 0$, the integration over y goes from positive value $-aL$ to a negative value aL.

$$\int_{-aL}^{aL} f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} = -\int_{aL}^{-aL} f\left(\frac{y}{a}\right)\delta(y)\frac{dy}{a} = \frac{f(0)}{|a|}. \tag{B.9}$$

To demonsttate property (B.6) first note that by $x^2 - b^2 = (x-b)(x+b)$ there are two roots: $x = b$ and $x = -b$. Near $x = b$ it behaves like $(x-b)2b$ near $x = b$, and like $-2b(x+b)$ near $x = -b$. Hence

$$\delta(x^2 - b^2) = \delta(2b(x-b)) + \delta(-2b(x+b)) \tag{B.10}$$

$\square$

B.2 Representions of Dirac's Delta Function

We wish to perform the integral

$$I = \int_{-\infty}^{\infty} dE_f \frac{4\sin^2(E_f - E_i)T/2}{(E_f - E_i)^2}. \tag{B.11}$$

We introduce the variable $x = (E_f - E_i)T/2$, then

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dx 4\frac{\sin^2 x}{\frac{4}{T^2}x^2}\frac{2}{T} \\ &= 2T\int_{-\infty}^{\infty} dx\frac{\sin^2 x}{x^2}. \end{aligned} \tag{B.12}$$

Since $\sin^2 x/x^2|_{x=0} = 1$, the integrand is continuous and bounded everywhere. Integration by parts yields

$$
\begin{aligned}
I = 2T\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx &= 2T\left[-\frac{1}{x}\sin^2 x\right]_{-\infty}^{\infty} + 2T\int_{-\infty}^{\infty}\frac{2\cos x\sin x}{x}dx \\
&= 2T\int_{-\infty}^{\infty}\frac{\sin 2x}{x}dx \\
&= 2T\int_{-\infty}^{\infty}\frac{\sin x}{x}dx.
\end{aligned} \tag{B.13}
$$

To evaluate this we start by defining

$$
I(s) = 4T\int_{0}^{\infty} e^{-sx}\frac{\sin x}{x}dx. \tag{B.14}
$$

Then

$$
I'(s) = -4T\int_{0}^{\infty} e^{-sx}\sin x dx \tag{B.15}
$$

where $I(\infty) = 0$. We can then find an algebraic equation for $I'(s)/4T$ by integrating by parts twice:

$$
\begin{aligned}
\frac{I'(s)}{4T} &= \left[\frac{e^{-sx}}{s}\sin x\right]_{0}^{\infty} - \frac{1}{s}\int_{0}^{\infty} e^{-sx}\cos x dx \\
&= \left[\frac{e^{-sx}}{s^2}\cos x\right]_{0}^{\infty} - \frac{1}{s^2}\int_{0}^{\infty} e^{-sx}\sin x dx \\
&= -\frac{1}{s^2} - \frac{1}{s^2}\frac{I'(s)}{4T}
\end{aligned} \tag{B.16}
$$

or

$$
\frac{I'(s)}{4T} = -\frac{1}{1+s^2}, \tag{B.17}
$$

which together with $I(\infty) = 0$ determines $I(s)$:

$$
\begin{aligned}
\frac{I(s)}{4T} &= -\int_s^\infty I'(s)ds \\
&= \int_s^\infty \frac{1}{1+s^2}ds \\
&= \left[\tan^{-1}(s)\right]_s^\infty \\
&= \tan^{-1}(\infty) - \tan^{-1}(s) \\
&= \frac{\pi}{2} - \tan^{-1}(s). \qquad \text{(B.18)}
\end{aligned}
$$

So that, finally

$$
I = I(0) = 4T\int_0^\infty \frac{\sin x}{x}dx = 2\pi T. \qquad \text{(B.19)}
$$

□

Appendix C

Elements of Complex Analysis

C.1 An Analytic Function

If $f(z)$ has a unique and finite derivative, it is said to be differentiable. If $f(z)$ is differentiable at z_0, and a small neighbourhood around z_0 in the complex plane, then $f(z)$ said to be an analytical function at $z = z_0$.

C.2 Laurent series

A Laurent series representation of a complex function $f(z)$, in contrast to a Taylor series expansion, is the inclusion of terms that involve negative powers.

A Laurent series can be written in the form

$$f(z) = c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + \frac{c_{-1}}{z - z_0} + \frac{c_{-2}}{(z - z_0)^2} + \cdots \tag{C.1}$$

C.3 A Pole

The function $f(z) = \sin z / z$ at $z = 0$ has what is called a removable singularity.

Suppose that the principal part of a Laurent series only has a finite number of terms

$$\frac{c_{-1}}{z - z_0} + \frac{c_{-2}}{(z - z_0)^2} + \cdots + \frac{c_{-n}}{(z - z_0)^n} \tag{C.2}$$

Then the pole at point $z = z_0$ is called a pole of order n. A pole causes the function to blow up at $z = z_0$ When c_{-1} is the only non-zero coefficient in the principal part of the series, we say that $z = z_0$ is a simple pole.

If the order n of the pole goes to infinity, the singularity is said to be an essential isolated singularity

A branch point $z = z_0$ is a point of a multivalued function where the function changes value when the curve winds around z_0.

C.4 A Branch Cut

A branch point belongs to two or more branches. It is a point in the neighbourhood of which the function is multivalued.

C.5 A Contour Integral

C.5.1 Residues

Let us expand $f(z)$ as a Laurent series about z_0

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n. \tag{C.3}$$

Integrating term by term, we find that the nth power term for $n \neq -1$

$$c_n \oint (z - z_0)^n dz = c_n \left. \frac{(z - z_0)^{n+1}}{n+1} \right|_{z_1}^{z_1} = 0. \tag{C.4}$$

When $n = -1$, the result is

$$\begin{aligned} c_{-1} \oint \frac{dz}{z - z_0} &= c^{-1} \oint \frac{re^{i\theta} id\theta}{re^{i\theta}} \\ &= c_{-1} 2\pi i. \end{aligned} \tag{C.5}$$

Hence

$$\frac{1}{2\pi i}\oint f(z)dz = c_{-1} = \mathrm{Res}\left[f(z_0)\right]. \tag{C.6}$$

This integral is called the *residue* $\mathrm{Res}\,[f(z_0)]$ of $f(z)$ at z_0.

C.5.2 The residue theorem

If $f(z)$ has singularities at points z_i, for a closed curve enclosing these points we have

$$\oint_C dz f(z) = 2\pi i \sum_{i=1}^{n} \mathrm{Res}[f(z_i)] \tag{C.7}$$

C.5.3 Calculation of residues

If we have a simple pole,

$$f(z) = \frac{c_{-1}}{(z-z_0)} + c_0 + c_1 z + \cdots \tag{C.8}$$

then we have

$$\mathrm{Res}[f(z_i)] = \lim_{z\to z_0}[(z-z_0)f(z)]. \tag{C.9}$$

Let $f(z)$ have a pole of order m at z_0,

$$f(z) = \frac{c_{-m}}{(z-z_0)^m} + \frac{c_{-m+1}}{(z-z_0)^{m-1}} + \cdots + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \cdots \tag{C.10}$$

Then we easily see that we can extract c_{-1} from

$$\mathrm{Res}[f(z_i)] = \frac{1}{(m-1)!}\lim_{z\to z_0}\left(\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right) \tag{C.11}$$

If the contour encloses more that one isolated singularity, c_{-1} gives the sum of the of the enclosed residues - see fig ().

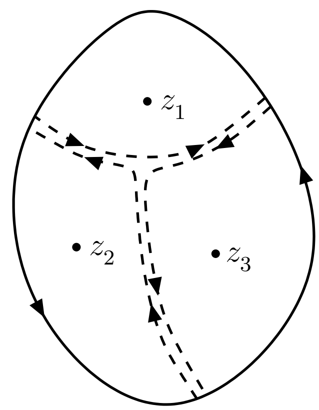


Figure C.1: Closing a contour around each isolated singularity.

C.6 Poles on the Contour

If there is a simple pole located right on a closed contour of integration the integral is not uniquely defined. Two different results are possible dependent on whether the small semicircle around the pole z_0 on the contour is completed in the positive (counterclockwse) or negative direction:

$$I_{\pm} = \oint_{\pm} f(z)dz = \tag{C.12}$$

We drive it by another procedure. We start with the identity

$$\frac{1}{\omega - \omega_0 \pm i\eta} = \frac{\omega - \omega_0}{(\omega - \omega_0)^2 + \eta^2} \mp i \frac{\eta}{(\omega - \omega_0)^2 + \eta^2}. \tag{C.13}$$

In the limit $\eta \rightarrow 0$

$$\lim_{\eta \rightarrow 0} \frac{1}{\omega - \omega_0 \pm i\eta} = \mathcal{P} \frac{1}{\omega - \omega_0} \mp i\pi D(\omega - \omega_0). \tag{C.14}$$

where the quantity

$$D(\omega - \omega_0) = \lim_{\eta \rightarrow 0} \frac{\eta/\pi}{(\omega - \omega_0)^2 + \eta^2} \tag{C.15}$$

has the properties that

$$D(\omega-\omega_0) \propto \begin{cases} \eta \to 0, & \omega \neq \omega_0 \\ 1/\eta \to \infty, & \omega = \omega_0, \end{cases} \tag{C.16}$$

and

$$\int_{-\infty}^{\infty} D(\omega-\omega_0)d\omega = \frac{\eta}{\pi}\oint \frac{dz}{z^2+\eta^2} = 1 \tag{C.17}$$

obtained by closing the contour in either the upper or the lower half plane (alternatively, we just use the substitution $\omega - \omega_0 = \eta \tan u$). These are the properties of a Dirac δ function. Hence

$$D(\omega-\omega_0) = \delta(\omega-\omega_0) \tag{C.18}$$

and

$$\lim_{\eta\to 0} \frac{1}{\omega-\omega_0 \pm i\eta} = \mathcal{P}\left(\frac{1}{\omega-\omega_0}\right) \mp i\pi\, \delta(\omega-\omega_0). \tag{C.19}$$

C.7 Complex integration and calculus of residues

Integrals along open contours, for example over intervals of the real line, may be evaluated by employing techniques for closing open contours. Of common interest are integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)dx. \tag{C.20}$$

Suppose $f(z)$ is the analytic continuation of $f(x)$, is a single-valued.

A first method of closing the contour is the following. If $|zf(z)| \to \infty$ as $|z| \to \infty$, the contour can be closed by a large semicircle in the upper, or in the lower, half-plane. In this case we have

$$I = \pm 2\pi i \sum_{enclosed} \mathrm{Res}(f) - I_R \tag{C.21}$$

where

$$
\begin{aligned}
I_R &= \lim_{R\to\infty} \int_0^{\pm\pi} [zf(z)] \frac{dz}{z} \\
&\le \lim_{R\to\infty} Max[zf(z)] \int_0^{\pm\pi} \frac{dz}{z} \\
&= \pm i\pi \lim_{R\to\infty} Max[zf(z)]
\end{aligned}
\tag{C.22}
$$

Hence

$$
I = \pm 2\pi i \sum_{enclosed} \mathrm{Res}(f) \tag{C.23}
$$

where the + (−) sign is used for the closure in the upper, or in the lower, half plane.

C.7.1 Fourier integrals via complex analysis

Complex Fourier integrals have the form

$$
I = \int_{-\infty}^{\infty} g(x) e^{i\lambda x} dx \tag{C.24}
$$

where λ is a real constant. The integration contour of these integrals can often be closed with the use of $Jordan's\ lemma$.

Jordan's lemma states: if $\lim_{|z|\to\infty} g(z) = 0$, the contour can be closed by a large semicircle in the upper half-plane if $\lambda > 0$, and in the lower half plane if $\lambda < 0$. The contribution from this semicircle will vanish, giving

$$
I = \pm 2\pi i \sum_{enclosed} \mathrm{Res}\left[g(z) e^{i\lambda z}\right]. \tag{C.25}
$$

Proof:

It is sufficient to consider the case of $\lambda > 0$. On the large semi-circle

$$
z = Re^{i\theta} = R\cos\theta + i\sin\theta, \qquad dz = Re^{i\theta} i d\theta. \tag{C.26}
$$

Let I_R be the integral along the upper semicircle, then we can write

$$I_R = \int_0^{\pi} g(z) e^{\lambda R(i\cos\theta - \sin\theta)} R e^{i\theta} i d\theta. \tag{C.27}$$

If $\epsilon = \text{Max} g(z = Re^{i\theta})$ is the upper bound of g on this semicircle

$$\begin{aligned} I_R &\leq \epsilon R \int_0^{\pi} e^{-\lambda R \sin\theta} d\theta \\ &= 2\epsilon R \int_0^{\pi/2} e^{-\lambda R \sin\theta} d\theta \end{aligned} \tag{C.28}$$

since $\sin\theta$ is symmetric about $\pi/2$. Between 0 and $\pi/2$, $\sin\theta > 2\theta/\pi$. As such, the upper bound on the integral can be simplified to

$$\begin{aligned} |I_R| &\leq 2\epsilon R \int_0^{\pi/2} e^{-\lambda R(2\theta/\pi)} d\theta \\ &= 2\epsilon R \frac{\pi}{2\lambda R}(1 - e^{-\lambda R}) \\ &\leq \frac{\pi\epsilon}{\lambda}. \end{aligned} \tag{C.29}$$

Hence

$$\lim_{R\to\infty} |I_R| \leq \frac{\pi}{\lambda} \lim_{|z|\to\infty} |g(z)| = 0. \tag{C.30}$$

□

C.7.2 Integral representation for the step function

We show that

$$\Theta(\tau) = -\frac{1}{2\pi i} \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega\tau}}{\omega + i\epsilon} \tag{C.31}$$

Proof: We can evaluate the integral by means of complex integration in the complex $\omega-$plane. This can be done if we can show if the contribution from the upper (lower), infinitely distant half circle vanishes. Let I_R be the integral along the upper (lower) semicircle, then we can write

$$
\begin{aligned}
I_R &= \lim_{R\to\infty} \int_0^{\pm\pi} [zf(z)] \frac{dz}{z} \\
&\le \lim_{R\to\infty} Max[zf(z)] \int_0^{\pm\pi} \frac{dz}{z} \\
&= \pm i\pi \lim_{R\to\infty} Max[zf(z)] \qquad \text{(C.32)}
\end{aligned}
$$

Hence, if we can show that $\lim_{R\to\infty} Max\,|[zf(z)]| \to 0$ then the integral over the semicircle can be ignored and the integral along the real line can be converted into a closed contour integral.

For $\tau < 0$ we show that the contribution from the upper, infinitely distant half circle vanishes

$$
\begin{aligned}
f(R,\theta) &= \frac{e^{-i\omega\tau}}{\omega} \\
&= \frac{e^{-iR\tau(\cos\theta + i\sin\theta)}}{re^{i\theta}} \\
&= e^{-iR\tau\cos\theta}\frac{e^{+R\tau\sin\theta}}{Re^{i\theta}} \qquad \text{(C.33)}
\end{aligned}
$$

and

$$
\lim_{R\to\infty} |Rf(R,\theta)| = e^{-R|\tau|\sin\theta} = 0
$$

For $\tau < 0$ we close the contour in the upper half plane. There is only a first order pole at $-i\epsilon$. Therefore this integral will be zero.

In the case $\tau > 0$ for similar reasons one can close the contour by means of an infinitely large half circle below the real axis. Cauchy's integral theorem says that the integrand at the pole

$$
\begin{aligned}
\Theta(\tau > 0) &= -\frac{1}{2\pi i}(-1)2\pi i \lim_{\epsilon\to 0} Res\left[\frac{e^{-i\omega\tau}}{\omega + i\epsilon}\right] \\
&= e^{-\epsilon\tau}\big|_{\epsilon=0} = 1. \qquad \text{(C.34)}
\end{aligned}
$$

where we have a minus sign coming from the clockwise direction of the integration.

□

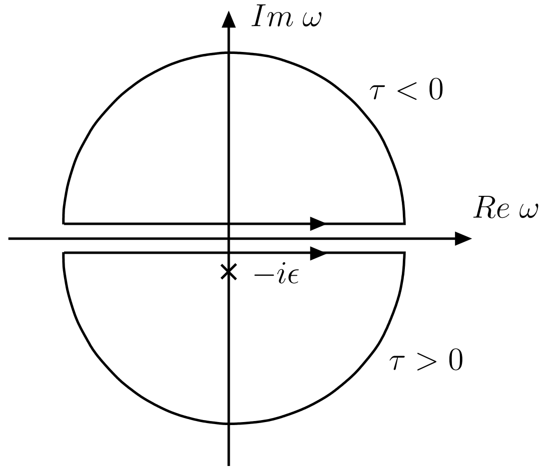


Figure C.2:

C.7.3 Another complex Fourier integral

Recall the result quoted in section 13.2:

$$\int \frac{d^3q}{(2\pi)^3}\frac{\exp(i\mathbf{q}\cdot\mathbf{x})}{\mathbf{q}^2+a^2} = \frac{1}{4\pi}\frac{\exp(-ar)}{r}. \tag{C.35}$$

Proof:

We fix $\mathbf{r}$ and choose q_z to be parallel to $\mathbf{r}$. We then rewrite the above integral in spherical polar coordinates as

$$I = \frac{1}{(2\pi)^3}\int_0^\infty dq \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta \frac{q^2\exp(iq\cos\theta)}{q^2+a^2} \tag{C.36}$$

We then make the substitution $u = \cos\theta$, with the result

$$\begin{aligned} I &= \frac{2\pi}{(2\pi)^3}\int_0^\infty dq \frac{q^2}{q^2+a^2}\int_{-1}^1 du\,\exp(iqu) \\ &= \frac{1}{(2\pi)^2}\int_0^\infty dq \frac{q^2}{q^2+a^2}\left[\frac{\exp(iqru)}{iqr}\right]_{-1}^1 \\ &= \frac{1}{(2\pi)^2}\int_0^\infty dq \frac{q}{i(q^2+a^2)}\left[\exp(iqr)-\exp(-iqr)\right]\frac{1}{r} \\ &= \frac{1}{(2\pi)^2}\int_{-\infty}^\infty dq \frac{q}{i(q^2+a^2)}\exp(iqr)\frac{1}{r} \end{aligned} \tag{C.37}$$

We can now perform this last integral using countour inegration. By a similar argument as in the previous section on the step function we can close the contour in the UHP,

$$
\begin{aligned}
I &= \frac{1}{(2\pi)^2}\oint \frac{dzz}{i(z+ia)(z-ia)}\exp(izr)\frac{1}{r} \\
&= 2\pi i \times \frac{1}{(2\pi)^2}\lim_{z\to ia}\frac{z}{i(z+ia)}\exp(izr)\frac{1}{r} \\
&= \frac{1}{4\pi}\frac{\exp(-ar)}{r}. \qquad \text{(C.38)}
\end{aligned}
$$

□

Evaluating an integral

Here we illustrate the utility of complex analysis by performing the integral in (1.57).

Proof:

Let

$$I = \int_0^\infty \frac{y^3 dy}{e^y - 1}. \qquad \text{(C.39)}$$

First we expand the integrand in a series

$$
\begin{aligned}
\frac{y^3 dy}{e^y-1} &= \frac{y^3 e^{-y}}{1-e^{-y}} \\
&= y^3 e^{-y}(1 + e^{-y} + e^{-2y} + \dots) \\
&= \sum_{n=1}^{\infty} e^{-ny} y^3 \qquad \text{(C.40)}
\end{aligned}
$$

Hence (C.39) becomes

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty}\int_0^\infty e^{-ny} y^3 dy \\
&= \sum_{n=1}^{\infty}\frac{1}{n^4}\int_0^\infty e^{-y} y^3 dy. \qquad \text{(C.41)}
\end{aligned}
$$

The integral in the above equation can be performed by repeated integration by parts

$$\begin{aligned}
\int_0^\infty e^{-y}y^3dy &= [-e^{-y}y^3]_0^\infty + 3\int_0^\infty e^{-y}y^2dy \\
&= 3\int_0^\infty e^{-y}y^2dy \\
&= 3\times 2\int_0^\infty e^{-y}ydy \\
&= 6\int_0^\infty e^{-y}dy \\
&= 6. \qquad \text{(C.42)}
\end{aligned}$$

So that

$$I = 6\sum_{n=1}^{\infty}\frac{1}{n^4}. \qquad \text{(C.43)}$$

This summation can be performed with the use of complex analysis. The function $(\tan\ \pi z)^{-1}$ has simple poles at all integral values, as such the summation (C.43)

$$\sum_{n=1}^{\infty} n^{-4} = \frac{1}{2i}\int_C \frac{dz}{z^4 \tan\ \pi z} \qquad \text{(C.44)}$$

where the contour is defined in fig ().

This follows from

$$\begin{aligned}
\text{Res}_{z_0=n}\left(\frac{1}{z^4\tan\ \pi z}\right) &= \lim_{z\to z_0=n}(z-z_0)\left(\frac{1}{z^4\tan\ \pi z}\right) \\
&= \frac{1}{n^4}\lim_{z\to n}\left(\frac{z-z_0}{\tan\ \pi z}\right) \\
&= \frac{1}{n^4}\lim_{z\to n}\left(\frac{1}{\frac{d}{dz}\tan\ \pi z}\right) \\
&= \frac{1}{n^4}\frac{1}{\pi}\lim_{z\to n}\cos^2\pi z \\
&= \frac{1}{n^4}\frac{1}{\pi}. \qquad \text{(C.45)}
\end{aligned}$$

Since the summation is unchanged by $n \to -n$, it can be expressed as a contour integral as along C'.

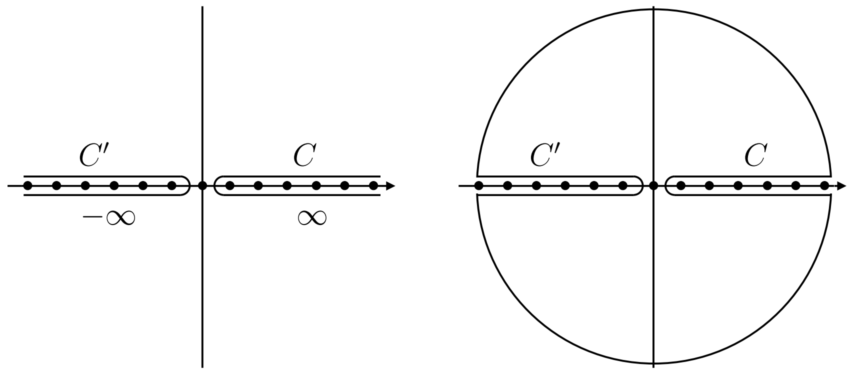


Figure C.3:

As the resulting enclosed are contains no singularities except at $z = 0$, we can the shrink this contour to an infintesimal circle C_0, surrounding the origin. Expanding the integrand in powers of z about $z = 0$

$$
\begin{aligned}
\frac{1}{z^4 \tan\ \pi z} &= \frac{1}{z^4(\pi z + \frac{1}{3}\pi^3 z^3 + \frac{2}{15}\pi^5 z^5 + \dots)} \\
&= \frac{1}{\pi z^5} \frac{1}{(1 + \frac{1}{3}\pi^2 z^2 + \frac{2}{15}\pi^4 z^4 + \dots)} \\
&= \frac{1}{\pi z^5}\left(1 - \left(\frac{1}{3}\pi^2 z^2 + \frac{2}{15}\pi^4 z^4 + \dots\right) + \left(\frac{1}{3}\pi^2 z^2 + \dots\right)^2 + \dots\right) \\
&= \frac{1}{\pi z^5}\left(\dots - \frac{1}{45}\pi^4 z^4 + \dots\right) \\
&= -\frac{\pi^3}{45}\frac{1}{z} + \dots . \qquad \text{(C.46)}
\end{aligned}
$$

only the term involving z^{-1} contributes to the integral

$$
\begin{aligned}
\sum_{n=1}^{\infty} n^{-4} &= -\frac{1}{2i}\int_{C_0} \frac{dz}{z^4 \tan\ \pi z} \\
&= 2\pi i \times \frac{1}{2i} \mathrm{Res}\left(\frac{\pi^3}{45}\frac{1}{z}\right) \\
&= \frac{\pi^4}{90} \qquad \text{(C.47)}
\end{aligned}
$$

As such

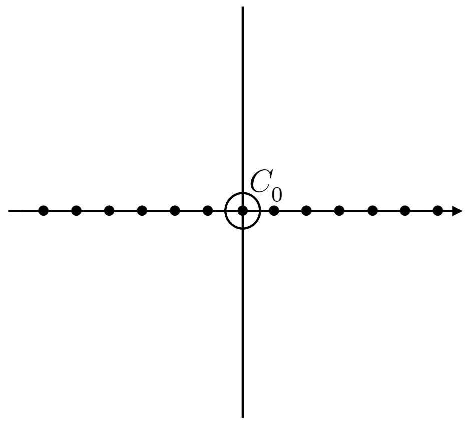


Figure C.4:

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{C.48}$$

and so

$$I = \int_0^{\infty} \frac{y^3 dy}{e^y - 1} = \frac{\pi^4}{15}. \tag{C.49}$$

□

Appendix D

Averaging over Spin

We have

$$\left(\overline{u}(f)\ \hat{\Gamma}_1 u(i)\right)\ \left(\overline{u}(f)\ \hat{\Gamma}_2 u(i)\right)^* = \left(\overline{u}(f)\ \hat{\Gamma}_1 u(i)\right)\ \left(\overline{u}(i)\ \overline{\hat{\Gamma}}_2 u(f)\right) \tag{D.1}$$

where

$$\overline{\hat{\Gamma}} = \gamma^0 \hat{\Gamma}^\dagger \gamma^0. \tag{D.2}$$

Proof:

First note what the complex conjugate $(\overline{u}(f)\hat{\Gamma}u(i))^*$ of the number $\overline{u}(f)\hat{\Gamma}u(i)$ is equal to

$$\begin{aligned}
(\overline{u}(f)\hat{\Gamma}u(i))^\dagger &= \left(\overline{u}^\dagger(f)\gamma^0\hat{\Gamma}u(i)\right)^\dagger \\
&= u(i)^\dagger\hat{\Gamma}^\dagger\gamma^{0\dagger}\ u^{\dagger\dagger}(f) \\
&= \overline{u}(i)(\gamma^0\hat{\Gamma}^\dagger\gamma^0)u(f) \\
&= \overline{u}(i)\overline{\hat{\Gamma}}u(f)
\end{aligned} \tag{D.3}$$

where we have used $\gamma^{0\dagger} = \gamma^0$ and $(\gamma^0)^2 = 1$.

$$\left(\overline{u}(f)\ \hat{\Gamma}_1 u(i)\right)\ \left(\overline{u}(f)\ \hat{\Gamma}_2 u(i)\right)^* = \left(\overline{u}(f)\ \hat{\Gamma}_1 u(i)\right)\ \left(\overline{u}(i)\ \overline{\hat{\Gamma}}_2 u(f)\right) \tag{D.4}$$

$\square$

The barred matrices $\overline{\hat{\Gamma}}$ can be directly calculated for a number of operators:

(i) $\overline{\gamma}^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$

(ii) $\overline{i\gamma^5} = i\gamma^5$

(iii) $\overline{\gamma^\mu \gamma^5} = \gamma^\mu \gamma^5$

(iv) $\overline{\gamma^\mu \gamma^\nu \ldots \gamma^\lambda} = \gamma^\lambda \ldots \gamma^\nu \gamma^\mu$

Proof:

(i) First

$$\gamma^0 \gamma^{0\dagger} \gamma^0 = \gamma^0 \gamma^0 \gamma^0 = \gamma^0$$

and secondly

$$\gamma^0 \gamma^{i\dagger} \gamma^0 = -\gamma^0 \gamma^i \gamma^0 = \gamma^i \gamma^0 \gamma^0 = \gamma^i.$$

(ii) As $i\gamma^5 = -\gamma^0\gamma^1\gamma^2\gamma^3$ and

$$\overline{i\gamma^5} = -\gamma^0 \gamma^{3\dagger} \gamma^{2\dagger} \gamma^{1\dagger} \gamma^{0\dagger} \gamma^0 = +\gamma^0 \gamma^3 \gamma^2 \gamma^1 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 = i\gamma^5.$$

(iii) Similar to (ii).

(iv) Proved using (i).

□

D.1 Spin summation of the general squared matrix element

$$\sum_{s_f\, s_i} \left(\overline{u}(p_f, s_f)\, \hat{\Gamma}_1\, u(p_i, s_i)\right) \left(\overline{u}(p_f, s_f)\, \hat{\Gamma}_2\, u(p_i, s_i)\right)^* = Tr\left[\hat{\Gamma}_1\, \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\, \hat{\overline{\Gamma}}_2\, \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right] \tag{D.5}$$

A special case is

$$\sum_{s_f\, s_i} |\overline{u}(p_f, s_f)\, \hat{\Gamma}\, u(p_i, s_i)|^2 = Tr\left[\hat{\Gamma}\, \frac{p_{i\mu}\gamma^\mu + m_0}{2m_0}\, \hat{\overline{\Gamma}}\, \frac{p_{f\nu}\gamma^\nu + m_0}{2m_0}\right] \tag{D.6}$$

Proof: We use Einstein's summation convention.

$$
\begin{aligned}
& \sum_{s_f\, s_i} \left(\overline{u}_\alpha(p_f,s_f)\,(\hat{\Gamma}_1)_{\alpha\beta}\,u_\beta(p_i,s_i)\right)\;\left(\overline{u}_\gamma(p_i,s_i)\,(\hat{\overline{\Gamma}}_2)_{\gamma\delta}\,u_\delta(p_f,s_f)\right) \\
= & \sum_{s_f} \overline{u}_\alpha(p_f,s_f)\,(\hat{\Gamma}_1)_{\alpha\beta}\left(\sum_{s_i} u_\beta(p_i,s_i)\overline{u}_\gamma(p_i,s_i)\right)\hat{\overline{\Gamma}}_{\gamma\tau}\,u_\tau(p_f,s_f) \\
= & \sum_{s_f} \overline{u}_\alpha(p_f,s_f)\,\left(\hat{\Gamma}_1\,\frac{p_{i\mu}\gamma^\mu+m_0}{2m_0}\,\hat{\overline{\Gamma}}_2\right)_{\alpha\beta} u_\beta(p_f,s_f) \\
= & \sum_{r=1}^{4} \epsilon_r\overline{\omega}^r_\alpha(p_f)\,\left(\hat{\Gamma}_1\frac{p_{i\mu}\gamma^\mu+m_0}{2m_0}\,\hat{\overline{\Gamma}}_2\right)_{\alpha\beta}\left(\frac{p_{f\mu}\gamma^\mu+m_0}{2m_0}\right)_{\beta\gamma}\omega^r_\gamma(p_f) \\
= & \; Tr\left[\hat{\Gamma}_1\,\frac{p_{i\mu}\gamma^\mu+m_0}{2m_0}\,\hat{\overline{\Gamma}}_2\,\frac{p_{f\nu}\gamma^\nu+m_0}{2m_0}\right]
\end{aligned}
\qquad (D.7)
$$

$\square$

D.2 Proof of Equations (3.51) and (3.52)

$$
\Theta(t-t')\psi^{(-E)}(x') = -i\int d^3x S_F(x'-x)\gamma_0\psi^{(-E)}(x)
$$

We prove (3.51):

$$
\Theta(t'-t)\psi^{(+E)}(x') = i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x)
$$

Proof:

Any wave packet of positive energy may be expressed in terms of normalised plane waves

$$
\psi^{(+E)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b(p,r)\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x) \qquad (D.8)
$$

where $E_p = \sqrt{\mathbf{p}^2+m_0^2}$ and $\epsilon_1 = \epsilon_2 = +1$.. We will need to make use of the orthogonality condition

$$
\omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E_p}{m_0}\delta_{rr'}. \qquad (D.9)
$$

We start with the plane-wave representation of the Feynman propagator

$$S_F(x'-x) = -i\Theta(t'-t)\int d^3p \sum_{r=1}^{2} \psi_p^r(x')\overline{\psi}_p^r(x) + i\Theta(t-t')\int d^3p \sum_{r=3}^{4} \psi_p^r(x')\overline{\psi}_p^r(x) \quad \text{(D.10)}$$

where

$$\psi_p^r = \sqrt{\frac{m_0}{E_p}}\frac{1}{(2\pi)^{3/2}}\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x). \quad \text{(D.11)}$$

Inserting the above into the RHS of ()

$$\begin{aligned}
& i\int d^3x S_F(x'-x)\gamma_0\psi^{(+E)}(x) \\
= \;& \Theta(t'-t)\int d^3x\int d^3p\sum_{r=1}^{2}\psi_p^r(x')\overline{\psi}_p^r(x)\gamma_0\psi^{(+E)}(x) \\
& -\Theta(t'-t)\int d^3x\int d^3p\sum_{r=3}^{4}\psi_p^r(x')\overline{\psi}_p^r(x)\gamma_0\psi^{(+E)}(x) \\
= \;& \Theta(t'-t)\int d^3x\int\frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p)\gamma_0\exp[-i\epsilon_r p\cdot(x'-x)] \\
& \times\int\frac{d^3p'}{(2\pi)^3}\frac{m_0}{E_{p'}}\sum_{r=1}^{2}b(p',r')\omega^{r'}(p')\exp(-i\epsilon_{r'}p'\cdot x) \\
& -\Theta(t'-t)\int d^3x\int\frac{d^3p}{(2\pi)^3}\frac{m_0}{E_p}\sum_{r=3}^{4}\omega^r(p)\overline{\omega}^r(p)\gamma_0\exp[-i\epsilon_r p\cdot(x'-x)] \\
& \times\int\frac{d^3p'}{(2\pi)^3}\frac{m_0}{E_{p'}}\sum_{r=1}^{2}b(p',r')\omega^{r'}(p')\exp(-i\epsilon_r p'\cdot x) \\
= \;& \Theta(t'-t)\int\frac{d^3pd^3p'}{(2\pi)^3}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_p}}\sum_{r=1,2;r'=1,2}\omega^r(p)\omega^{r\dagger}(p)\omega^{r'}(p')b(p',r')\exp(-i\epsilon_{r'}p'\cdot x) \\
& \quad\times\int\frac{d^3x}{(2\pi)^3}\exp[i(\epsilon_r p-\epsilon_{r'}p')\cdot x] \\
- \;& \Theta(t'-t)\int\frac{d^3pd^3p'}{(2\pi)^3}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_{p'}}}\sum_{r=3,4;r'=1,2}\omega^r(p)\omega^{r\dagger}(p)\omega^{r'}(p')b(p',r')\exp(-i\epsilon_r p\cdot x') \\
& \quad\times\int\frac{d^3x}{(2\pi)^3}\exp[i(\epsilon_r p-\epsilon_{r'}p')\cdot x]
\end{aligned} \quad \text{(D.12)}$$

Performing the x integration in the Θ term yields

$$\exp[i(E_p - E_{p'})t]\delta^3(\mathbf{p} - \mathbf{p}') \to \delta^3(\mathbf{p} - \mathbf{p}') \tag{D.13}$$

Performing the x integration in the Θ term yields

$$\exp[-i(E_p + E_{p'})t]\delta^3(\mathbf{p} + \mathbf{p}') \to \exp(-2iE_p t)\delta^3(\mathbf{p} + \mathbf{p}'). \tag{D.14}$$

Integrating over $\mathbf{p}$ and relabelling $\mathbf{p}'$ as $\mathbf{p}$ we find

$$\begin{aligned}
& i\int d^3x S_F(x' - x)\gamma_0 \psi^{(+E)}(x) \\
= \; & \Theta(t' - t)\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2} \sum_{r=1,2;r'=1,2} \omega^r(\mathbf{p})\omega^{r\dagger}(\mathbf{p})\omega^{r'}(\mathbf{p})b(p,r')\exp(i\epsilon_r p\cdot x') \\
& -\Theta(t - t')\int \frac{d^3p}{(2\pi)^{3/2}}\left(\frac{m_0}{E_p}\right)^{3/2} \sum_{r=3,4;r'=1,2} \omega^r(-\mathbf{p})\omega^{r\dagger}(-\mathbf{p})\omega^{r'}(+\mathbf{p})b(p,r') \\
& \times \exp(i\epsilon_r p\cdot x')\exp(-2iE_p t)
\end{aligned} \tag{D.15}$$

Now we make use of the orthogonality relation. For $r, r' = 1, 2$

$$\omega^{r\dagger}(\mathbf{p})\omega^{r'}(\mathbf{p}) = \omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = \frac{E_p}{m_0}\delta_{rr'} \tag{D.16}$$

and for $r = 3, 4$ and $r' = 1, 2$,

$$\omega^{r\dagger}(-\mathbf{p})\omega^{r'}(\mathbf{p}) = \omega^{r\dagger}(\epsilon_r\mathbf{p})\omega^{r'}(\epsilon_{r'}\mathbf{p}) = 0 \tag{D.17}$$

The second term vanishes. The remaining term gives

$$\begin{aligned}
i\int d^3x S_F(x' - x)\gamma_0\psi^{(+E)}(x) &= \Theta(t' - t)\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b(p,r)\omega^r(\mathbf{p})\exp(-i\epsilon_r p\cdot x') \\
&= \Theta(t' - t)\psi^{(+E)}(x')
\end{aligned} \tag{D.18}$$

$\square$

Similar relations can be deduced the propagation of adjoint spinors $\overline{\psi}^{(+E)}(x), \overline{\psi}^{(-E)}(x)$:

$$\Theta(t-t')\overline{\psi}^{(+E)}(x') = i\int d^3x\overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \tag{D.19}$$

and

$$\Theta(t'-t)\overline{\psi}^{(-E)}(x') = -i\int d^3x\overline{\psi}^{(-E)}(x)\gamma_0 S_F(x'-x) \tag{D.20}$$

Proof:

Any adjoint wave packet of positive energy may be expressed in terms of normalised plane waves

$$\overline{\psi}^{(+E)}(x) = \int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b^*(p,r)\overline{\omega}^r(\mathbf{p})\exp(+ip\cdot x) \tag{D.21}$$

Consider the integral

$$\begin{aligned}
& i\int d^3x\overline{\psi}^{(+E)}(x)\gamma_0 S_F(x'-x) \\
= \quad & i\int d^3x \int \frac{d^3p'}{(2\pi)^{3/2}}\frac{m_0}{E_p}\sqrt{\frac{m_0}{E_{p'}}}\int \frac{d^3p}{(2\pi)^3}\sum_{r'=1}^{2}\overline{\omega}^{r'}(p')\exp(ip'\cdot x)\gamma_0 \\
& \times\Bigg\{-i\Theta(t-t')\sum_{r=1}^{2}\omega^r(p)\overline{\omega}^r(p)\ \exp[-ip\cdot(x-x')] \\
& \quad + i\Theta(t-t')\sum_{r=3}^{4}\omega^r(p)\overline{\omega}^r(p)\ \exp[+ip\cdot(x-x')]\Bigg\}
\end{aligned} \tag{D.22}$$

Again we do the x integration and use the orthogonality relations for spinors. We obtain

$$\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m_0}{E_p}}\sum_{r=1}^{2} b^*(p,r)\overline{\omega}(\mathbf{p})\exp(ip\cdot x)\Theta(t-t'). \tag{D.23}$$

This is just the expansion of the adjoint spinor $\overline{\psi}^{(+E)}$ multiplied by the step function $\Theta(t-t')$.

□

Appendix E

Feynman Parameterisation

Equation (12.8) generalises to

$$\frac{1}{a_1 a_2 \dots a_n} = \Gamma(n) \int_0^1 du_1 \cdots \int_0^1 du_n \frac{\delta(u_1 + \cdots + u_n - 1)}{[u_1 a_1 + u_2 a_2 + \cdots + u_n a_n]^n} \tag{E.1}$$

We can prove this by induction,

$$\begin{aligned}
&\frac{1}{a_1 \dots a_{m+1}} = \frac{1}{a_{m+1}} \frac{1}{a_1 \dots a_m} \\
= \ & \frac{1}{a_{m+1}} \Gamma(m) \int_0^1 du_1 \cdots \int_0^1 du_n \frac{\delta(u_1 + \cdots + u_m - 1)}{[u_1 a_1 + u_2 a_2 + \cdots + u_m a_m]^m} \\
= \ & \Gamma(m) \int_0^1 du_1 \cdots \int_0^1 du_m \delta(\sum_{i=1}^m u_i - 1) \frac{1}{a_{m+1}} \frac{1}{[\sum_{i=1}^m u_i a_i]^m} \\
= \ & \Gamma(m) \int_0^1 du_1 \cdots \int_0^1 du_m \delta(\sum_{i=1}^m u_i - 1) \int_0^1 \frac{dw_{m+1}(1 - w_{m+1})^{m-1}}{[a_{m+1} w_{m+1} + (\sum_{i=1}^m u_i a_i)(1 - w_{m+1})]^{m+1}}
\end{aligned} \tag{E.2}$$

where we have used (12.9). Now making the substitution

$$w_i := (1 - w_{m+1}) u_i, \qquad dw_i = (1 - w_{m+1}) du_i$$

and noting the property that $\delta(f(x)/a) = a\delta(f(x))$ then

$$
\begin{aligned}
&\frac{1}{a_1 \dots a_{m+1}} \\
= \;& \Gamma(m+1) \int_0^1 dw_{m+1} \int_0^1 du_1 \cdots \int_0^1 du_m \delta(\sum_{i=1}^m u_i - 1) \\
& \frac{(1-w_{m+1})^{m-1}}{[a_{m+1} w_{m+1} + (\sum_{i=1}^m u_i a_i)(1-w_{m+1})]^{m+1}} \\
= \;& \Gamma(m+1) \int_0^1 dw_{m+1} \int_0^{1-w_{m+1}} dw_1 \cdots \int_0^{1-w_{m+1}} dw_m \delta\left(\sum_{i=1}^m \frac{w_i}{1-w_{m+1}} - 1\right) \frac{1}{(1-w_{m+1})^m} \\
& \frac{(1-w_{m+1})^{m-1}}{[w_i a_1 + \cdots + w_m a_m + w_{m+1} a_{m+1}]^{m+1}} \\
= \;& \Gamma(m+1) \int_0^1 dw_{m+1} \int_0^{1-w_{m+1}} dw_1 \cdots \int_0^{1-w_{m+1}} dw_m \frac{\delta\left(\sum_{i=1}^{m+1} w_i - 1\right)}{\left[w_1 a_1 + \cdots + w_{m+1} a_{m+1}\right]^{m+1}}
\end{aligned}
\tag{E.3}
$$

Now because of the delta function the expression vanishes unless we have:

$$
1 - w_{m+1} = \sum_{i=1}^m w_i
$$

implying that the delta function vanishes for

$$
w_i > 1 - w_{m+1} \quad \text{for } i = 1, 2, \dots, m.
$$

As such we are able to extend the upper limit of integration in (E.3) from $1 - w_{m+1}$ to 1, so that

$$
\frac{1}{a_1 \dots a_{m+1}} = \Gamma(m+1) \int_0^1 dw_1 \cdots dw_{m+1} \frac{\delta(\sum_{i=1}^{m+1} w_i - 1)}{\left[w_1 a_1 + \cdots + w_{m+1} a_{m+1}\right]^{m+1}} \tag{E.4}
$$

By repeated differentiation of (E.1), we obtain the more general result

$$
\frac{1}{a_1^{m_1} a_2^{m_2} \dots a_n^{m_n}} = \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1) \dots \Gamma(m_n)} \int_0^1 du_1 \cdots \int_0^1 du_n \delta(\sum_{i=1}^n u_i - 1) \frac{u_1^{m_1 - 1} \dots u_n^{m_n - 1}}{[\sum_{i=1}^n u_i a_i]^{\sum m_i}} \tag{E.5}
$$

This formula is true even when the m_i's are not integers.

E.0.1 Alternative forumla

We can generalise the original form

$$\frac{1}{a_1 a_2} = \int_0^1 \frac{du_1}{[a_1 + (a_2 - a_1)u_1]^2} \tag{E.6}$$

generalises to three factors,

$$\frac{1}{a_1 a_2 a_3} = 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{[a_1 + (a_2 - a_1)u_1 + (a_3 - a_2)u_2]^3}. \tag{E.7}$$

We prove this by integrating with respect to u_2,

$$\begin{aligned}
& 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{[a_1 + (a_2 - a_1)u_1 + (a_3 - a_2)u_2]^3} \\
= \;& 2\int_0^1 du_1 \int_0^{u_1} du_2 \frac{1}{(a_3 - a_2)^3 \left[\frac{a_1}{a_3 - a_2} + \frac{(a_2 - a_1)}{a_3 - a_2}u_1 + u_2\right]^3} \\
= \;& 2\int_0^1 du_1 \frac{1}{(a_3 - a_2)^3} \left[-\frac{1}{2}\frac{1}{\left[\frac{a_1}{a_3 - a_2} + \frac{(a_2 - a_1)}{a_3 - a_2}u_1 + u_2\right]^2}\right]_0^{u_1} \\
= \;& \frac{1}{a_3 - a_2}\int_0^1 du_1 \left(\frac{1}{[a_1 + (a_2 - a_1)u_1]^2} - \frac{1}{[a_1 + (a_2 - a_1)u_1 + (a_3 - a_2)u_1]^2}\right) \\
= \;& \frac{1}{a_3 - a_2}\left(\frac{1}{a_1 a_2} - \frac{1}{a_1 a_3}\right) \\
= \;& \frac{1}{a_1 a_2 a_3}
\end{aligned} \tag{E.8}$$

where we used (E.6). This generalises to an arbitrary number of factors

$$\begin{aligned}
\frac{1}{a_1 a_2 \dots a_n} \; &= \; \Gamma(n)\int_0^1 du_1 \int_0^{u_1} du_2 \cdots \int_0^{u_{n-2}} du_{n-1} \\
& \qquad \times \frac{1}{[a_1 + (a_2 - a_1)u_1 + \cdots + (a_n - a_{n-1})u_{n-1}]^n}
\end{aligned} \tag{E.9}$$

We prove this by induction. First we integrate with respect to u_{n-1}

$$
\begin{aligned}
& \Gamma(m+1)\int_0^1 du_1\int_0^{u_1} du_2\cdots\int_0^{u_{m-1}} du_m \frac{1}{[a_1+(a_2-a_1)u_1+\cdots+(a_{m+1}-a_m)u_m]^{m+1}} \\
= \;& \frac{\Gamma(m+1)}{(a_{m+1}-a_m)^{m+1}}\int_0^1 du_1\int_0^{u_1} du_2\cdots\int_0^{u_{m-1}} du_m \frac{1}{\left[\frac{a_1}{a_{m+1}-a_m}+\frac{(a_2-a_1)}{a_{m+1}-a_m}u_1+\cdots+u_m\right]^{m+1}} \\
= \;& \frac{\Gamma(m+1)}{(a_{m+1}-a_m)^{m+1}}\int_0^1 du_1\int_0^{u_1} du_2\cdots\int_0^{u_{m-2}} du_{m-1} \\
& \quad \times\left[-\frac{1}{m}\frac{1}{\left[\frac{a_1}{a_{m+1}-a_m}+\frac{(a_2-a_1)}{a_{m+1}-a_m}u_1+\cdots+\frac{a_m-a_{m-1}}{a_{m+1}-a_m}u_{m-1}+u_m\right]^{m}}\right]_0^{u_{m-1}} \\
= \;& \frac{\Gamma(m+1)}{m(a_{m+1}-a_m)}\int_0^1 du_1\int_0^{u_1} du_2\cdots\int_0^{u_{m-2}} du_{m-1} \\
& \quad \times\left(\frac{1}{[a_1+(a_2-a_1)u_1+\cdots+(a_m-a_{m-1})u_{m-1}]^m}\right. \\
& \qquad \left. -\frac{1}{[a_1+(a_2-a_1)u_1+\cdots+(a_{m+1}-a_{m-1})u_{m-1}]^m}\right) \\
= \;& \frac{1}{a_{m+1}-a_m}\left(\frac{1}{a_1\ldots a_{m-1}a_m}-\frac{1}{a_1\ldots a_{m-1}a_{m+1}}\right) \\
= \;& \frac{1}{a_1a_2\ldots a_{m+1}}
\end{aligned} \tag{E.10}
$$

where we have used (E.9) with $n=m$.

By repeated differentiation of (E.9), we obtain the more general result

$$
\begin{aligned}
\frac{1}{a_1^{m_1}a_2^{m_2}\ldots a_n^{m_n}} = \;& \frac{\Gamma(m_1+\cdots+m_n)}{\Gamma(m_1)\ldots\Gamma(m_n)}\int_0^1 du_1\int_0^{u_1} du_2\cdots\int_0^{u_{n-2}} du_{n-1} \\
& \times\frac{(1-u_1)^{m_1-1}\cdots(u_{n-2}-u_{n-1})^{m_{n-2}-1}u_{n-1}^{m_{n-1}-1}}{[a_1(1-u_1)+\cdots+a_{n-1}(u_{n-2}-u_{n-1})+a_nu_{n-1}]^{\sum_i m_i}}
\end{aligned} \tag{E.11}
$$

Appendix F

$d-$Dimensional Integration

polar coordinates in two dimensions

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1 \end{aligned} \tag{F.1}$$

In three dimensions, we use the transformation to spherical coordinates

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_2 &= r\sin\theta_1\sin\theta_2 \end{aligned} \tag{F.2}$$

The $d-$dimensional transformation is

$$\begin{aligned} x_1 &= r\cos\theta_1 \\ x_2 &= r\sin\theta_1\cos\theta_2 \\ x_3 &= r\sin\theta_1\sin\theta_2\cos\theta_3 \\ &\vdots \\ x_{d-2} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\cos\theta_{d-1} \\ x_{d-1} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{d-2}\sin\theta_{d-1} \end{aligned} \tag{F.3}$$

We calculate the Jacobian row by row

$$
\begin{aligned}
J &= \left(\frac{\partial(x_1, x_2, \cdots, x_d)}{\partial(r, \theta_1, \theta_2, \cdot, \theta_{d-1})}\right) \\
&= \begin{pmatrix} \cos\theta_1 & -r\sin\theta_1 & 0 & \cdots \\ \sin\theta_1\cos\theta_2 & r\cos\theta_1\cos\theta_2 & -r\sin\theta_1\sin\theta_2 & \cdots \\ \sin\theta_1\sin\theta_2\cos\theta_3 & r\cos\theta_1\sin\theta_2\cos\theta_3 & r\sin\theta_1\cos\theta_2\cos\theta_3\cdots & \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\
&= r^{d-1}\cos\theta_1 \times \cos\theta_1 \det\begin{pmatrix} \cos\theta_2 & -\sin\theta_1\sin\theta_2 & \cdots \\ \sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_2\cos\theta_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
&+ r^{d-1}\sin\theta_1 \times \sin\theta_1 \det\begin{pmatrix} \cos\theta_2 & -\sin\theta_1\sin\theta_2 & \cdots \\ \sin\theta_2\cos\theta_3 & \sin\theta_1\cos\theta_2\cos\theta_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
&= r^{d-1}[\cos^2\theta_1(\sin\theta_1)^{d-2} + \sin^2\theta_1(\sin\theta_1)^{d-2}] \\
&\quad \det\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & \cdots \\ \sin\theta_2\cos\theta_3 & \cos\theta_2\cos\theta_3 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \\
&= (\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\dots(\sin\theta_{d-3})^2 \\
&\quad \det\begin{pmatrix} \cos\theta_{d-2} & -\sin\theta_{d-2} & 0 \\ \sin\theta_{d-2}\cos\theta_{d-1} & \cos\theta_{d-2}\cos\theta_{d-1} & -\sin\theta_{d-2}\sin\theta_{d-1} \\ \sin\theta_{d-2}\sin\theta_{d-1} & \cos\theta_{d-2}\sin\theta_{d-1} & \sin\theta_{d-2}\cos\theta_{d-1} \end{pmatrix} \\
&= (\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\dots(\sin\theta_{d-3})^2 \det\begin{pmatrix} \cos\theta_{d-1} & -\sin\theta_{d-1} \\ \sin\theta_{d-1} & \cos\theta_{d-1} \end{pmatrix} \\
&= r^{d-1}(\sin\theta_1)^{d-2}(\sin\theta_2)^{d-3}\dots(\sin\theta_{d-2}) \qquad \text{(F.4)}
\end{aligned}
$$

The Gaussian integral is then

$$
\int_0^\infty e^{-\vec{x}^2} d^d x = \int_0^\infty x^{d-1} e^{-x^2} dx \int_0^\pi (\sin\theta_1)^{d-1} d\theta_1 \dots \int_0^\pi \sin\theta_{d-2} d\theta_{d-2} \int_0^\pi d\theta_{d-1}. \qquad \text{(F.5)}
$$

To evaluate the integrals we employ integration by parts,

$$
\begin{aligned}
I_n &= \int_0^\pi (\sin\theta)^{n-1} d\theta \\
&= [\sin\theta)^{n-1}(-\cos\theta)]_0^\pi - (n-1)\int_0^\pi (\sin\theta)^{n-2}\cos\theta(-\cos\theta)d\theta \\
&= (n-1)\int_0^\pi (\sin\theta)^{n-2}(1-\sin^2\theta)d\theta \\
&= (n-1)\int_0^\pi \left[(\sin\theta)^{n-2} - (\sin\theta)^n\right] d\theta \\
&= (n-1)(I_{n-2} - I_n)
\end{aligned}
\tag{F.6}
$$

or

$$
\int_0^\pi (\sin\theta)^n d\theta = \frac{n-1}{n}\int_0^\pi (\sin\theta)^{n-2} d\theta.
$$

For even $n \geq 2$ we thus get

$$
\int_0^\pi (\sin\theta)^n d\theta = \frac{(n-1)!!}{n!!}\int_0^\pi = \frac{(n-1)!!}{n!!}\pi
\tag{F.7}
$$

and for $n = 0$ we have

$$
\int_0^\pi d\theta = \pi.
\tag{F.8}
$$

For odd n we find

$$
\begin{aligned}
\int_0^\pi (\sin\theta)^n d\theta &= \frac{(n-1)!!}{n!!}\int_0^\pi \sin\theta d\theta \\
&= \frac{(n-1)!!}{n!!} \times 2.
\end{aligned}
\tag{F.9}
$$

We have

$$
\begin{aligned}
\int_0^\infty e^{-\vec{x}^2} d^d x = \int_0^\infty dx x^{d-1} e^{-x^2} I_d \frac{1}{(d-2)!!} \\
I_d = \begin{cases} (2\pi)^{\frac{d}{2}} & d\ even \\ 2 \times (2\pi)^{\frac{d-1}{2}} & d\ odd \end{cases}
\end{aligned}
\tag{F.10}
$$

On the other hand, integration by parts gives

$$\begin{aligned}
\int_0^\infty x^{d-1}e^{-x^2}dx &= -\frac{1}{2}\int_0^\infty x^{d-2}(-2xe^{-x^2})dx \\
&= -\frac{1}{2}\left[x^{d-2}e^{-x^2}\right]_0^\infty + \frac{d-2}{2}\int_0^\infty x^{d-3}e^{-x^2}dx \\
&= \frac{d-2}{2}\int_0^\infty x^{d-3}e^{-x^2}dx \\
&= (d-2)!!J_d
\end{aligned} \tag{F.11}$$

where for d even

$$\begin{aligned}
J_d &= \left(\frac{1}{2}\right)^{\frac{d-2}{2}}\int_0^\infty xe^{-x^2}dx \\
&= \left(\frac{1}{2}\right)^{\frac{d}{2}}
\end{aligned} \tag{F.12}$$

and d odd

$$\begin{aligned}
J_d &= \left(\frac{1}{2}\right)^{\frac{d-2}{2}}\int_0^\infty e^{-x^2}dx \\
&= \left(\frac{1}{2}\right)^{\frac{d-1}{2}}\frac{\sqrt{\pi}}{2}.
\end{aligned} \tag{F.13}$$

Putting these together we have

$$\int_0^\infty e^{-\vec{x}^2}d^dx = \frac{1}{(d-2)!!}(d-2)!!I_dJ_d = \pi^{\frac{d}{2}}. \tag{F.14}$$

Appendix G

Dimensional Regularisation

G.1 Dimensional Regularised Feynman-integrals

$$\int d^d q \frac{1}{(q^2 + 2q \cdot p - m^2)^\alpha} = \frac{i\pi^{\frac{d}{2}} \Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)(-p^2 - m^2)^{\alpha - d/2}}. \tag{G.1}$$

$$\int d^d q \frac{q_\mu}{(q^2 + 2q \cdot p - m^2)^\alpha} = \frac{i\pi^{\frac{d}{2}} \Gamma\left(\alpha - \frac{d}{2}\right)}{\Gamma(\alpha)(-p^2 - m^2)^{\alpha - d/2}} (-p_\mu). \tag{G.2}$$

$$\begin{aligned} \int d^d q \frac{q_\mu q_\nu}{(q^2 + 2q \cdot p - m^2)^\alpha} &= \frac{i\pi^{\frac{d}{2}}}{(-p^2 - m^2)^{\alpha - d/2}} \frac{1}{\Gamma(\alpha)} \left[\Gamma\left(\alpha - \frac{d}{2}\right) p_\mu p_\nu \right. \\ &\quad \left. + \Gamma\left(\alpha - 1 - \frac{d}{2}\right) \frac{1}{2} \delta_{\mu\nu} (-p^2 - m^2)\right]. \end{aligned} \tag{G.3}$$

$$\begin{aligned} \int d^d q \frac{q^2}{(q^2 + 2q \cdot p - m^2)^\alpha} &= \frac{i\pi^{\frac{d}{2}}}{(-p^2 - m^2)^{\alpha - d/2}} \frac{1}{\Gamma(\alpha)} \left[\Gamma\left(\alpha - \frac{d}{2}\right) p^2 \right. \\ &\quad \left. + \Gamma\left(\alpha - 1 - \frac{d}{2}\right) \frac{d}{2} (-p^2 - m^2)\right]. \end{aligned} \tag{G.4}$$

$$\begin{aligned} \int d^d q \frac{q_\mu q_\nu q_\lambda}{(q^2 + 2q \cdot p - m^2)^\alpha} &= \frac{i\pi^{\frac{d}{2}}}{(-p^2 - m^2)^{\alpha - d/2}} \frac{1}{\Gamma(\alpha)} \left[-\Gamma\left(\alpha - \frac{d}{2}\right) p_\mu p_\nu p_\lambda \right. \\ &\quad \left. - \Gamma\left(\alpha - 1 - \frac{d}{2}\right) (-p^2 - m^2) \frac{1}{2} (\delta_{\mu\nu} p_\lambda + \delta_{\mu\lambda} p_\nu + \delta_{\nu\lambda} p_\mu)\right]. \end{aligned} \tag{G.5}$$

G.2 Gamma Function Expansion

The gamma function $\Gamma(z)$ is a function of a complex variable which, for positive integers k, has the property $\Gamma(k+1) = k!$. We define $\Gamma(z)$ for Re $z > 0$ by Euler's formula

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{G.6}$$

To find the behaviour near $d = 4$, define $\epsilon = 4 - d$. We will prove that

$$\Gamma(2 - \frac{d}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon), \tag{G.7}$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. Also that

$$\Gamma(1 - \frac{d}{2}) = \Gamma(\frac{\epsilon}{2} - 1) = -\frac{2}{\epsilon} - \gamma + 1 + \mathcal{O}(\epsilon). \tag{G.8}$$

These follow from the infinite product representation

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-z/k}. \tag{G.9}$$

Proof:

First we establish (G.9). We have

$$e^{-t} = \lim_{k\to\infty} \left(1 - \frac{t}{k}\right)^k$$

so that

$$\begin{aligned}
\Gamma(z) &= \lim_{k\to\infty} \int_0^k \left(1 - \frac{t}{k}\right)^k t^{z-1} dt \\
&= \lim_{k\to\infty} k^z \int_0^1 (1-\tau)^k \tau^{z-1} d\tau
\end{aligned} \tag{G.10}$$

where we used $t = k\tau$. For integral k, integrating by parts now gives

$$
\begin{aligned}
\int_0^1 (1-\tau)^k \, \tau^{z-1} d\tau &= \left[\frac{1}{z}\tau^z (1-\tau)^k\right]_0^1 + \frac{k}{z}\int_0^1 (1-\tau)^{k-1}\, \tau^z d\tau \\
&= \frac{k}{z}\left[\frac{1}{z+1}\tau^{z+1}(1-\tau)^{k-1}\right]_0^1 + \frac{k(k-1)}{z(z+1)}\int_0^1 (1-\tau)^{k-2}\, \tau^{z+1} d\tau \\
&= \cdots \\
&= \frac{k(k-1)\cdots(2)}{z(z+1)\cdots(z+k-2)}\left[\frac{1}{z+k-1}\tau^{z+k-1}(1-\tau)\right]_0^1 \\
&\qquad + \frac{k(k-1)\cdots(1)}{z(z+1)\cdots(z+k-1)}\int_0^1 \tau^{z+k-1} d\tau \\
&= \frac{k(k-1)\cdots(1)}{z(z+1)\cdots(z+k)} \qquad \text{(G.11)}
\end{aligned}
$$

So that

$$
\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{k\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{k}\right)k^{-z}\right] \\
&= \lim_{k\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{k}\right)e^{-z\ln k}\right] \qquad \text{(G.12)}
\end{aligned}
$$

Define

$$
\begin{aligned}
\gamma &:= \lim_{k\to\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{k}\right)-\ln k\right] \\
&= 0.5772\ldots \qquad \text{(G.13)}
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{1}{\Gamma(z)} &= \lim_{k\to\infty}\left[z\left(1+\frac{z}{1}\right)\left(1+\frac{z}{2}\right)\cdots\left(1+\frac{z}{k}\right)k^{-z}\right] \\
&= \lim_{k\to\infty}\left[z\left(1+\frac{z}{1}\right)e^{-z}\left(1+\frac{z}{2}\right)e^{-(1/2)z}\cdots\left(1+\frac{z}{k}\right)e^{-(1/k)z}e^{-[1+(1/2)+\cdots+(1/k)-\ln k]z}\right] \\
&= ze^{\gamma z}\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-(1/k)z} \qquad \text{(G.14)}
\end{aligned}
$$

Now we move onto establishing expansions of the gamma function.

Integration by parts of (G.6) establishes that

$$\Gamma(z+1) = z\Gamma(z). \tag{G.15}$$

Consider

$$\begin{aligned}
\Psi_1(z) &:= \frac{d}{dz}\ln[\Gamma(z)] \\
&= -\frac{d}{dz}\left[\ln z + \gamma z + \sum_{k=1}^{\infty}\left\{\ln\left(1+\frac{z}{k}\right) - \frac{z}{k}\right\}\right] \\
&= -\gamma - \frac{1}{z} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{k}{z+k}\right) \\
&= -\gamma - \frac{1}{z} + z\sum_{k=1}^{\infty}\frac{1}{k(z+k)}
\end{aligned} \tag{G.16}$$

We will need

$$\begin{aligned}
\Psi_1(1) &= -\gamma - 1 + \sum_{k=1}^{\infty}\frac{1}{k(k+1)} \\
&= -\gamma - 1 + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= -\gamma - 1 + (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + \ldots \\
&= -\gamma.
\end{aligned} \tag{G.17}$$

Now look at the Taylor-expansion of the $\Gamma-$function around the regular point $z = 1$:

$$\begin{aligned}
\Gamma(1+\epsilon/2) &= 1 + \frac{\epsilon}{2}\Gamma'(1) + \mathcal{O}(\epsilon^2) \\
&= 1 + \frac{\epsilon}{2}\frac{\Gamma'(1)}{\Gamma(1)} + \mathcal{O}(\epsilon^2) \\
&= 1 + \frac{\epsilon}{2}\Psi_1(1) + \mathcal{O}(\epsilon^2) \\
&= 1 - \frac{\epsilon}{2}\gamma + \mathcal{O}(\epsilon^2)
\end{aligned} \tag{G.18}$$

Then, using $z\Gamma(z) = \Gamma(z+1)$,

$$
\begin{aligned}
\Gamma(\epsilon/2) &= \frac{1}{\epsilon/2}\Gamma(1+\epsilon/2) \\
&= \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon).
\end{aligned} \tag{G.19}
$$

which establishes (G.7). We use this to prove (G.8):

$$
\begin{aligned}
\Gamma(\epsilon/2-1) &= \frac{\Gamma(\epsilon/2)}{\epsilon/2-1} \\
&= (-1)\left[1+\frac{\epsilon}{2}+\mathcal{O}(\epsilon^2)\right]\left[\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right] \\
&= -\frac{2}{\epsilon}+\gamma-1+\mathcal{O}(\epsilon).
\end{aligned} \tag{G.20}
$$

G.3 Dimensional Regularised Dirac Matrices

We think of the gamma matrices in dimensional regularisation as a set of d matrices obeying

$$
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1} \tag{G.21}
$$

where $g_{\mu\nu}g^{\mu\nu} = d$ and

$$
\mathrm{Tr}\mathbb{1} = 4. \tag{G.22}
$$

In particular we have the identities

$$
\begin{aligned}
\gamma_\mu\gamma^\nu\gamma^\mu &= -(2-\epsilon)\gamma^\nu \\
\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\mu &= 4g^{\nu\sigma}\mathbb{1} - \epsilon\gamma^\nu\gamma^\sigma \\
\gamma_\mu\gamma^\nu\gamma^\sigma\gamma^\gamma\gamma^\mu &= 2\gamma^\gamma\gamma^\sigma\gamma^\nu + \epsilon\gamma^\nu\gamma^\sigma\gamma^\gamma.
\end{aligned} \tag{G.23}
$$

Proof:

First note

$$
\begin{aligned}
\gamma_\mu \gamma^\mu &= \frac{1}{2}(\gamma_\mu \gamma^\mu + \gamma^\mu \gamma_\mu) = \frac{1}{2} 2 g_{\mu\nu} g^{\mu\nu} \mathbb{1} \\
&= d\mathbb{1} \\
&= (4-\epsilon)\mathbb{1}.
\end{aligned} \tag{G.24}
$$

Then

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\mu &= \gamma_\mu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) \\
&= 2\gamma^\nu - (4-\epsilon)\gamma^\nu \\
&= -(2-\epsilon)\gamma^\nu.
\end{aligned} \tag{G.25}
$$

Next

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\mu &= \gamma_\mu \gamma^\nu (2g^{\mu\sigma} - \gamma^\mu \gamma^\sigma) \\
&= 2\gamma^\sigma \gamma^\nu - \gamma_\mu \gamma^\nu \gamma^\mu \gamma^\sigma \\
&= 2\gamma^\sigma \gamma^\nu + (2-\epsilon)\gamma^\nu \gamma^\sigma \\
&= 2(2g^{\nu\sigma}\mathbb{1} - \gamma^\nu \gamma^\sigma) + (2-\epsilon)\gamma^\nu \gamma^\sigma \\
&= 4g^{\nu\sigma}\mathbb{1} - \epsilon \gamma^\nu \gamma^\sigma.
\end{aligned} \tag{G.26}
$$

Lastly

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\gamma \gamma^\mu &= \gamma_\mu \gamma^\nu \gamma^\sigma (2g^{\mu\gamma} - \gamma^\mu \gamma^\gamma) \\
&= 2\gamma^\gamma \gamma^\nu \gamma^\sigma - \gamma_\mu \gamma^\nu \gamma^\sigma \gamma^\mu \gamma^\gamma \\
&= 2\gamma^\gamma \gamma^\nu \gamma^\sigma - (4g^{\nu\sigma}\mathbb{1} - \epsilon \gamma^\nu \gamma^\sigma)\gamma^\gamma \\
&= 2\gamma^\gamma (2g^{\nu\sigma} - \gamma^\nu \gamma^\sigma) - 4g^{\nu\sigma}\gamma^\gamma + \epsilon \gamma^\nu \gamma^\sigma \gamma^\gamma \\
&= 2\gamma^\gamma \gamma^\sigma \gamma^\nu + \epsilon \gamma^\nu \gamma^\sigma \gamma^\gamma.
\end{aligned} \tag{G.27}
$$

Appendix H

Bessel Function Representations

H.1 Solutions of the Modified Bessel Equation

The differential equation for the modified Bessel function is

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0. \tag{H.1}$$

First consider the case where ν is an integer.

$$x^2y'' + xy' - (x^2 + n^2)y = 0. \tag{H.2}$$

The solution

$$I_n(z) = \left(\frac{z}{2}\right)^n \sum_{m=0}^{\infty} \frac{1}{m!(m+n)!} \left(\frac{z}{2}\right)^{2m} \tag{H.3}$$

For general values

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m} \tag{H.4}$$

We define the modified Bessel function of the second kind by

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}. \tag{H.5}$$

For integer vales of ν we use L'hopital's rule,

$$
\begin{aligned}
K_\nu(z) &= \frac{\pi}{2}\frac{\frac{\partial}{\partial\nu}I_{-\nu}(z)-\frac{\partial}{\partial\nu}I_\nu(z)}{\frac{\partial}{\partial\nu}\sin(\nu\pi)}\bigg|_{\nu=n} \\
&= \frac{1}{2\cos(n\pi)}\left(\frac{\partial}{\partial\nu}I_{-\nu}(z)-\frac{\partial}{\partial\nu}I_\nu(z)\right)\bigg|_{\nu=n}. \qquad \text{(H.6)}
\end{aligned}
$$

We wish to in particular obtain the expression for $K_0(z)$. Using $\frac{\partial}{\partial\nu}(z/2)^\nu = \frac{\partial}{\partial\nu}e^{\nu\ln(z/2)} = \ln(z/2)(z/2)^\nu$, we have

$$
\begin{aligned}
K_0(z) &= \frac{1}{2}\Bigg(-\ln\left(\frac{z}{2}\right)I_{-\nu}(z)-\ln\left(\frac{z}{2}\right)I_\nu(z) \\
&+ \left(\frac{z}{2}\right)^{-\nu}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\frac{\partial}{\partial\nu}\Gamma(m-\nu+1)}{[\Gamma(m-\nu+1)]^2}\left(\frac{z}{2}\right)^{2m}+\left(\frac{z}{2}\right)^{\nu}\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\frac{\partial}{\partial\nu}\Gamma(m+\nu+1)}{[\Gamma(m+\nu+1)]^2}\left(\frac{z}{2}\right)^{2m}\Bigg)\Bigg|_{\nu=0} \\
&= -\ln\left(\frac{z}{2}\right)I_0(z)+\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\psi(m+1)\left(\frac{z}{2}\right)^{2m} \qquad \text{(H.7)}
\end{aligned}
$$

so that

$$
K_0(z) = \sum_{m=0}^{\infty}(\psi(m+1)+\ln 2-\ln z)\frac{z^{2m}}{2^{2m}(m!)^2} \qquad \text{(H.8)}
$$

H.2 Intergal Represetaions

The modified Bessel function of the second kind has the integral representation

$$
K_0(z) = \int_0^\infty e^{-z\cosh t}dt. \qquad \text{(H.9)}
$$

It is easy to verify that it satisfies the modified Bessel equation,

$$
\begin{aligned}
x^2K_0(t)'' + xK_0(x)' - x^2K_0(x) &= \int_0^\infty e^{-x\cosh t}(x^2\cosh^2 t - x\cosh t - x^2)dt \\
&= \int_0^\infty e^{-x\cosh t}(x^2\sinh^2 t - x\frac{d}{dt}\sinh t)dt \\
&= \left[x\sinh t\, e^{-x\cosh t}\right]_0^\infty \\
&\quad + \int_0^\infty (e^{-x\cosh t}x^2\sinh^2 t + x\sinh t\frac{d}{dt}e^{-x\cosh t})dt \\
&= 0. \qquad \text{(H.10)}
\end{aligned}
$$

This is linearly independent of $I_0(z)$.

We will need for the Uehling potential

$$
\begin{aligned}
K_{i_2}(z) &= \int_0^\infty e^{-z\cosh t}\frac{1}{\cosh^2 t}dt \\
&= \int_z^\infty \left(\int_{z_2}^\infty \left(\int_0^\infty e^{-z_1\cosh t}dt\right)dz_1\right)dz_2 \qquad \text{(H.11)}
\end{aligned}
$$

and

$$
\begin{aligned}
K_{i_4}(z) &= \int_0^\infty e^{-z\cosh t}\frac{1}{\cosh^4 t}dt \\
&= \int_z^\infty \left(\int_{z_2}^\infty K_{i_2}(z_1)dz_1\right)dz_2. \qquad \text{(H.12)}
\end{aligned}
$$

H.3 Rigorous Derivation of Integral Representation

The Gamma function $\Gamma(z)$ is defined for all values of z by the contour integral

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C t^{-z}e^t dt. \qquad \text{(H.13)}$$

It is easy to prove the well known property of the Gamma function (i.e. $\Gamma(z) = (z-1)\Gamma(z-1)$) by considering integration by parts

$$
\begin{aligned}
\frac{1}{\Gamma(z)} &= \frac{1}{2\pi i}\left(\left[\frac{-e^t}{(z-1)t^{z-1}}\right]_{-\infty-i\epsilon}^{-\infty+i\epsilon} + \frac{1}{z-1}\int_C t^{-z-1}e^t dt\right) \\
&= \frac{1}{z-1}\frac{1}{\Gamma(z-1)}. \qquad \text{(H.14)}
\end{aligned}
$$

Employing (H.13) in (H.4)

$$
\begin{aligned}
I_\nu(z) &= \frac{(z/2)^\nu}{2\pi i}\sum_{m=0}^{\infty}\frac{1}{m!}\left(\int_C t^{-(m+\nu+1)}e^t dt\right)\left(\frac{z}{2}\right)^{2m} \\
&= \frac{(z/2)^\nu}{2\pi i}\int_C t^{-\nu-1}e^t\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{z^2}{4t}\right)^m dt \\
&= \frac{(z/2)^\nu}{2\pi i}\int_C t^{-\nu-1}e^{t+z^2/4t}dt. \qquad \text{(H.15)}
\end{aligned}
$$

Make the change of variables $u = e^w$

$$
I_\nu(z) = \frac{1}{2\pi i}\int_{C'} e^{z\cosh w-\nu w}dw \qquad \text{(H.16)}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_0^\pi e^{z\cosh(i\theta)-\nu i\theta}id\theta \\
&= \frac{1}{2\pi}\int_0^\pi e^{z\cos\theta}[\cos(\nu\theta) - i\sin(\nu\theta)]d\theta \qquad \text{(H.17)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_\pi^0 e^{z\cosh(i\theta)-\nu i\theta}i(-d\theta) \\
&= \frac{1}{2\pi i}\int_0^\pi e^{z\cosh(-i\theta)+\nu i\theta}d\theta \\
&= \frac{1}{2\pi}\int_0^\pi e^{z\cos\theta}[\cos(\nu\theta) + i\sin(\nu\theta)]d\theta. \qquad \text{(H.18)}
\end{aligned}
$$

Adding these two integrals

$$
\frac{1}{2\pi}\int_0^\pi e^{z\cos\theta}\cos(\nu\theta) \qquad \text{(H.19)}
$$

$w = t \pm i\pi$

Using

$$\begin{aligned} \cosh(t \pm i\pi) &= \cosh(t)\cosh(\pm i\pi) + \sinh(t)\sinh(\pm i\pi) \\ &= -\cosh(t) \end{aligned} \tag{H.20}$$

Then

$$I_\nu(z) = \frac{1}{\pi}\int_0^\pi e^{z\cos(t)}\cos(\nu t)dt - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh(t)-\nu t}dt. \tag{H.21}$$

From the definition (H.5)

$$\begin{aligned} K_\nu(z) &= \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)} \\ &= \frac{1}{2}\left(\int_0^\infty e^{-z\cosh(t)+\nu t}dt + \int_0^\infty e^{-z\cosh(t)+\nu t}dt\right) \\ &= \int_0^\infty e^{-z\cosh(t)}\cosh(\nu t)dt \end{aligned} \tag{H.22}$$